PRICING AND ASSORTMENT SELECTION WITH DEMAND UNCERTAINTY

by

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To my family,
With love

To my husband, Raúl E. Báez-Toro,
For his support and sacrifice

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Chapter I

Introduction

Many firms make pricing and assortment decisions on an ongoing basis. Furthermore, such decisions are often made in the face of demand uncertainty and must take into account associated inventory costs. The assortment, prices and inventory decisions interact in ways that should not be overlooked when making decisions. However, such interactions are often ignored both in practice and research. This dissertation studies a series of problems, in which there is much to gain by making assortment, pricing and inventory decisions simultaneously.

The growing importance of assortment planning is evidenced by the increasing number of retail chains who have adopted sophisticated solutions that perform assortment planning at the individual store level. In a survey conducted in 2005, 64% of retail chains reported that they either adopted, or would adopt within two years, state-of the-art assortment planning solutions.[1] One retail chain that did so is Stop & Shop, which announced in 2008 that it was expanding its use of an assortment planning software (IRI Loyalty Analytics Assortment Planner).[2] Another example is Bakers Footwear, which invested in Marketmax Assortment Planning software.[3]

[1] "An RIS White Paper: Driving Profitability through Assortment Optimization," last retrieved from http://www.businessobjects.com/pdf/solutions/retail/wp_profitibility_through_assortment_optimization.pdf on 10/7/08.

[2] "Stop & Shop Expands Customer-Centric Assortment Planning Using IRI Loyalty Analytics," May 2008, *Reuters*, last retrieved from http://www.reuters.com/article/pressRelease/idUS163109+13-May-2008+BW20080513 on 11/13/09.

[3] "One step ahead of the competition," last retrieved from http://www.sas.com/success/bakers.html on 11/12/09.

It appears that price optimization software has gained as much traction among retailers as assortment planning software. For example, in 2008, Super S Foods, and operator of grocery stores, selected KSS Retail as its provider of price optimization software, after stating that retailers that have used this software consistently show positive results in terms of increasing their market competitiveness.[4]

As retailers are becoming more sophisticated in their assortment planning and pricing, the manufacturers are following suit. For example, SignalDemand, a recent entry into the pricing software market, now boasts food manufacturers such as Cargill, Hormel Foods, and Chiquita among its customers. Cargill uses SignalDemand's product to price many different cuts of meat in response to popularity and production capacity.[5] Another example is Seagate Technologies, a hard drive supplier, who in 2008, joined the growing list of companies using price optimization software. Seagate's Director of Global Pricing, argues that companies that implement price optimization software typically gain a minimum of a 2-percent improvement in their margins. Interestingly, the software Seagate is implementing takes into account operational considerations by identifying the cost to serve particular customers which includes freight, fuel surcharges and logistics support.[6]

Given the growing interest in assortment and pricing software among retailers and manufacturers, it is not surprising that operations management researchers have become increasingly interested in assortment planning and pricing problems in the presence of operational costs. In this dissertation we contribute to the stream of operations management literature that studies assortment planning and pricing in the

[4] Super S Foods selects KSS Retail to provide predictive pricing using price and promotion optimization," July 2008, *Business Wire*, last retrieved from `http://findarticles.com/p/articles/mi_m0EIN/is_2008_July_1/ai_n27876436/?tag=content;col1` on 11/21/09.

[5] Daniel Bogoslaw, "Food Companies: Recipes for Tough Times," Business Week, 7/18/08, last retrieved from `http://www.businessweek.com/investor/content/jul2008/pi20080717_554529.htm` on 10/7/08.

[6] Doug Henschen, "Price Optimization Software Helps Seagate Boost Profits," *Intelligent Enterprise*, July 2009, last retrieved from `http://intelligent-enterprise.informationweek.com/showArticle.jhtml;jsessionid=GTQ4ULLVF3KEXQE1GHOSKH4ATMY32JVN?articleID=218501352` on 11/12/09.

presence of operational costs. In particular, the dissertation addresses two important practical considerations that have received little attention so far: (1) the presence of complementary products whose assortments and prices influence each other's demand, and (2) a manufacturer's use of dual sales channels that sell overlapping assortments.

1.1. Research Questions

We next describe the research questions, addressed by this dissertation, pertaining to the two important practical considerations discussed above.

1.1.1 Single firm with complementary categories of products

Joint assortment and pricing decisions in the presence of inventory considerations have recently received attention in the operations literature, e.g. Aydın and Porteus (2008) and Maddah and Bish (2007). However, this work addresses the pricing and inventory decisions for a single-category assortment. The coordination of multiple-category assortments has been typically addressed as a marketing problem, where it is common to find this type of work without inventory considerations. See Russell et al. (1997) for a review. In this dissertation, we address a problem where a single firm must choose the assortments of two distinct components that go into a final product configured by the customer, which can be thought of as a two-category problem with complementary categories. Together with the assortment decision, the firm must also decide on the prices in the presence of stochastic demand. The questions we address in this setting are: (1) How do several characteristics of the variants (e.g., customer appeal, unit purchase cost, unit holding and shortage costs and demand variability) combine to determine which variants should be included in the assortment? (2) Can we develop practical pricing rules to capture the optimal pricing behavior? (3) What effects do inventory-related parameters have on the optimal assortment and prices? (4) What influence does the marketing strategy for a component (in particular, whether or not the component is marketed as an impulse purchase) has on the optimal assortment and pricing decision?

1.1.2 Two-echelon, dual-channel supply chain with a single category

In certain supply chains, the manufacturer sells not only through a retailer, but also through a direct sales channel, which makes the manufacturer both a supplier to and a competitor of the retailer. This is what we refer to as a two-echelon, dual-channel supply chain. In any supply chain, the assortment and pricing decisions are crucial not only for the retailer, but also for its supplier. These decisions require even more scrutiny in a two-echelon, dual-channel supply chain, given the complicated nature of the relationship between the manufacturer and the retailer.

There is a growing amount of work in the economics, marketing, and operations literatures that addresses the pricing strategies of the manufacturer and the retailer in a two-echelon, dual-channel supply chain. The overwhelming majority of this work, especially in the operations literature, focuses on a supply chain that sells a single item. This dissertation contributes to this stream of research by addressing the assortment and pricing decisions that arise when the two channels are selling an overlapping assortment of substitutable products.

This dissertation addresses the following research questions in the context of a two-echelon, dual-channel supply chain: (1) What practical pricing rules, if any, characterize the optimal retail and wholesale pricing behavior? (2) What types of conflicts, if any, may arise between the manufacturer and the retailer when it comes to the assortment offered through the retailer? (3) What are the benefits, if any, for the manufacturer to use a dual-channel strategy? (4) What role do inventory-related parameters play in the optimal assortment and pricing?

1.2. Organization of the Dissertation

The rest of the dissertation is organized into four chapters as follows.

Chapter II analyzes the pricing and assortment problem with inventory considerations for a single firm selling a configurable product (e.g. a computer). This product is a combination of two components, one of which is a required component in that it must be bought for the product to function (e.g. hard drive), and the other is an optional component that the customer may wish to add to the product (e.g. speakers). The firm must decide the assortment of variants to be offered for each component as well as the variants' prices.

The chapter finds that, in the optimal solution, the prices of the required component's variants must be such that all variants will share the same effective profit margin, which is the unit gross margin net of inventory-related costs. On the other hand, all variants of the optional component will be sold at effective cost to drive store traffic. As for the assortment selection, the chapter provides a specific measure of a variant's profitability, labeled surplus, which brings together many supply-side and demand-side parameters. The surpluses of the variants that go into a product configuration combine together to determine the attractiveness of the configuration. The chapter shows how a firm can choose between two variants by comparing the attractiveness of the product configurations enabled by those variants.

Chapter III extends Chapter II in three ways: First, it explores the pricing and assortment decisions in a scenario where only the required component drives store traffic and the optional component is an impulse purchase. Here, in contrast to the model in Chapter II, the optional component will now be a profit driver for the retailer, which highlights the importance that the marketing strategy has on the pricing decision. Second, this chapter presents a model that considers two customers segments, one of which is interested in buying only the optional component, perhaps as an add-on for a product they bought earlier. In this case, the chapter identifies a price-discrimination strategy that the firm can use to improve its profit and that can be implemented through the use of discounts. Third, a generalization of the model in Chapter II is presented where the two components are still complementary but each can function without the other component. The chapter shows that, under this generalization, it remains optimal to charge the same effective margin for all the variants of a given component.

Chapter IV presents the pricing and assortment problems in a two-echelon, dual-channel supply chain. In this chapter's model, a manufacturer sells through its own direct channel as well as through a retailer. The chapter shows that the manufacturer's wholesale prices must be such that all variants have the same 'weighted' wholesale price, where the weight is a measure of the retailer's service level and the variant's demand variability. One managerial implication of this result is that variants with larger demand variability will carry lower wholesale price tags. As for the assortment decision, the chapter studies the assortment that will be offered through the retailer under various scenarios, which differ in who chooses the retailer's assortment (the manufacturer or the retailer itself) and the timing of the assortment decision relative to pricing decisions. The chapter characterizes the scenarios that lead to a conflict between the manufacturer and the retailer in terms of the assortment to be offered through the retailer. For example, the chapter finds that, in certain cases, the retailer may want to offer a variant with low demand variability while the manufacturer prefers an item with higher demand variability. In addition, the manufacturer prefers an assortment that is larger than the one preferred by the retailer.

Chapter V concludes and summarizes the contributions of the dissertation. The main contributions are: (1) modeling and analyzing assortment planning and pricing problems for two complementary items in the presence of inventory considerations, (2) characterizing the effect of inventory-related costs on the optimal pricing policy for two complementary assortments, (3) obtaining a practical measure of a variant's profitability, which can guide assortment selection for complementary items, (4) highlighting the pricing implications of marketing an item as an impulse purchase, (5) characterizing the optimal wholesale pricing policy in a two-echelon, dual-channel supply chain, where the manufacturer is both a supplier to and competitor of the retailer, and (6) identifying conflicts that may arise between the manufacturer and the retailer with regard to the assortment offered through the retailer.

Chapter II

Assortment Selection and Pricing for Configurable Products

2.1. Introduction

Consider a configurable product, formed by putting together several components, each of which gives the customer several choices as to color, size, performance, etc. Many different products, ranging from modular furniture to cars, fit this general description, but perhaps the prototypical example is the personal computer: A customer configures a computer by choosing from several processor speeds, memory sizes, monitor types, etc. This chapter explores the assortment and pricing decisions for such configurable products in the presence of demand uncertainty.

A firm offering a configurable product must decide what level of variety to offer for each component. For example, a computer assembler does not necessarily carry the entire array of processors in the market or offer all possible variants of monitors. Of course, one component's assortment influences another component's demand. For instance, if a computer assembler shrinks its assortment of processors, the demand for monitors may decrease, because some customers who do not find their ideal processor may choose to shop elsewhere. The model presented in this chapter recognizes such complementarity among component demands.

For most configurable products, the price of a configuration is determined by the components that go into it, each of which is individually priced. For example, this is typically the case when ordering a personal computer. Accordingly, in the model presented, the firm chooses the price of each choice offered for every component. These prices then add up to determine the price of a configuration.

One can draw an important distinction between two types of components that go into a configurable product: required versus optional components. Some components are required in that the product could not function without them, for example, a computer's processor. Others are optional in that the product could still be complete without them, for example, an external speaker. A customer who decides to make a purchase will necessarily buy the required components, but may choose not to purchase the optional components. The model includes these two distinct types of components.

The firm's assortment and pricing decisions play out in an environment where inventory considerations are important. Returning to the personal computer example, the possibility of a stockout is very real, especially when the demand for computers hits seasonal highs (for example, during the back-to-school period). On the flipside, a firm also incurs holding costs for on-hand inventory, e.g. due to obsolescence risk in the case of personal computers. Hence, the possibility of overage and underage needs to be taken into account when making the assortment and pricing decisions. The model incorporates inventory-related costs due to overage and underage.

The model's ability to incorporate inventory-related costs produces a number of useful managerial insights. For example, we find that the optimal prices are such that all variants of a component share the same *effective profit margin*, which is defined as the selling price net of unit purchase cost and unit inventory cost, where the unit inventory cost itself is a function of a variant's unit underage and overage cost, service level and demand variability. This result offers support for certain pricing practices such as higher quality variants typically carrying higher gross margins (that is, unit selling price minus unit purchase cost) or out-of-ordinary colors of a component carrying higher price tags. The former practice makes sense, because using the same effective margin for all variants would result in higher gross margins for high quality variants, assuming that such variants have higher overage costs. The latter practice makes sense, because, even if the color makes no discernible difference in terms of the firm's purchase cost or the customer's willingness to pay, out-of-ordinary colors would still command higher prices if their demand variability and, hence, their inventory costs, are larger than the standard colors.

The results about assortment selection underscore the importance of the variant's *surplus*, which is a measure of the variant's profitability that depends not only on the customer's utility from the variant, but also on the variant's unit purchase cost, unit underage and overage costs, service level and demand variability. The results reveal that when choosing from two variants of a given component, the firm should pick the one with the higher surplus. However, when choosing from two variants that belong to different components, it is not enough to compare the variants' surpluses. In such a case, one must take into account the complementarity between the demands of two components. In particular, one could choose from two variants that belong to different components by comparing the attractiveness of the new configurations that are enabled by the addition of each variant. To that end, the results provide a precise definition for the attractiveness of a configuration, which draws upon the variants' surpluses.

This chapter is organized as follows. Next section contains a review of the related literature. The model is introduced in Section 2.3 and Section 2.4 provides its analysis. All of the results in Section 2.4 are obtained under the assumption that the demand is a multiplicative random perturbation of the expected demand. Section 2.5 provides a discussion on how the results change when the randomness is additive. Finally, Section 2.6 summarizes the managerial implications and contributions. The proofs are included in Appendix A.

2.2. Literature Review

In its broadest sense the assortment problem is choosing what subset of a given set of products to offer. Pentico (2008) reviews the body of work on the assortment problem in this broad sense. This chapter is related to the subset of this research that deals with assortment planning in retail operations. Such problems have been studied in the economics, marketing and operations literature. For a recent review, see Kök, Fisher and Vaidyanathan (2008). Here we focus on the stream of the operations literature that studies the tradeoff between the higher revenues achieved by larger assortments and the inventory burden of offering such broad assortments. The inclusion of inventory-related costs is what separates this body of work from that in economics and marketing. van Ryzin and Mahajan (1999) and Smith and Agrawal (2000) seem to be among the first to consider such problems. When modeling how the customer demand is split among several alternatives, van Ryzin and Mahajan (1999) use the multinomial logit (MNL) choice model while Smith and Agrawal (2000) use an exogenously-specified substitution pattern. Subsequently, both types of demand models have been utilized to further analyze assortment planning in the presence of inventory considerations. Using the MNL model, Li (2007) extends van Ryzin and Mahajan (1999) by allowing the price and cost parameters to differ across variants. Using an exogenously-specified substitution pattern, Kök and Fisher (2007) propose a methodology to estimate the substitution parameters from sales data and they develop a heuristic method to select the assortment and to set the inventory levels in the presence of shelf space constraints. Yücel, Karaesmen, Salman and Türkay (2009) also use an exogenously specified substitution pattern, and they study supplier selection and assortment planning in the presence of shelf space limitations. Departing from both the MNL choice model and exogenously-specified substitution patterns, Gaur and Honhon (2006) study the retail assortment planning problem using a locational choice model. Rajaram (2001), on the other hand, analyzes a catalog retailer's assortment planning problem, in which the consumer choice is of secondary importance in that the demand for an item does not depend on what other items are offered in the assortment. This chapter is in the spirit of van Ryzin and Mahajan (1999) and Li (2007) in that it also uses the MNL model. However, this chapter's work differs in two important ways from the earlier work cited so far. First, the work cited so far studies assortment planning in the presence of exogenously fixed prices whereas this chapter studies a joint pricing and assortment selection problem. In addition, all of the earlier work cited so far study single-category assortment planning, while this chapter addresses a problem where the firm must choose the assortments of two distinct components with complementary demands.

Other recent work that studies joint inventory and pricing decisions for assortment planning include Maddah and Bish (2007) and Aydın and Porteus (2008). The latter studies inventory and pricing decisions for a given assortment, whereas the former studies joint inventory, assortment and pricing decisions. Both Aydın and Porteus (2008) and Maddah and Bish (2007) study single-category assortment planning, while here the focus is on assortment planning for two components with complementary demands.

The two-component assortment planning problem in this chapter is akin to multiple category assortment planning, studied in the marketing literature. See Russell et al. (1997) for a review. Assortment planning for complementary products has received scant attention in operations management. One exception is Agrawal and Smith (2003) who deal with such a problem in the absence of pricing. In their model, products are complementary in the sense that there exist customers who wish to purchase a set of products, and if one product is out-of-stock, then the demand for other products in the customer's set may also be lost. Cachon and Kök (2007) also study assortment planning for complementary products. Specifically, they consider multiple competing firms, each of which is selling multiple categories whose demands are complementary. Cachon and Kök (2007) focus on the equilibrium assortments that arise in their competition model. We drop competition. Furthermore, in this chapter's setting the product categories have a specific interpretation: the categories are the required and optional components that come together in a configurable product. The focus is on the optimal pricing of the required versus optional components. In addition, this chapter deals explicitly with the effect of inventory-related parameters on the optimal assortment.

2.3. Model Description

Consider a product made up of two components. One of these components is required in the sense that the product cannot function without it. The other component is optional in the sense that it adds some functionality to the product or enhances the use of it, but the product can function even without this component. Hereafter, refer to these components as the *required* component (e.g., an iPhone) and the *optional* component (e.g., a carrying case). For both the required and optional components, the firm offers multiple variants (e.g., 8GB versus 16GB iPhone, a slider case versus a flipper case). Let S_R denote the set of variants for the required component and S_O the set of variants for the optional component. The product is configurable, that is, the customer can configure the product to her taste by combining a variant of the required component with a variant of the optional component. The model allows the possibility that the customer chooses not to purchase the optional component. In addition, if there is no configuration that beats the "no-purchase" alternative, then the customer may choose not to purchase the product at all.

In keeping with the setting of configurable products, we model the case where the firm stocks each variant separately. In other words, the firm stocks the components, but not the assembled, finished product. Furthermore, the price for a given configuration is the sum of the prices of the variants that make up the configuration. Let p_{Ri} denote the price of required component's variant $i \in S_R$, and p_{Oj} the price of optional component's variant $j \in S_O$.

2.3.1 The Choice Model

Consider an individual customer who is presented with an assortment of variants for the required component and an assortment of variants for the optional component. The customer must decide whether to purchase the product and, if so, what configuration to purchase. As described next, the customer's decision is modeled using the multinomial logit (MNL) choice model. Let $U_{[ij]}$ denote the customer's random utility from the configuration that combines variant i of the required component with variant j of the optional component.[1] In addition, let $U_{[i0]}$ denote the customer's utility from the configuration that consists of required component's variant i only (i.e., a configuration that excludes the optional component). The customer's utility from configuration $[ij]$ is given by

$$U_{[ij]} = \alpha_{Ri} + \alpha_{Oj} - p_{Ri} - p_{Oj} + \xi_{[ij]}, i \in S_R \text{ and } j \in S_O,$$

where α_{nk} is the utility contribution of variant k of component $n \in \{R, O\}$ and $\xi_{[ij]}$ is a random error term. One could easily incorporate different weights for the utility contributions of the required and optional components. This utility model is in the spirit of models widely used for conjoint analysis, where a customer's utility from a multi-attribute product is modeled as the summation of part-worth, each of which is the customer's utility from a certain attribute. See, in particular, Louviere and Woodworth (1983) who use a similar utility model to fit the aggregate choice data for multi-attribute alternatives. Similarly, the customer's utility from configuration $[i0]$, which consists of required component's variant i, but excludes the optional component, is given by

$$U_{[i0]} = \alpha_{Ri} + \alpha_{O\circ} - p_{Ri} + \xi_{[i0]}, i \in S_R,$$

where $\alpha_{O\circ}$ is the utility contribution of not purchasing the optional component. In

[1] As a notational convention, brackets are used around subscripts to indicate a configuration that brings together two variants, one of each component.

addition, the customer may choose not to purchase from the firm. Let U_0 denote the customer's utility from not purchasing. Here, U_0 can be thought of as the aggregate utility from a number of choices exogenous to the model, such as the utility from purchasing the product from an alternate firm or the utility from purchasing a different product altogether. This utility is also given by an expected utility, α_0, plus a random error term ξ_0:

$$U_0 = \alpha_0 + \xi_0.$$

An implicit assumption of this utility model is that the utility contribution of component n's variant k is given by α_{nk}, and it is independent of what variant the customer chooses for the other component. One limitation of this assumption is that it does not capture the case where a variant's appeal may depend on what variant it will be matched with. Here the focus is on a simpler complementarity relationship, which nonetheless captures many practical applications. For example, the utility of having a slider case for an iPhone is unlikely to depend on the memory size of the phone.

Following the MNL model, this model assumes that $\xi_{[ij]}, i \in S_R$ and $j \in S_O \cup \{0\}$ and ξ_0 follow a Gumbel distribution whose mean is zero and scale parameter is one, and these error terms are independent across products. The customer then makes the choice that maximizes her utility. (For more on the MNL model, see, for example, Ben-Akiva and Lerman, 1985.) According to the MNL model, the probability that the customer will pick configuration $[ij]$ is

$$q_{[ij]}(S_R, S_O, \boldsymbol{p}_R, \boldsymbol{p}_O) = \frac{v_{[ij]}(p_{Ri}, p_{Oj})}{\exp(\alpha_0) + \sum_{i \in S_R} \sum_{j \in S_O \cup \{0\}} v_{[ij]}(p_{Ri}, p_{Oj})}, \tag{2.1}$$

where $\boldsymbol{p}_n$ denotes the vector of prices for component $n \in \{R, O\}$, $v_{[ij]}(p_{Ri}, p_{Oj}) := \exp(\alpha_{Ri} + \alpha_{Oj} - p_{Ri} - p_{Oj})$ for $i \in S_R$ and $j \in S_O$ and $v_{[i0]} := \exp(\alpha_{Ri} + \alpha_{Oo} - p_{Ri})$.

Likewise the probability that the customer will not purchase any of the products offered by the firm is

$$q_0(S_R, S_O, \boldsymbol{p}_R, \boldsymbol{p}_O) = \frac{\exp(\alpha_0)}{\exp(\alpha_0) + \sum_{i \in S_R} \sum_{j \in S_O \cup \{0\}} v_{[ij]}(p_{Ri}, p_{Oj})}. \tag{2.2}$$

The probability that a customer chooses variant i of the required component, denoted by q_{Ri}, is then given by the sum of the probabilities that a customer chooses a configuration that contains variant i of the required component:

$$q_{Ri}(S_R, S_O, \boldsymbol{p}_R, \boldsymbol{p}_O) = \sum_{j \in S_O \cup \{0\}} q_{[ij]}(S_R, S_O, \boldsymbol{p}_R, \boldsymbol{p}_O), i \in S_R. \tag{2.3}$$

Similarly, the probability that a customer chooses variant j of the optional component, denoted by q_{Oj}, is

$$q_{Oj}(S_R, S_O, \boldsymbol{p}_R, \boldsymbol{p}_O) = \sum_{i \in S_R} q_{[ij]}(S_R, S_O, \boldsymbol{p}_R, \boldsymbol{p}_O), j \in S_O. \tag{2.4}$$

2.3.2 The Aggregate Demand Model

The purchase probabilities in (2.3) and (2.4) serve as the starting point to model the aggregate demand for a variant. Specifically, it is assumed that the aggregate, random demand for variant k of component $n \in \{R, O\}$, denoted by $D_{nk}(S_R, S_O, \boldsymbol{p}_R, \boldsymbol{p}_O)$, is given by

$$D_{nk}(S_R, S_O, \boldsymbol{p}_R, \boldsymbol{p}_O) \sim q_{nk}(S_R, S_O, \boldsymbol{p}_R, \boldsymbol{p}_O)\epsilon_{nk} \text{ for } k \in S_n, n \in \{R, O\},$$

where ϵ_{nk}'s are i.i.d normal random variables with mean one and standard deviation σ_{nk}. Notice that σ_{nk} amounts to the coefficient of variation of demand for variant k of component n. Assuming that the mean of ϵ_{nk} is one amounts to a normalization of the demand size.

One important consequence of multiplicative randomness is that the coefficient of variation of demand is constant with respect to prices. To capture other cases where the coefficient of variation depends on prices, one could use a combination of both additive and multiplicative perturbations. However, such a model leads to an analytically intractable problem. Instead, we first assume a multiplicative perturbation. Then in Section 2.5, the results for the case with additive perturbation are discussed. For a comparison of multiplicative versus additive randomness in single-product inventory and pricing problems, see Petruzzi and Dada (1999).

2.3.3 The Firm's Problem

The firm's problem is modeled as a one-period problem. The firm chooses its assortment, $S_R \cup S_O$, the variants' prices and the stock levels at the beginning of the period. Let y_{nk} denote the stock level of variant k of component n, and c_{nk} its unit purchase cost. The demands for all variants, D_{nk}'s are then realized. The leftover inventory of variant k of component n incurs an overage cost of c^o_{nk} per unit. Excess demand for variant k of component n is backordered at a unit cost of c^u_{nk}.

The assumption of backordering is plausible for configurable products that are assembled after an order is placed, for example, personal computers. There are other retail settings where backordering may be a viable option. For example, if a customer chooses to purchase a desk and a hutch from Officemax.com, but the hutch is out-of-stock, the customer can still purchase the desk and have the hutch delivered at a later time. Admittedly, however, not all customers will choose to backorder. To the extent that customers are willing to substitute when their favorite product is out of stock, one would have to model such switching behavior. Unfortunately, in a model that allows such stockout-induced substitutions, the effective demand for a product is a function of the stock levels for other products, which in turn makes for an intractable inventory and pricing problem.

Notice the implicit assumption that the unit cost of backordering does not depend on the prices. This is likely to be the case when the cost of backordering is dominated by supply-side activities such as expedited production or emergency shipments. This assumption helps analytical tractability by removing the effect of the selling price on the unit underage cost. In comparison, in a model with lost sales, the unit underage cost will boil down to the unit profit margin, which would then be a function of the selling price.

The firm's expected profit is

$$\Pi(S_R, S_O, \boldsymbol{p}_{\scriptscriptstyle R}, \boldsymbol{p}_{\scriptscriptstyle O}) = \sum_{n\in\{R,O\}} \sum_{k\in S_n} \Big[(p_{nk} - c_{nk})E\big[D_{nk}(S_R, S_O, \boldsymbol{p}_{\scriptscriptstyle R}, \boldsymbol{p}_{\scriptscriptstyle O})\big] - L_{nk}(S_R, S_O, \boldsymbol{p}_{\scriptscriptstyle R}, \boldsymbol{p}_{\scriptscriptstyle O})\Big] \tag{2.5}$$

where

$$L_{nk}(S_R, S_O, \boldsymbol{p}_{\scriptscriptstyle R}, \boldsymbol{p}_{\scriptscriptstyle O}) = c^u_{nk}E\big[D_{nk}(S_R, S_O, \boldsymbol{p}_{\scriptscriptstyle R}, \boldsymbol{p}_{\scriptscriptstyle O}) - y_{nk}\big]^+ + c^o_{nk}E\big[y_{nk} - D_{nk}(S_R, S_O, \boldsymbol{p}_{\scriptscriptstyle R}, \boldsymbol{p}_{\scriptscriptstyle O})\big]^+. \tag{2.6}$$

Here, the function $L_{nk}(S_R, S_O, \boldsymbol{p}_{\scriptscriptstyle R}, \boldsymbol{p}_{\scriptscriptstyle O})$ represents the expected overage and underage cost associated with variant k of component n. Let $\mu^D_{nk}(S_R, S_O, \boldsymbol{p}_{\scriptscriptstyle R}, \boldsymbol{p}_{\scriptscriptstyle O})$ and $\sigma^D_{nk}(S_R, S_O, \boldsymbol{p}_{\scriptscriptstyle R}, \boldsymbol{p}_{\scriptscriptstyle O})$ denote the mean and standard deviation of the demand for variant k of component n. Define

$$z_{nk} := \frac{y_{nk} - \mu^D_{nk}(S_R, S_O, \boldsymbol{p}_R, \boldsymbol{p}_O)}{\sigma^D_{nk}(S_R, S_O, \boldsymbol{p}_{\scriptscriptstyle R}, \boldsymbol{p}_{\scriptscriptstyle O})},$$

which can be interpreted as the service level for variant k of component n, measured in standard deviations above the mean. In fact, given the assumption that the demand for variant k of component n is normally distributed with mean $\mu^D_{nk}(S_R, S_O, \boldsymbol{p}_{\scriptscriptstyle R}, \boldsymbol{p}_{\scriptscriptstyle O})$ and standard deviation $\sigma^D_{nk}(S_R, S_O, \boldsymbol{p}_{\scriptscriptstyle R}, \boldsymbol{p}_{\scriptscriptstyle O})$, the value of z_{nk} uniquely determines the probability that variant $k \in S_n$ will not run out of stock, which is typically referred to as *type-1 service level*. More precisely, the type-1 service level for variant $k \in S_n$ is $\Phi_N(z_{nk})$, where $\Phi_N(\cdot)$ is the cdf of the standard normal distribution. In this paper, it is assumed that z_{nk}'s are exogenously fixed. This is equivalent to assuming that the firm fixes a certain type-1 service level for each variant. Alternatively, for a given assortment and prices, one could assume that the firm picks the stock level for each variant to minimize its expected inventory costs. In such a case, the optimal stock level would satisfy the standard critical fractile solution where $P(D_{nk}(S_R, S_O, \boldsymbol{p}_{_R}, \boldsymbol{p}_{_O}) \leq y_{nk}) = \frac{c^u_{nk}}{c^u_{nk}+c^o_{nk}}$, which boils down to choosing y_{nk} so that $z_{nk} = \Phi_N^{-1}\left(\frac{c^u_{nk}}{c^u_{nk}+c^o_{nk}}\right)$. All subsequent analysis would carry over for z_{nk} values fixed at those optimal levels. Nonetheless, it's assumed here that z_{nk}'s are fixed exogenously in order to investigate the effect of service levels on the assortment decisions.

Given the assumption that the demand for each variant is normally distributed, the expected underage and overage cost function in (2.6), $L_{nk}(S_R, S_O, \boldsymbol{p}_{_R}, \boldsymbol{p}_{_O})$ can be simplified as follows:[2]

$$L_{nk}(S_R, S_O, \boldsymbol{p}_{_R}, \boldsymbol{p}_{_O}) = \sigma^D_{nk}(S_R, S_O, \boldsymbol{p}_{_R}, \boldsymbol{p}_{_O})\left[c^o_{nk}z_{nk} + (c^o_{nk} + c^u_{nk})I_N(z_{nk})\right] \tag{2.7}$$

where I_N is the unit normal loss function defined as $I_N(z) := \phi_N(z) - z(1 - \Phi_N(z))$ and, $\phi_N(\cdot)$ and $\Phi_N(\cdot)$ are the standard normal density and distribution functions respectively. Substituting (2.7) in (2.5) and recalling that $\mu^D_{nk} = q_{nk}(S_R, S_O, \boldsymbol{p}_{_R}, \boldsymbol{p}_{_O})$ and $\sigma^D_{nk} = q_{nk}(S_R, S_O, \boldsymbol{p}_{_R}, \boldsymbol{p}_{_O})\sigma_{nk}$, one obtains

$$\Pi(S_R, S_O, \boldsymbol{p}_{_R}, \boldsymbol{p}_{_O}) = \sum_{n\in\{R,O\}}\sum_{k\in S_n}\left[\begin{array}{l}(p_{nk} - c_{nk})q_{nk}(S_R, S_O, \boldsymbol{p}_{_R}, \boldsymbol{p}_{_O}) \\ -\,\sigma_{nk}\big(c^o_{nk}z_{nk} + (c^o_{nk} + c^u_{nk})I_N(z_{nk})\big)q_{nk}(S_R, S_O, \boldsymbol{p}_{_R}, \boldsymbol{p}_{_O})\end{array}\right].$$

Define $\gamma_{nk} := \sigma_{nk}\left[c^o_{nk}z_{nk} + (c^o_{nk} + c^u_{nk})I_N(z_{nk})\right]$. Notice from above that γ_{nk} works as a unit cost incurred by the firm and it depends only on unit overage cost, c^o_{nk}, unit underage cost, c^u_{nk}, the service level z_{nk} and demand's coefficient of variation, σ_{nk}. In

[2]See Section 1.2 of Porteus (2002) for more details. The c^o_{nk}, $c^u_{nk} + c^o_{nk}$ and σ_{nk} here correspond to c, p and σ, respectively, in Porteus (2002).

other words, all of the inventory-related costs are incorporated to the firm's objective function in the form of parameter γ_{nk}. Using this definition, one can now write the firm's expected profit function as follows:

$$\Pi(S_R, S_O, \boldsymbol{p}_R, \boldsymbol{p}_O) = \sum_{n \in \{R,O\}} \sum_{k \in S_n} (p_{nk} - c_{nk} - \gamma_{nk}) q_{nk}(S_R, S_O, \boldsymbol{p}_R, \boldsymbol{p}_O). \qquad (2.8)$$

This simple form of the profit function is a culmination of, in particular, the assumption that the demand for each variant is normally distributed and the assumption that the excess demand is backordered. The firm's problem is to choose the assortments S_R and S_O and the corresponding price vectors $\boldsymbol{p}_R$ and $\boldsymbol{p}_O$ to maximize its profit, that is,

$$\max_{S_R, S_O, \boldsymbol{p}_R, \boldsymbol{p}_O} \Pi(S_R, S_O, \boldsymbol{p}_R, \boldsymbol{p}_O).$$

As the subsequent section and the next chapter will illustrate, this model is simple enough to be analytically tractable, yet sophisticated enough to capture the interactions among the assortment, prices and inventory levels.

2.4. Results for the Base Model: Multiplicative Demand Model

The pricing problem is first analyzed given an assortment of variants. Given that the firm will price any assortment optimally, the firm's assortment selection problem and the effect of inventory-related parameters on the assortment decisions will follow.

2.4.1 Optimal Pricing of a Given Assortment

In preparation for the next result, consider a make-to-order firm offering an assortment of substitutable products, denoted by S. Suppose that the unit retail price of variant k is p_k and the firm's only cost is the purchase cost, c_k per unit of variant k. Furthermore, suppose that the customers are choosing from the assortment according to the MNL model. It is well known that the optimal prices for this problem satisfy the *equal margin property*; i.e., $p_k - c_k$ is the same for all $k \in S$ at the optimal prices. (See, for example, Besanko et al., 1998.) The pricing problem that arises in this chapter's model embellishes in a number of ways the simpler problem outlined above: the product is a combination of two components, which are to be priced separately; the customer purchase decision depends on the entire assortment and prices of all variants; the firm makes to stock, which results in inventory costs when the demand does not match the stock level. Nonetheless, the following result shows that a modified version of the equal margin property is maintained in this model.

Proposition 1. *Consider component* $n \in \{R, O\}$*. Let* $m_{nk} := p_{nk} - c_{nk} - \gamma_{nk}$ *denote the effective profit margin of variant* k *of component* n*. At any pair of optimal price vectors* $\boldsymbol{p}_R$ *and* $\boldsymbol{p}_O$*, all variants* k *of component* n *will have the same effective profit margin, i.e.,* $m_{nk} = m_n$ *for all* $k \in S_n$*.*

There is one crucial difference between this result and the "equal margin property" obtained by earlier work on assortment pricing under logit-type choice models (e.g., Besanko et al., 1998, Anderson and de Palma, 1992, Aydın and Ryan, 2000): The earlier work finds that the unit *gross margin* (i.e., unit retail price minus unit purchase cost) should be the same across all variants while here the effective margins (unit gross margin net of the inventory cost parameter, γ_{nk}) must be the same. In practice, gross margins are hardly ever equal; in fact, higher quality variants tend to have higher gross margins. For example, data from Cars.com show that the gross margin (the difference between the manufacturer suggested retail price and the invoice price) for a 2009 Honda Civic increases progressively as one moves up the ladder of trim levels, from DX to LX to EX to EX-L to Si. This model's result provides one possible motivation for the attraction of applying higher gross margins to higher quality variants: According to Proposition 1, what should be the same across variants is the effective margin, not the gross margin. Because it costs more to produce a higher quality variant and because part of the unit overage cost is the cost of capital tied in inventory, such variants tend to have higher unit overage costs as well. Hence, the inventory cost parameter, γ_{nk}, tends to be higher for higher quality variants. Consequently, if the firm charges the same effective margin for all variants, the gross margins will be larger for high quality variants.

The equal margin property described in Proposition 1 allows recasting the pricing problem: Instead of choosing price vectors $\boldsymbol{p}_R$ and $\boldsymbol{p}_O$, the pricing problem can be simplified to choosing two margins: m_R for the required component and m_O for the optional component. In order to help with such a reformulation, first define the *unit surplus* of variant $k \in S_n$:

$$\eta_{nk} := \alpha_{nk} - c_{nk} - \gamma_{nk}. \tag{2.9}$$

In addition, define $\eta_{O_0} := \alpha_{O_0}$ as the surplus of the no-purchase option for the optional component. Notice that η_{nk} is the difference between the customer's expected utility from variant $k \in S_n$, α_{nk}, and the unit cost incurred by the firm for variant $k \in S_n$, $c_{nk} + \gamma_{nk}$. Hence, one could interpret η_{nk} as the total surplus created by one unit of the variant exchanging hands from the firm to the customer. Taking advantage of the equal effective margin property stated in Proposition 1 and using the parameter η_{nk}, one can rewrite the product's purchase probability, $q_{[ij]}(S_R, S_O, \boldsymbol{p}_R, \boldsymbol{p}_O)$, given by (2.1), as:

$$\begin{aligned}
&q_{[ij]}(S_R, S_O, m_R, m_O) = \\
&\qquad \frac{\exp(\eta_{Ri} - m_R)\exp(\eta_{Oj} - m_O)}{\exp(\alpha_0) + \sum_{i \in S_R} \exp(\eta_{Ri} - m_R)\left[\sum_{j \in S_O} \exp(\eta_{Oj} - m_O) + \exp(\eta_{O0})\right]}, \\
&\qquad i \in S_R, j \in S_O, \\
&q_{[i0]}(S_R, S_O, m_R, m_O) = \\
&\qquad \frac{\exp(\eta_{Ri} - m_R)\exp(\eta_{O0})}{\exp(\alpha_0) + \sum_{i \in S_R} \exp(\eta_{Ri} - m_R)\left[\sum_{j \in S_O} \exp(\eta_{Oj} - m_O) + \exp(\eta_{O0})\right]}, i \in S_R.
\end{aligned}$$

Likewise, the probability of not purchasing from the firm, $q_0(S_R, S_O, \boldsymbol{p}_{\scriptscriptstyle R}, \boldsymbol{p}_{\scriptscriptstyle O})$, given by (2.2), can be rewritten as:

$$q_0(S_R, S_O, m_{\scriptscriptstyle R}, m_{\scriptscriptstyle O}) \;\; = \frac{\exp(\alpha_0)}{\exp(\alpha_0)+\sum_{i\in S_R}\exp(\eta_{Ri}-m_R)\left[\sum_{j\in S_O}\exp(\eta_{Oj}-m_O)+\exp(\eta_{O0})\right]}. \quad (2.10)$$

A variant's choice probability is then obtained by adding together the choice probabilities of configurations that contain the variant, i.e.,

$$\begin{aligned} q_{\scriptscriptstyle Rk}(S_R, S_O, m_{\scriptscriptstyle R}, m_{\scriptscriptstyle O}) &= \sum_{j\in S_O\cup 0} q_{[kj]}(S_R, S_O, m_{\scriptscriptstyle R}, m_{\scriptscriptstyle O}) \text{ for all } k \in S_R, \\ q_{\scriptscriptstyle Ok}(S_R, S_O, m_{\scriptscriptstyle R}, m_{\scriptscriptstyle O}) &= \sum_{i\in S_R} q_{[ik]}(S_R, S_O, m_{\scriptscriptstyle R}, m_{\scriptscriptstyle O}) \text{ for all } k \in S_O \cup \{0\}. \quad (2.11) \end{aligned}$$

Using the redefined choice probabilities, one can now write the firm's expected profit, $\Pi(S_R, S_O, \boldsymbol{p}_R, \boldsymbol{p}_O)$, given by (2.8), as a function of the margins instead of price vectors:

$$\Pi(S_R, S_O, m_{\scriptscriptstyle R}, m_{\scriptscriptstyle O}) = m_{\scriptscriptstyle R}[1 - q_0(S_R, S_O, m_{\scriptscriptstyle R}, m_{\scriptscriptstyle O})] + m_{\scriptscriptstyle O} \begin{bmatrix} 1 - q_0(S_R, S_O, m_{\scriptscriptstyle R}, m_{\scriptscriptstyle O}) \\ - q_{\scriptscriptstyle O_0}(S_R, S_O, m_{\scriptscriptstyle R}, m_{\scriptscriptstyle O}) \end{bmatrix}. \quad (2.12)$$

In preparation for the next result, rewrite the firm's profit in (2.12) as

$$\Pi(S_R, S_O, m_{\scriptscriptstyle R}, m_{\scriptscriptstyle O}) = (m_{\scriptscriptstyle R} + m_{\scriptscriptstyle O})\left[1 - q_0(S_R, S_O, m_{\scriptscriptstyle R}, m_{\scriptscriptstyle O})\right] - m_{\scriptscriptstyle O} q_{\scriptscriptstyle O_0}(S_R, S_O, m_{\scriptscriptstyle R}, m_{\scriptscriptstyle O}).$$

Notice from above that the firm's profit is given by the profit collected from all customers who purchase, $(m_{\scriptscriptstyle R} + m_{\scriptscriptstyle O})\left[1 - q_0(S_R, S_O, m_{\scriptscriptstyle R}, m_{\scriptscriptstyle O})\right]$, adjusted downward by $m_{\scriptscriptstyle O} q_{\scriptscriptstyle O_0}(S_R, S_O, m_{\scriptscriptstyle R}, m_{\scriptscriptstyle O})$, to account for the customers who choose to purchase the product without the optional component. Hence, it would seem that it is more attractive for the firm to shift some of its margin from the optional component to the required component so as to reduce the "loss" in profit due to customers who choose not to purchase the optional component. Owing to this property, at the optimal solution, it is best to allocate zero margin to the optional component while making money on the required component. The next proposition formalizes this observation.

Proposition 2. *The unique optimal margin for the optional component is zero. The optimal margin for the required component is the unique* m_R *that satisfies the following equation:*

$$m_R = \frac{1}{q_0(S_R, S_O, m_R, m_o)}.$$

The proposition indicates that all the variants of the optional component should be priced at cost, where the cost includes not just the purchase cost c_{O_j} of the variant $j \in S_O$, but also the inventory-related cost γ_{O_j}. Proposition 2 may contradict the popular belief that firms make money on accessories, which correspond to optional components in the context of this model. As discussed in the next chapter, the firm will no longer price the optional component at cost when the optional component is an impulse purchase.

2.4.2 Assortment Selection

This section explores the firm's assortment selection problem, given that the firm will price any chosen assortment optimally. In order to highlight the substantial effect of optimal pricing on the assortment selection, consider first the example shown in Table 2.1, where the prices are fixed exogenously. In this example it is optimal for the firm to offer only variant 1 of the required component (R1). To see why R3 and O1 are not offered, notice that while the gross margin is positive for those two variants (i.e., $p_{R_3} - c_{R_3} > 0$ and $p_{O_1} - c_{O_1} > 0$), the effective margin is negative for both of them, due to the relatively high inventory-related cost (i.e., $p_{R_3} - c_{R_3} - \gamma_{R_3} < 0$ and $p_{O_1} - c_{O_1} - \gamma_{O_1} < 0$). As for $R2$, it is not carried in the optimal assortment even though its effective margin, $p_{R_2} - c_{R_2} - \gamma_{R_2}$, is positive. This is because offering $R2$ would cannibalize the demand of the more profitable $R1$. As this example shows,

	Assortment			
	Required			Optional
	Variant 1 (R1)	Variant 2 (R2)	Variant 3 (R3)	Variant 1 (O1)
Unit purchase cost, c_{nk}	1	1	1.6	1.5
Unit inventory cost, γ_{nk}	0.428	0.4383	0.5775	0.8456
Total unit cost, $c_{nk}+\gamma_{nk}$	1.4280	1.5775	2.0383	2.3456
Unit price, p_{nk}	2	2	2	2
Expected utility, α_{nk}	7	8.5	9	9
Included in the optimal assortment?	Yes	No	No	No

Table 2.1: Optimal assortment for a firm that does not price its assortment optimally. In this example, $\alpha_{O_o} = 3$, $\alpha_0 = 5$.

when the prices are fixed exogenously, certain variants may be left out of the optimal assortment due to inventory-related costs or due to concerns about the cannibalization effect. However, as the following proposition states, once the firm is allowed to set its prices optimally, it becomes optimal for the firm to offer all variants.

Proposition 3. *Suppose the firm is currently offering assortment $S_R \cup S_O$. The firm is always better off after adding a new variant to the assortment $S_R \cup S_O$.*

When prices are set optimally, there is no longer any concern about a less profitable variant cannibalizing the demand for a more profitable variant, because all variants have the same effective margin and, thus, are equally profitable. Furthermore, the optimal prices will always more than cover the inventory-related costs and no variant will be left out because of inventory costs. In summary, the concerns about cannibalization and inventory-related costs, which limit the optimal assortment when prices are fixed exogenously, do not play a role when the firm optimizes over prices.

Even when the firm is able to price its assortment optimally, in most (if not all) cases the firm either faces constraints on the assortment size (e.g., shelf space or warehouse space limitations) or incurs fixed costs for carrying a product (e.g., administrative costs of adding a product to the database, cost of putting the product on the store and warehouse shelves, etc.) In the presence of such limiting factors, the firm must choose which variants to offer. Next, assuming the existence of a limit on

Number of Variants Allowed	Assortment	
	Required Component	Optional Component
1	$\{1\}$	$\{\}$
2	$\{1,2\}$	$\{\}$
3	$\{1,2\}$	$\{1\}$
4	$\{1,2,3\}$	$\{1\}$
5	$\{1,2,3\}$	$\{1,2\}$
6	$\{1,2,3,4\}$	$\{1,2\}$

Table 2.2: Optimal assortment for several different limits on the assortment size. In this example, $\eta_{R_1} = 7.1696$, $\eta_{R_2} = 7.1255$, $\eta_{R_3} = 6.5533$, $\eta_{R_4} = 6.5335$, $\eta_{O_1} = 7.6492$, $\eta_{O_2} = 7.1544$, $\eta_{O_3} = 6.8725$, $\eta_{O_4} = 6.8235$, $\alpha_0 = 5$, $\alpha_{O_0} = 8$.

the size of the optimal assortment, we explore how the firm decides what variant to add.

Table 2.2 illustrates an example where there is a limit on how many variants the firm can offer in its assortment. In this example, there are four potential variants for each of the required and optional components and the variants are indexed in decreasing order of surplus, that is, $\eta_{n_1} \geq \eta_{n_2} \geq \eta_{n_3} \geq \eta_{n_4}, n \in \{R, O\}$. Notice from the table that when the firm can offer only one variant, the firm offers variant 1 of the required component (R1), that is, the required component with the highest surplus. As the firm is allowed to offer more variants, the firm first adds variant 2 of the required component (R2), followed by variant 1 of the optional component (O1). As the limit on the assortment size is relaxed further, R3 and O2 are added, in that order.

One observation from this table is that whenever the firm adds one more variant of the required component to the assortment, it adds the one with the highest surplus among all remaining variants of the required component. Likewise, whenever the firm adds a new variant of the optional component, it adds the one with the highest surplus among all remaining variants of the optional component. To formalize this observation, consider a firm who is planning to expand a given assortment by choosing one variant from a set of candidates that all belong to the same component. The following proposition states that the firm will pick the variant with the largest unit surplus.

Proposition 4. *Consider component $n \in \{R, O\}$. Suppose that there is a set of variants of component n, denoted by V, that may be added to the current assortment. If the firm is allowed to add only one of these candidates, then the firm will add the candidate with the highest surplus, i.e., the firm will add variant k such that $\eta_{nk} \geq \eta_{nl}$ for all $l \in V$.*

Because η_{nk} is the total surplus created by one unit of the variant exchanging hands from the firm to the customer, a larger η_{nk} not only allows the firm to keep a larger profit margin but also allows the firm to leave a higher surplus to the customer, thereby increasing the customer's purchase probability. Hence, the firm prefers to add the variant k with the highest η_{nk}.

Given the component whose assortment is to be expanded, Proposition 4 describes which variant to add. However, the proposition is silent about whether to expand the required component's assortment or the optional component's assortment. For instance, in the example shown in Table 2.2, if the firm is allowed to offer four variants instead of three, why is it that the firm chooses to add R3 over O2? This decision might be more surprising given that the surplus of O2 is higher than the surplus of R3. The explanation lies in the new product configurations enabled by the addition of R3 versus O2. It turns out that the new configurations enabled by the addition of R3 (that is, R3 with no optional component and R3 with O1) are more profitable for the firm than the new configurations enabled by the addition of O2 (that is, R1 with O2 and R2 with O2). Motivated by this observation, define the *attraction* of a product configuration that combines variant i of the required component with variant j of the optional component, denoted by $a_{[ij]}$:

$$a_{[ij]} := \exp(\eta_{Ri} + \eta_{Oj}). \tag{2.13}$$

Proposition 5 uses this definition and provides a precise description of how a firm chooses between a required component and an optional component.

Proposition 5. *Suppose the firm is currently offering assortment $S_R \cup S_O$ and must decide whether to add variant i of the required component or variant j of the optional component. The firm will add variant i of the required component if and only if the total attraction of all product configurations that contain required component i exceeds the total attraction of all product configurations that contain optional component j, i.e., if and only if $\sum_{k \in S_O \cup \{0\}} a_{[ik]} \geq \sum_{k \in S_R} a_{[kj]}$.*

The lesson from Proposition 5 is that, when deciding whether to add a required or an optional component, the firm must avoid making decisions solely on the basis of the variants' own surpluses, because the profitability of each candidate depends on what variants it can be coupled with to create new product configurations.

2.4.3 Effect of Service Level, Demand Variability and Inventory Costs

Propositions 4 and 5 illustrate the importance of the surplus when deciding what variants to include in the assortment. The surplus is affected not only by the customer's expected utility from the variant and the firm's unit purchase cost, but also the inventory-related parameters such as underage and overage costs, the demand's coefficient of variation, and the service level. Proposition 6 describes how the assortment decisions depend on such inventory-related parameters.

Proposition 6. *Consider component $n \in \{R, O\}$. Suppose that there are two variants of component n, variants l and k, that may be added to the current assortment, but the firm is allowed to add only one of these candidates. If the two variants are identical in all respects, but:*
(a) The demand's coefficient of variation is larger for variant l than for variant k, i.e. $\sigma_{nl} > \sigma_{nk}$, then the firm will add the variant with the lower coefficient of variation, variant k.

(b) The unit underage (overage) cost is larger for variant l than variant k, i.e., $c^u_{nl} > c^u_{nk}$ ($c^o_{nl} > c^o_{nk}$), then the firm will add the variant with the lower unit underage (overage) cost, variant k.

(c) The service level is larger for variant l than variant k, i.e., $z_{nl} > z_{nk}$, then there is a critical service level $\overline{z}$ such that if $z_{nk} < z_{nl} < \overline{z}$, then the firm will add the variant with the higher service level, variant l, and if $\overline{z} < z_{nk} < z_{nl}$, then the firm will add the variant with the lower service level, variant k.

The results in Proposition 6(a),(b) confirm the intuition that the higher the variant's demand variability and the higher the variant's unit underage and overage costs, the more costly it is to carry that variant. Hence, the firm will add variants with lower demand variability and lower underage and overage costs. It is evident from Proposition 6(c) that the effect of the service level is not as straightforward. A higher service level does not necessarily increase a variant's surplus. In fact, for each variant, there is an optimal service level that balances overage and underage costs, and the further the service level moves from this optimal level in either direction, the smaller the variant's surplus becomes. This observation explains the result stated in Proposition 6(c). If two variants have the same optimal service level and both variants' service levels exceed that optimal level, the firm would pick the variant with the lower service level. Likewise, if both variants' service levels are below the optimal level, then the firm will pick the variant with the higher service level.

2.5. Additive Demand Model

In the base model in Section 2.3, randomness in demand is introduced using a multiplicative perturbation of the expected demand. This section explores the pricing and assortment decisions when randomness is introduced as an additive perturbation of the expected demand.

Using the purchase probabilities $q_{_{Ri}}(S_R, S_O, \boldsymbol{p}_{_R}, \boldsymbol{p}_{_O})$ and $q_{_{Oj}}(S_R, S_O, \boldsymbol{p}_{_R}, \boldsymbol{p}_{_O})$, given by (2.3) and (2.4), respectively, as the starting point to model the aggregate demand for a variant, the random demand for variant k of component $n \in \{R, O\}$ is now given by,

$$D_{nk}(S_R, S_O, \boldsymbol{p}_{\scriptscriptstyle R}, \boldsymbol{p}_{\scriptscriptstyle O}) \sim q_{nk}(S_R, S_O, \boldsymbol{p}_{\scriptscriptstyle R}, \boldsymbol{p}_{\scriptscriptstyle O}) + \epsilon_{nk} \text{ for } k \in S_n, n \in \{R, O\},$$

where ϵ_{nk}'s are i.i.d normal random variables with mean 0 and standard deviation σ_{nk}. Hence, the demand for variant k of component $n \in \{R, O\}$ follows a normal distribution with a mean of $q_{nk}(S_R, S_O, \boldsymbol{p}_{\scriptscriptstyle R}, \boldsymbol{p}_{\scriptscriptstyle O})$ and a standard deviation of σ_{nk}.[3] Following the same steps as before, it can be shown that the firm's expected profit is:

$$\overline{\Pi}(S_R, S_O, \boldsymbol{p}_{\scriptscriptstyle R}, \boldsymbol{p}_{\scriptscriptstyle O}) = \sum_{n \in \{R,O\}} \sum_{k \in S_n} \left[(p_{nk} - c_{nk}) q_{nk}(S_R, S_O, \boldsymbol{p}_{\scriptscriptstyle R}, \boldsymbol{p}_{\scriptscriptstyle O}) - \gamma_{nk}) \right], \qquad (2.14)$$

where $\gamma_{nk} := \sigma_{nk}\big(c^o_{nk} z_{nk} + (c^o_{nk} + c^u_{nk}) I_N(z_{nk})\big)$. Observe from (2.14) that the firm's expected profit separates into two terms: The first term captures the gross profit (which depends on prices) and the second term captures inventory-related costs (which do not depend on prices). Consequently, the optimal prices do not depend on parameters such as unit overage and underage costs, service levels and standard deviations of demand. The following proposition characterizes the optimal prices.

Proposition 7. *Consider component $n \in \{R, O\}$. Let $\overline{m}_{nk} := p_{nk} - c_{nk}$ denote the gross margin of variant k of component n. At any pair of optimal price vectors $\boldsymbol{p}_{\scriptscriptstyle R}$ and $\boldsymbol{p}_{\scriptscriptstyle O}$, all variants k of component n will have the same gross margin, i.e., $\overline{m}_{nk} = \overline{m}_n$ for all $k \in S_n$. Moreover, the optimal $\overline{m}_{\scriptscriptstyle O} = 0$ and $\overline{m}_{\scriptscriptstyle R} = \frac{1}{q_0(S_R, S_O, \overline{m}_R, \overline{m}_O)}$, where*

$$q_0(S_R, S_O, \overline{m}_{\scriptscriptstyle R}, \overline{m}_{\scriptscriptstyle O}) := \frac{\exp(\alpha_0)}{\exp(\alpha_0) + \sum_{i \in S_R} \exp(\overline{\eta}_{\scriptscriptstyle Ri} - \overline{m}_{\scriptscriptstyle R}) \left[\sum_{j \in S_O} \exp(\overline{\eta}_{\scriptscriptstyle Oj} - \overline{m}_{\scriptscriptstyle O}) + \exp(\overline{\eta}_{\scriptscriptstyle Oo}) \right]}$$

[3] One drawback of the additive model is that, when σ_{nk} is sufficiently large, the probability of negative demand is non-negligible.

and $\overline{\eta}_{Ri} := \alpha_{Ri} - c_{Ri}$ *for* $i \in S_R$, $\overline{\eta}_{Oj} := \alpha_{Oj} - c_{Oj}$ *for* $j \in S_O$ *and* $\overline{\eta}_{Oo} := \alpha_{Oo}$.

According to Proposition 7, the gross margin is the same across all variants of a given component. This is markedly different from the result under the model with multiplicative demand, where the effective margin was the same across all variants of a given component. In the case of additive demand, the optimal prices are not affected by parameters that drive inventory costs, namely, unit overage and underage costs, service levels and standard deviations of demand. In contrast, under multiplicative randomness, the optimal prices do depend on those parameters. In a more general model that allows both additive and multiplicative randomness, one would expect that the optimal prices will continue to depend on inventory-related parameters. Given such general models pose significant analytical challenges, the purely additive and purely multiplicative models are two compromises that achieve analytical tractability. Of these two alternative models, the model with multiplicative demand appears to be more advantageous in that it does capture the effect of inventory-related parameters on the optimal prices, which the additive model fails to do.

Notice from Proposition 7 that one important insight from the model with multiplicative demand continues to hold under additive demand: the optional component is sold at zero gross margin under additive demand; it was sold at zero effective margin under multiplicative demand.

As for the assortment decision, observe from the profit expression, given by (2.14), that the inventory costs captured in the form of γ_{nk} now play the role of a 'fixed cost' of carrying a variant. Hence, when deciding what items to offer, the firm will weigh the item's contribution to gross profit against the fixed cost of carrying the item. The next proposition formalizes this observation.

Proposition 8. *Suppose the firm is currently offering assortment* $S_R \cup S_O$. *Let* $\overline{m}^*_R(S_R, S_O)$ *denote the optimal margin for the required component as defined by Proposition 7. Given a new variant k of the required component, the firm will add variant k* *if and only if* $\overline{m}_{R}^{*}(S_R \cup \{k\}, S_O) - \overline{m}_{R}^{*}(S_R, S_O) > \gamma_{Rk}$. *Likewise, a new variant* k *of the optional component will be added if and only if* $\overline{m}_{R}^{*}(S_R, S_O \cup \{k\}) - \overline{m}_{R}^{*}(S_R, S_O) > \gamma_{Ok}$.

Contrast Proposition 8 with its analog in the multiplicative case, Proposition 3. In the model with multiplicative demand, the firm was always better off after adding a new variant. In the model with additive demand, this is no longer true, because the firm incurs a fixed cost, γ_{nk}, for adding variant k of component n, and there is no guarantee that the gross profit from the item will cover its fixed costs. The ability to incorporate fixed costs is an advantage of the additive demand, but if one wanted to model fixed costs, it would not be hard to incorporate them to the model with multiplicative demand either.

Proposition 8 deals with whether or not to add a variant. The next proposition, on the other hand, describes how to pick from variants that are competing for inclusion in the assortment.

Proposition 9. *Consider component* $n \in \{R, O\}$. *Suppose that there is a set of variants of component* n*, denoted by* V*, that may be added to the current assortment and all candidates have the same inventory-related cost,* γ*, i.e.* $\gamma_{nl} = \gamma$ *for all* $l \in V$. *If the firm is allowed to add only one of these candidates, then the firm will add the candidate with the highest* $\alpha_{nl} - c_{nl}$ *for all* $l \in V$.

The above proposition provides an insight similar to that in Proposition 4, which stated that, given a set of variants, V, that may be added to the assortment, the firm should add the variant with the highest surplus, i.e., the variant that has the highest $\alpha_{nl} - c_{nl} - \gamma_{nl}$ for all $l \in V$. For the additive demand case, Proposition 9 shows that a special case of this result continues to hold: if all the variants have the same fixed cost, γ, the firm uses a redefined surplus, $\alpha_{nl} - c_{nl}$, and adds the variant with the highest redefined surplus.

2.6. Conclusion

This chapter deals with a firm's assortment and pricing decisions for configurable products under demand uncertainty. A model for a configurable product as a combination of a required component and an optional component is presented. In this model, each component's assortment allows the consumer to choose from several variants. The demands for the required and optional components are complementary in the sense that expanding the assortment of one component, or decreasing its price, will increase the demand for the other component. In this model, the customer decides what product to purchase based on the entire assortment and prices for that assortment.

This chapter finds that the optimal prices are such that all variants of a component share the same *effective profit margin* (i.e., the selling price net of unit purchase cost and unit inventory cost). In the absence of inventory costs, logit-type choice models typically lead to optimal pricing rules that require the *gross margins* (selling price net of unit purchase cost) to be equal (e.g., Besanko et al., 1998). However, in practice, gross margins are rarely ever equal across variants. In fact, in many cases, the gross margin is larger for higher quality variants. The results of this chapter offer one possible explanation: What should be equal across variants is not the gross margin, but the effective margin, which is the gross margin net of unit inventory cost. Assuming that higher quality variants have higher overage and underage costs, it would follow that such variants will also have higher gross margins. Furthermore, these results may explain why some firms choose to charge a premium on variants that are only artificially different from others in the same assortment. For example, Dell's XPS laptops come in one of two standard colors for the cover (black and red), but one can purchase them in blue, white or pink covers by paying a premium. One may argue that this is because Dell pays higher unit prices for such colors or that customers are willing to pay higher prices for them. This chapter contains an alternative explanation: Even if the color of the cover makes no discernible difference in terms of Dell's purchase cost or customer's willingness to pay, pink or white covers would still command higher prices if their demand variability (measured in terms of coefficient of variation) is larger than the standard colors.

As for assortment selection, the importance of a variant's *surplus* is highlighted. The surplus is a measure of the variant's profitability that depends not only on the customer's utility from the variant, but also on the variant's unit purchase cost, unit underage and overage costs, service level and demand variability. The surplus combines all these demand-side and supply-side parameters in one convenient package. When choosing from two variants of a given component, the firm should pick the one with the higher surplus. However, this chapter shows that when choosing from two variants that belong to different components, it is not enough to compare the variants' surpluses. In such a case, one must take into account the complementarity and must be mindful of how adding one variant to a component's assortment will influence the demand for the other component. Another measure is provided in this chapter, *product attractiveness*, that serves well for this type of decision. The product attractiveness measures the profitability of a product configuration that brings together a variant of the required component and a variant of the optional component (as opposed to surplus, which measures the profitability of a variant by itself). One could choose from two variants that belong to different components by comparing the product attractiveness of the new configurations that are enabled by the addition of each variant.

In the base model, randomness in demand is introduced using a multiplicative perturbation of the expected demand. The chapter also explores the pricing and assortment decisions when the randomness is introduced as an additive perturbation. The main difference between the additive and multiplicative models is that, under additive randomness, the optimal pricing rule is to use the same gross margin for all variants of a given component, as opposed to using the same effective margin.

The inventory-related costs of a variant, on the other hand, boil down to a fixed cost of carrying the variant, which depends on the variant's unit underage and overage costs, service level and demand variability. Hence, the optimal assortment selection requires a balance between the revenue contributions of the variants and fixed costs of carrying them.

Chapter III

Assortment Selection and Pricing for Configurable Products: Extensions and Generalizations

3.1. Introduction

This chapter explores three different extensions to the model presented in Chapter II: First, it explores a model where the required component is the only driver of store traffic. Second, it allows for a customer segment in the population that is interested in purchasing the optional component only. Third, it explores a generalization where the components are complementary and can be functional when purchased separately. Throughout this chapter, the term *base model* refers to the model presented in Chapter II Section 2.3.

The required and optional components may differ in their ability to draw store traffic. For example, according to Johnson (2004), "Consumer Electronics Association research shows that a consumer will decide where to buy the computer based on advertised pricing, yet will have little or no point of reference for pricing on accessories once they enter a store." In Chapter II, it is assumed that customers take into account the assortments and prices of both the required and optional components when making their decision to purchase from the firm. However, this chapter explores a model variation where the optional component, which could be labeled as an accessory, is an impulse purchase. Such different modeling assumption allows a comparison, which highlights the different roles played by the optional component in the optimal pricing of configurable products. In Chapter II, where the customer's decision to purchase from the firm depends on the prices and assortments of both the required and optional components, the optimal effective margin is zero for the optional component, and the firm makes money on the required component. In contrast, this chapter finds that when the customer's decision to purchase from the firm depends only on the required component's assortment and prices (e.g., when the optional component is an impulse purchase), the optional component is now sold at a positive effective margin.

In Chapter II, it is assumed that a customer purchases the optional component only if she purchases the required component. However, in a more general setting, there may be customers shopping for the optional component only, maybe because they bought the required component earlier and they would now like to purchase the optional component as well. For example, a customer who bought a cell phone at an earlier time may now be in the market to buy a carrying case. This chapter presents a model where the firm will observe an additional segment of customers who are interested in purchasing the optional component only (e.g. customers interested in the cell phone case only). In such a setting, this chapter shows that the firm will charge lower prices on the required component and higher prices on the optional components compared to the setting described in Chapter II. Moreover, the firm will be better off if it is able to price discriminate between those customers who are purchasing the optional component only and those customers who are purchasing the optional component in addition to the required component.

One of the important contributions of Chapter II is the structure of the pricing policy namely, the optimality of using the same effective margin for all variants of a component. In this chapter it is demonstrated that the same structure holds when the model is generalized so that both components are optional (instead of one of them being required).

The rest of the chapter is organized as follows. Section 3.2, modifies the model in Chapter II to explore the case where the optional component is an impulse purchase.

Section 3.3 builds on the model presented in the previous chapter to allow for a new customer segment: One interested in purchasing only the optional component. Section 3.4 presents a generalization of the model in Chapter II. The concluding remarks are presented in Section 3.5. The proofs are included in Appendix B.

3.2. The Optional Component as an Impulse Purchase

In Chapter II, the customer chooses a configuration based on the assortment and prices of both the required and optional components. This chapter explores an extension where the customer's decision to purchase from the firm is based on only the assortment and prices of the required component. Only if the customer decides to purchase the required component, will he assess the price and assortment of the optional component and choose what variant of the optional component to purchase, if any. This captures a scenario where customers buy the optional component on impulse, which is likely to be the case when, for example, customers learn about the accessories of a product only after arriving at the store.

In keeping with the notation of the previous chapter, let S_R denote the set of variants for the required component and S_O the set of variants for the optional component. In addition, let p_{Ri} and p_{Oj} denote the price of required component's variant $i \in S_R$ and the price of optional component's variant $j \in S_O$, respectively.

First, the customer makes the utility-maximizing choice from the variants of the required component, including the choice of not purchasing at all. Let $\widehat{\alpha}_0$ be the customer's expected utility of not purchasing from the firm. Let $\widehat{q}_{Rk}(S_R, \boldsymbol{p}_R)$ denote the probability that a customer decides to purchase variant k of the required component and $\widehat{q}_0(S_R, \boldsymbol{p}_R)$ the probability that the customer chooses not to purchase. Following the MNL model, $\widehat{q}_{Rk}(S_R, \boldsymbol{p}_R)$ and $\widehat{q}_0(S_R, \boldsymbol{p}_R)$ are given by:

$$\widehat{q}_{Rk}(S_R, \boldsymbol{p}_R) = \frac{\exp(\alpha_{Rk} - p_{Rk})}{\exp(\widehat{\alpha}_0) + \sum_{i \in S_R} \exp(\alpha_{Ri} - p_{Ri})} \text{ for all } k \in S_R, \tag{3.1}$$

$$\widehat{q}_0(S_R, \boldsymbol{p}_R) = \frac{\exp(\widehat{\alpha}_0)}{\exp(\widehat{\alpha}_0) + \sum_{i \in S_R} \exp(\alpha_{Ri} - p_{Ri})}. \tag{3.2}$$

The aggregate random demand for variant k is now modeled as the purchase probability multiplied by a random error term:

$$D_{Rk}(S_R, \boldsymbol{p}_R) = \widehat{q}_{Rk}(S_R, \boldsymbol{p}_R)\epsilon_{Rk},$$

where ϵ_{Rk}'s are i.i.d normal random variables with mean one and standard deviation σ_{Rk}.

As for the optional component, recall that the customer purchases it only if he has already bought a variant of the required component. Therefore, the probability that a customer purchases variant k of the optional component is the probability that the customer purchases a variant of the required component, multiplied by the probability that variant $k \in S_O$ is the customer's utility-maximizing choice among all variants of the optional component, i.e., for $k \in S_O$:

$$\widehat{q}_{Ok}(S_R, S_O, \boldsymbol{p}_R, \boldsymbol{p}_O) = \left[\frac{\sum_{i \in S_R} \exp(\alpha_{Ri} - p_{Ri})}{\exp(\widehat{\alpha}_0) + \sum_{i \in S_R} \exp(\alpha_{Ri} - p_{Ri})}\right] \frac{\exp(\alpha_{Ok} - p_{Ok})}{\exp(\alpha_{Oo}) + \sum_{j \in S_O} \exp(\alpha_{Oj} - p_{Oj})}. \tag{3.3}$$

Then, the aggregate random demand for variant k of the optional component is

$$D_{Ok}(S_R, S_O, \boldsymbol{p}_R, \boldsymbol{p}_O) = \widehat{q}_{Ok}(S_R, S_O, \boldsymbol{p}_R, \boldsymbol{p}_O)\epsilon_{Ok},$$

where ϵ_{Ok}'s are i.i.d normal random variables with mean one and standard deviation σ_{Ok}.

Given this new demand model, one could follow the same steps as in the previous section to show that the firm's expected profit (revenue minus purchase costs minus overage and underage costs), denoted by $\widehat{\Pi}(S_R, S_O, \boldsymbol{p}_{\scriptscriptstyle R}, \boldsymbol{p}_{\scriptscriptstyle O})$, simplifies to:

$$\widehat{\Pi}(S_R, S_O, \boldsymbol{p}_{\scriptscriptstyle R}, \boldsymbol{p}_{\scriptscriptstyle O}) =$$
$$\sum_{k\in S_R}(p_{\scriptscriptstyle Rk} - c_{\scriptscriptstyle Rk} - \gamma_{\scriptscriptstyle Rk})\widehat{q}_{\scriptscriptstyle Rk}(S_R, \boldsymbol{p}_{\scriptscriptstyle R}) + \sum_{k\in S_O}(p_{\scriptscriptstyle Ok} - c_{\scriptscriptstyle Ok} - \gamma_{\scriptscriptstyle Ok})\widehat{q}_{\scriptscriptstyle Ok}(S_R, S_O, \boldsymbol{p}_{\scriptscriptstyle R}, \boldsymbol{p}_{\scriptscriptstyle O}). \quad (3.4)$$

Next we explore the firm's pricing decisions under a given assortment, $S_R \cup S_O$. As the following proposition states, the optimal solution continues to exhibit the equal effective margin property as defined in the previous chapter.

Proposition 10. *Given an assortment $S_R \cup S_O$, the optimal pricing policy is such that all variants of a given component $n \in \{R, O\}$ have the same effective margin. The optimal effective margin for the optional component is the unique $m_{\scriptscriptstyle O}$ that satisfies*

$$m_{\scriptscriptstyle O} = \frac{\exp(\alpha_{\scriptscriptstyle Oo}) + \sum_{j\in S_O}\exp(\eta_{\scriptscriptstyle Oj} - m_{\scriptscriptstyle O})}{\exp(\alpha_{\scriptscriptstyle Oo})}.$$

Given the optimal value of $m_{\scriptscriptstyle O}$, the optimal effective margin for the required component is the unique $m_{\scriptscriptstyle R}$ that satisfies the following equality:

$$m_{\scriptscriptstyle R} = \frac{1}{\widehat{q}_0(S_R, m_{\scriptscriptstyle R})} - m_{\scriptscriptstyle O} + 1,$$

where $\widehat{q}_0(S_R, m_{\scriptscriptstyle R}) := \frac{\exp(\widehat{\alpha}_0)}{\exp(\widehat{\alpha}_0)+\sum_{k\in S_R}\exp(\eta_{Rk}-m_R)}$.

Notice from the proposition that the optimal margin for the optional component is no longer zero, unlike the result in Proposition 2. In fact, the optimal margin prescribed for the optional component is the margin that optimizes the firm's profit from a customer who is already at the store and is now deciding what variant of the optional component to purchase, if any.

As for the required component, it is interesting to observe that the firm may now

	Assortment			
	Required Component		**Optional Component**	
	Variant 1	Variant 2	Variant 1	Variant 2
η (approx.)	3.4294	3.4294	6.7860	7.7860
margins	-1.5731		5.5780	

Table 3.1: When the optional component is an impulse purchase, the required component's margin may be negative at optimality. Parameters: $\alpha_{o_o} = 1$, $\widehat{\alpha}_0 = 5$.

choose to sell it at a loss. Such an example is provided in Table 3.1. In this example, the optional component is very lucrative for the firm, because the expected surpluses of the optional components are much higher than those of the required components, giving the firm more room for higher margins on the optional components. Now that the optional component is so lucrative, the firm sells the required component at a loss in a bid to attract more customers and to drive even more demand toward the optional component. The firm chooses to do so even though selling the required component at a loss means that the firm will lose money on the customers who eventually decide not to purchase the optional component. This type of setting where the optional component is the profit driver is not uncommon. For example, it is reported that motorcycle dealers in the UK are selling the bikes themselves at small gross margins, but they are driving up their profits by carrying accessories, alarm systems, protective gear and apparel, which are "far more lucrative" with "far higher" profit margins than selling or servicing vehicles.[1]

Next, we explore the firm's assortment selection problem given that the firm will price any chosen assortment optimally. As before, in the absence of any constraints on the size of the assortment, the firm would be better off by offering all possible variants. When the firm has to choose between two variants of the same component, the same decision rule as in the base model (in Chapter II) applies: Add the variant

[1] AM-online.com, "Motorcycles Accessories: Dealers make the most of those little extras," July 8, 2008. Retrieved from http://www.am-online.com/law/story/?nID=42897361 on September 25, 2008.

	Current Assortment				Available Variants to Add	
	Required Component		Optional Component		Required Component	Optional Component
	Variant 1	Variant 2	Variant 1	Variant 2	Variant A	Variant B
η	5.5720	4.1544	2.5720	1.6544	1.9772	1.5772

Table 3.2: In this example the firm would add Variant B, an optional component, under the base model, but switches to Variant A, a required component, when the optional component is an impulse purchase. Parameters: $\widehat{\alpha}_0 = 1$, and $\alpha_{O_o} = 2$.

with the higher surplus, as stated in Proposition 11.

Proposition 11. *Suppose the firm faces the problem described in Section 3.2, that is, the firm's expected profit is* $\widehat{\Pi}(S_R, S_O, \boldsymbol{p}_R, \boldsymbol{p}_O)$, *given by* (3.4). *Consider component* $n \in \{R, O\}$. *Suppose that there is a set of variants of component* n, *denoted by* V, *that may be added to the current assortment. If the firm is allowed to add only one of these candidates, then the firm will add the candidate with the highest surplus, i.e., the firm will add variant* k *such that* $\eta_{nk} \geq \eta_{nl}$ *for all* $l \in V$.

In other words, Proposition 4 continues to hold in the case where the optional component is an impulse purchase. When the firm has to choose between a variant of the required component and a variant of the optional component, the decision rule that was optimal in the base model (see Proposition 5) is no longer optimal. For instance, in the example shown in Table 3.2, the firm is currently offering an assortment with two variants each of the required and the optional component. When choosing between Variant A of the required component and Variant B of the optional component, this firm would add Variant B under the base model, because the total attraction of the product configurations that are enabled by Variant B is larger. However, the firm switches to Variant A when the optional component is an impulse purchase, because of the asymmetry that now exists between the optional and required components: The required component contributes to the probability of a customer purchasing from the firm, but the optional component does not. Hence, the required component gains an edge that makes it more desirable even when the total product attraction favors the optional component.

3.3. Customers Purchasing the Optional Component Only

The model considered in this section allows a segment of customers that are shopping for the optional component exclusively. This is likely to be the case, for example, when the purchase of the optional component is separated in time from the purchase of the required component.

Let $\beta \in (0, 1)$ denote the fraction of customers that are shopping for both the required and optional components (indexed as segment 1) and $1 - \beta$ the fraction of customers that are shopping for the optional component exclusively (indexed as segment 2). The entire demand for the required component originates from segment 1. For an individual who belongs to segment 1, the decision model is the one described in the base model. Therefore, the probability that a segment-1 customer purchases variant k of the required component is $q_{Rk}(S_R, S_O, \boldsymbol{p}_R, \boldsymbol{p}_O)$. Given that an individual belongs to segment-1 with probability β, the stochastic, aggregate demand for variant k of the required component, $D_{Rk}(S_R, S_O, \boldsymbol{p}_R, \boldsymbol{p}_O)$, is:

$$D_{Rk}(S_R, S_O, \boldsymbol{p}_R, \boldsymbol{p}_O) = \beta q_{Rk}(S_R, S_O, \boldsymbol{p}_R, \boldsymbol{p}_O)\epsilon_{Rk},$$

where ϵ_{Rk}'s are i.i.d normal random variables with mean one and standard deviation σ_{Rk}. Notice that the change from the previous chapter's model is that the demand for a required component is now a fraction β of what it used to be, reflecting the fact that only a fraction β of the consumer population is shopping for the required component.

The demand for variant k of the optional component originates from both segments. A segment-1 customer purchases variant k of the optional component with probability $q_{Ok}(S_R, S_O, \boldsymbol{p}_R, \boldsymbol{p}_O)$. On the other hand, an individual who belongs to segment 2 decides whether or not to purchase from the firm on the basis of the assortment and prices of the optional component only (without any regard to the prices and assortment of the required component). Letting $\widetilde{\alpha}_0$ denote the expected utility of an exogenous alternative for customers only interested in the optional component (i.e., the expected utility of not purchasing from the firm), the MNL choice model yields the following expression for the probability that a segment-2 customer purchases variant k of the optional component:

$$\widetilde{q}_{Ok}(S_O, \boldsymbol{p}_O) = \frac{\exp(\alpha_{Ok} - p_{Ok})}{\exp(\widetilde{\alpha}_0) + \sum_{k \in S_O} \exp(\alpha_{Ok} - p_{Ok})}. \tag{3.5}$$

Weighing the purchase probabilities, $q_{Ok}(S_R, S_O, \boldsymbol{p}_R, \boldsymbol{p}_O)$ and $\widetilde{q}_{Ok}(S_O, \boldsymbol{p}_O)$, with the fractions of segments 1 and 2, β and $1 - \beta$, respectively, a randomly chosen customer purchases variant k of the optional component with probability

$$\beta q_{Ok}(S_R, S_O, \boldsymbol{p}_R, \boldsymbol{p}_O) + (1 - \beta)\widetilde{q}_{Ok}(S_O, \boldsymbol{p}_O).$$

Once again, we use a multiplicative error term to model stochastic, aggregate demand for variant k of the optional component, denoted by $D_{Ok}(S_R, S_O, \boldsymbol{p}_R, \boldsymbol{p}_O)$:

$$D_{Ok}(S_R, S_O, \boldsymbol{p}_R, \boldsymbol{p}_O) = \Big(\beta q_{Ok}(S_R, S_O, \boldsymbol{p}_R, \boldsymbol{p}_O) + (1 - \beta)\widetilde{q}_{Ok}(S_O, \boldsymbol{p}_O)\Big)\epsilon_{Ok},$$

where ϵ_{Ok}'s are i.i.d normal random variables with mean one and standard deviation σ_{Ok}.

Given the demand model described above, the firm's expected profit is given by:

$$\begin{aligned}\widetilde{\Pi}(S_R, S_O, \boldsymbol{p}_R, \boldsymbol{p}_O) =& \beta \sum_{n \in \{R,O\}} \sum_{k \in S_n} (p_{nk} - c_{nk} - \gamma_{nk}) q_{nk}(S_R, S_O, \boldsymbol{p}_R, \boldsymbol{p}_O) \\ &+ (1 - \beta) \sum_{k \in S_O} (p_{Ok} - c_{Ok} - \gamma_{Ok}) \widetilde{q}_{Ok}(S_O, \boldsymbol{p}_O). \end{aligned} \tag{3.6}$$

The following proposition states that any optimal price vector must satisfy the equal effective margin property as was the case in the previous chapter.

Proposition 12. *The optimal prices must satisfy first-order-conditions of the firm's expected profit,* $\widetilde{\Pi}(S_R, S_O, \boldsymbol{p}_{\scriptscriptstyle R}, \boldsymbol{p}_{\scriptscriptstyle O})$*, and are such that, for a given component* $n \in \{R, O\}$*, all variants* k *of component* n *will have the same effective profit margin, i.e.,* $m_{nk} := p_{nk} - c_{nk} - \gamma_{nk} = m_n$ *for all* $k \in S_n$*.*

Using the property in Proposition 12 we rewrite the firm's profit function, given by (3.6), as a function of the margins, denoted by $m_{\scriptscriptstyle R}$ and $m_{\scriptscriptstyle O}$, instead of a function of the price vectors:

$$\widetilde{\Pi}(S_R, S_O, m_{\scriptscriptstyle R}, m_{\scriptscriptstyle O}) = \beta \left[m_{\scriptscriptstyle R} \sum_{k \in S_R} q_{\scriptscriptstyle Rk}(S_R, S_O, m_{\scriptscriptstyle R}, m_{\scriptscriptstyle O}) + m_{\scriptscriptstyle O} \sum_{k \in S_O} q_{\scriptscriptstyle Ok}(S_R, S_O, m_{\scriptscriptstyle R}, m_{\scriptscriptstyle O}) \right] + (1-\beta) m_{\scriptscriptstyle O} \sum_{k \in S_O} \widetilde{q}_{\scriptscriptstyle Ok}(S_O, m_{\scriptscriptstyle O}), \tag{3.7}$$

where $\widetilde{q}_{\scriptscriptstyle Ok}(S_O, m_{\scriptscriptstyle O}) = \frac{\exp(\eta_{Ok} - m_O)}{\exp(\widetilde{\alpha}_0) + \sum_{k \in S_O} \exp(\eta_{Ok} - m_O)}$. The profit function in (3.7) is not well behaved in that there may exist multiple margin pairs that satisfy the first-order conditions. In other words, the first-order conditions are not sufficient for optimality. See Figure 3.1 for a numerical example with two local maxima.

Even though the profit function is not necessarily well-behaved, it is still tractable enough to obtain a number of comparisons with the model considered in Chapter II. The following proposition describes important properties of the optimal margins for the optional and required components.

Proposition 13. *For any given assortment,* $S_R \cup S_O$*, the optimal margin for the optional component,* $m_{\scriptscriptstyle O}$*, is always strictly positive. On the other hand, the optimal margin for the required component may be negative.*

In the model where the optional component was bought only as part of a product configuration, the firm sold the optional component at effective cost. Now that there

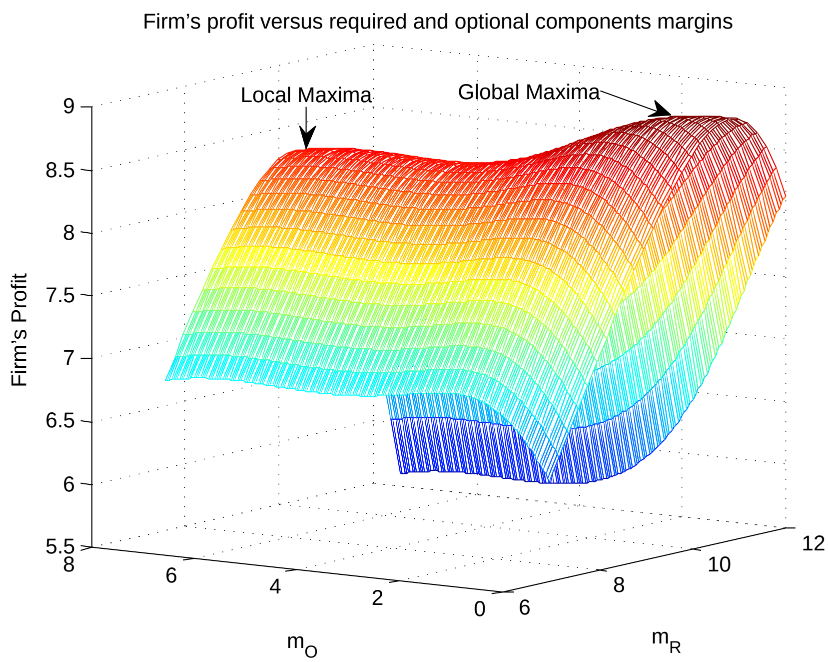

Figure 3.1: The firm's expected profit as a function of the two margins, m_R and m_O. This example shows that the profit function may have multiple stationary points, each of which is a local optimum. Hence, the first order conditions are not sufficient for optimality. Parameters: $\beta = 0.9$, $\alpha_{O_0} = 8$, $\alpha_0 = 5$, $\widetilde{\alpha}_0 = 2$, $\eta_{R_1} = 8.57$, $\eta_{R_2} = 7.57$, $\eta_{O_1} = 8.57$ and $\eta_{O_2} = 7.57$.

	Assortment			
	Required Component		**Optional Component**	
	Variant 1	Variant 2	Variant 1	Variant 2
η (approx.)	2.15	2.15	8.78	9.28
margins	-0.3829		7.6483	

Table 3.3: When there is a segment of customers shopping for the optional component only, the required component's margin may be negative at optimality. Parameters: $\beta = 0.1$, $\alpha_{o_o} = 2$, $\alpha_0 = 5$ and $\widetilde{\alpha}_0 = 1$.

is a segment of customers that are shopping for the optional component only (segment 2), this is no longer the case: The firm would like to charge a positive margin on the optional component to make money from segment-2 customers.

The firm may choose to sell the required component at a loss. Such an example is provided in Table 3.3. In this example, the optional component is very lucrative for the firm, not only because the expected surpluses of the optional components are much higher than those of the required components, but also because segment 2, which consists of customers shopping only for the optional component, is very large (90% of the population) compared to segment 1. Therefore, the firm sells the required component at a loss so as to drive more demand toward the optional component.

The behavior shown in Table 3.3 is an especially colorful example of a more general behavior: The existence of segment 2 drives up the margin for the optional component, compared to the model considered in Chapter II. Thus, the firm now wishes to sell more of the optional component, which can be achieved by slightly reducing the margin of the required component compared to what it was under the base model. The next proposition formalizes this result.

Proposition 14. *For a given assortment, $S_R \cup S_O$, when there is a segment shopping for the optional component only, the optimal margin for the required component is lower than that in the base model.*

In the presence of two segments, the firm would benefit from price discriminating between the two segments, if it were possible. In particular, given that segment 1 customers exhibit the same behavior as those under the model with only segment-1 customers, the firm would like to sell the optional component at cost to segment 1, but it would like to charge a positive margin on the optional component when selling it to segment-2 customers. Such price discrimination can be implemented by offering discounts on the optional component to customers who also buy the required component. This type of price discrimination is common in practice. For example, in June 2008, Best Buy was offering a discount on printers (optional component) with the purchase of a personal computer (required component).

Next, consider the assortment selection of a firm in the presence of two segments. Under the base model, given two variants of the same component, the firm always chooses to add the one with the higher expected surplus (see Proposition 4). The same result is expected to hold when there are two segments. Even though numerical experiments suggest this to be the case, the result is not easy to prove due to the complicated nature of the pricing problem for a given assortment. In addition, under the model with only segment-1, given one variant of each component, the firm chooses to add the variant that has a higher total attraction across the product configurations that it enables (see Proposition 5). This is no longer true under the two-segment model. Consider the numerical example in Table 3.4. In this example, the firm can add either variant A, a required component, or variant B, an optional component. Even though the total product attraction (as defined in Proposition 5) favors variant A of the required component, the firm is better off by adding variant B of the optional component. This is not surprising, because adding another variant of the required component is not going to improve the profits from the second segment who is shopping for the optional component only.

3.4. Generalization to an assortment with two complementary components

	Current Assortment				Available Variants to Add	
	Required Component		Optional Component		Required Component	Optional Component
	Variant 1	Variant 2	Variant 1	Variant 2	Variant A	Variant B
η	3.4294	3.4294	4.5720	4.5720	5.577	6.577

Table 3.4: In this example the firm would add Variant A, a required component, but switches to Variant B, an optional component, when there is a segment shopping for the optional component only. Parameters: $\beta = 0.5$, $\alpha_0 = 1$, $\widetilde{\alpha}_0 = 2$ and $\alpha_{O0} = 1$.

Chapter II considers a product formed by putting together a required and an optional component. This section presents a more generalized problem where a product is a combination of two complementary components that can be purchased separately from each other. In other words, there is no required component in this model; the customer chooses whether or not to purchase each component. In such a setting the probability that the customer will pick configuration $[ij]$, $q^G_{[ij]}(S_R, S_O, \boldsymbol{p}_R, \boldsymbol{p}_O)$ is now given by,

$$q^G_{[ij]}(S_R, S_O, \boldsymbol{p}_R, \boldsymbol{p}_O) = \frac{v_{[ij]}(p_{Ri}, p_{Oj})}{\exp(\alpha^G_0) + \sum_{i \in S_R} \sum_{j \in S_O \cup \{0\}} v_{[ij]}(p_{Ri}, p_{Oj}) + \sum_{j \in S_O} v_{[0j]}(p_{Ri}, p_{Oj})}, \tag{3.8}$$

for $i \in S_R \cup \{0\}$, $j \in S_O \cup \{0\}$, where $v_{[0j]} := \exp(\alpha_{Ro} + \alpha_{Oj} - p_{Oj})$ and α_{Ro} is the utility contribution of the not purchasing the required component. Likewise the probability that the customer will not purchase any of the products offered by the firm is

$$q_0(S_R, S_O, \boldsymbol{p}_R, \boldsymbol{p}_O) = \frac{\exp(\alpha^G_0)}{\exp(\alpha^G_0) + \sum_{i \in S_R} \sum_{j \in S_O \cup \{0\}} v_{[ij]}(p_{Ri}, p_{Oj}) + \sum_{j \in S_O} v_{[0j]}(p_{Ri}, p_{Oj})}. \tag{3.9}$$

The probability that a customer chooses variant i of the required component, denoted by q_{Ri}, is then given by the sum of the probabilities that a customer chooses a configuration that uses variant i of the required component:

$$q^{G}_{Ri}(S_R, S_O, \boldsymbol{p}_R, \boldsymbol{p}_O) \;=\; \sum_{j \in S_O \cup \{0\}} q_{[ij]}(S_R, S_O, \boldsymbol{p}_R, \boldsymbol{p}_O), i \in S_R. \tag{3.10}$$

Similarly, the probability that a customer chooses variant j of the optional component, denoted by q_{Oj}, is

$$q^{G}_{Oj}(S_R, S_O, \boldsymbol{p}_R, \boldsymbol{p}_O) \;= \textstyle\sum_{i \in S_R \cup \{0\}} q_{[ij]}(S_R, S_O, \boldsymbol{p}_R, \boldsymbol{p}_O), j \in S_O. \tag{3.11}$$

We use the purchase probabilities in (3.10) and (3.11) as the starting point to model the aggregate demand for a variant. Specifically, it is assumed that the aggregate, random demand for variant k of component $n = \{R, O\}$, denoted by $D^{G}_{nk}(S_R, S_O, \boldsymbol{p}_R, \boldsymbol{p}_O)$, is given by

$$D^{G}_{nk}(S_R, S_O, \boldsymbol{p}_R, \boldsymbol{p}_O) \sim q^{G}_{nk}(S_R, S_O, \boldsymbol{p}_R, \boldsymbol{p}_O)\epsilon_{nk} \text{ for } k \in S_n, n \in \{R, O\},$$

where ϵ_{nk}'s are i.i.d normal random variables with mean one and standard deviation σ_{nk}. The firm's profit function is similar to that in the previous chapter (see (2.8)) but uses $q_{nk}(S_R, S_O, \boldsymbol{p}_R, \boldsymbol{p}_O)$ instead of $q^{G}_{nk}(S_R, S_O, \boldsymbol{p}_R, \boldsymbol{p}_O)$

$$\Pi^{G}(S_R, S_O, \boldsymbol{p}_R, \boldsymbol{p}_O) = \sum_{n \in \{R,O\}} \sum_{k \in S_n} (p_{nk} - c_{nk} - \gamma_{nk}) q^{G}_{nk}(S_R, S_O, \boldsymbol{p}_R, \boldsymbol{p}_O). \tag{3.12}$$

We next analyze the pricing problem given an assortment of variants.

The following proposition states that the equal effective margin property continues to hold.

Proposition 15. *Consider component $n \in \{R, O\}$. Let $m_{nk} := p_{nk} - c_{nk} - \gamma_{nk}$ denote the effective profit margin of variant k of component n. At any pair of optimal price vectors $\boldsymbol{p}_R$ and $\boldsymbol{p}_O$, all variants k of component n will have the same effective profit* *margin, i.e.,* $m_{nk} = m_n$ *for all* $k \in S_n$*.*

Proposition 15 reduces the pricing decision to only two margins. In contrast to the previous chapter, in this generalized version both components are now sold at positive margin.

3.5. Conclusion

This chapter extends Chapter II by exploring alternative models and generalizations. Here three alternative scenarios are presented and the optimal pricing and assortment policies are explored for a two-category assortment under demand uncertainty.

In Chapter II, the customer decides what product to purchase based on the entire assortment and prices. This chapter studies an extension where the customer's decision to purchase from the firm depends solely on the required component's assortment and prices, thereby rendering the optional component an impulse purchase. In the model where the customer's decision to purchase from the firm depends on the prices and assortments of both the required and optional components (Chapter II), the optimal effective margin is zero for the optional component, and the firm makes money on the required component. In contrast, when the customer's decision to purchase from the firm depends only on the required component's assortment and prices (e.g., when the optional component is an impulse purchase), the optional component is now sold at a positive effective margin. In such a case, if the optional component is lucrative enough, the firm may even choose to sell the required component at a loss so as to drive more demand toward the optional component. Therefore, when selling configurable products, the firm's optimal pricing strategy depends very much on how the firm is marketing the optional component.

In another model variation presented in this chapter, a new customer segment is added to the customer population described in Chapter II. This new customer segment is assumed to be shopping only for the optional component. When such a segment exists, the optional component is sold at a positive effective margin. However, the firm would ideally sell the optional component at zero effective margin to the segment that is shopping for both components. This suggests that the firm could benefit from a price discrimination strategy where all customers who purchase the required component receive a discount on the optional component.

The motivation of Chapter II was to model the configurable nature of some products. In such a case, the definition of two distinct types of components (i.e. required and optional components) was useful. However, this chapter explores an alternative formulation that removes this distinction and allows the customer to purchase or not to purchase either component. Under this generalized model, this chapter finds that the optimal pricing rule is the same as before: All variants within the same category should be priced so that all have the same "equal effective margin".

Chapter IV

Assortment Selection and Pricing in the presence of Dual Sales Channels

4.1. Introduction

The previous two chapters studied assortment selection and pricing problems faced by a single seller. In a supply chain setting, the pricing of an assortment is a critical decision not only for the seller itself, but also for its supplier. This pricing question becomes even more critical in supply chains where the manufacturer is both a supplier to and competitor of the retailer. Take the relationship between Sony and Best Buy as an example; specifically the Sony VAIO BZ560 line of laptop computers. SonyStyle.com sells, at the bare minimum, 45 different configurations for the Sony VAIO BZ560 computer. In contrast, Best Buy offers the customer only one VAIO BZ560 configuration. Motivated by such channel relationships, in this chapter we consider the pricing and assortment selection problems that arise in a supply chain where the manufacturer uses dual channels.

The marketing literature suggests that a store's assortment is almost as important as its price profile and location in driving the store traffic, see, for example, Zhang et al. (2009) and the references therein. Hence, in this chapter, we model the customer's choice of channel as a function of the assortment and prices offered by both channels. In particular, we use the nested logit model to capture the consumer choice: The customer first chooses the channel she wants to purchase from (if any) and, subsequently, decides which product to purchase from her chosen channel. This demand model allows us to account for the effect of both channels' assortments and prices on the demand observed by each channel.

There are several assumptions one can make regarding who carries inventory in this dual-channel structure and where. In keeping with the motivating example, this chapter considers a manufacturer (e.g. Sony) who sells a build-to-order product through its direct channel while meeting the orders from the retailer. As for the retailer (e.g. Best Buy), it is assumed that it keeps inventory of the final (assembled) products and meets the observed demand with this inventory. Because the retailer must make stock level decisions before observing the customer demand, there exists the possibility of demand-inventory mismatch at the retailer. Hence, the model accounts for the inventory-related costs associated with the demand-inventory mismatch at the retailer. In the case where the demand at the retailer exceeds the inventory level, the retailer is allowed to procure additional products from the manufacturer. For example, if the demand of a specific VAIO BZ560 computer is greater than the amount Best Buy had in stock, then Best Buy could order additional units from Sony to meet the excess demand.

A strength of this model is that it allows us to analyze the effect of inventory-related costs on the pricing decisions. The selling prices charged by the manufacturer's direct channel and the retailer follow an equal effective margin property similar to that described in Chapter II. In addition, this chapter characterizes the optimal wholesale prices that the manufacturer charges to the retailer. For example, we find that, everything else being equal, the manufacturer will charge lower wholesale prices for variants with larger demand variability.

This setting where the manufacturer is able to sell through two channels, can be used to study if the manufacturer benefits from selling through two separate channels. In practice one can find both success and failure stories about engaging in dual (or hybrid) sales channel strategies. For example, although Dell has been very successful selling directly to customers, in 2006 they saw their profits and market share decline significantly. The reaction to this decline came in 2007, when Dell successfully embraced a hybrid strategy by adding resellers to their channel mix.[1] In contrast, by 2008, Gateway, another computer business, moved from engaging in dual sales channels to only selling indirectly to customers.[2] Inspired by these examples, in this chapter we investigate the benefits of engaging in dual-sales channels.

For a build-to-order manufacturer and a retailer engaging in dual sales channels, another relevant question is what assortment to offer through the retailer. More often than not, the retailer offers only a subset of what the manufacturer's direct channel offers as indicated by the Sony VAIO example discussed earlier. Depending on the power structure in the supply chain, the retailer's assortment can be decided by the retailer itself or it could be dictated by the manufacturer. We study both options. Moreover, we study different sequences of decision-making that allow various scenarios regarding the timing of assortment and pricing decisions.

We first explore problems where there is no fixed cost for offering a product variant and there is no capacity limitation on the number of variants to carry. In such cases, we find that if the manufacturer's pricing decisions precede the retailer's assortment selection, both the retailer and the manufacturer will be best off by offering every available product. However, if the retailer's assortment selection precedes the manufacturer's pricing decisions, then the retailer may strategically leave certain variants out of its assortment.

The chapter also studies cases where there is a fixed cost for carrying a variant or where there is a limit on the size of the assortment. When there is a limit on the

[1] Kellogg Insight: Focus on Research, A new strategy for Dell. Retrieved from `http://insight.kellogg.northwestern.edu/index.php/Kellogg/article/a_new_channel_strategy_for_dell` on 9/29/2009.

[2] Betanews.com, End of an era: Gateway stops selling PCs directly to customers. Retrieved from `http://www.betanews.com/article/End-of-an-era-Gateway-stops-selling-PCs-directly-to-customers/1217271457` on 9/29/2009.

size of the retailer's assortment, we find that the manufacturer and the retailer may disagree about which product to sell through the retailer, because the manufacturer prefers products with higher demand variability while the retailer prefers products with lower demand variability. When both the manufacturer and the retailer incur a fixed cost for offering a product through the retailer, we find that the manufacturer's preferred assortment is larger than the retailer's, even when the manufacturer's fixed cost per product is slightly higher.

The rest of the chapter is organized as follows. In the following section we review the related literature. Section 4.3 describes the model. The pricing problem is analyzed in Section 4.4 and the benefit of adding an indirect channel is analyzed in Section 4.5. In Section 4.6 we explore the assortment decisions. Finally, Section 4.7 summarizes the results and contributions. The proofs are included in Appendix C.

4.2. Literature Review

The analysis of distribution systems has received considerable attention in the operations and marketing literature. In the context of the broad literature on distribution systems, the problem studied in this chapter belongs to the subset that deals with multiple-channel distribution systems, in which a supplier sells through more than one channel. Cattani et al. (2004) present a recent and extensive literature survey on the coordination of multiple channels.

The multiple-channel distribution system studied in this chapter belongs to the narrower subset of dual-channel systems, in which the supplier sells through two channels only. The interest in dual channel systems (which have also been labeled as "hybrid distribution") dates back to as early as 1965 (Preston and Schramm, 1965). However, the interest in dual-channel systems has been revived in recent years due to the tradeoffs presented by e-commerce. Agatz et al. (2008), Swaminathan and Tayur (2003) and Tsay and Agrawal (2004) review the literature dealing with multiple-channels that arise in the e-business setting.

One could separate between two streams of work on the dual-channel distribution systems. The first stream of work deals with questions surrounding how much to stock and where to keep that stock in the distribution system, see for example Boyaci (2005), Alptekinoğlu and Tang (2005), Chiang and Monahan (2005), Moinzadeh (2003) and Seifert et al. (2006). Another example is Zhao (2008), which adds the pricing problem to the inventory decision. In commonality with this work, this chapter takes into consideration inventory costs of the products offered. However, we simplify the inventory aspect of the problem by assuming that the inventory levels are chosen to satisfy an exogenously fixed service level and the only stock-keeping location is the retailer. These assumptions are in line with our motivating examples, which revolve around build-to-order manufacturers adding a retailer to their channel mix.

The second stream of work in dual-channel systems deals with how the prices should be set and/or coordinated in this distribution system., e.g. Cattani et al. (2006), Chiang and Chhajed (2005), Kumar and Ruan (2006) and Rhee and Park (2000). This chapter is related to this second stream of research in that we study, among other things, the pricing decisions in a dual-channel system. Earlier work that is particularly related to the type of pricing problems that arise in this chapter are Chiang et al. (2003) and Tsay and Agrawal (2004), who treat the manufacturer's channel structure as a decision variable, i.e., manufacturer decides whether or not to use a dual-sales channel. They study the effects of the channel structure on the pricing strategies and profits. Tsay and Agrawal (2004) build on Chiang et al. (2003) by incorporating sales effort and the unit cost of supplying an item; however, they restrict the selling prices to be the same in both channels. This chapter differs from Chiang et al. (2003) and Tsay and Agrawal (2004) in a number of ways. In particular, this chapter incorporates demand uncertainty and explicitly models the inventory costs associated with demand-stock mismatches at the retailer.

This chapter also addresses the question of whether it is always beneficial to sell through dual-channels. There has been some work on the question of channel design, especially when considering the distribution costs, e.g. Rangan (1987) and Chiang et al. (2003). In our case, we do not explicitly model distribution costs but we incorporate inventory costs and compare the expected profits for the manufacturer under both scenarios.

What separates this chapter from all the work that came before it is that we model a dual-channel system in which each channel sells an assortment of substitutable products, and we analyze the assortment decisions.

4.3. Model Description

Consider a product that can be purchased through two channels: directly from the manufacturer and through an independent retailer. Take a Dell Inspiron desktop computer as an example: A customer can purchase a Dell Inspiron computer directly from Dell.com (the manufacturer's direct channel) where the customer configures the computer by choosing from at least 2 models, 8 colors, 7 processors, 3 operating systems, 5 memory choices, 5 hard drive capacities (at least 8,400 different variants of the Inspiron). On the other hand, the customer may choose to purchase a Dell Inspiron from Best Buy (retailer) by choosing from five different pre-configured Inspiron computers. Notice that Best Buy is offering a subset of the variants that could be purchased from Dell.com. In keeping with this scenario, we model a build-to-order manufacturer, who offers an assortment of all possible variants; we denote this set of variants with S^M. The retailer in our model, on the other hand, offers a subset of the variants in S^M and keeps stock of the variants it carries. Let S^R denote the set of variants carried by the retailer.

In our model, the pricing decisions available to the manufacturer are the prices for the direct channel, p_k^M for variant $k \in S^M$ (i.e. prices charged to the customers who purchase from the manufacturer), and the wholesale price charged to the retailer, w_k

for variant $k \in S^R$. On the other hand, the retailer chooses the price it charges to its customers. Let p_k^R denote the retailer's selling price for variant $k \in S^R$.

4.3.1 Customer Demand Model

Consider an individual customer. We model the individual customer's decision building on the nested-logit model. In the nested-logit model the customer choice is modeled as a sequential process, where the customer first chooses one of many 'nests' of items and, conditional on the choice of the nest, the customer chooses what specific item to purchase from the nest. The choices in each stage follow a multinomial logit choice model (MNL). The nested logit model leads to a closed form expression for the probability that a customer purchases a specific item in a given nest. For more on the nested-logit model see Anderson and de Palma (1992). In our model, the nests are the manufacturer's assortment, the retailer's assortment and an external alternative. If the customer decides to purchase from the manufacturer or the retailer, then she decides on the specific product to purchase. For the sake of exposition, we first describe the customer's variant choice given that the customer already decided what channel to purchase from (the retailer or the manufacturer's direct channel).

4.3.1.1 Deciding what variant to purchase

In this stage, the customer chooses which variant to purchase from the assortment offered by the channel she chose in the first stage. Consider an individual customer who decided to purchase from channel $n \in \{M, R\}$, where $n = M$ refers to the manufacturer's direct channel and $n = R$ refers to the retailer. Let $\boldsymbol{p}^n$ denote the vector of prices for the variants offered by channel n.[3] In keeping with the nested logit choice model, we model the customer's choice of variant using the multinomial logit (MNL) choice model. (For details on the MNL model, see, for example, Ben-Akiva

[3] As a notational convention; we use bold symbols to denote vectors, e.g. $\boldsymbol{p}^n$ is the vector of prices charged by channel n to its customers.

and Lerman, 1985.) Let U_k^n denote the customer's utility from the variant $k \in S^n$. Following the MNL model, U_k^n is given by

$$U_k^n = \alpha_k - p_k^n + \xi_k^n, \text{ for } k \in S^n, n \in \{M, R\},$$

where α_k is the customer's expected utility from variant $k \in S^n$, p_k^n is the price of variant k in channel n and ξ_k^n is a random error term with a Gumbel distribution with mean zero and scale parameter $\mu_2 > 0$. The scale parameter μ_2 can be interpreted as the degree of heterogeneity across the variants offered by a given channel; when μ_2 is close to zero the variants become perfect substitutes. In this setting, the customer chooses the variant that maximizes his utility and the probability that the customer will choose variant k is

$$q_k^n(S^n, \boldsymbol{p}^n) = \frac{v_k(p_k^n)}{\sum_{k \in S^n} v_k(p_k^n)} \text{ for all } k \in S^n, \tag{4.1}$$

where

$$v_k(p_k^n) := \exp([\alpha_k - p_k^n]/\mu_2) \text{ for } k \in S^n. \tag{4.2}$$

4.3.1.2 Channel choice

Let U^M and U^R denote the customer's random utility of purchasing from the manufacturer and from the retailer, respectively. The nested logit model posits that the utility of purchasing from a nest is the expected utility of the utility-maximizing choice in that nest plus a Gumbel error term. Hence, following the nested logit model, $U^n := E\left[\max_k U_k^n\right] + \xi_n$ for $n = R, M$ where ξ_n is a Gumbel random term with mean zero and scale parameter μ_1, where $\mu_1 > 0$. The parameter μ_1, in contrast to μ_2, represents the heterogeneity of assortments across the two channels. Hence, we expect that $\mu_1 > \mu_2$. Given that U_k^n's are Gumbel random variables (refer to Section

4.3.1.1) and the Gumbel distribution is closed under maximization, $\max_k U_k^n$ is again Gumbel, and its expected value is,

$$E\left[U^n\right] = E\left[\max_k U_k^n\right] = \mu_2 \ln\left[\sum_{k\in S^n} v_k(p_k^n)\right]. \tag{4.3}$$

Similarly, let U^0 denote the random utility of an external alternative. Here, U^0 can be thought of as the aggregate utility from a number of choices exogenous to our model, such as the utility from purchasing the product from an alternate firm or the utility from purchasing a different product altogether. This utility of the external alternative, U^0, is itself Gumbel with a mean of $\alpha_0\mu_1$.

The nested logit model then yields the following expression for the probability that the customer purchases from channel n, denoted by $\tau^n(S^M, S^R, \boldsymbol{p}^{\scriptscriptstyle M}, \boldsymbol{p}^{\scriptscriptstyle R})$:

$$\tau^n(S^M, S^R, \boldsymbol{p}^{\scriptscriptstyle M}, \boldsymbol{p}^{\scriptscriptstyle R}) = \frac{\exp(E[U^n]/\mu_1)}{\exp(E\left[U^0\right]/\mu_1) + \exp(E\left[U^R\right]/\mu_1) + \exp(E[U^M]/\mu_1)}$$

for $n \in \{M, R\}$. Using (4.3) to substitute for $E[U^n]$ in the above equation, we obtain:

$$\tau^n(S^M, S^R, \boldsymbol{p}^{\scriptscriptstyle M}, \boldsymbol{p}^{\scriptscriptstyle R}) = \frac{\left[\sum_{k\in S^n} v_k(p_k^n)\right]^{\mu_2/\mu_1}}{\exp\left(\alpha_0\right) + \left[\sum_{k\in S^R} v_k(p_k^{\scriptscriptstyle R})\right]^{\mu_2/\mu_1} + \left[\sum_{k\in S^M} v_k(p_k^{\scriptscriptstyle M})\right]^{\mu_2/\mu_1}}. \tag{4.4}$$

4.3.1.3 The Aggregate Demand

According to the model we have described so far, an individual customer purchases from the channel $n \in \{M, R\}$ with probability $\tau^n(S^M, S^R, \boldsymbol{p}^{\scriptscriptstyle M}, \boldsymbol{p}^{\scriptscriptstyle R})$ and chooses variant $k \in S^n$ with probability $q_k^n(S^n, \boldsymbol{p}^n)$. Thus, the probability that an individual purchases variant k from channel n is

$$\tau^n(S^M, S^R, \boldsymbol{p}^{\scriptscriptstyle M}, \boldsymbol{p}^{\scriptscriptstyle R}) q_k^n(S^n, \boldsymbol{p}^n).$$

We use the probability above as the starting point to model the aggregate demand for a product. Specifically, we assume that the aggregate demand observed by channel n for product k, denoted by $D_k^n(S^M, S^R, \boldsymbol{p}^{\scriptscriptstyle M}, \boldsymbol{p}^{\scriptscriptstyle R})$, is

$$D_k^n(S^M, S^R, \boldsymbol{p}^{\scriptscriptstyle M}, \boldsymbol{p}^{\scriptscriptstyle R}) \sim \tau^n(S^M, S^R, \boldsymbol{p}^{\scriptscriptstyle M}, \boldsymbol{p}^{\scriptscriptstyle R}) q_k^n(S^n, \boldsymbol{p}^n)\epsilon_k \text{ for } k \in S^n,$$

where ϵ_k's are i.i.d normal random variables with mean one and standard deviation σ_k. Notice that σ_k amounts to the coefficient of variation of customer's demand for variant $k \in S^n$.

4.3.2 The Firms' Profit Functions

We treat the manufacturer's assortment as fixed and hence we drop S^M from the argument lists of the functions. On the other hand, we treat the retailer's assortment, S^R, as a decision variable. This modeling choice is aligned with our motivating example, in which the manufacturer offers all possible variants while the retailer offers only a subset of them. We consider both the case where the retailer chooses its own assortment, and the case where the manufacturer decides what to offer through the retailer.

We assume that the manufacturer has two pricing decisions to make: the direct channel prices, $p_k^{\scriptscriptstyle M}$, and the wholesale price for each variant, denoted by w_k for $k \in S^R$. The retailer, on the other hand, needs to determine its own selling prices, $p_k^{\scriptscriptstyle R}$ for variant $k \in S^R$. We consider several scenarios regarding the sequencing of decisions, including the pricing decisions and the assortment decision. For the sake of exposition, we next describe the retailer's and manufacturer's profit functions given the prices and assortments and delay an explanation of the sequence of events until the next section.

4.3.2.1 Retailer's Expected Profit

We consider a one-period problem. Recall that we model a make-to-stock retailer who procures the finished product from the manufacturer and faces random demand. Hence, inventory decisions at the retailer are taken into account and we assume these decisions are made after all pricing and assortment decisions have been made by both the manufacturer and retailer.

Let y_k denote the retailer's stock level of variant $k \in S^R$. We assume that the retailer will stock y_k units at the beginning of the period to satisfy a type-1 service level objective, which, as in previous chapters, is uniquely determined by the exogenously fixed parameter, z_k, where

$$z_k := \frac{y_k - \mu_k^{R}(S^R, \boldsymbol{p}^{M}, \boldsymbol{p}^{R})}{\widehat{\sigma}_k^{R}(S^R, \boldsymbol{p}^{M}, \boldsymbol{p}^{R})}, \tag{4.5}$$

and $\mu_k^{R}(S^R, \boldsymbol{p}^{M}, \boldsymbol{p}^{R})$ and $\widehat{\sigma}_k^{R}(S^R, \boldsymbol{p}^{M}, \boldsymbol{p}^{R})$ denote the mean and standard deviation of the demand for variant k observed by the retailer.

At the beginning of the period, the retailer places the orders, y_k's, with the manufacturer. Next, the demand at the retailer, $D_k^R(S^R, \boldsymbol{p}^{M}, \boldsymbol{p}^{R})$ for $k \in S^R$, is realized. For each variant, two outcomes may arise in this setting for the retailer: either the variant's stock is sufficient to meet the demand or not. In the former case, the retailer incurs a cost co_k for each unit of leftover inventory. In the latter case, we assume that the retailer places an additional order with the manufacturer to meet the excess demand, at a unit cost cu_k, which is in addition to the wholesale price. The cost cu_k may be interpreted as the additional cost associated with expediting a shipment in order to meet the excess demand. In this setting, the retailer's expected profit is

$$\Pi^R(S^R, \boldsymbol{p}^{M}, \boldsymbol{w}, \boldsymbol{p}^{R}) = \sum_{k \in S^R} \left[(p_k^{R} - w_k) E\left[D_k^R(S^R, \boldsymbol{p}^{M}, \boldsymbol{p}^{R}) \right] - L_k^R(S^R, \boldsymbol{p}^{M}, \boldsymbol{p}^{R}) \right], \tag{4.6}$$

where

$$L_k^R(S^R, \boldsymbol{p}^{\scriptscriptstyle M}, \boldsymbol{p}^{\scriptscriptstyle R}) = cu_k E\big[D_k^R(S^R, \boldsymbol{p}^{\scriptscriptstyle M}, \boldsymbol{p}^{\scriptscriptstyle R}) - y_k\big]^+ + co_k E\big[y_k - D_k^R(S^R, \boldsymbol{p}^{\scriptscriptstyle M}, \boldsymbol{p}^{\scriptscriptstyle R})\big]^+. \tag{4.7}$$

Here, the function $L_k^R(S^R, \boldsymbol{p}^{\scriptscriptstyle M}, \boldsymbol{p}^{\scriptscriptstyle R})$ represents the expected overage and underage cost associated with variant $k \in S^R$. Following the same approach as in Section 2.3.3, the profit function in (4.6) can be simplified to the following:

$$\Pi^R(S^R, \boldsymbol{p}^{\scriptscriptstyle M}, \boldsymbol{w}, \boldsymbol{p}^{\scriptscriptstyle R}) =$$

$$\tau^{\scriptscriptstyle R}(S^R, \boldsymbol{p}^{\scriptscriptstyle M}, \boldsymbol{p}^{\scriptscriptstyle R}) \sum_{k \in S^R} \begin{bmatrix} (p_k^{\scriptscriptstyle R} - w_k) q_k^{\scriptscriptstyle R}(S^R, \boldsymbol{p}^{\scriptscriptstyle R}) \\ - \sigma_k \big[co_k z_k + (co_k + cu_k) I_N(z_k)\big] q_k^{\scriptscriptstyle R}(S^R, \boldsymbol{p}^{\scriptscriptstyle R}) \end{bmatrix}.$$

Let us define $\gamma_k := \sigma_k\big[co_k z_k + (co_k + cu_k) I_N(z_k)\big]$. Notice that γ_k works as a unit cost incurred by the retailer and it depends only on unit overage cost, co_k, unit underage cost, cu_k, the demand's coefficient of variation, σ_k, and the service level z_k. Using this definition, we can now write the retailer's expected profit function as follows:

$$\Pi^R(S^R, \boldsymbol{p}^{\scriptscriptstyle M}, \boldsymbol{w}, \boldsymbol{p}^{\scriptscriptstyle R}) = \tau^{\scriptscriptstyle R}(S^R, \boldsymbol{p}^{\scriptscriptstyle M}, \boldsymbol{p}^{\scriptscriptstyle R}) \sum_{k \in S^R} (p_k^{\scriptscriptstyle R} - w_k - \gamma_k) q_k^{\scriptscriptstyle R}(S^R, \boldsymbol{p}^{\scriptscriptstyle R}). \tag{4.8}$$

4.3.2.2 Manufacturer's Expected Profit

The manufacturer builds to order and faces two sources of demand: one from the direct channel and the other from the retailer. The sequence of events for the manufacturer is the following: At the beginning of the period, the manufacturer receives an order from the retailer and meets that order. Throughout the period, the manufacturer continues to build to order to meet the demand from its direct channel. At the end of the period, the retailer may observe excess demand over the initial stocking quantity. When that happens, the retailer will backorder excess demand

and will procure the needed quantity from the manufacturer. The manufacturer will meet the retailer's additional demand charging the same wholesale price as before.

Given that the retailer backorders the excess demand from the manufacturer, the quantity that the manufacturer sells to the retailer is the maximum of the retailer's initial order quantity, y_k, and the demand observed by the retailer, $D_k^R(S^R, \boldsymbol{p}^{\scriptscriptstyle M}, \boldsymbol{p}^{\scriptscriptstyle R})$. The quantity that the manufacturer sells through the direct channel, on the other hand, is simply the demand from the direct channel, $D_k^M(S^R, \boldsymbol{p}^{\scriptscriptstyle M}, \boldsymbol{p}^{\scriptscriptstyle R})$. Let c_k denote the unit production cost of variant k. The manufacturer's expected profit is

$$\begin{aligned}\Pi^M(S^R, \boldsymbol{p}^{\scriptscriptstyle M}, \boldsymbol{w}, \boldsymbol{p}^{\scriptscriptstyle R}) = &\sum_{k\in S^M} (p_k^{\scriptscriptstyle M} - c_k) E\big[D_k^M(S^R, \boldsymbol{p}^{\scriptscriptstyle M}, \boldsymbol{p}^{\scriptscriptstyle R})\big] \\ &+ \sum_{k\in S^R} (w_k - c_k) \Big(y_k + E\left[D_k^R(S^R, \boldsymbol{p}^{\scriptscriptstyle M}, \boldsymbol{p}^{\scriptscriptstyle R}) - y_k\right]^+ \Big). \end{aligned} \tag{4.9}$$

Notice from (4.9) that the first term corresponds to the expected profit from the direct channel and the second term corresponds to the profit collected by selling to the retailer. We substitute for y_k in (4.9) using the definition in (4.5). Note that, by the assumption of normal demand, we can replace $E\left[D_k^R(S^R, \boldsymbol{p}^{\scriptscriptstyle M}, \boldsymbol{p}^{\scriptscriptstyle R}) - y_k\right]^+$ with $\widehat{\sigma}_k^{\scriptscriptstyle R}(S^R, \boldsymbol{p}^{\scriptscriptstyle M}, \boldsymbol{p}^{\scriptscriptstyle R}) I_N(z_k)$. Substituting $\mu_k^{\scriptscriptstyle R}(S^R, \boldsymbol{p}^{\scriptscriptstyle M}, \boldsymbol{p}^{\scriptscriptstyle R}) = \tau^{\scriptscriptstyle R}(S^R, \boldsymbol{p}^{\scriptscriptstyle M}, \boldsymbol{p}^{\scriptscriptstyle R}) q_k^{\scriptscriptstyle R}(S^R, \boldsymbol{p}^{\scriptscriptstyle R})$, $\widehat{\sigma}_k(S^R, \boldsymbol{p}^{\scriptscriptstyle M}, \boldsymbol{p}^{\scriptscriptstyle R}) = \tau^{\scriptscriptstyle R}(S^R, \boldsymbol{p}^{\scriptscriptstyle M}, \boldsymbol{p}^{\scriptscriptstyle R}) q_k^{\scriptscriptstyle R}(S^R, \boldsymbol{p}^{\scriptscriptstyle R}) \sigma_k$, and

$$E\big[D_k^M(S^R, \boldsymbol{p}^{\scriptscriptstyle M}, \boldsymbol{p}^{\scriptscriptstyle R})\big] = \tau^{\scriptscriptstyle M}(S^R, \boldsymbol{p}^{\scriptscriptstyle M}, \boldsymbol{p}^{\scriptscriptstyle R}) q_k^{\scriptscriptstyle M}(\boldsymbol{p}^{\scriptscriptstyle M}),$$

we obtain that the profit for the manufacturer is given by

$$\begin{aligned}\Pi^M(S^R, \boldsymbol{p}^{\scriptscriptstyle M}, \boldsymbol{w}, \boldsymbol{p}^{\scriptscriptstyle R}) = &\,\tau^{\scriptscriptstyle M}(S^R, \boldsymbol{p}^{\scriptscriptstyle M}, \boldsymbol{p}^{\scriptscriptstyle R}) \sum_{k\in S^M} (p_k^{\scriptscriptstyle M} - c_k) q_k^{\scriptscriptstyle M}(\boldsymbol{p}^{\scriptscriptstyle M}) \\ &+ \tau^{\scriptscriptstyle R}(S^R, \boldsymbol{p}^{\scriptscriptstyle M}, \boldsymbol{p}^{\scriptscriptstyle R}) \sum_{k\in S^R} (w_k - c_k) \theta_k q_k^{\scriptscriptstyle R}(S^R, \boldsymbol{p}^{\scriptscriptstyle R}), \end{aligned} \tag{4.10}$$

where

$$\theta_k := \sigma_k \left(z_k + I_N(z_k) \right) + 1. \tag{4.11}$$

We refer to θ_k as the "safety stock factor" for variant k. Notice the significance of the safety stock factor: For each variant, the expected quantity sold by the manufacturer to the retailer is the variant's expected demand times its safety stock factor, which itself depends on the service level and demand's coefficient of variation, as shown in (4.11). This factor is always larger than one. It captures the fact that the total quantity that the retailer buys from the manufacturer is an amplification of the expected demand. To understand the intuition behind this amplification, recall that the retailer backorders from the manufacturer whenever there is a shortage. Therefore, the total quantity the retailer buys from the manufacturer is never below the demand, but it may exceed the demand when the retailer overstocks. The factor θ_k captures this effect.

4.4. The Pricing Problem in the Dual Channel

In this section we explore the retailer's and manufacturer's pricing problem assuming a Stackelberg game where the manufacturer is the leader. To examine the retailer's pricing decision, we assume that the retailer's assortment, S^R, is exogenously fixed. The sequence of decisions for this game is the following: (1) the manufacturer picks the wholesale prices, w_k, and direct channel prices, p_k^{M}, (2) the retailer sets its prices, p_k^{R}. We first analyze the retailer's pricing decision in response to the manufacturer's wholesale prices, $\boldsymbol{w}$, and direct channel prices, $\boldsymbol{p}^{M}$. As stated in the following proposition, we find that the retailer will price its products following an equal effective margin property.

Proposition 16. *Consider variant k in the retailer's assortment S^R. Let $m_k^{R} := p_k^{R} - w_k - \gamma_k$ denote the effective profit margin of variant k. Given the vector of wholesale* *prices, $\boldsymbol{w}$, the vector of direct channel prices, $\boldsymbol{p}^{M}$, and the retailer's assortment, S^R, any price vector that is optimal for the retailer is such that all variants have the same effective profit margin, i.e., $m_k^{R} = m^{R}$ for all $k \in S^R$.*

Proposition 16 reduces the retailer's pricing decision to picking a single effective margin, m^{R}, for all the variants in its assortment S^R. This result shows the same pricing structure as Proposition 1 in Chapter II.

Recall the expression for the retailer's profit, $\Pi^R(S^R, \boldsymbol{p}^{M}, \boldsymbol{w}, \boldsymbol{p}^{R})$, given by (4.8). Using the result of Proposition 16, we obtain:

$$\sum_{k \in S^R} (p_k^{R} - w_k - \gamma_k) q_k^{R}(S^R, \boldsymbol{p}^{R}) = m^{R} \sum_{k \in S^R} q_k^{R}(S^R, \boldsymbol{p}^{R}) = m^{R}.$$

Using the equality above, we can rewrite the retailer's profit, $\Pi^R(S^R, \boldsymbol{p}^{M}, \boldsymbol{p}^{R})$, as

$$\Pi^R(S^R, \boldsymbol{p}^{M}, \boldsymbol{p}^{R}) = \tau^{R}(S^R, \boldsymbol{p}^{M}, \boldsymbol{p}^{R}) m^{R}. \tag{4.12}$$

Furthermore, he probability that the customer chooses the retailer, $\tau^{R}(S^R, \boldsymbol{p}^{M}, \boldsymbol{p}^{R})$, can be rewritten as a function of m^{R} instead of the price vector $\boldsymbol{p}^{R}$. To this end, for variant k in retailer's assortment S^R, define the retailer's surplus associated with variant k as

$$\eta_k^{R} := \alpha_k - w_k - \gamma_k \text{ for } k \in S^R. \tag{4.13}$$

Note that the surplus is the customer's expected utility from the variant minus the costs the retailer incurs for carrying that variant. Using the above definition, we can define the function $v_k^{R}(m^{R})$ as follows:

$$v_k^{R}(m^{R}) := \exp([\eta_k^{R} - m^{R}]/\mu_2) \text{ for } k \in S^R. \tag{4.14}$$

Using (4.13) and (4.14), we can now write $\tau^{R}(S^R, \boldsymbol{p}^{M}, \boldsymbol{p}^{R})$ in (4.4) as a function of the retailer's effective margin, m^R, instead of the retailer's price vector, $\boldsymbol{p}^R$,

$$\tau^R(S^R, \boldsymbol{p}^M, m^R) = \frac{\left[\sum_{k\in S^R} v_k^R(m^R)\right]^{\mu_2/\mu_1}}{\exp(\alpha_0) + \left[\sum_{k\in S^M} v_k(p_k^M)\right]^{\mu_2/\mu_1} + \left[\sum_{k\in S^R} v_k^R(m^R)\right]^{\mu_2/\mu_1}}. \tag{4.15}$$

Using the expression above, we are now ready to rewrite the retailer's expected profit, $\Pi^R(S^R, \boldsymbol{p}^M, \boldsymbol{p}^R)$ in (4.12), as a function of the effective margin, m^R instead of the price vector, $\boldsymbol{p}^R$

$$\Pi^R(S^R, \boldsymbol{p}^M, m^R) = \tau^R(S^R, \boldsymbol{p}^M, m^R)m^R. \tag{4.16}$$

The following proposition uses the redefined profit function in (4.16) to characterize the retailer's optimal margin as a function of the wholesale prices, $\boldsymbol{w}$, direct channel prices, $\boldsymbol{p}^M$, and the retailer's assortment, S^R.

Proposition 17. *Given the vector of wholesale prices, $\boldsymbol{w}$, the vector of direct channel prices, $\boldsymbol{p}^M$, and the retailer's assortment, S^R, the retailer's optimal effective margin is the unique value of $m^R(S^R, \boldsymbol{p}^M, \boldsymbol{w})$ that satisfies*

$$m^R(S^R, \boldsymbol{p}^M, \boldsymbol{w}) = \frac{\mu_1}{1 - \tau^R(S^R, \boldsymbol{p}^M, m^R(S^R, \boldsymbol{p}^M, \boldsymbol{w}))}.$$

Proposition 17 characterizes the retailer's optimal pricing response to any vector of wholesale prices, $\boldsymbol{w}$, and direct channel prices, $\boldsymbol{p}^M$, chosen by the manufacturer. Given the retailer's optimal pricing response, we next analyze the manufacturer's optimal pricing decisions.

In the manufacturer's profit function, the retailer's price vector can now be replaced with the retailer's effective margin. The details of this simplification are provided in Appendix C. This simplification facilitates further analysis of the manufacturer's pricing problem. The next proposition describes the properties of the optimal direct channel and wholesale prices.

Proposition 18. *Define* $m_k^M := p_k^M - c_k$ *as the manufacturer's effective profit margin for* $k \in S^M$ *sold through the direct channel and* $\overline{w}_k := (w_k - c_k - \mu_2)\theta_k$ *as the weighted wholesale price for variant* $k \in S^R$*. At any optimal vector of wholesale prices,* $\boldsymbol{w}$*, and optimal vector of direct channel prices,* $\boldsymbol{p}^M$*:*

(a) All variants $k \in S^M$ *will have the same effective margin, i.e.* $m_k^M = m^M$ *for all* $k \in S^M$*.*

(b) All variants $k \in S^R$ *will have the same weighted wholesale price, i.e.* $\overline{w}_k = \overline{w}$ *for all* $k \in S^R$*.*

The result in Proposition 18 (a) does not comes as a surprise given that we found the same structure in different settings, where the effective margin (i.e. gross margin net of unit inventory cost) must be the same for all variants in the channel. In this case, what is the same across variants is the gross margin because the manufacturer's direct channel has no inventory-related costs.

Proposition 18 (b) has important consequences regarding the effect of demand variability on the wholesale prices, as we state in the following proposition.

Proposition 19. *Suppose that all variants are the same in all respects but* θ_k*, i.e.* $\alpha_k = \alpha$*,* $c_k = c$ *and* $\gamma_k = \gamma$ *for all* $k \in S^R$*. Let* i *and* j *be two of the variants carried in the retailer's assortment. If* $\theta_i < \theta_j$ *then* $w_i > w_j$*. Furthermore, if* i *and* j *are identical in all respects but the demand's coefficient of variation, with* $\sigma_i < \sigma_j$*, then* $w_i > w_j$*.*

Proposition 19 states that the larger the demand variability of a variant (measured by demand's coefficient of variation), the lower the variant's wholesale price. In general, the proposition indicates that variants with higher safety stock factors will prompt lower prices from the manufacturer. The intuition behind this result lies in the safety stock factor, θ_k. Suppose we start with a number of products identical in all respects, and we then increase the coefficient of variation for one of them, say variant h. Now, the safety stock factor for variant h also increases. In other words, for variant h, the retailer's order quantity is going to be a larger amplification of the expected demand compared to other variants. Hence, this variant now becomes more attractive for the manufacturer, so the manufacturer decreases the wholesale price to increase the market share of this variant among others.

Using Proposition 18 we can recast the manufacturer's profit as a function of a single effective wholesale price (instead of a wholesale price vector) and a single effective margin for the direct channel (instead of a price vector for the direct channel). These simplifications, which are presented in detail in the appendix, allow us to further analyze assortment selection and pricing in a dual-channel setting.

4.5. Effects of Adding a Retailer to a Direct Channel on the Manufacturer's Profit

In this section, we explore how adding an indirect channel (i.e. retailer) influences a manufacturer currently selling directly to the customer. In order to understand the benefits of dual-sales channels, we first explore the effect of adding an indirect channel on the pricing decisions.

Recall that the manufacturer decides the wholesale prices and direct-channel prices. We find that the manufacturer will always increase the gross margin used by its direct channel when it adds an indirect channel.

Proposition 20. *Let m^M_{direct} be the manufacturer's optimal direct channel margin when selling through the direct channel only and m^M_{dual} be the manufacturer's optimal direct channel margin when selling through dual channels. Then, $m^M_{dual} > m^M_{direct}$.*

It is interesting that the manufacturer increases its direct channel margin despite introducing a competing source of product, the retailer, into the channel mix. To understand the intuition behind this result, note that the manufacturer's choice of direct margin is ultimately a trade-off between its sales volume and unit margin. Once the retailer is introduced to the mix, it is true that the volume sold through the direct channel will decrease as some customers will switch to the retailer. However, the manufacturer's total sales volume, combined across both channels, will now be higher than it was when the manufacturer had only the direct channel, because fewer customers will now choose to go with the external alternative. Given that the manufacturer's total sales volume will increase after the addition of the retailer, the manufacturer will respond by increasing its direct channel margin to take advantage of the larger sales volume.

Given that adding a retailer results in an increase in both the total sales volume and the direct channel's effective margin, the manufacturer will always benefit from using a dual-channel strategy.

Proposition 21. *Let Π^M_{direct} be the manufacturer's optimal expected profit when selling through the direct channel only and Π^M_{dual} be the manufacturer's optimal expected profit when selling through dual channels. Then, for any given assortments S^M and S^R we have that $\Pi^M_{dual} \geq \Pi^M_{direct}$.*

While the manufacturer always benefits from adding a retailer to the channel mix, the size of the benefit does depend on the assortment offered through the retailer. In Table 4.1, we present an example with three alternatives: The first example is one where the manufacturer is only selling directly to the customer, i.e. $S^R = \{\emptyset\}$, in the second example the manufacturer is selling variant 1 through the retailer, i.e. $S^R = \{1\}$, and in the third example the manufacturer is selling variant 2 through the retailer, i.e. $S^R = \{2\}$. Notice that the manufacturer's expected profit, Π^M, increases as the indirect-channel is added as predicted by Proposition 21. However, notice that carrying variant 1 through the retailer is more profitable than carrying variant 2. Also notice that variants 1 and 2 are identical in all respects except for the difference between γ_1 and γ_2, the parameters that capture the inventory-related unit costs associated with variants 1 and 2, respectively. This example highlights two aspects of the problem we are studying: First, the assortment decision on what to offer through the retailer plays an important role on the manufacturer's profit.

S^R	m^M	$\overline{w}$	m^R	Π^M	Π^R
$\{\emptyset\}$	3.0992	—	—	1.5992	0
$\{1\}$	3.2839	3.1954	1.6586	1.8041	0.1586
$\{2\}$	3.2489	3.1163	1.6278	1.7620	0.1278

Table 4.1: Manufacturer's profit improvements of adding an indirect-channel to a direct-channel. In this example, $\mu_1 = 1.5$, $\mu_2 = 0.5$, $\alpha_0 = 10$, $\alpha_1 = \alpha_2 = 5$, $c_1 = c_2 = 1$, $\gamma_1 = 0.4034$, $\gamma_2 = 0.08068$ and $\theta_1 = \theta_2 = 1.2008$.

Second, even when the manufacturer builds to order (as is the case in our model), the inventory costs that the retailer incurs for its assortment will influence not only the retailer's profit, but also the manufacturer's profit. Hence, we next explore the assortment decision on what to offer through the retailer.

4.6. Assortment Results

In this section, we explore the retailer's assortment decision. Depending on the power structure in the supply chain, the retailer may choose its own assortment or the manufacturer may decide on the assortment it is going to offer through the retailer. Hence, in this section we explore both cases. In terms of the timing of the assortment decision, we consider three alternative scenarios. In Scenario 1, the manufacturer picks the assortment while it is choosing its direct channel and wholesale prices. In Scenario 2, the retailer picks the assortment after the manufacturer makes its pricing decisions. In Scenario 3, the retailer picks the assortment before the manufacturer makes its pricing decisions.

Next we explore Scenario 1. Suppose that the manufacturer can pick the assortment offered through the retailer, S^R. Then the manufacturer will offer every available variant as stated in the next proposition.

Proposition 22. *In Scenario 1, where the manufacturer is choosing the retailer's assortment in addition to setting the direct channel and wholesale prices, the manufacturer will always choose to offer all variants through the retailer.*

This result may seem surprising given the fact that the retailer's assortment is going to compete with the direct channel's assortment. However, the manufacturer can always utilize its ability to manipulate the retailer's demand through its wholesale price choice. Hence, the manufacturer can always price the products to get a favorable combination of market share and margin.

Now consider Scenario 2, where the retailer picks its assortment after the manufacturer makes its pricing decisions. In this scenario, we find that the retailer will always choose to carry all available variants.

Proposition 23. *In Scenario 2, where the retailer picks its assortment after the manufacturer makes its pricing decision, the retailer will choose to offer every available variant.*

Proposition 23 is analogous to Proposition 3 in Section 2.4.2. As discussed earlier in Chapter II, the driver of this result is the fact that the retailer uses its pricing decision to balance the gross profit and inventory-related costs of its variants so that all variants are equally attractive to be carried in the assortment.

The common ground that joins scenarios 1 and 2 is that they both assume that the assortment decision is made between the manufacturer's pricing decisions and the retailer's pricing decision. In contrast, Scenario 3 assumes that the retailer picks its assortment before the pricing decisions for either firm are made.

Under Scenario 3, we find that even if the retailer has no constraints on the number of variants to carry nor has a fixed cost for carrying a variant, the retailer may still choose not to carry all possible variants in its assortment. We illustrate this result with an example in Table 4.2. In this example, the manufacturer has an assortment of two variants and the retailer can choose to carry only one variant, two variants or neither. Observe that when the retailer carries variant 2 in its assortment (either on its own or along with variant 1) the retailer charges larger prices to the customer. The reason for increase in prices is that variant 2 shows higher demand variability and therefore, the inventory related costs associated with carrying this variant are large

	Profit		Prices					
S^R	*Manufacturer*	*Retailer*	$\boldsymbol{p}_1^M$	$\boldsymbol{p}_2^M$	w_1	w_2	p_1^R	p_2^R
$\{1\}$	1.9738	0.1731	4.6535	4.6535	4.5827	–	7.0810	–
$\{2\}$	1.9347	0.1175	4.6231	4.6231	–	4.1709	–	7.7250
$\{1,2\}$	1.9812	0.1722	4.6603	4.6603	4.6535	4.0084	7.1509	7.6172

Table 4.2: Manufacturer and Retailer's Profit and Margins for a Two-Variant Assortment. In this example, the variant 1 and 2 are the same in all respects but the demand's coefficient of variation σ_k, where, $\sigma_1 = 0.09$, $\sigma_2 = 0.25$, $\mu_1 = 1.7$, $\mu_2 = 0.3$, $\alpha_0 = 10$, $\alpha_1 = \alpha_2 = 5$, $c_1 = c_2 = 1$, $co_1 = co_2 = 4$, $cu_1 = cu_2 = 4$ and $z_1 = z_2 = 1.7$.

for the retailer. It is interesting that this will happen even when the manufacturer will choose a lower wholesale price of variant 2 compared to variant 1, as discussed in Proposition 19. In this example, the retailer would choose to leave variant 2 out of its assortment in order to push the inventory related costs down.

4.6.1 Assortment Selection Problem

An important assumption made in the previous section for Propositions 22 and 23 is that there are no constraints on the number of variants that the retailer can offer and there are no fixed costs for carrying a variant. In this section we relax this assumption for two simplified versions of the problem. First, we explore a problem where the retailer is constrained to offer only one variant. This setting allows us to explore what product characteristics present conflicts between the retailer and the manufacturer when it comes to the assortment offered by the retailer. Second, we explore the case where all variants are identical and there is a fixed cost associated with the number of variants carried at the retailer. This setting allows us to explore conflicts between the manufacturer and the retailer in terms of the size of the retailer's assortment.

4.6.1.1 Variant Preference

Consider the case where the retailer can offer a single variant. The next proposition compares Scenarios 1 and 2.

Proposition 24. *Suppose that the retailer can offers only one variant from* S^M*. Then the variant that would be offered through the retailer under Scenario 1 is the same as the variant that would be offered under Scenario 2.*

Recalling the definitions of Scenarios 1 and 2, Proposition 24 states that as long as the retailer's assortment is chosen after the manufacturer sets its prices but before the retailer does, the manufacturer and retailer are in agreement about what variant the retailer should offer.

We next analyze the case where the retailer chooses its assortment before the manufacturer sets its prices: Do the manufacturer and the retailer still agree on what variant to offer through the retailer? In other words, we compare Scenario 1 (where the manufacturer chooses what to offer through the retailer) and Scenario 3 (where the retailer chooses what to offer, but before the manufacturer sets its prices). Proposition 25 identifies certain conditions under which the same variant is chosen under both Scenarios 1 and 3.

Proposition 25. *Consider a manufacturer's assortment that consists of only variants 1 and 2, i.e.* $S^M = \{1, 2\}$*. Let variants 1 and 2 be the same in all respects but the overage and/or underage costs, i.e.* $\alpha_1 = \alpha_2$*,* $c_1 = c_2$*,* $\theta_1 = \theta_2$*. Under both Scenario 1 and Scenario 3, the variant that will be offered through the retailer is the one with the lower underage/overage cost.*

Proposition 25 presents a setting in which the retailer and the manufacturer will not face a conflict in terms of what should be carried through the retailer. However, the same is not true for variants that differ only in the demand's coefficient of variation. In such a case, we find that there exist situations where the manufacturer,

S^R	Π^M	Π^R	p_1^M	p_2^M	w_1	w_2	p_1^R	p_2^R
{ 1}	5.3526	0.2864	7.9969	7.9969	7.6224	–	10.2340	–
{ 2}	5.3889	0.2597	8.0322	8.0322	–	6.7291	–	10.4254

Table 4.3: Profits of a Retailer's Single-Variant Assortment. In this example, the variant 1 and 2 are the same in all respects but the demand's coefficient of variation σ_k, where $\mu_1 = 1.7$, $\mu_2 = 0.3$, $\sigma_1 = 0.09$, $\sigma_2 = 0.25$, $\alpha_0 = 10$, $\alpha_1 = \alpha_2 = 10$, $c_1 = c_2 = 1$, $co_1 = co_2 = 4$, $cu_1 = cu_2 = 4$ and $z_1 = z_2 = 1.7$; $\theta_1 = 1.1546$, $\theta_2 = 1.4296$, $\gamma_1 = 0.6252$, and $\gamma_2 = 1.7366$.

in Scenario 1, will prefer to offer through the retailer the variant with the higher demand's coefficient of variation whereas the retailer, in Scenario 3, will choose the variant that has the lower coefficient of variation. In Table 4.3, we provide an example showing such a conflict. In this example there are two variants, labeled 1 and 2. They are the same in all respects but the demand's coefficient of variation, σ_k. The retailer decides according to Scenario 3. Here the coefficient of variation for variant 2 is significantly larger than the one for variant 1. Observe that the retailer's expected profit is higher when it carries only variant 1, whereas the manufacturer's expected profit is higher when the retailer carries variant 2.

The most significant driver of this conflict is the fact that variant 2 has a higher safety factor, i.e. $\theta_2 > \theta_1$. Hence the manufacturer will show preference for variant 2. However, variant 2 also has higher inventory costs, i.e. $\gamma_2 > \gamma_1$ which makes it less attractive for the retailer. Hence, the retailer prefers variant 1, but the manufacturer prefers variant 2.

4.6.1.2 Assortment Size

In this section we explore the assortment size preference of the manufacturer and retailer for a simplified version of the base model. Suppose that both the retailer and the manufacturer incur a fixed cost for every product carried in the retailer's assortment. Furthermore, suppose that all variants are identical, i.e., α_k, c_k, γ_k and θ_k are all the same for all $k \in S^M$. We assume that the assortment decision is made after the manufacturer picks its prices, i.e. Scenarios 1 and 2. In this setting, we compare the retailer's and manufacturer's preference for the size of the assortment offered through the retailer. We find that the manufacturer prefers a larger assortment than the retailer.

Proposition 26. *Suppose $\theta_k < 2$. The optimal size of the retailer's assortment is larger under Scenario 1 than under Scenario 2.*

Proposition 26 implies that, when the assortment decision is made after the manufacturer's pricing decision but before the retailer's pricing decision, the manufacturer will always prefer a broader assortment than the one preferred by the retailer. There are two reasons for this discrepancy. First, the retailer incurs inventory costs for each variant in its assortment while the manufacturer doesn't. Second, when the demand falls short of stock level, the retailer sells only up to demand, but the manufacturer's sales quantity is what the retailer stocked. Thus, the manufacturer always sells at least as much as the retailer, and sometimes strictly more, thereby benefiting more from each variant than the retailer does.

4.7. Conclusion

This chapter studies a supply chain structure where the upper echelon supplies and competes with the lower echelon. We study this setting by modeling a build-to-order manufacturer selling directly to the customer and also through a retailer. The customer is assumed to be able to purchase from either firm and the manufacturer will offer all available variants, whereas the retailer will offer only a subset of variants.

There are three main challenges that this type of problem poses: First we have the manufacturer's decision to set direct channel prices and wholesale prices, while taking into account the competition from the retailer. Second, the retailer's manages not only prices but also inventory levels. Third, both the retailer and the manufacturer must be concerned about the effects of the retailer's assortment on their profits.

In this setting, we first look at the firms' pricing problems. We find that for the retailer, the pricing structure will be the same as the one found in earlier chapters, where the prices should follow an equal effective margin. The optimal wholesale prices on the other hand show a different structure: The wholesale price net of unit production and inventory costs, weighted by the item's safety stock factor, must be the same across all items. Here, the safety stock factor is a function of the item's service level and demand variability. This structure has some immediate implications. For example, if all items are the same but some show higher demand variability, then those items with higher demand variability will have a lower wholesale price attached to them. The rationale behind this result is that, for these items, the retailer's order quantity will represent a larger amplification of the expected demand, compared to other items with lower demand variability. Hence, the manufacturer will want these items to have a larger market share, which can be induced by keeping the wholesale price low.

Given the increasing diversification of channels, we also explore if a manufacturer, who is currently selling through a direct channel, benefits from adding a retailer to its channel mix. This decision is non-trivial because the two channels will be competing against each other. We find that even though the direct channel will observe a decrease in its market share, the manufacturer will increase its total market share (where total market share includes the demand from both channels). Hence, the manufacturer will always benefit from adding a channel, giving it another venue to capture more customers.

One of the main contributions of this chapter is that is explores the retailer's assortment decision. First, we assume that there are no limitations on the number of variants to offer through the retailer and that there are no fixed costs of carrying a variant. In this setting, we explore scenarios that differ in mainly two aspects: who decides the retailer's assortment (the retailer or the manufacturer itself) and when the assortment decision is made relative to pricing decisions. We find that depending on the sequence of decisions, the manufacturer's preference may conflict with the retailer's. In particular, in certain cases, the manufacturer wishes the retailer to offer items with high demand variability while the retailer opts for low demand variability.

In addition, when both the retailer and the manufacturer incur a fixed cost for the variants carried by the retailer, we find that the manufacturer prefers a broader assortment than the retailer even when the manufacturer's fixed cost is slightly higher than the retailer.

Chapter V

Conclusion

In its broadest sense, this dissertation studies joint pricing and assortment problems with inventory considerations. Two main settings are studied. One of these settings considers a single firm selling a configurable product. The configurable product is formed by putting together two components: one required for product functionality (i.e. required component) and one that enhances or complements the product but is not essential for the product to work (i.e. optional component). Several scenarios are considered and are presented in Chapters II and III. The second setting is one where there is one firm (manufacturer) who sells directly to the customers and through an independent retailer. The retailer is then a collaborator and a competitor to the manufacturer. This setting is treated in Chapter IV. Next, we discuss the contributions of this dissertation, grouped into several themes.

Effects of inventory-related parameters on pricing: Throughout this dissertation there is evidence that inventory considerations play an important role in the pricing decisions.

Chapters II and III find that the optimal prices for the variants of a component should be such that they all have the same *effective profit margin* (i.e., the selling price net of unit purchase cost and unit inventory cost). This result extends previous work, which ignored inventory costs and found that *gross margins* (selling price net of unit purchase cost) should be equal across variants. This result may explain why variants with higher overage and underage costs also have higher gross margins.

Chapter IV adds to the pricing results in Chapters II and III by finding the optimal structure for wholesale pricing. In particular, the chapter finds that the wholesale price net of unit production and inventory costs, weighted by the item's safety stock factor, must be the same across all items. The safety stock factor is a function of the item's service level and demand variability.

Assortment decisions - Importance of inventory related parameters and channel structures: Chapter II highlights the importance of a variant's *surplus*. The surplus is a measure of the variant's profitability that depends on the customer's utility from the variant, the variant's unit purchase cost, unit underage and overage costs, service level and demand variability. This surplus measure combines all demand-side and supply-side parameters in a clear-cut measure that translates into variant profitability within a component. By incorporating demand-side and supply-side parameters, it inherently accounts for the effect of inventory costs on assortment selection.

When choosing between two variants, each of which belongs to a different component, we find that the surplus itself is not enough to determine which variant to carry in the assortment. Chapter II also sets the ground for making such decisions and shows the importance of taking into account the complementary relationship between the components. To that end, the chapter establishes a measure of the attractiveness of a product configuration. This attractiveness measure draws upon the surpluses of the component variants that go into a configuration. When choosing from two variants that belong to different components, one must weigh the attractiveness of the new configurations enabled by the addition of each variant, and pick the variant that leads to the more attractive set of new configurations.

In Chapter IV, we explore the assortment that should be offered through the retailer, when the manufacturer has a competing direct channel. We develop several scenarios that differ from each other in terms of who is making the assortment decision (manufacturer or retailer) and the sequence of the decision making process (when the assortment decision is made relative to the pricing decision). We find that some of the scenarios will trigger conflicts on the assortment decision, where the manufacturer will prefer the retailer to offer all available variants and the retailer will prefer a narrower assortment. When there are restrictions on the number of variants that can be offered through the retailer, a particular conflict may arise, in which the retailer prefers to offer the item with lower demand variability (as measured by the demand's coefficient of variation), while the manufacturer prefers the item with higher demand variability. In addition, when both the retailer and the manufacturer incur a fixed cost for the variants carried by the retailer, we find that the manufacturer prefers a broader assortment than the retailer even when the manufacturer's fixed cost is slightly higher than the retailer.

Benefits of selling through dual channels: In Chapter IV, we studied the manufacturer's benefits from adding a retailer to an existing direct sales channel. In this setting we find that increasing diversification of the channels is always beneficial to the manufacturer. This result may be surprising because the retailer will compete with the manufacturer's direct channel. However, we find that even though the manufacturer's direct channel will observe a decrease in its market share, the manufacturer will increase its total market share (where the total market share includes the demand from both channels).

Effect of the marketing strategy on the assortment and pricing: In Chapter II the customer decides what product to purchase based on the entire assortment and prices. In such a case, we find that the optimal effective margin is zero for the optional component. In Chapter III, the model is extended to consider the possibility that only one of the components (the required component) is the driver of store traffic, i.e. the customer's decision to purchase from the retailer depends only on the assortment and prices of the required component. In this setting, the optional component is now sold at a positive effective margin. Therefore, when selling configurable products, the firm's optimal pricing strategy depends very much on how the firm is marketing the optional component.

Price discrimination strategy in the presence of different customer segments: Chapter III explores a problem where there are two segments in the customer population. One segment is assumed to be shopping for the required and, possibly, the optional component, while the other segment is shopping only for the optional component (maybe because they bought the required component at an earlier time). In this setting, we find that the firm will benefit from a price discrimination strategy in which all customers who purchase the required component receive a discount on the optional component while those buying only the optional component pay the full price for it.

5.1. Future work

First, the model presented in Chapter II considers a configurable product, whose components are stocked separately and are combined only after the retailer receives the order from the customer (e.g. Dell computers). In other settings, we may observe that the retailer will carry a set of pre-configured products, that is, the retailer may produce some product configurations on a make-to-stock basis and others on a make-to-order basis (e.g. vehicles sold by a dealer). The models presented in this dissertation set the ground to study the supply chain implications of such strategies.

Another question related to configurable products is the *degree* to which two variants of different components are compatible. There may be situations where the customer's utility from a variant depends on what variant of the other component it will be matched with. Take for example a laptop computer that can be sold with or without an integrated video camera. Consider now an external video camera. One would expect that a customer's utility from the external camera would depend very much on whether the customer is buying a computer with or without an integrated camera. This generalization of the model presented in Chapter II, might result in different assortment and pricing strategies for different types of variants. In particular, one may expect that the optimal assortment will favor variants that are a *universal fit* for any variant of the other component, that is, variants that are highly valuable no matter what variant of the other component they are matched with. In contrast, one may also expect that items adding special or critical value to the most profitable variants may be the ones carried in the assortment.

In the dual-channel setting (Chapter IV), we consider a retailer who sells only a subset of the variants sold through the manufacturer's direct channel. A scenario which is aligned with the motivating examples. However, we can find situations in practice where the assortments of the manufacturer do not overlap with the assortment sold through the retailer. For example, this is the case for the Hewlett Packard (HP) desktop computers sold through Wal-Mart. HP product lines can be offered through both channels, direct or through Wal-Mart, but the models offered through these two channels are not the same. It will be interesting to study the settings that push for this kind of model differentiation and how this behavior affects the profits of the chain.

While the manufacturer's direct channel may be selling only its own brands, the same exclusive structure will not be likely to hold for the retailer. The retailer is more than likely to sell multiple, competing brands. We expect that this type of competition within the retailer's own assortment will affect the pricing and assortment decisions for both the manufacturer and the retailer. Moreover, such competition may alter the manufacturer's decision to sell through dual channels in the first place. Hence, the assortment and pricing decisions in this setting are worthy of future research.

Appendices

Appendix A

Proofs of Propositions in Chapter II

This appendix, as well as subsequent appendices, utilizes the Lambert-W function, denoted by $W(\cdot)$. Given a number a, the Lambert-W function is defined as the value of x that solves $x\exp(x) = a$. In other words, the Lambert-W function is implicitly defined as

$$W(a)\exp(W(a)) = a.$$

The function $W(a)$ is increasing in a for all $a \geq 0$, which is a property used in the proofs. For more on the properties of the Lambert-W function, see, for example, Corless et al. (1996).

To help with the exposition of this Appendix, define $v_{nk}(p_{nk})$ as follows:

$$v_{nk}(p_{nk}) := \exp(\alpha_{nk} - p_{nk}), k \in S_n, n = R, O. \tag{A.1}$$

Notice that v_{nk} is a measure of the gap between the expected utility and the price of variant k of component $n \in \{R, O\}$. In addition, recall that $v_{[ij]}$ was defined in

Section 2.3 as follows:

$$v_{[ij]}(p_{Ri}, p_{Oj}) = \exp(\alpha_{Ri} + \alpha_{Oj} - p_{Ri} - p_{Oj})$$

Notice that $v_{[ij]}$ is a measure of the gap between the expected utility and the price of the configuration that brings together variant i of the required component and variant j of the optional component.

Proofs of Section 2.4

In preparation for the proofs that follow, the derivatives of purchase probabilities defined in Section 2.3 are noted below. Given the assortments of the required and optional components, S_R and S_O, the following derivatives can be verified through some algebra using the definitions of purchase probabilities in (2.1), (2.3) and (2.4):

$$\frac{\partial q_{nz}(\boldsymbol{p}_R, \boldsymbol{p}_O)}{\partial p_{nz}} = -q_{nz}(\boldsymbol{p}_R, \boldsymbol{p}_O)[1 - q_{nz}(\boldsymbol{p}_R, \boldsymbol{p}_O)], z \in S_n, n \in \{R, O\}, \tag{A.2}$$

$$\frac{\partial q_{nk}(\boldsymbol{p}_R, \boldsymbol{p}_O)}{\partial p_{nz}} = q_{nk}(\boldsymbol{p}_R, \boldsymbol{p}_O) q_{nz}(\boldsymbol{p}_R, \boldsymbol{p}_O), k, z \in S_n, k \neq z, n \in \{R, O\}, \tag{A.3}$$

$$\frac{\partial q_{Rk}(\boldsymbol{p}_R, \boldsymbol{p}_O)}{\partial p_{Oz}} = -q_{[kz]}(\boldsymbol{p}_R, \boldsymbol{p}_O) + q_{Rk}(\boldsymbol{p}_R, \boldsymbol{p}_O) q_{Oz}(\boldsymbol{p}_R, \boldsymbol{p}_O), k \in S_R, z \in S_O \text{ and} \tag{A.4}$$

$$\frac{\partial q_{Ok}(\boldsymbol{p}_R, \boldsymbol{p}_O)}{\partial p_{Rz}} = -q_{[zk]}(\boldsymbol{p}_R, \boldsymbol{p}_O) + q_{Rz}(\boldsymbol{p}_R, \boldsymbol{p}_O) q_{Ok}(\boldsymbol{p}_R, \boldsymbol{p}_O), z \in S_R, k \in S_O. \tag{A.5}$$

Proof of Proposition 1: Suppose that the assortments, S_R and S_O, are fixed as required by the proposition. In this proof, S_R and S_O are dropped from the argument list of the functions, since they are exogenously fixed for the purposes of this proof. The firm's problem of maximizing the expected profit in (2.8) reduces to picking the price vectors $\boldsymbol{p}_R$ and $\boldsymbol{p}_O$ to maximize the profit function $\Pi(\boldsymbol{p}_R, \boldsymbol{p}_O)$.

We first prove that any optimal price vector $\boldsymbol{p}_R$ and $\boldsymbol{p}_O$ must satisfy the first-order-conditions (FOC) of the firm's objective function $\Pi(\boldsymbol{p}_R, \boldsymbol{p}_O)$. To this end, we start by taking the derivative of the firm's objective function, $\Pi(\boldsymbol{p}_R, \boldsymbol{p}_O)$, given by (2.8), with respect to the price of variant z of component $n \in \{R, O\}$, p_{nz}. The following derivatives can be verified through algebra and using the derivatives of the purchase probabilities, provided in (A.2), (A.3), (A.4) and (A.5) in the preamble to Appendix A.

$$\frac{\partial \Pi(\boldsymbol{p}_R, \boldsymbol{p}_O)}{\partial p_{Rz}} = q_{Rz}(\boldsymbol{p}_R, \boldsymbol{p}_O) \left[\begin{array}{l} \sum_{n \in \{R,O\}} \sum_{k \in S_n} (p_{nk} - c_{nk} - \gamma_{nk}) q_{nk}(\boldsymbol{p}_R, \boldsymbol{p}_O) - (p_{Rz} - c_{Rz} - \gamma_{Rz}) + 1 \\ - \sum_{k \in S_O} (p_{Ok} - c_{Ok} - \gamma_{Ok}) \frac{v_{Ok}(p_{Ok})}{\sum_{j \in S_O} v_{Oj}(p_{Oj}) + \exp(\alpha_{Oo})} \end{array} \right], \tag{A.6}$$

where $v_{nk}(p_{nk})$ is as defined in (A.1), and

$$\frac{\partial \Pi(\boldsymbol{p}_R, \boldsymbol{p}_O)}{\partial p_{Oz}} = q_{Oz}(\boldsymbol{p}_R, \boldsymbol{p}_O) \left[\begin{array}{l} \sum_{n \in \{R,O\}} \sum_{k \in S_n} (p_{nk} - c_{nk} - \gamma_{nk}) q_{nk}(\boldsymbol{p}_R, \boldsymbol{p}_O) - (p_{Oz} - c_{Oz} - \gamma_{Oz}) + 1 \\ - \sum_{k \in S_R} (p_{Rk} - c_{Rk} - \gamma_{Rk}) \frac{v_{Rk}(p_{Rk})}{\sum_{i \in S_R} v_{Ri}(p_{Ri})} \end{array} \right]. \tag{A.7}$$

It can be shown that (A.6) is negative (positive) when p_{Rz} is very large (very small), i.e.,

$$\lim_{p_{Rz} \to \infty} \frac{\partial \Pi(\boldsymbol{p}_R, \boldsymbol{p}_O)}{\partial p_{Rz}} < 0 \text{ and } \lim_{p_{Rz} \to -\infty} \frac{\partial \Pi(\boldsymbol{p}_R, \boldsymbol{p}_O)}{\partial p_{Rz}} > 0.$$

Likewise, it can be shown that (A.7) is negative (positive) when p_{Oz} is very large (very small), i.e.,

$$\lim_{p_{Oz} \to \infty} \frac{\partial \Pi(\boldsymbol{p}_R, \boldsymbol{p}_O)}{\partial p_{Oz}} < 0 \text{ and } \lim_{p_{Oz} \to -\infty} \frac{\partial \Pi(\boldsymbol{p}_R, \boldsymbol{p}_O)}{\partial p_{Oz}} > 0.$$

Hence, it is never optimal to set one of the prices at a boundary, and the optimal price vectors must satisfy the FOC.

To conclude the proof of the proposition, we next prove that any pair of price vectors $\boldsymbol{p}_{R}$ and $\boldsymbol{p}_{O}$ that satisfy the FOC in (A.6) and (A.7) must also satisfy the property that $p_{Rz} - c_{Rz} - \gamma_{Rz}$ is the same for all $z \in S_R$ and $p_{Oz} - c_{Oz} - \gamma_{Oz}$ is the same for all $z \in S_O$.

For the case of $z \in S_R$, note that setting (A.6) equal to zero (to obtain the FOC with respect to $z \in S_R$) and re-arranging the terms yields the following:

$$
\begin{aligned}
p_{Rz} - c_{Rz} - \gamma_{Rz} = & \sum_{n \in \{R,O\}} \sum_{k \in S_n} (p_{nk} - c_{nk} - \gamma_{nk}) q_{nk}(\boldsymbol{p}_{R}, \boldsymbol{p}_{O}) + 1 \\
& - \sum_{k \in S_O} (p_{Ok} - c_{Ok} - \gamma_{Ok}) \frac{v_{Ok}(p_{Ok})}{\sum_{j \in S_O} v_{Oj}(p_{Oj}) + \exp(\alpha_{O_o})}
\end{aligned}
$$

Observe that the right-hand side (RHS) of the above equation is the same for all $z \in S_R$. Therefore, $p_{Rz} - c_{Rz} - \gamma_{Rz}$ is the same for all $z \in S_R$. Likewise, using (A.7), one can verify that $p_{Oz} - c_{Oz} - \gamma_{Oz}$ is the same for all $z \in S_O$. Therefore, the optimization over prices reduces to optimizing over two margins, m_R and m_O, where $p_{nz} - c_{nz} - \gamma_{nz} = m_n$ for all $z \in S_n$, $n \in \{R, O\}$.

Proof of Proposition 2: Recall that the assortments S_R and S_O are fixed for the purposes of this proposition, hence they are dropped from the argument list of the functions. One could check that partial derivative of $q_0(m_R, m_O)$, given by (2.10), and the partial derivative of $q_{O_o}(m_R, m_O)$, given by (2.11), with respect to m_R and m_O are:

$$
\frac{\partial q_0(m_R, m_O)}{\partial m_R} = q_0(m_R, m_O)[1 - q_0(m_R, m_O)] > 0, \tag{A.8}
$$

$$
\frac{\partial q_0(m_R, m_O)}{\partial m_O} = q_0(m_R, m_O)[1 - q_0(m_R, m_O) - q_{O_o}(m_R, m_O)] > 0, \tag{A.9}
$$

$$
\frac{\partial q_{O_o}(m_R, m_O)}{\partial m_R} = -q_{O_o}(m_R, m_O) q_0(m_R, m_O), \tag{A.10}
$$

$$
\frac{\partial q_{O_o}(m_R, m_O)}{\partial m_O} = q_{O_o}(m_R, m_O) \left[1 - q_0(m_R, m_O) - q_{O_o}(m_R, m_O)\right]. \tag{A.11}
$$

Using the derivatives above and through some algebra one could verify that the partial derivative of $\Pi(m_R, m_O)$, given by (2.12), with respect to m_R and m_O, yields,

$$\frac{\partial \Pi(m_R, m_O)}{\partial m_R} = 1 - q_0(m_R, m_O) - m_R q_0(m_R, m_O)[1 - q_0(m_R, m_O)]$$
$$- m_O q_0(m_R, m_O)\left[1 - q_0(m_R, m_O) - q_{O_o}(m_R, m_O)\right], \tag{A.12}$$
$$\frac{\partial \Pi(m_R, m_O)}{\partial m_O} = \frac{1 - q_0(m_R, m_O) - q_{O_o}(m_R, m_O)}{1 - q_0(m_R, m_O)}\left[\frac{\partial \Pi(m_R, m_O)}{\partial m_R} - m_O q_{O_o}(m_R, m_O)\right]. \tag{A.13}$$

Recall from Proposition 1 that the optimal prices must be interior solutions, that is, $-\infty < p_{nz} < \infty$ for $z \in S_n, n \in \{R, O\}$. In addition, by Proposition 1, the margin, m_n, and the price, p_{nz}, differ by a finite constant $c_{nz} + \gamma_{nz}$. Therefore, the optimal margins m_R and m_O must also be interior solutions, i.e., $-\infty < m_R, m_O < \infty$. Thus, m_R and m_O must satisfy the FOC of $\Pi(m_R, m_O)$. Now, observe from (A.13) that at any pair of m_R and m_O where the FOC are satisfied, i.e., at any pair of m_R and m_O such that $\frac{\partial \Pi(m_R, m_O)}{\partial m_R} = \frac{\partial \Pi(m_R, m_O)}{\partial m_O} = 0$, we must have $m_O q_{O_o}(m_R, m_O) = 0$. The only value of $m_O < \infty$ that satisfies this last equality is $m_O = 0$. Thus, at the optimal solution, $m_O = 0$. Substituting $m_O = 0$ in (A.12) and setting $\frac{\partial \Pi(m_R, m_O)}{\partial m_R} = 0$ yields

$$m_R = \frac{1}{q_0(m_R, m_O = 0)}.$$

It remains to show that there is a unique m_R that satisfies the above equality. Using the definition of $q_0(m_R, m_O)$ provided in (2.10) and substituting $m_O = 0$, one can rewrite the above equality as:

$$(m_R - 1)\exp(m_R - 1) = \frac{\sum_{i \in S_R} \sum_{j \in S_O \cup \{0\}} \exp(\eta_{Ri} + \eta_{Oj})}{\exp(\alpha_0 + 1)}.$$

Applying the definition of the Lambert-W function to the equality above, one can write

$$m_{R} - 1 = W\left(\frac{\sum_{i\in S_R}\sum_{j\in S_O\cup\{0\}}\exp(\eta_{Ri}+\eta_{Oj})}{\exp(\alpha_0+1)}\right). \tag{A.14}$$

Observe that the right-hand side of the above equality does not depend on m_{R} and the left-hand side is strictly increasing in m_{R}. Therefore, there exists a unique m_{R} that satisfies the FOC.

Proofs of Propositions 3, 4 and 5: Given assortments S_R and S_O for the required and optional components, let $m^*_{R}(S_R, S_O)$ and $m^*_{O}(S_R, S_O)$ denote the optimal margins for the required and optional components, respectively. Let $q^*_0(S_R, S_O)$ and $\Pi^*(S_R, S_O)$ denote, respectively, the probability that the customer will not purchase from the firm and the firm's expected profit, given that the margins are set equal to their optimal values $m^*_{R}(S_R, S_O)$ and $m^*_{O}(S_R, S_O)$. Recall from Proposition 2 that $m^*_{O}(S_R, S_O) = 0$ and $m^*_{R}(S_R, S_O) = \frac{1}{q^*_0(S_R,S_O)}$. Substituting these values for $m^*_{R}(S_R, S_O)$ and $m^*_{O}(S_R, S_O)$ in the firm's profit function given by (2.12), one obtains

$$\Pi^*(S_R, S_O) = \frac{1 - q^*_0(S_R, S_O)}{q^*_0(S_R, S_O)} = \frac{1}{q^*_0(S_R, S_O)} - 1 = m^*_{R}(S_R, S_O) - 1.$$

Observe from above that the larger the optimal margin for the required component, $m^*_{R}(S_R, S_O)$, the larger the firm's optimal expected profit, $\Pi^*(S_R, S_O)$. Therefore, any change to assortment S_R or S_O will increase the profit if and only if it increases $m^*_{R}(S_R, S_O)$. Furthermore, the larger the increase in $m^*_{R}(S_R, S_O)$ due to a change in assortment, the larger the increase in $\Pi^*(S_R, S_O)$. This observation is used next to prove Propositions 3, 4 and 5. In addition, recall from the proof of Proposition 2 that $m^*_{R}(S_R, S_O)$ is the unique solution to (A.14). Therefore:

$$m^*_{R}(S_R, S_O) = W\left(\frac{\sum_{i\in S_R}\exp(\eta_{Ri})\sum_{j\in S_O\cup\{0\}}\exp(\eta_{Oj})}{\exp(\alpha_0+1)}\right) + 1. \tag{A.15}$$

To prove Proposition 3, notice that if a new variant is added to S_R or S_O, the RHS of (A.15) will become strictly larger, which means that $m^*_{_R}(S_R, S_O)$ will also become strictly larger.

To prove Proposition 4, observe from (A.15) that if $\eta_{nk} > \eta_{nl}$ for variants k and l of component n, then adding variant k to S_n leads to a larger optimal margin for the required component than adding variant l to S_n, i.e., if $\eta_{_{Ok}} > \eta_{_{Ol}}$, then $m^*_{_R}(S_R, S_O \cup \{k\}) > m^*_{_R}(S_R, S_O \cup \{l\})$; if $\eta_{_{Rk}} > \eta_{_{Rl}}$, then $m^*_{_R}(S_R \cup \{k\}, S_O) > m^*_{_R}(S_R \cup \{l\}, S_O)$. The result now follows.

To prove Proposition 5, first note that, given variant k for the required component and variant z for the optional component, it is better to add variant k instead of variant z if and only if $m^*_{_R}(S_R \cup \{k\}, S_O) > m^*_{_R}(S_R, S_O \cup \{z\})$. One can use the definition of product attractiveness, $a_{[ij]}$, given by equation (2.13), to rewrite (A.15) as follows:

$$m^*_{_R}(S_R, S_O) = W\left(\frac{\sum_{i \in S_R} \sum_{j \in S_O \cup \{0\}} a_{[ij]}}{\exp(\alpha_0 + 1)}\right) + 1.$$

Notice from the above equation that $m^*_{_R}(S_R \cup \{k\}, S_O) > m^*_{_R}(S_R, S_O \cup \{z\})$ if and only if

$$\sum_{j \in S_O \cup \{0\}} a_{[kj]} > \sum_{i \in S_R} a_{[iz]}.$$

This concludes the proof of Proposition 5.

Proof of Proposition 6: From Proposition 4, we know that, given a set of new variants of component $n \in \{R, O\}$ to choose from, it is always better to add the variant with the largest surplus, i.e., choose variant k with the largest value of η_{nk}. Recall that $\eta_{nk} := \alpha_{nk} - c_{nk} - \gamma_{nk}$ where $\gamma_{nk} := \sigma_{nk}\big(c^{o}_{nk} z_{nk} + (c^{o}_{nk} + c^{u}_{nk}) I_N(z_{nk})\big)$. Now, parts (a) and (b) follow because γ_{nk} increases in each of σ_{nk}, c^{o}_{nk}, and c^{u}_{nk}. For part (c), note that first and second derivatives of γ_{nk} with respect to z_{nk} are

$$\gamma'_{nk} = \sigma_{nk}\left(c^{o}_{nk} - (c^{o}_{nk} + c^{u}_{nk})(1 - \Phi_N(z_{nk}))\right),$$
$$\gamma''_{nk} = \sigma_{nk}\phi_N(z_{nk})(c^{o}_{nk} + c^{u}_{nk}) > 0.$$

Observe that γ_{nk} is strictly convex in z_{nk} and it is minimized at $z_{nk} = \Phi_N^{-1}\left(\frac{c^u_{nk}}{c^o_{nk}+c^u_{nk}}\right)$. Therefore, η_{nk} is strictly concave in z_{nk} and it is maximized at $z_{nk} = \Phi_N^{-1}\left(\frac{c^u_{nk}}{c^o_{nk}+c^u_{nk}}\right)$, which also defines the critical value $\overline{z}$. Part (c) now follows.

Proofs of Section 2.5

Proof of Proposition 7: Suppose that the assortments, S_R and S_O, are fixed as required by the proposition. Then, the firm's problem in (2.14) reduces to picking the price vectors $\boldsymbol{p}_R$ and $\boldsymbol{p}_O$ to maximize the profit function $\overline{\Pi}(\boldsymbol{p}_R, \boldsymbol{p}_O)$. In this proof, S_R and S_O are dropped from the argument list of the functions. Let $v_{nk}(p_{nk})$ be defined as in (A.1). Using the derivatives of the purchase probabilities $q_{nk}(\boldsymbol{p}_R, \boldsymbol{p}_O)$ provided in (A.2), (A.3), (A.4) and (A.5), and through some algebra, the following partial derivatives of $\overline{\Pi}(\boldsymbol{p}_R, \boldsymbol{p}_O)$ (given by (2.14)) can be verified:

$$\frac{\partial \overline{\Pi}(\boldsymbol{p}_R, \boldsymbol{p}_O)}{\partial p_{Ri}} = q_{Ri}(\boldsymbol{p}_R, \boldsymbol{p}_O)\left[\begin{array}{l}\displaystyle\sum_{n\in\{R,O\}}\sum_{k\in S_n}(p_{nk} - c_{nk})q_{nk}(\boldsymbol{p}_R, \boldsymbol{p}_O) - (p_{Ri} - c_{Ri}) + 1 \\ \displaystyle -\frac{\sum_{j\in S_O}(p_{Oj} - c_{Oj})v_{Oj}(p_{Oj})}{\exp(\alpha_{Oo}) + \sum_{j\in S_O} v_{Oj}(p_{Oj})}\end{array}\right], \tag{A.16}$$

and

$$\frac{\partial \overline{\Pi}(\boldsymbol{p}_{R}, \boldsymbol{p}_{O})}{\partial p_{Oj}} = q_{Oj}(\boldsymbol{p}_{R}, \boldsymbol{p}_{O}) \left[\begin{array}{l} \sum_{n\in\{R,O\}} \sum_{k\in S_n} (p_{nk} - c_{nk}) q_{nk}(\boldsymbol{p}_{R}, \boldsymbol{p}_{O}) - (p_{Oj} - c_{Oj}) + 1 \\ - \frac{\sum_{i\in S_R} (p_{Ri} - c_{Ri}) v_{Ri}(p_{Ri})}{\sum_{i\in S_R} v_{Ri}(p_{Ri})} \end{array} \right]. \tag{A.17}$$

It can be shown that (A.16) is negative (positive) when p_{Ri} is very large (very small), i.e.,

$$\lim_{p_{Ri}\to\infty} \frac{\partial \overline{\Pi}(\boldsymbol{p}_{R}, \boldsymbol{p}_{O})}{\partial p_{Ri}} < 0 \text{ and } \lim_{p_{Ri}\to-\infty} \frac{\partial \overline{\Pi}(\boldsymbol{p}_{R}, \boldsymbol{p}_{O})}{\partial p_{Ri}} > 0.$$

Also, it can be shown that (A.17) is negative (positive) when p_{Oj} is very large (very small), i.e.,

$$\lim_{p_{Oj}\to\infty} \frac{\partial \overline{\Pi}(\boldsymbol{p}_{R}, \boldsymbol{p}_{O})}{\partial p_{Oj}} < 0 \text{ and } \lim_{p_{Oj}\to-\infty} \frac{\partial \overline{\Pi}(\boldsymbol{p}_{R}, \boldsymbol{p}_{O})}{\partial p_{Oj}} > 0.$$

Hence, it is never optimal to set one of the prices at a boundary, and the optimal price vectors must satisfy the FOC.

We next prove that any optimal price vector $\boldsymbol{p}_{R}$ and $\boldsymbol{p}_{O}$ must satisfy the property that $p_{Rk} - c_{Rk}$ are the same for all $k \in S_R$ and $p_{Ok} - c_{Ok}$ are the same for all $k \in S_O$. Setting (A.16) and (A.17) equal to zero and rearranging the terms yields the following FOCs for $\overline{\Pi}(\boldsymbol{p}_{R}, \boldsymbol{p}_{O})$:

$$p_{Ri} - c_{Ri} = 1 - \frac{\sum_{j\in S_O} (p_{Oj} - c_{Oj}) v_{Oj}(p_{Oj})}{\exp(\alpha_{Oo}) + \sum_{j\in S_O} v_{Oj}(p_{Oj})} + \sum_{n\in\{R,O\}} \sum_{k\in S_n} (p_{nk} - c_{nk}) q_{nk}(\boldsymbol{p}_{R}, \boldsymbol{p}_{O}),$$

$$p_{Oj} - c_{Oj} = 1 - \frac{\sum_{i\in S_R} (p_{Ri} - c_{Ri}) v_{Ri}(p_{Ri})}{\sum_{i\in S_R} v_{Ri}(p_{Ri})} + \sum_{n\in\{R,O\}} \sum_{k\in S_n} (p_{nk} - c_{nk}) q_{nk}(\boldsymbol{p}_{R}, \boldsymbol{p}_{O}).$$

Observe that the right-hand side (RHS) of the above FOCs are the same for all $k \in S_n$. Therefore, $p_{nk} - c_{nk} = \overline{m}_n$ for all $k \in S_n$, $n \in \{R, O\}$, where $\overline{m}_n$ is the gross profit margin of component n.

Next, the equal margin property is used to rewrite the firm's profit function, $\overline{\Pi}(\boldsymbol{p}_{R}, \boldsymbol{p}_{O})$, given by (2.14), in terms of the gross margins, $\overline{m}_{R}$ and $\overline{m}_{O}$. In the remainder of the proof, define $\overline{\eta}_{Ri} := \alpha_{Ro} - c_{Ri}$ for $i \in S_R$, $\overline{\eta}_{Oj} := \alpha_{Oj} - c_{Oj}$ for $j \in S_O$ and $\overline{\eta}_{Oo} := \alpha_{Oo}$. In addition, define:

$$q_{Ri}(\overline{m}_{R}, \overline{m}_{O}) = \frac{\exp(\overline{\eta}_{Ri} + \overline{\eta}_{Oo} - \overline{m}_{R}) + \sum_{j \in S_O} \exp(\overline{\eta}_{Ri} + \overline{\eta}_{Oj} - \overline{m}_{R} - \overline{m}_{O})}{\exp(\alpha_0) + \sum_{i \in S_R} \exp(\overline{\eta}_{Ri} - \overline{m}_{R}) \left[\sum_{j \in S_O} \exp(\overline{\eta}_{Oj} - \overline{m}_{O}) + \exp(\overline{\eta}_{Oo})\right]},$$

and

$$q_{Oj}(\overline{m}_{R}, \overline{m}_{O}) = \frac{\sum_{i \in S_R} \exp(\overline{\eta}_{Ri} + \overline{\eta}_{Oj} - \overline{m}_{R} - \overline{m}_{O})}{\exp(\alpha_0) + \sum_{i \in S_R} \exp(\overline{\eta}_{Ri} - \overline{m}_{R}) \left[\sum_{j \in S_O} \exp(\overline{\eta}_{Oj} - \overline{m}_{O}) + \exp(\overline{\eta}_{Oo})\right]}.$$

Using the above definitions, one can write (2.14) as

$$\overline{\Pi}(\overline{m}_{R}, \overline{m}_{O}) = \overline{m}_{R} \sum_{i \in S_R} q_{Ri}(\overline{m}_{R}, \overline{m}_{O}) + \overline{m}_{O} \sum_{j \in S_O} q_{Oj}(\overline{m}_{R}, \overline{m}_{O}) - \sum_{n \in \{R,O\}} \sum_{k \in S_n} \gamma_{nk}. \quad \text{(A.18)}$$

To further simplify (A.18), define

$$q_0(\overline{m}_{R}, \overline{m}_{O}) := \frac{\exp(\alpha_0)}{\exp(\alpha_0) + \sum_{i \in S_R} \exp(\overline{\eta}_{Ri} - \overline{m}_{R}) \left[\sum_{j \in S_O} \exp(\overline{\eta}_{Oj} - \overline{m}_{O}) + \exp(\overline{\eta}_{Oo})\right]}$$

and

$$q_{Oo}(\overline{m}_{R}, \overline{m}_{O}) := \frac{\sum_{i \in S_R} \exp(\overline{\eta}_{Ri} + \overline{\eta}_{Oo} - \overline{m}_{R})}{\exp(\alpha_0) + \sum_{i \in S_R} \exp(\overline{\eta}_{Ri} - \overline{m}_{R}) \left[\sum_{j \in S_O} \exp(\overline{\eta}_{Oj} - \overline{m}_{O}) + \exp(\overline{\eta}_{Oo})\right]}.$$

Using these definitions, one can now write the expression for $\overline{\Pi}(\overline{m}_{R}, \overline{m}_{O})$ in (A.18) as follows:

$$\overline{\Pi}(\overline{m}_{R}, \overline{m}_{O}) =$$

$$\overline{m}_{R}\left(1 - q_0(\overline{m}_{R}, \overline{m}_{O})\right) + \overline{m}_{O}(1 - q_0(\overline{m}_{R}, \overline{m}_{O}) - q_{O_o}(\overline{m}_{R}, \overline{m}_{O})) - \sum_{n\in\{R,O\}}\sum_{k\in S_n}\gamma_{nk}. \quad \text{(A.19)}$$

Because the optimal prices cannot be at the boundaries (i.e., they must be finite), the optimal margins $\overline{m}_{R}$ and $\overline{m}_{O}$ cannot be at the boundaries either and, hence, they must satisfy the FOC of (A.19). Taking the derivative of (A.19) with respect to the margins $\overline{m}_{R}$ and $\overline{m}_{O}$ yield the following FOC. (The following derivatives can be verified through some algebra after noting that $\frac{\partial q_0(\overline{m}_R,\overline{m}_O)}{\partial \overline{m}_R} = q_0(\overline{m}_{R}, \overline{m}_{O})\left(1 - q_0(\overline{m}_{R}, \overline{m}_{O})\right)$ and $\frac{\partial q_{O0}(\overline{m}_R,\overline{m}_O)}{\partial \overline{m}_R} = -q_0(\overline{m}_{R}, \overline{m}_{O})q_{O0}(\overline{m}_{R}, \overline{m}_{O})$.)

$$\begin{aligned}\frac{\partial\overline{\Pi}(\overline{m}_{R}, \overline{m}_{O})}{\partial\overline{m}_{R}} &= -\overline{m}_{R}q_0(\overline{m}_{R}, \overline{m}_{O})\left[1 - q_0(\overline{m}_{R}, \overline{m}_{O})\right] + 1 - q_0(\overline{m}_{R}, \overline{m}_{O}) \\ &\quad - \overline{m}_{O}q_0(\overline{m}_{R}, \overline{m}_{O})\left[1 - q_0(\overline{m}_{R}, \overline{m}_{O}) - q_{O_o}(\overline{m}_{R}, \overline{m}_{O})\right] \qquad \text{(A.20)}\end{aligned}$$

and

$$\frac{\partial\overline{\Pi}(\overline{m}_{R}, \overline{m}_{O})}{\partial\overline{m}_{O}} = \frac{1 - q_0(\overline{m}_{R}, \overline{m}_{O}) - q_{O_o}(\overline{m}_{R}, \overline{m}_{O})}{1 - q_0(\overline{m}_{R}, \overline{m}_{O})}\left[\frac{\partial\overline{\Pi}(\overline{m}_{R}, \overline{m}_{O})}{\partial\overline{m}_{R}} - \frac{\overline{m}_{O}q_{O_o}(\overline{m}_{R}, \overline{m}_{O})}{1 - q_0(\overline{m}_{R}, \overline{m}_{O})}\right]. \quad \text{(A.21)}$$

Notice from (A.21) that when the FOC are satisfied, i.e., when

$$\frac{\partial\overline{\Pi}(\overline{m}_{R}, \overline{m}_{O})}{\partial\overline{m}_{O}} = \frac{\partial\overline{\Pi}(\overline{m}_{R}, \overline{m}_{O})}{\partial\overline{m}_{R}} = 0,$$

it must be that $\overline{m}_{O} = 0$. Hence, the only $\overline{m}_{O}$ that satisfies the FOC is $\overline{m}_{O} = 0$. Substituting for $\overline{m}_{O} = 0$ in (A.20) yields

$$\overline{m}_{R} = \frac{1}{q_0(\overline{m}_{R}, \overline{m}_{O} = 0)}.$$

It remains to show that there is a unique $\overline{m}_{R}$ that satisfies the above equality. Substituting for $q_0(\overline{m}_{R}, 0)$ in the above equation and algebraically manipulating the terms, one can rewrite the above equality as:

$$(\overline{m}_{R} - 1)\exp(\overline{m}_{R} - 1) = \frac{\sum_{i\in S_R}\sum_{j\in S_O\cup\{0\}}\exp(\overline{\eta}_{Ri} + \overline{\eta}_{Oj})}{\exp(\alpha_0 + 1)}.$$

Applying the definition of the Lambert-W function to the equality above, one can write

$$\overline{m}_{R} - 1 = W\left(\frac{\sum_{i\in S_R}\sum_{j\in S_O\cup\{0\}}\exp(\overline{\eta}_{Ri} + \overline{\eta}_{Oj})}{\exp(\alpha_0 + 1)}\right). \tag{A.22}$$

Observe that the right-hand side of the above equality does not depend on $\overline{m}_{R}$ and the left-hand side is strictly increasing in $\overline{m}_{R}$. Therefore, there exists a unique $\overline{m}_{R}$ that satisfies the FOC.

Proof of Propositions 8 and 9: Given assortments S_R and S_O for the required and optional components, let $\overline{m}^*_{R}(S_R, S_O)$ and $\overline{m}^*_{O}(S_R, S_O)$ denote the optimal margins for the required and optional components, respectively. Let $q^*_0(S_R, S_O)$ and $\overline{\Pi}^*(S_R, S_O)$ denote the no-purchase probability and the expected profit at the optimal margins $\overline{m}^*_{R}(S_R, S_O)$ and $\overline{m}^*_{O}(S_R, S_O)$. Recall from Proposition 7 that $\overline{m}^*_{O}(S_R, S_O) = 0$ and $\overline{m}^*_{R} = \frac{1}{q^*_0(S_R,S_O)}$. Substituting these values for $\overline{m}^*_{R}(S_R, S_O)$ and $\overline{m}^*_{O}(S_R, S_O)$ in the firm's profit function given by (2.14), one obtains

$$\begin{aligned}\overline{\Pi}^*(S_R, S_O) &= \frac{1 - q^*_0(S_R, S_O)}{q^*_0(S_R, S_O)} - \sum_{n\in\{R,O\}}\sum_{k\in S_n}\gamma_{nk} = \frac{1}{q^*_0(S_R, S_O)} - 1 - \sum_{n\in\{R,O\}}\sum_{k\in S_n}\gamma_{nk}\\ &= \overline{m}^*_{R}(S_R, S_O) - 1 - \sum_{n\in\{R,O\}}\sum_{k\in S_n}\gamma_{nk}.\end{aligned}$$

Observe from above that the larger the optimal margin for the required component, $\overline{m}^*_{R}(S_R, S_O)$, the larger the firm's optimal expected profit, $\overline{\Pi}^*(S_R, S_O)$. Observe also that any addition to assortment S_R or S_O will increase the inventory related costs.

In particular, adding a new variant k to the assortment of component n will increase the inventory cost by exactly γ_{nk}. Hence, adding a new variant will improve the firm's profit if and only if the improvement in the margin outweighs the increase in inventory costs, i.e. if $\overline{m}_{R}^{*}(S_R \cup k, S_O) - \overline{m}_{R}^{*}(S_R, S_O) > \gamma_{Rk}$ in the case of expanding the required component's assortment, and $\overline{m}_{R}^{*}(S_R, S_O \cup k) - \overline{m}_{R}^{*}(S_R, S_O) > \gamma_{Ok}$ in the case of expanding the optional component's assortment. This concludes the proof of Proposition 8.

As for Proposition 9, first recall from the proof of Proposition 7 that the optimal $\overline{m}_{R}$ must satisfy (A.22). Observe from (A.22) that if $\overline{\eta}_{nk} > \overline{\eta}_{nl}$ for variants k and l of component n, then adding variant k to S_n leads to a larger optimal margin for the required component than adding variant l to S_n, i.e., if $\overline{\eta}_{Ok} > \overline{\eta}_{Ol}$, then $\overline{m}_{R}^{*}(S_R, S_O \cup \{k\}) > \overline{m}_{R}^{*}(S_R, S_O \cup \{l\})$; if $\overline{\eta}_{Rk} > \overline{\eta}_{Rl}$, then $m_{R}^{*}(S_R \cup \{k\}, S_O) > m_{R}^{*}(S_R \cup \{l\}, S_O)$. Recalling the definition that $\overline{\eta}_{nk} = \alpha_{nk} - c_{nk}$, the result now follows.

Appendix B

Proofs of Propositions in Chapter III

Proofs of Section 3.2

Proof of Proposition 10: Suppose that the assortments, S_R and S_O, are fixed as required by the proposition; hence we will drop S_R and S_O from the argument list of the functions. Then, the firm's problem is to pick the price vectors $\boldsymbol{p}_{_R}$ and $\boldsymbol{p}_{_O}$ to maximize its expected profit, $\widehat{\Pi}(\boldsymbol{p}_{_R}, \boldsymbol{p}_{_O})$, given by (3.4).

Given $n \in \{R, O\}$, we first prove that any optimal price vector $\boldsymbol{p}_{_R}$ and $\boldsymbol{p}_{_O}$ must satisfy the property that $p_{nk} - c_{nk} - \gamma_{nk}$ is the same for all $k \in S_n$. Let $v_{nk}(p_{nk})$ be defined as in (A.1). The following partial derivatives of $\widehat{q}_{_{Rk}}(\boldsymbol{p}_{_R})$, given by (3.1), and $\widehat{q}_{_{Ok}}(\boldsymbol{p}_{_R}, \boldsymbol{p}_{_O})$, given by (3.3), can be verified through some algebra:

$$\frac{\partial \widehat{q}_{Rk}(\boldsymbol{p}_R)}{\partial p_{Rz}} = \widehat{q}_{_{Rk}}(\boldsymbol{p}_{_R})\widehat{q}_{_{Rz}}(\boldsymbol{p}_{_R}) \text{ for } k, z \in S_R, k \neq z,$$

$$\frac{\partial \widehat{q}_{Rz}(\boldsymbol{p}_R)}{\partial p_{Rz}} = -\widehat{q}_{_{Rz}}(\boldsymbol{p}_{_R})\left[1 - \widehat{q}_{_{Rz}}(\boldsymbol{p}_{_R})\right] \text{ for } z \in S_R,$$

$$\frac{\partial \widehat{q}_{Ok}(\boldsymbol{p}_R, \boldsymbol{p}_O)}{\partial p_{Rz}} = -\widehat{q}_{_{Rz}}(\boldsymbol{p}_{_R})\widehat{q}_0(\boldsymbol{p}_{_R})\frac{v_{Ok}(p_{Ok})}{\exp(\alpha_{Oo}) + \sum_{j \in S_O} v_{Oj}(p_{Oj})}, \text{ for } k \in S_O, z \in S_R$$

$$\frac{\partial \widehat{q}_{Ok}(\boldsymbol{p}_R, \boldsymbol{p}_O)}{\partial p_{Oz}} = \left[1 - \widehat{q}_0(\boldsymbol{p}_{_R})\right]\frac{v_{Ok}(p_{Ok})v_{Oz}(p_{Oz})}{\left[\exp(\alpha_{Oo}) + \sum_{j \in S_O} v_{Oj}(p_{Oj})\right]^2}, \text{ for } k, z \in S_O, k \neq z,$$

$\frac{\partial \widehat{q}_{Oz}(\boldsymbol{p}_R,\boldsymbol{p}_O)}{\partial p_{Oz}} = -\left[1-\widehat{q}_0(\boldsymbol{p}_{R})\right] \frac{v_{Oz}(p_{Oz})}{\exp(\alpha_{Oo})+\sum_{j\in S_O} v_{Oj}(p_{Oj})} \left[1-\frac{v_{Oz}(p_{Oz})}{\exp(\alpha_{Oo})+\sum_{j\in S_O} v_{Oj}(p_{Oj})}\right]$, for $z \in S_O$.

Using these derivatives and some algebra, the following derivatives of $\widehat{\Pi}(\boldsymbol{p}_{R}, \boldsymbol{p}_{O})$, given by (3.4), with respect to p_{nz} can be verified:

$$\frac{\partial \widehat{\Pi}(\boldsymbol{p}_{R}, \boldsymbol{p}_{O})}{\partial p_{Rz}} = \widehat{q}_{Rz}(\boldsymbol{p}_{R}) \left[\begin{array}{l} \sum_{k\in S_R} (p_{Rk} - c_{Rk} - \gamma_{Rk})\widehat{q}_{Rk}(\boldsymbol{p}_{R}) - (p_{Rz} - c_{Rz} - \gamma_{Rz}) + 1 \\ -\widehat{q}_0(\boldsymbol{p}_{R}) \sum_{k\in S_O} (p_{Ok} - c_{Ok} - \gamma_{Ok}) \frac{v_{Ok}(p_{Ok})}{\exp(\alpha_{Oo}) + \sum_{j\in S_O} v_{Oj}(p_{Oj})} \end{array} \right], \tag{B.1}$$

and

$$\frac{\partial \widehat{\Pi}(\boldsymbol{p}_{R}, \boldsymbol{p}_{O})}{\partial p_{Oz}} = \left[1-\widehat{q}_0(\boldsymbol{p}_{R})\right] \frac{v_{Oz}(p_{Oz})}{\exp(\alpha_{Oo}) + \sum_{j\in S_O} v_{Oj}(p_{Oj})} \times \left[\begin{array}{l} \sum_{k\in S_O} (p_{Ok} - c_{Ok} - \gamma_{Ok}) \frac{v_{Ok}(p_{Ok})}{\exp(\alpha_{Oo}) + \sum_{j\in S_O} v_{Oj}(p_{Oj})} \\ -(p_{Oz} - c_{Oz} - \gamma_{Oz}) + 1 \end{array} \right]. \tag{B.2}$$

It can be shown that (B.1) is negative (positive) when p_{Rz} is very large (very small), i.e.,

$$\lim_{p_{Rz}\to\infty} \frac{\partial \widehat{\Pi}(\boldsymbol{p}_{R}, \boldsymbol{p}_{O})}{\partial p_{Rz}} < 0 \text{ and } \lim_{p_{Rz}\to-\infty} \frac{\partial \widehat{\Pi}(\boldsymbol{p}_{R}, \boldsymbol{p}_{O})}{\partial p_{Rz}} > 0.$$

Also, it can be shown that (B.2) is negative (positive) when p_{Oz} is very large (very small), i.e.,

$$\lim_{p_{Oz}\to\infty} \frac{\partial \widehat{\Pi}(\boldsymbol{p}_{R}, \boldsymbol{p}_{O})}{\partial p_{Oz}} < 0 \text{ and } \lim_{p_{Oz}\to-\infty} \frac{\partial \widehat{\Pi}(\boldsymbol{p}_{R}, \boldsymbol{p}_{O})}{\partial p_{Oz}} > 0.$$

Hence, it is never optimal to set one of the prices at the boundary, and the optimal price vectors must satisfy the FOC. Setting the derivatives in (B.1) and (B.2) equal to zero yield the following FOC for $\widehat{\Pi}(\boldsymbol{p}_R, \boldsymbol{p}_O)$ with respect to p_{Rz} and p_{Oz}, respectively:

$$p_{Rz} - c_{Rz} - \gamma_{Rz} = \sum_{k \in S_R} (p_{Rk} - c_{Rk} - \gamma_{Rk}) \widehat{q}_{Rk}(\boldsymbol{p}_R) + 1 - \widehat{q}_0(\boldsymbol{p}_R) \sum_{k \in S_O} (p_{Ok} - c_{Ok} - \gamma_{Ok}) \frac{v_{Ok}(p_{Ok})}{\exp(\alpha_{Oo}) + \sum_{j \in S_O} v_{Oj}(p_{Oj})}, \tag{B.3}$$

and

$$p_{Oz} - c_{Oz} - \gamma_{Oz} = \sum_{k \in S_O} (p_{Ok} - c_{Ok} - \gamma_{Ok}) \frac{v_{Ok}(p_{Ok})}{\exp(\alpha_{Oo}) + \sum_{j \in S_O} v_{Oj}(p_{Oj})} + 1. \tag{B.4}$$

Observe that the RHS of (B.3) is the same for all $z \in S_R$ and the RHS of (B.4) is the same for all $z \in S_O$. Therefore, fixing $n \in \{R, O\}$, we have $p_{nk} - c_{nk} - \gamma_{nk} = m_n$ for all $k \in S_n$, where m_n is the effective profit margin of component n.

Next, we use the equal margin property to rewrite the firm's profit function, $\widehat{\Pi}(\boldsymbol{p}_R, \boldsymbol{p}_O)$, given by (3.4), in terms of the effective margins, m_R and m_O. Using η_{nk}, defined in (2.9), we first rewrite $\widehat{q}_{Rk}(\boldsymbol{p}_R)$ and $\widehat{q}_{Ok}(\boldsymbol{p}_R, \boldsymbol{p}_O)$, given respectively by (3.1) and (3.3), as functions of m_R and m_O:

$$\widehat{q}_{Rk}(m_R) = \frac{\exp(\eta_{Rk} - m_R)}{\exp(\widehat{\alpha}_0) + \sum_{i \in S_R} \exp(\eta_{Ri} - m_R)}, k \in S_R, \text{ and} \tag{B.5}$$

$$\widehat{q}_{Ok}(m_R, m_O) = \left[\frac{\sum_{i \in S_R} \exp(\eta_{Ri} - m_R)}{\exp(\widehat{\alpha}_0) + \sum_{i \in S_R} \exp(\eta_{Ri} - m_R)} \right] \frac{\exp(\eta_{Ok} - m_O)}{\exp(\alpha_{Oo}) + \sum_{j \in S_O} \exp(\eta_{Oj} - m_O)}, k \in S_O. \tag{B.6}$$

Using (B.5) and (B.6), and letting

$$\widehat{q}_0(m_R) := \frac{\exp(\widehat{\alpha}_0)}{\exp(\widehat{\alpha}_0) + \sum_{k \in S_R} \exp(\eta_{Rk} - m_R)}, \tag{B.7}$$

one can check that (3.4) can be rewritten as:

$$\begin{aligned}\widehat{\Pi}(m_R, m_O) &= \sum_{k\in S_R} m_R \widehat{q}_{Rk}(m_R) + \sum_{k\in S_O} m_O \widehat{q}_{Ok}(m_R, m_O) \\ &= [1-\widehat{q}_0(m_R)]\left\{m_R + m_O\left[1 - \frac{\exp(\alpha_{O_o})}{\exp(\alpha_{O_o}) + \sum_{k\in S_O}\exp(\eta_{Ok} - m_O)}\right]\right\}.\end{aligned} \tag{B.8}$$

The following derivatives of $\widehat{\Pi}(m_R, m_O)$ with respect to m_R and m_O can be verified through some algebra and using the observation that $\frac{\partial \widehat{q}_0(m_R)}{\partial m_R} = \widehat{q}_0(m_R)\left[1-\widehat{q}_0(m_R)\right]$:

$$\frac{\partial\widehat{\Pi}(m_R, m_O)}{\partial m_R} = \\ [1-\widehat{q}_0(m_R)]\left\{1 - \left[m_R + m_O\left(1 - \frac{\exp(\alpha_{O_o})}{\exp(\alpha_{O_o}) + \sum_{k\in S_O}\exp(\eta_{Ok} - m_O)}\right)\right]\widehat{q}_0(m_R)\right\}, \tag{B.9}$$

$$\frac{\partial\widehat{\Pi}(m_R, m_O)}{\partial m_O} = \\ \frac{[1-\widehat{q}_0(m_R)]\sum_{k\in S_O}\exp(\eta_{Ok} - m_O)}{\exp(\alpha_{O_o}) + \sum_{k\in S_O}\exp(\eta_{Ok} - m_O)}\left[1 - \frac{m_O\exp(\alpha_{O_o})}{\exp(\alpha_{O_o}) + \sum_{k\in S_O}\exp(\eta_{Ok} - m_O)}\right]. \tag{B.10}$$

We now explore the the Hessian of $\widehat{\Pi}(m_R, m_O)$. Taking further derivatives of (B.9) and (B.10) yields:

$$\begin{aligned}\frac{\partial^2\widehat{\Pi}(m_R,m_O)}{\partial m_R^2} &= \frac{\partial\widehat{\Pi}(m_R,m_O)}{\partial m_R}\left(1-2\widehat{q}_0(m_R)\right) - \left(1-\widehat{q}_0(m_R)\right), \\ \frac{\partial^2\widehat{\Pi}(m_R,m_O)}{\partial m_O^2} &= -2\frac{\exp(\alpha_{O_o})}{\exp(\alpha_{O_o})+\sum_{k\in S_O}\exp(\eta_{Ok}-m_O)}\frac{\partial\widehat{\Pi}(m_R,m_O)}{\partial m_O} \\ &\quad - [1-\widehat{q}_0(m_R)]\frac{\exp(\alpha_{O_o})\sum_{k\in S_O}\exp(\eta_{Ok}-m_O)}{\left[\exp(\alpha_{O_o})+\sum_{k\in S_O}\exp(\eta_{Ok}-m_O)\right]^2}\left[m_O+1\right], \text{ and} \\ \frac{\partial^2\widehat{\Pi}(m_R,m_O)}{\partial m_R\partial m_O} &= -\widehat{q}_0(m_R)\frac{\partial\widehat{\Pi}(m_R,m_O)}{\partial m_O}.\end{aligned}$$

Using the second derivatives and the cross derivative listed above, we obtain the following for the Hessian of $\widehat{\Pi}(m_R, m_O)$ at any pair of m_R and m_O that satisfy the FOC, i.e., at any pair of m_R and m_O such that (B.9) and (B.10) are both zero. (To help verify the following Hessian, note that $m_O = 1 + \frac{\sum_{k\in S_O}\exp(\eta_{Ok}-m_O)}{\exp(\alpha_{O_o})}$ when (B.10) is zero.)

$$\begin{bmatrix} -[1-\widehat{q}_0(m_R)] & 0 \\ 0 & -[1-\widehat{q}_0(m_R)]\frac{\sum_{k\in S_O}\exp(\eta_{Ok}-m_O)}{\exp(\alpha_{Oo})+\sum_{k\in S_O}\exp(\eta_{Ok}-m_O)} \end{bmatrix}.$$

Observe that the above Hessian is negative definite. Therefore, at any (m_R, m_O) pair that satisfies the FOC of $\widehat{\Pi}(m_R, m_O)$, the function $\widehat{\Pi}(m_R, m_O)$ has a local maxima. This observation can be used to prove that there is a unique pair of (m_R, m_O) that satisfy the FOC of $\widehat{\Pi}(m_R, m_O)$, and this unique pair maximizes $\widehat{\Pi}(m_R, m_O)$. (Here is a sketch of this claim's proof: If there were two such pairs, then there would have to be two local maxima, which would then require the presence of a local minimum in between. However, such a local minimum would contradict the earlier observation that any stationary point is a local maxima.) In other words, the FOC are necessary and sufficient for optimality. Hence, m_O that satisfies $m_O = 1 + \frac{\sum_{k\in S_O}\exp(\eta_{Ok}-m_O)}{\exp(\alpha_{Oo})}$ is the unique optimal margin for the optional component. As for m_R, setting (B.9) equal to zero yields the following:

$$1-\left[m_R + m_O\left(1-\frac{\exp(\alpha_{Oo})}{\exp(\alpha_{Oo})+\sum_{k\in S_O}\exp(\eta_{Ok}-m_O)}\right)\right]\widehat{q}_0(m_R) = 0.$$

Setting m_O equal to its optimal value, i.e. substituting $m_O = 1+\frac{\sum_{k\in S_O}\exp(\eta_{Ok}-m_O)}{\exp(\alpha_{Oo})}$, and rearranging the terms, we find that the optimal m_R must satisfy $m_R = \frac{1}{\widehat{q}_0(m_R)} - m_O + 1$. This concludes the proof.

Proof of Proposition 11: We use the equal margin property, proved in Proposition 10, to rewrite the firm's profit function, $\widehat{\Pi}(S_R, S_O, \boldsymbol{p}_R, \boldsymbol{p}_O)$, given by (3.4), in terms of the effective margins, m_R and m_O:

$$\widehat{\Pi}(S_R, S_O, m_R, m_O) =$$
$$[1 - \widehat{q}_0(S_R, m_R)] \left[m_R + m_O \left[1 - \frac{\exp(\alpha_{O_0})}{\exp(\alpha_{O_0}) + \sum_{k \in S_O} \exp(\eta_{O_k} - m_O)} \right] \right],$$

where $\widehat{q}_0(S_R, m_R) := \frac{\exp(\widehat{\alpha}_0)}{\exp(\widehat{\alpha}_0) + \sum_{k \in S_R} \exp(\eta_{Rk} - m_R)}$. Given assortment $S_R \cup S_O$, we will first prove that the profit $\widehat{\Pi}(S_R, S_O, m_R, m_O)$ can be written as:

$$W\left(\frac{\sum_{k \in S_R} \exp(\eta_{Rk})}{\exp(\widehat{\alpha_0} + 1)} \exp\left(W\left(\frac{\sum_{k \in S_O} \exp(\eta_{Ok})}{\exp(\alpha_{O0} + 1)} \right) \right) \right).$$

We will then use this expression for the profit to conclude the proof of the proposition.

Let us denote the optimal margins for the required and optional components as m_R^* and m_O^*, respectively. From Proposition 10, observe that

$$m_O^* = \frac{\exp(\alpha_{O_0}) + \sum_{k \in S_O} \exp(\eta_{O_k} - m_O)}{\exp(\alpha_{O_0})},$$
$$\widehat{q}_0(S_R, m_R) = \frac{1}{m_R^* + m_O^* - 1}. \tag{B.11}$$

Substituting these two equalities in (B.8), we can write the firm's optimal expected profit as

$$\widehat{\Pi}(S_R, S_O, m_R, m_O) = m_R^* + m_O^* - 2.$$

Using (B.11) and using $\widehat{q}_0(S_R, m_R) := \frac{\exp(\widehat{\alpha}_0)}{\exp(\widehat{\alpha}_0) + \sum_{k \in S_R} \exp(\eta_{Rk} - m_R)}$, we write:

$$m_R^* + m_O^* - 1 = \frac{1}{\widehat{q}_0(S_R, m_R)} = \frac{\exp(\widehat{\alpha}_0) + \sum_{k \in S_R} \exp(\eta_{Rk} - m_R)}{\exp(\widehat{\alpha}_0)}.$$

Algebraic manipulation of the above equality yields:

$$(m_R^* + m_O^* - 2)\exp(m_R^* + m_O^* - 2) = \frac{\sum_{k \in S_R} \exp(\eta_{Rk})}{\exp(\widehat{\alpha}_0)} \exp(m_O^* - 2).$$

Consequently, using the definition of the Lambert-W function, we can write

$$m_R^* + m_O^* - 2 = W\left(\frac{\sum_{k \in S_R} \exp(\eta_{Rk})}{\exp(\widehat{\alpha}_0)} \exp(m_O^* - 2)\right). \tag{B.12}$$

Now, algebraic manipulation of the expression for m_O^*, given in Proposition 10 yields:

$$(m_O^* - 1)\exp(m_O^* - 1) = \frac{\sum_{k \in S_O} \exp(\eta_{Ok})}{\exp(\alpha_{O_o} + 1)}.$$

Consequently, using the definition of the Lambert-W function, we can write

$$m_O^* = W\left(\frac{\sum_{k \in S_O} \exp(\eta_{Ok})}{\exp(\alpha_{O_o} + 1)}\right) + 1. \tag{B.13}$$

By substituting (B.13) in (B.12), we obtain:

$$m_R^* + m_O^* - 2 = W\left(\frac{\sum_{k \in S_R} \exp(\eta_{Rk})}{\exp(\widehat{\alpha}_0 + 1)} \exp\left(W\left(\frac{\sum_{k \in S_O} \exp(\eta_{Ok})}{\exp(\alpha_{O_o} + 1)}\right)\right)\right).$$

Observe now that the LHS of the above equality is the firm's profit, and the RHS depends only on problem parameters. In particular, since the Lambert-W function is increasing in its argument, the RHS of the above equality is increasing in each of η_{Rk} and η_{Ok}. Hence, given a set of new variants that all belong to component $n \in \{R, O\}$, the firm will choose to add candidate k with the highest η_{nk}.

Proofs of Section 3.3

To help with the exposition of the proofs of Section 3.3, define $\widetilde{\Pi}_O(S_O, \boldsymbol{p}_O)$ as follows:

$$\widetilde{\Pi}_O(S_O, \boldsymbol{p}_{\scriptscriptstyle O}) := \sum_{j\in S_O} (p_{\scriptscriptstyle Oj} - c_{\scriptscriptstyle Oj} - w_{\scriptscriptstyle Oj})\widetilde{q}_{\scriptscriptstyle Oj}(S_O, \boldsymbol{p}_{\scriptscriptstyle O}) \tag{B.14}$$

where $\widetilde{q}_{\scriptscriptstyle Oj}(S_O, \boldsymbol{p}_{\scriptscriptstyle O})$ is the probability that a segment-2 customer (who shops for the optional component only) chooses variant j of the optional component in the second stage and is defined in (3.5). Observe that $\widetilde{\Pi}_O(S_O, \boldsymbol{p}_{\scriptscriptstyle O})$ is the firm's profit from a customer that belongs to segment 2, given that the customer decided to purchase from the firm.

Proof of Proposition 12: Suppose that the assortments, S_R and S_O, are fixed as required by the proposition; hence we drop S_R and S_O from the argument list of the functions. Then, the firm's problem is to pick the price vectors $\boldsymbol{p}_{\scriptscriptstyle R}$ and $\boldsymbol{p}_{\scriptscriptstyle O}$ that maximize the profit function $\widetilde{\Pi}(S_R, S_O, \boldsymbol{p}_{\scriptscriptstyle R}, \boldsymbol{p}_{\scriptscriptstyle O})$, given by (3.6). We start by proving that any optimal price vector $\boldsymbol{p}_{\scriptscriptstyle R}$ and $\boldsymbol{p}_{\scriptscriptstyle O}$ must satisfy the first-order-conditions (FOC) of the firm's objective function $\widetilde{\Pi}(\boldsymbol{p}_{\scriptscriptstyle R}, \boldsymbol{p}_{\scriptscriptstyle O})$.

Let $v_{nk}(p_{nk})$ be defined as in (A.1). Taking the derivative of $\widetilde{\Pi}(\boldsymbol{p}_{\scriptscriptstyle R}, \boldsymbol{p}_{\scriptscriptstyle O})$, given by (3.6), with respect to p_{nz} yields:

$$\frac{\partial\widetilde{\Pi}(\boldsymbol{p}_{\scriptscriptstyle R}, \boldsymbol{p}_{\scriptscriptstyle O})}{\partial p_{\scriptscriptstyle Rz}} = \beta q_{\scriptscriptstyle Rz}(\boldsymbol{p}_{\scriptscriptstyle R}, \boldsymbol{p}_{\scriptscriptstyle O}) \left[\begin{array}{c} \displaystyle\sum_{n\in\{R,O\}} \sum_{k\in S_n} (p_{nk} - c_{nk} - \gamma_{nk}) q_{nk}(\boldsymbol{p}_{\scriptscriptstyle R}, \boldsymbol{p}_{\scriptscriptstyle O}) - (p_{\scriptscriptstyle Rz} - c_{\scriptscriptstyle Rz} - \gamma_{\scriptscriptstyle Rz}) + 1 \\ \displaystyle - \sum_{k\in S_O} (p_{\scriptscriptstyle Ok} - c_{\scriptscriptstyle Ok} - \gamma_{\scriptscriptstyle Ok}) \frac{v_{\scriptscriptstyle Ok}(p_{\scriptscriptstyle Ok})}{\sum_{k\in S_O} v_{\scriptscriptstyle Ok}(p_{\scriptscriptstyle Ok}) + \exp(\alpha_{\scriptscriptstyle Oo})} \end{array} \right], \tag{B.15}$$

and

$$\frac{\partial\widetilde{\Pi}(\boldsymbol{p}_{R},\boldsymbol{p}_{O})}{\partial p_{Oz}} = \beta q_{Oz}(\boldsymbol{p}_{R},\boldsymbol{p}_{O})\left[\begin{array}{l}\sum_{n\in\{R,O\}}\sum_{k\in S_n}(p_{nk}-c_{nk}-\gamma_{nk})q_{nk}(\boldsymbol{p}_{R},\boldsymbol{p}_{O})-(p_{Oz}-c_{Oz}-\gamma_{Oz})+1\\ -\sum_{k\in S_R}(p_{Rk}-c_{Rk}-\gamma_{Rk})\frac{v_{Rk}(p_{Rk})}{\sum_{k\in S_R}v_{Rk}(p_{Rk})}\end{array}\right] + (1-\beta)\widetilde{q}_{Oz}(\boldsymbol{p}_{R},\boldsymbol{p}_{O})\left[\sum_{k\in S_O}(p_{Ok}-c_{Ok}-\gamma_{Ok})\widetilde{q}_{Ok}(\boldsymbol{p}_{R},\boldsymbol{p}_{O})-(p_{Oz}-c_{Oz}-\gamma_{Oz})+1\right]. \tag{B.16}$$

The above derivatives can be verified by using the derivatives provided in the preamble of Appendix A, i.e. (A.2), (A.3), (A.4) and (A.5) and the following partial derivatives of $\widetilde{q}_{Ok}(S_O,\boldsymbol{p}_{O})$, given by (3.5), with respect to p_{nk}: $\frac{\partial\widetilde{q}_{Oz}(\boldsymbol{p}_O)}{\partial p_{Oz}} = -\widetilde{q}_{Oz}(\boldsymbol{p}_{O})\left[1-\widetilde{q}_{Oz}(\boldsymbol{p}_{O})\right]$ for $z\in S_O$, and $\frac{\partial\widetilde{q}_{Ok}(\boldsymbol{p}_O)}{\partial p_{Oz}} = \widetilde{q}_{Ok}(\boldsymbol{p}_{O})\widetilde{q}_{Oz}(\boldsymbol{p}_{O})$ for $k\in S_O; k\neq z$.

It can be shown that (B.15) is negative (positive) when p_{Rz} is very large (very small), i.e. $\lim_{p_{Rz}\to\infty}\frac{\partial\widetilde{\Pi}(\boldsymbol{p}_R,\boldsymbol{p}_O)}{\partial p_{Rz}} < 0$ ($\lim_{p_{Rz}\to-\infty}\frac{\partial\widetilde{\Pi}(\boldsymbol{p}_R,\boldsymbol{p}_O)}{\partial p_{Rz}} > 0$). Also, it can be shown that (B.16) is negative (positive) when p_{Oz} is very large (very small), i.e. $\lim_{p_{Oz}\to\infty}\frac{\partial\widetilde{\Pi}(\boldsymbol{p}_R,\boldsymbol{p}_O)}{\partial p_{Oz}} < 0$ ($\lim_{p_{Oz}\to-\infty}\frac{\partial\widetilde{\Pi}(\boldsymbol{p}_R,\boldsymbol{p}_O)}{\partial p_{Oz}} > 0$). Hence, it is never optimal to set one of the prices at a boundary, and the optimal price vectors must satisfy the FOC.

To conclude the proof of the proposition, we next prove that any optimal price vector $\boldsymbol{p}_{R}$ and $\boldsymbol{p}_{O}$ must satisfy the property that $p_{nk}-c_{nk}-\gamma_{nk}$ are the same for all $k\in S_n$. Setting the first derivatives of $\widetilde{\Pi}(\boldsymbol{p}_{R},\boldsymbol{p}_{O})$ with respect to p_{Rk} and p_{Ok}, given by (B.15) and (B.16), to zero yield the following FOCs:

$$p_{Rz}-c_{Rz}-\gamma_{Rz} = \sum_{n\in\{R,O\}}\sum_{k\in S_n}(p_{nk}-c_{nk}-\gamma_{nk})q_{nk}(\boldsymbol{p}_{R},\boldsymbol{p}_{O})+1 - \sum_{k\in S_O}(p_{Ok}-c_{Ok}-\gamma_{Ok})\frac{v_{Ok}(p_{Ok})}{\sum_{k\in S_O}v_{Ok}(p_{Ok})+\exp(\alpha_{Oo})}$$

and

$$(p_{O_z} - c_{O_z} - \gamma_{O_z}) = \left[\frac{\beta}{\exp(\alpha_0) + \sum_{i \in S_R} \sum_{j \in S_O \cup \{0\}} v_{[ij]}(p_{Ri}, p_{Oj})} + \frac{(1-\beta)}{\exp(\widetilde{\alpha}_0) + \sum_{k \in S_O} v_{Ok}(p_{Ok})} \right]^{-1}$$

$$\times \left[\frac{\beta \sum_{k \in S_R} v_{Rk}(p_{Rk})}{\exp(\alpha_0) + \sum_{i \in S_R} \sum_{j \in S_O \cup \{0\}} v_{[ij]}(p_{Ri}, p_{Oj})} \times \left[\sum_{n \in \{R,O\}} \sum_{k \in S_n} (p_{nk} - c_{nk} - \gamma_{nk}) q_{nk}(\boldsymbol{p}_R, \boldsymbol{p}_O) + 1 - \sum_{k \in S_R} (p_{Rk} - c_{Rk} - \gamma_{Rk}) \frac{v_{Rk}(p_{Rk})}{\sum_{k \in S_R} v_{Rk}(p_{Rk})} \right] + \frac{(1-\beta)\left[\sum_{k \in S_O} (p_{Ok} - c_{Ok} - \gamma_{Ok}) \widetilde{q}_{Ok}(\boldsymbol{p}_R, \boldsymbol{p}_O) + 1\right]}{\exp(\widetilde{\alpha}_0) + \sum_{k \in S_O} v_{Ok}(p_{Ok})} \right].$$

Observe that fixing $n \in \{R, O\}$, the right-hand side (RHS) of the above equations are the same for all $z \in S_n$. Therefore, $p_{nz} - c_{nz} - \gamma_{nz} = m_n$ for all $z \in S_n$. This concludes the proof.

Proof of Proposition 13: Recall that the assortments S_R and S_O are fixed for the purposes of this proposition; hence we drop them from the argument list of the functions. The firm's expected profit as a function of margins is given by (3.7). From Proposition 12 we know that the optimal prices must satisfy FOC (hence the optimal margins must satisfy FOC). Note that (3.7) can be written as:

$$\widetilde{\Pi}(m_R, m_O) = \beta \left[m_R \sum_{k \in S_R} [1 - q_0(m_R, m_O)] + m_O[1 - q_0(m_R, m_O) - q_{O_o}(m_R, m_O)] \right] + (1-\beta) m_O \sum_{k \in S_O} \widetilde{q}_{Ok}(m_O),$$

where $\widetilde{q}_{Ok}(m_O) = \frac{\exp(\eta_{Ok} - m_O)}{\exp(\widetilde{\alpha}_0) + \sum_{k \in S_O} \exp(\eta_{Ok} - m_O)}$. Taking the first derivatives of $\widetilde{\Pi}(m_R, m_O)$

with respect to m_R and m_O yields,

$$\frac{\partial\widetilde{\Pi}(m_R,m_O)}{\partial m_R} = \beta\left[1-q_0(m_R,m_O)\left[\begin{array}{l} m_O\left[1-q_0(m_R,m_O)-q_{O_o}(m_R,m_O)\right] \\ +1+m_R[1-q_0(m_R,m_O)] \end{array}\right]\right]$$

$$\frac{\partial\widetilde{\Pi}(m_R,m_O)}{\partial m_O} = \beta\left[\begin{array}{l} 1-q_0(m_R,m_O) \\ -q_{O_o}(m_R,m_O) \end{array}\right]\left[\begin{array}{l} -m_R q_0(m_R,m_O)+1 \\ -m_O\left[q_0(m_R,m_O)+q_{O_o}(m_R,m_O)\right] \end{array}\right]$$
$$-(1-\beta)\left[1-\widetilde{q}_0(m_O)\right]\left[m_O\widetilde{q}_0(m_O)-1\right].$$

The above derivatives can be verified using the derivatives provided for $q_0(m_R,m_O)$ and $q_{O_o}(m_R,m_O)$ in (A.8), (A.9), (A.10), (A.11) and the partial derivative

$$\frac{\partial\widetilde{q}_0(m_O)}{\partial m_O} = \widetilde{q}_0(m_O)\left[1-\widetilde{q}_0(m_O)\right].$$

By setting the derivatives of $\widetilde{\Pi}(m_R,m_O)$ equal to zero and rearranging, we obtain the following FOC:

$$m_R = \frac{1-q_0(m_R,m_O)-m_O q_0(m_R,m_O)\left[1-q_0(m_R,m_O)-q_{O_o}(m_R,m_O)\right]}{q_0(m_R,m_O)[1-q_0(m_R,m_O)]}.$$

and

$$m_O = \frac{(1-\beta)(1-\widetilde{q}_0(m_O))}{\beta(1-q_0(m_R,m_O)-q_{O_o}(m_R,m_O))\frac{q_{O_o}(m_R,m_O)}{1-q_0(m_R,m_O)}+(1-\beta)(1-\widetilde{q}_0(m_O))\widetilde{q}_0(m_O)}.$$

Observe from the second equality that the optimal m_O will always be positive. As for the optimal m_R, it is easy to find examples where it is negative. We provide such example in Table 3.3.

Proof of Proposition 14: Recall that the assortments S_R and S_O are fixed for the purposes of this proposition; hence we drop them from the argument list of the functions. Recall from Proposition 2 that the optimal margin pair m_R^* and m_O^* is the unique pair of m_R and m_O that satisfy the FOC of $\Pi(m_R, m_O)$ and, furthermore, $m_O^* = 0$. Also recall that there may be more than one pair of m_R and m_O that satisfy the FOC of $\widetilde{\Pi}(m_R, m_O)$, but the optimal pair of m_R and m_O must satisfy the FOC of $\widetilde{\Pi}(m_R, m_O)$. We will conclude the proof by showing that $m_R^* > \widetilde{m}_R$.

We first provide the derivative of $\Pi(m_R, m_O)$ with respect to m_R, i.e.

$$\begin{aligned}\frac{\partial \Pi(m_R, m_O)}{\partial m_R} =& 1 - q_0(m_R, m_O)\left[1 + m_R[1 - q_0(m_R, m_O)]\right] \\ & - m_O q_0(m_R, m_O)\left[1 - q_0(m_R, m_O) - q_{O_o}(m_R, m_O)\right].\end{aligned}$$

Recall from Proposition 2 that the m_O that satisfy both FOC for $\Pi(m_R, m_O)$ is $m_O = 0$. Hence the FOC of $\Pi(m_R, m_O)$ with respect to m_R becomes

$$\left.\frac{\partial \Pi(m_R, m_O)}{\partial m_R}\right|_{m_O=0} = 1 - q_0(m_R, m_O)\left[1 + m_R[1 - q_0(m_R, m_O)]\right] = 0. \tag{B.17}$$

To conclude the proof we will show that any m_R that satisfy FOC of $\widetilde{\Pi}(m_R, m_O)$ will result in $\frac{\partial \Pi(m_R, m_O)}{\partial m_R} > 0$, which will imply that the m_R needs to be increased. Let $\widetilde{m}_R$ and $\widetilde{m}_O$ be a pair of margins that satisfy the FOC of $\widetilde{\Pi}(m_R, m_O)$, i.e. $\widetilde{m}_R$ and $\widetilde{m}_O$ satisfy the following

$$\begin{aligned}\left.\frac{\partial \widetilde{\Pi}(m_R, m_O)}{\partial m_R}\right|_{\substack{m_R=\widetilde{m}_R,\\ m_O=\widetilde{m}_O}} &= \beta\left[1 - q_0(m_R, m_O)\begin{bmatrix} m_O\left[1 - q_0(m_R, m_O) - q_{O_o}(m_R, m_O)\right] \\ +1 + m_R[1 - q_0(m_R, m_O)]\end{bmatrix}\right] \\ &= 0.\end{aligned}$$

From Proposition 13, recall that $\widetilde{m}_O > 0$. It then follows from the equality above that

$$\beta\left\{1 - q_0(m_R, m_O)\left[1 + m_R[1 - q_0(m_R, m_O)]\right]\right\} > 0.$$

Now we rewrite the expression above as:

$$\beta\left[1-q_0(m_R,m_O)\right]\left[1-m_R q_0(m_R,m_O)\right]\right] > 0$$

One can check form the derivative of the no-purchase probability $q_0(m_R, m_O)$ that it increases in the margin m_O. Hence, we can use the above equality to write:

$$\beta\left[1-q_0(m_R,0)\right]\left[1-m_R q_0(m_R,0)\right]\right] > 0.$$

Observe from (B.17) that the left-hand side of the above inequality is equal to $\left.\frac{\partial\Pi(m_R,m_O)}{\partial m_R}\right|_{\substack{m_R=\widetilde{m}_R\\ m_O=0}}$. Therefore, we conclude that

$$\left.\frac{\partial\Pi(m_R,m_O)}{\partial m_R}\right|_{m_R=\widetilde{m}_R, m_O=0} > 0.$$

Now, to conclude the proof, we note that m_R^* and $m_O^*=0$ is the unique (m_R, m_O) pair that satisfies the FOC of $\Pi(m_R, m_O)$. This implies that, when $m_O=0$, $\Pi(m_R, m_O)$ is unimodal in m_R. Since $\left.\frac{\partial\Pi(m_R,m_O)}{\partial m_R}\right|_{m_R=\widetilde{m}_R, m_O=0} > 0$, it now follows that $m_R^* > \widetilde{m}_O$.

Proofs for Section 3.4

Proof of Proposition 15: Suppose that the assortments, S_R and S_O, are fixed as required by the proposition. Then, the firm's problem in (3.12) reduces to picking the price vectors $\boldsymbol{p}_R$ and $\boldsymbol{p}_O$ to maximize the profit function $\Pi^G(S_R, S_O, \boldsymbol{p}_R, \boldsymbol{p}_O)$. In this proof, S_R and S_O are dropped from the argument list of the functions.

In preparation for the rest of the proof, a list of the partial derivatives of (3.10) and (3.11) are provided next.

$$
\begin{aligned}
\frac{\partial q^{G}_{nz}(\boldsymbol{p}_{R},\boldsymbol{p}_{O})}{\partial p_{Rz}} &= -q^{G}_{nz}(\boldsymbol{p}_{R},\boldsymbol{p}_{O})\left[1-q^{G}_{nz}(\boldsymbol{p}_{R},\boldsymbol{p}_{O})\right] \text{ for } z\in S_n, n\in\{R,O\},\\
\frac{\partial q^{G}_{nk}(\boldsymbol{p}_{R},\boldsymbol{p}_{O})}{\partial p_{Rz}} &= q^{G}_{nk}(\boldsymbol{p}_{R},\boldsymbol{p}_{O})q^{G}_{Rz}(\boldsymbol{p}_{R},\boldsymbol{p}_{O}) \text{ for } k\in S_n; k\neq z,\\
\frac{\partial q^{G}_{Ok}(\boldsymbol{p}_{R},\boldsymbol{p}_{O})}{\partial p_{Rz}} &= -q^{G}_{[zk]}(\boldsymbol{p}_{R},\boldsymbol{p}_{O})+q^{G}_{Ok}(\boldsymbol{p}_{R},\boldsymbol{p}_{O})q^{G}_{Rz}(\boldsymbol{p}_{R},\boldsymbol{p}_{O}),\\
\frac{\partial q^{G}_{Rk}(\boldsymbol{p}_{R},\boldsymbol{p}_{O})}{\partial p_{Oz}} &= -q^{G}_{[kz]}(\boldsymbol{p}_{R},\boldsymbol{p}_{O})+q^{G}_{Rk}(\boldsymbol{p}_{R},\boldsymbol{p}_{O})q^{G}_{Oz}(\boldsymbol{p}_{R},\boldsymbol{p}_{O}).
\end{aligned}
$$

Using the derivatives above, one can show that the partial derivatives of $\Pi^{G}(\boldsymbol{p}_{R},\boldsymbol{p}_{O})$, given by (3.12), with respect to p_{Rz} and p_{Oz} are:

$$
\begin{aligned}
\frac{\partial \Pi^{G}(\boldsymbol{p}_{R},\boldsymbol{p}_{O})}{\partial p_{Rz}} = \; & q^{G}_{Rz}(\boldsymbol{p}_{R},\boldsymbol{p}_{O})\left[\begin{array}{l}\displaystyle\sum_{n\in\{R,O\}}\sum_{k\in S_n}(p_{nk}-c_{nk}-\gamma_{nk})q^{G}_{nk}(\boldsymbol{p}_{R},\boldsymbol{p}_{O})\\ -(p_{Rz}-c_{Rz}-\gamma_{Rz})+1\end{array}\right]\\
& -\sum_{k\in S_O}(p_{Ok}-c_{Ok}-\gamma_{Ok})q^{G}_{\flat zk\sharp}(\boldsymbol{p}_{R},\boldsymbol{p}_{O}),
\end{aligned} \tag{B.18}
$$

$$
\begin{aligned}
\frac{\partial \Pi^{G}(\boldsymbol{p}_{R},\boldsymbol{p}_{O})}{\partial p_{Oz}} = \; & q^{G}_{Oz}(\boldsymbol{p}_{R},\boldsymbol{p}_{O})\left[\begin{array}{l}\displaystyle\sum_{n\in\{R,O\}}\sum_{k\in S_n}(p_{nk}-c_{nk}-\gamma_{nk})q^{G}_{nk}(\boldsymbol{p}_{R},\boldsymbol{p}_{O})\\ -(p_{Oz}-c_{Oz}-\gamma_{Oz})+1\end{array}\right]\\
& -\sum_{k\in S_R}(p_{Rk}-c_{Rk}-\gamma_{Rk})q^{G}_{\flat kz\sharp}(\boldsymbol{p}_{R},\boldsymbol{p}_{O}).
\end{aligned} \tag{B.19}
$$

It can be shown that (B.18) is negative (positive) when p_{Rz} is very large (very small), i.e.,

$$
\lim_{p_{Rz}\to\infty}\frac{\partial \Pi^{G}(\boldsymbol{p}_{R},\boldsymbol{p}_{O})}{\partial p_{Rz}}<0 \text{ and } \lim_{p_{Rz}\to-\infty}\frac{\partial \Pi^{G}(\boldsymbol{p}_{R},\boldsymbol{p}_{O})}{\partial p_{Rz}}>0.
$$

Likewise, it can be shown that (B.19) is negative (positive) when p_{Oz} is very large (very small), i.e.,

$$
\lim_{p_{Oz}\to\infty}\frac{\partial \Pi^{G}(\boldsymbol{p}_{R},\boldsymbol{p}_{O})}{\partial p_{Oz}}<0 \text{ and } \lim_{p_{Oz}\to-\infty}\frac{\partial \Pi^{G}(\boldsymbol{p}_{R},\boldsymbol{p}_{O})}{\partial p_{Oz}}>0.
$$

Hence, it is never optimal to set one of the prices at a boundary, and the optimal price vectors must satisfy the FOC.

Next is shown that any optimal price vector $\boldsymbol{p}_R$ and $\boldsymbol{p}_O$ must satisfy the property that $p_{Rk} - c_{Rk} - \gamma_{Rk}$ are the same for all $k \in S_R$ and $p_{Ok} - c_{Ok} - \gamma_{Ok}$ are the same for all $k \in S_O$.

Setting the derivative in (B.18) equal to zero and rearranging the terms yields the following FOCs for $\Pi^G(\boldsymbol{p}_R, \boldsymbol{p}_O)$ with respect to p_{Rz}:

$$p_{Rz} - c_{Rz} - \gamma_{Rz} =$$
$$\sum_{n\in\{R,O\}} \sum_{k\in S_n} (p_{nk} - c_{nk} - \gamma_{nk}) q^G_{nk}(\boldsymbol{p}_R, \boldsymbol{p}_O) + 1 - \frac{\sum_{k\in S_O}(p_{Ok} - c_{Ok} - \gamma_{Ok}) \exp(\alpha_{Ok} - p_{Ok})}{\sum_{j\in S_O} \exp(\alpha_{Oj} - p_{Oj}) + \exp(\alpha_{Oo})}.$$

Observe that the right-hand side (RHS) of the above FOC are the same for all $z \in S_R$. Therefore, $p_{Rk} - c_{Rk} - \gamma_{Rk} = m_R$ for all $k \in S_R$. Now, set the derivative in (B.19) equal to zero and rearranging the terms yields the following FOCs for $\Pi^G(\boldsymbol{p}_R, \boldsymbol{p}_O)$ with respect to p_{Oz}:

$$p_{Oz} - c_{Oz} - \gamma_{Oz} =$$
$$\sum_{n\in\{R,O\}} \sum_{k\in S_n} (p_{nk} - c_{nk} - \gamma_{nk}) q^G_{nk}(\boldsymbol{p}_R, \boldsymbol{p}_O) + 1 - \frac{\sum_{k\in S_R}(p_{Rk} - c_{Rk} - \gamma_{Rk}) \exp(\alpha_{Rk} - p_{Rk})}{\sum_{i\in S_R} \exp(\alpha_{Ri} - p_{Ri}) + \exp(\alpha_{Ro})}.$$

Observe that the right-hand side (RHS) of the above FOC are the same for all $z \in S_O$. Therefore, $p_{Ok} - c_{Ok} - \gamma_{Ok} = m_O$ for all $k \in S_O$.

Appendix C

Proofs of Propositions in Chapter IV

Proofs for Section 4.4

In preparation for the proofs that follow, we note below the derivatives of purchase probabilities defined in Section 4.3.1. The following derivatives can be verified through some algebra using the definitions of purchase probabilities in (4.1) and (4.4):

$$\frac{\partial q_z^n(S^n, \boldsymbol{p}^n)}{\partial p_k^n(S^n, \boldsymbol{p}^n)} = \frac{q_k^n(S^n, \boldsymbol{p}^n)}{\mu_2} q_z^n(S^n, \boldsymbol{p}^n), z \in S^n, z \neq k, \tag{C.1}$$

$$\frac{\partial q_k^n(S^n, \boldsymbol{p}^n)}{\partial p_k^n} = -\frac{q_k^n(S^n, \boldsymbol{p}^n)}{\mu_2}[1 - q_k^n(S^n, \boldsymbol{p}^n)], k \in S^n, \tag{C.2}$$

$$\frac{\partial \tau^{R}(S^R, \boldsymbol{p}^{M}, \boldsymbol{p}^{R})}{\partial p_k^{R}} = -\frac{q_k^{R}(S^R, \boldsymbol{p}^{R})}{\mu_1} \tau^{R}(S^R, \boldsymbol{p}^{M}, \boldsymbol{p}^{R})[1 - \tau^{R}(S^R, \boldsymbol{p}^{M}, \boldsymbol{p}^{R})]. \tag{C.3}$$

The proofs of the propositions in Section 4.4 are for any given S^R; hence S^R is dropped from the argument list of the functions.

Proof of Proposition 16: The retailer's problem in (4.8) reduces to picking the price vector $\boldsymbol{p}^{R}$ to maximize its expected profit function $\Pi^R(\boldsymbol{p}^{M}, \boldsymbol{w}, \boldsymbol{p}^{R})$. We start by proving that any optimal price vector $\boldsymbol{p}^{R}$ must satisfy the first order conditions (FOC) of the retailer's objective function $\Pi^R(\boldsymbol{p}^{M}, \boldsymbol{w}, \boldsymbol{p}^{R})$.

Using the derivatives of the functions provided in (C.1), (C.2) and (C.3) it can be verified that the derivative of retailer's expected profit, $\Pi^R(\boldsymbol{p}^{\scriptscriptstyle M}, \boldsymbol{w}, \boldsymbol{p}^{\scriptscriptstyle R})$, given by (4.8), is:

$$\frac{\partial \Pi^R(\boldsymbol{p}^{\scriptscriptstyle M}, \boldsymbol{w}, \boldsymbol{p}^{\scriptscriptstyle R})}{\partial p_k^{\scriptscriptstyle R}} = \frac{\tau^{\scriptscriptstyle R}(\boldsymbol{p}^{\scriptscriptstyle M}, \boldsymbol{p}^{\scriptscriptstyle R}) q_k^{\scriptscriptstyle R}(\boldsymbol{p}^{\scriptscriptstyle R})}{\mu_2} \left[\begin{array}{l} \displaystyle\sum_{k \in S^R} (p_k^{\scriptscriptstyle R} - w_k - \gamma_k)\, q_k^{\scriptscriptstyle R}(\boldsymbol{p}^{\scriptscriptstyle R}) \left[1 - \frac{\mu_2 (1 - \tau^{\scriptscriptstyle R}(\boldsymbol{p}^{\scriptscriptstyle M}, \boldsymbol{p}^{\scriptscriptstyle R}))}{\mu_1} \right] \\ - (p_k^{\scriptscriptstyle R} - w_k - \gamma_k) + \mu_2 \end{array} \right]. \quad \text{(C.4)}$$

It can be shown that (C.4) is negative (positive) when $p_k^{\scriptscriptstyle R}$ is very large (very small), i.e.,

$$\lim_{p_k^{\scriptscriptstyle R} \to \infty} \frac{\partial \Pi^R(\boldsymbol{p}^{\scriptscriptstyle M}, \boldsymbol{w}, \boldsymbol{p}^{\scriptscriptstyle R})}{\partial p_k^{\scriptscriptstyle R}} < 0, \text{ and } \lim_{p_k^{\scriptscriptstyle R} \to -\infty} \frac{\partial \Pi^R(\boldsymbol{p}^{\scriptscriptstyle M}, \boldsymbol{w}, \boldsymbol{p}^{\scriptscriptstyle R})}{\partial p_k^{\scriptscriptstyle R}} > 0.$$

Hence, it is never optimal to set one of the prices at a boundary, and the optimal price vector must satisfy the FOC.

To conclude the proof of the proposition, we next prove that any optimal price vector $\boldsymbol{p}^{\scriptscriptstyle R}$ must satisfy the property that $p_k^{\scriptscriptstyle R} - w_k - \gamma_k$ is the same for all $k \in S^R$. Setting the derivative of $\Pi^R(\boldsymbol{p}^{\scriptscriptstyle M}, \boldsymbol{w}, \boldsymbol{p}^{\scriptscriptstyle R})$, provided in (C.4), equal to zero and re-arranging terms yields the following FOC for $\Pi^R(\boldsymbol{p}^{\scriptscriptstyle M}, \boldsymbol{w}, \boldsymbol{p}^{\scriptscriptstyle R})$:

$$p_k^{\scriptscriptstyle R} - w_k - \gamma_k = \sum_{k \in S^R} (p_k^{\scriptscriptstyle R} - w_k - \gamma_k)\, q_k^{\scriptscriptstyle R}(\boldsymbol{p}^{\scriptscriptstyle R}) \left[1 - \frac{\mu_2 (1 - \tau^{\scriptscriptstyle R}(\boldsymbol{p}^{\scriptscriptstyle M}, \boldsymbol{p}^{\scriptscriptstyle R}))}{\mu_1} \right] + \mu_2.$$

Observe that the right-hand side (RHS) of the above equation is the same for all $k \in S^R$. Therefore, $p_k^{\scriptscriptstyle R} - w_k - \gamma_k = m^{\scriptscriptstyle R}$ for all $k \in S^R$, where $m^{\scriptscriptstyle R}$ is the effective profit margin for the retailer.

Proof of Proposition 17: In preparation for the proof, we note below that the derivative of (4.15) is:

$$\frac{\partial \tau^{\scriptscriptstyle R}(\boldsymbol{p}^{\scriptscriptstyle M}, m^{\scriptscriptstyle R})}{\partial m^{\scriptscriptstyle R}} = -\frac{\tau^{\scriptscriptstyle R}(\boldsymbol{p}^{\scriptscriptstyle M}, m^{\scriptscriptstyle R})}{\mu_1} [1 - \tau^{\scriptscriptstyle R}(\boldsymbol{p}^{\scriptscriptstyle M}, m^{\scriptscriptstyle R})] < 0.$$

Using this derivative it can be verified that the derivative of $\Pi^R(\boldsymbol{p}^M, m^R)$, given in (4.16), with respect to m^R is given by:

$$\frac{\partial \Pi^R(\boldsymbol{p}^M, m^R)}{\partial m^R} = \tau^R(\boldsymbol{p}^M, m^R)\left[1 - \frac{m^R}{\mu_1}[1 - \tau^R(\boldsymbol{p}^M, m^R)]\right]. \tag{C.5}$$

Recall from Proposition 16 that the optimal prices must be interior solutions, that is, $-\infty < p_k^R < \infty$ for $k \in S^R$. In addition, observe from Proposition 16, the margin, m^R, and the price, p_k^R, differ by $w_k + \gamma_k$. Therefore, for finite $\boldsymbol{w}$, the optimal margin m^R must also be an interior solution, i.e., $-\infty < m^R < \infty$. Thus, m^R must satisfy the FOC of $\Pi^R(\boldsymbol{p}^M, m^R)$.

Setting the derivative of $\Pi^R(\boldsymbol{p}^M, m^R)$ (provided in (C.5)) equal to zero and rearranging terms, yields the following FOC for $\Pi^R(\boldsymbol{p}^M, m^R)$ with respect to m^R:

$$m^R = \frac{\mu_1}{1 - \tau^R(\boldsymbol{p}^M, m^R)}. \tag{C.6}$$

To conclude, we next prove that the m^R that satisfies this FOC is in fact unique. Substituting for $\tau^R(\boldsymbol{p}^M, m^R)$ in (C.6) with the expression provided in (4.15) and algebraically manipulating the terms, we can rewrite the above equality as:

$$\left(\frac{m^R}{\mu_1} - 1\right)\exp\left(\frac{m^R}{\mu_1} - 1\right) = \frac{\exp(-1)\left[\sum_{k\in S^R}\exp(\eta_k^R/\mu_2)\right]^{\mu_2/\mu_1}}{\exp(\alpha_0) + \left[\sum_{k\in S^M} v_k(p_k^M)\right]^{\mu_2/\mu_1}}.$$

Applying the definition of the Lambert-W function (provided in the preamble of Appendix A) and with some algebra we can rewrite the above expression as,

$$m^R(\boldsymbol{p}^M, \boldsymbol{w}) = \mu_1\left[1 + W\left(\frac{\exp(-1)\left[\sum_{k\in S^R}\exp(\eta_k^R/\mu_2)\right]^{\mu_2/\mu_1}}{\exp(\alpha_0) + \left[\sum_{k\in S^M} v_k(p_k^M)\right]^{\mu_2/\mu_1}}\right)\right], \tag{C.7}$$

Observe that the right-hand side of the above equality does not depend on m^R and the left-hand side is strictly increasing in m^R. Therefore, there exists a unique m^R

that satisfies the FOC.

Proof of Proposition 18: This proof will proceed in two steps. In Step 1, we will rewrite the manufacturer's profit, given in (4.10), as a function of the optimal effective margin chosen by the retailer in response to the manufacturer's wholesale and direct channel price vector, i.e., $m^R(\boldsymbol{p}^M, \boldsymbol{w})$ as provided in (C.7). In Step 2, we will use this version of the manufacturer's profit function to prove the properties that the manufacturer's optimal price vector must satisfy.

Step 1: We will show that the manufacturer's profit in (4.10) can be written as

$$\Pi^M(S^R, \boldsymbol{p}^M, \boldsymbol{w}, m^R(S^R, \boldsymbol{p}^M, \boldsymbol{w})) = \tau^M(S^R, \boldsymbol{p}^M, m^R(S^R, \boldsymbol{p}^M, \boldsymbol{w})) \sum_{k \in S^M} (p_k^M - c_k) q_k^M(\boldsymbol{p}^M)$$
$$+ \tau^R(S^R, \boldsymbol{p}^M, m^R(S^R, \boldsymbol{p}^M, \boldsymbol{w})) \sum_{k \in S^R} (w_k - c_k)\theta_k q_k^R(S^R, m^R(S^R, \boldsymbol{p}^M, \boldsymbol{w})),$$

where $\tau^R(\boldsymbol{p}^M, m^R(\boldsymbol{p}^M, \boldsymbol{w}))$, $\tau^M(\boldsymbol{p}^M, m^R(\boldsymbol{p}^M, \boldsymbol{w}))$ and $q_k^R(m^R(\boldsymbol{p}^M, \boldsymbol{w}))$ are defined appropriately.

$$\tau^R(\boldsymbol{p}^M, m^R(\boldsymbol{p}^M, \boldsymbol{w})) = \frac{W(\Omega(\boldsymbol{p}^M, \boldsymbol{w}))}{1 + W(\Omega(\boldsymbol{p}^M, \boldsymbol{w}))},$$

$$\tau^M(\boldsymbol{p}^M, m^R(\boldsymbol{p}^M, \boldsymbol{w})) = \frac{\left[\sum_{k \in S^M} v_k(p_k^M)\right]^{\mu_2/\mu_1}}{\left[\exp(\alpha_0) + \left[\sum_{k \in S^M} v_k(p_k^M)\right]^{\mu_2/\mu_1}\right] \left[1 + W(\Omega(\boldsymbol{p}^M, \boldsymbol{w}))\right]},$$

and

$$q_k^R(m^R(\boldsymbol{p}^M, \boldsymbol{w})) = \frac{v_k^R(m^R(\boldsymbol{p}^M, \boldsymbol{w}))}{\sum_{k \in S^n} v_k^R(m^R(\boldsymbol{p}^M, \boldsymbol{w}))} \text{ for all } k \in S^R.$$

Let $\Omega(\boldsymbol{p}^{M}, \boldsymbol{w})$ be defined as:

$$\Omega(\boldsymbol{p}^{M}, \boldsymbol{w}) := \frac{\exp(-1)\left[\sum_{k \in S^R} \exp([\alpha_k - w_k - \gamma_k]/\mu_2)\right]^{\mu_2/\mu_1}}{\exp(\alpha_0) + \left[\sum_{k \in S^M} v_k(p_k^{M})\right]^{\mu_2/\mu_1}}. \tag{C.8}$$

Using the expression above we can write $m^{R}(\boldsymbol{p}^{M}, \boldsymbol{w})$ in (C.7) as

$$m^{R}(\boldsymbol{p}^{M}, \boldsymbol{w}) = \mu_1 \left[1 + W(\Omega(\boldsymbol{p}^{M}, \boldsymbol{w}))\right].$$

Observe from Proposition 17 that $\tau^{R}(\boldsymbol{p}^{M}, m^{R}(\boldsymbol{p}^{M}, \boldsymbol{w})) = 1 - \frac{\mu_1}{m^{R}(\boldsymbol{p}^{M}, \boldsymbol{w})}$. Substituting in this expression for $m^{R}(\boldsymbol{p}^{M}, \boldsymbol{w})$ with $\mu_1 \left[1 + W(\Omega(\boldsymbol{p}^{M}, \boldsymbol{w}))\right]$ we can write $\tau^{R}(\boldsymbol{p}^{M}, m^{R}(\boldsymbol{p}^{M}, \boldsymbol{w}))$ as,

$$\tau^{R}(\boldsymbol{p}^{M}, m^{R}(\boldsymbol{p}^{M}, \boldsymbol{w})) = \frac{W(\Omega(\boldsymbol{p}^{M}, \boldsymbol{w}))}{1 + W(\Omega(\boldsymbol{p}^{M}, \boldsymbol{w}))}. \tag{C.9}$$

Next we recast $\tau^{M}(\boldsymbol{p}^{M}, \boldsymbol{p}^{R})$ in (4.4) in terms of $m^{R}(\boldsymbol{p}^{M}, \boldsymbol{w})$ instead of $\boldsymbol{p}^{R}$. To that end, we replace $v_k(p_k^{R})$ in the denominator of (4.4) with $v_k^{R}(m^{R}(\boldsymbol{p}^{M}, \boldsymbol{w}))$, where $v_k^{R}(m^{R}(\boldsymbol{p}^{M}, \boldsymbol{w}))$ is defined by (4.14):

$$\tau^{M}(\boldsymbol{p}^{M}, m^{R}(\boldsymbol{p}^{M}, \boldsymbol{w})) = \frac{\left[\sum_{k \in S^M} v_k(p_k^{M})\right]^{\mu_2/\mu_1}}{\exp(\alpha_0) + \left[\sum_{k \in S^M} v_k(p_k^{M})\right]^{\mu_2/\mu_1} + \left[\sum_{k \in S^R} v_k^{R}(m^{R}(\boldsymbol{p}^{M}, \boldsymbol{w}))\right]^{\mu_2/\mu_1}}. \tag{C.10}$$

Given that $m^{R}(\boldsymbol{p}^{M}, \boldsymbol{w}) = \mu_1 \left[1 + W(\Omega(\boldsymbol{p}^{M}, \boldsymbol{w}))\right]$ the definition $\Omega(\boldsymbol{p}^{M}, \boldsymbol{w})$ and the fact that for any value $a > 0$ we have that $\exp(W(a)) = \frac{a}{W(a)}$, we can write the following equality:

$$\left[\sum_{k\in S^R} v_k^{R}(m^{R}(\boldsymbol{p}^{M},\boldsymbol{w}))\right]^{\mu_2/\mu_1} = \left[\sum_{k\in S^R} \exp([\eta_k^{R} - m^{R}(\boldsymbol{p}^{M},\boldsymbol{w})]/\mu_2)\right]^{\mu_2/\mu_1}$$
$$= \left[\sum_{k\in S^R} \exp([\eta_k^{R} - \mu_1\left[1 + W(\Omega(\boldsymbol{p}^{M},\boldsymbol{w}))\right]/\mu_2)\right]^{\mu_2/\mu_1}$$
$$= W(\Omega(\boldsymbol{p}^{M},\boldsymbol{w}))\left[\exp(\alpha_0) + \left[\sum_{k\in S^M} v_k(p_k^{M})\right]^{\mu_2/\mu_1}\right].$$

Hence, we can write $\tau^{M}(\boldsymbol{p}^{M}, m^{R}(\boldsymbol{p}^{M},\boldsymbol{w}))$, given by (C.10),

$$\tau^{M}(\boldsymbol{p}^{M}, m^{R}(\boldsymbol{p}^{M},\boldsymbol{w})) = \frac{\left[\sum_{k\in S^M} v_k(p_k^{M})\right]^{\mu_2/\mu_1}}{\left[\exp(\alpha_0) + \left[\sum_{k\in S^M} v_k(p_k^{M})\right]^{\mu_2/\mu_1}\right]\left[1 + W(\Omega(\boldsymbol{p}^{M},\boldsymbol{w}))\right]}. \quad \text{(C.11)}$$

As for $q_k^{R}(\boldsymbol{p}^{R})$ given in (4.1), we use the definition of $v_k^{R}(m^{R}(S^R,\boldsymbol{p}^{M},\boldsymbol{w}))$, in (4.14), to write,

$$q_k^{R}(m^{R}(\boldsymbol{p}^{M},\boldsymbol{w})) = \frac{v_k^{R}(m^{R}(\boldsymbol{p}^{M},\boldsymbol{w}))}{\sum_{k\in S^n} v_k^{R}(m^{R}(\boldsymbol{p}^{M},\boldsymbol{w}))} \text{ for all } k \in S^R. \quad \text{(C.12)}$$

Provided the expressions above for the purchase probabilities, i.e. (C.9), (C.11) and (C.12), we can now recast the manufacturer's expected profit in (4.10) as,

$$\Pi^{M}(S^R,\boldsymbol{p}^{M},\boldsymbol{w},m^{R}(S^R,\boldsymbol{p}^{M},\boldsymbol{w})) = \tau^{M}(S^R,\boldsymbol{p}^{M},m^{R}(S^R,\boldsymbol{p}^{M},\boldsymbol{w}))\sum_{k\in S^M}(p_k^{M}-c_k)q_k^{M}(\boldsymbol{p}^{M})$$
$$+\,\tau^{R}(S^R,\boldsymbol{p}^{M},m^{R}(S^R,\boldsymbol{p}^{M},\boldsymbol{w}))\sum_{k\in S^R}(w_k-c_k)\theta_k q_k^{R}(S^R,m^{R}(S^R,\boldsymbol{p}^{M},\boldsymbol{w})). \quad \text{(C.13)}$$

Step 2: To continue with the proof, let us now show that any optimal price vector $\boldsymbol{p}^{M}$ and $\boldsymbol{w}$ must satisfy the first order conditions (FOC) of the manufacturer's objective function $\Pi^{M}(\boldsymbol{p}^{M},\boldsymbol{w},m^{R}(\boldsymbol{p}^{M},\boldsymbol{w}))$, given by (C.13). Below are the partial derivatives of $\Omega(\boldsymbol{p}^{M},\boldsymbol{w})$, given in (C.8), $\tau^{R}(\boldsymbol{p}^{M},m^{R}(\boldsymbol{p}^{M},\boldsymbol{w}))$, given in (C.9), $\tau^{M}(\boldsymbol{p}^{M},m^{R}(\boldsymbol{p}^{M},\boldsymbol{w}))$, given in (C.11), and $q_k^{R}(m^{R}(\boldsymbol{p}^{M}, \boldsymbol{w}))$, given in (C.12).

$$
\begin{aligned}
\frac{\partial \Omega(\boldsymbol{p}^{M}, \boldsymbol{w})}{\partial p_k^{M}} &= \frac{q_k^{M}(\boldsymbol{p}^{M})\Omega(\boldsymbol{p}^{M}, \boldsymbol{w})}{\mu_1}\tau^{M}(\boldsymbol{p}^{M}, m^{R}(\boldsymbol{p}^{M}, \boldsymbol{w}))[1 + W(\Omega(\boldsymbol{p}^{M}, \boldsymbol{w}))],\\
\frac{\partial \tau^{R}(\boldsymbol{p}^{M}, m^{R}(\boldsymbol{p}^{M}, \boldsymbol{w}))}{\partial p_k^{M}} &= \frac{q_k^{M}(\boldsymbol{p}^{M})}{\mu_1}\tau^{R}(\boldsymbol{p}^{M}, m^{R}(\boldsymbol{p}^{M}, \boldsymbol{w}))\\
&\quad \times [1 - \tau^{R}(\boldsymbol{p}^{M}, m^{R}(\boldsymbol{p}^{M}, \boldsymbol{w}))]\tau^{M}(\boldsymbol{p}^{M}, m^{R}(\boldsymbol{p}^{M}, \boldsymbol{w})),\\
\frac{\partial \tau^{M}(\boldsymbol{p}^{M}, m^{R}(\boldsymbol{p}^{M}, \boldsymbol{w}))}{\partial p_k^{M}} &= \frac{-q_k^{M}(\boldsymbol{p}^{M})\tau^{M}(\boldsymbol{p}^{M}, m^{R}(\boldsymbol{p}^{M}, \boldsymbol{w}))}{\mu_1}\\
&\quad \times \left[\begin{array}{c} 1 - \tau^{M}(\boldsymbol{p}^{M}, m^{R}(\boldsymbol{p}^{M}, \boldsymbol{w}))[1 + W(\Omega(\boldsymbol{p}^{M}, \boldsymbol{w}))] \\ + \tau^{M}(\boldsymbol{p}^{M}, m^{R}(\boldsymbol{p}^{M}, \boldsymbol{w}))\tau^{R}(\boldsymbol{p}^{M}, m^{R}(\boldsymbol{p}^{M}, \boldsymbol{w})) \end{array}\right],\\
\frac{\partial \Omega(\boldsymbol{p}^{M}, \boldsymbol{w})}{\partial w_k} &= \frac{-q_k^{R}(\boldsymbol{p}^{R})\Omega(\boldsymbol{p}^{M}, \boldsymbol{w})}{\mu_1},\\
\frac{\partial \tau^{R}(\boldsymbol{p}^{M}, m^{R}(\boldsymbol{p}^{M}, \boldsymbol{w}))}{\partial w_k} &= \frac{-q_k^{R}(\boldsymbol{p}^{R})}{\mu_1}\tau^{R}(\boldsymbol{p}^{M}, m^{R}(\boldsymbol{p}^{M}, \boldsymbol{w}))[1 - \tau^{R}(\boldsymbol{p}^{M}, m^{R}(\boldsymbol{p}^{M}, \boldsymbol{w}))]^2,\\
\frac{\partial \tau^{M}(\boldsymbol{p}^{M}, m^{R}(\boldsymbol{p}^{M}, \boldsymbol{w}))}{\partial w_k} &= \frac{q_k^{R}(\boldsymbol{p}^{R})}{\mu_1}\tau^{M}(\boldsymbol{p}^{M}, m^{R}(\boldsymbol{p}^{M}, \boldsymbol{w}))\tau^{R}(\boldsymbol{p}^{M}, m^{R}(\boldsymbol{p}^{M}, \boldsymbol{w}))\\
&\quad \times [1 - \tau^{R}(\boldsymbol{p}^{M}, m^{R}(\boldsymbol{p}^{M}, \boldsymbol{w}))],\\
\frac{\partial q_l^{R}(m^{R}(\boldsymbol{p}^{M}, \boldsymbol{w}))}{\partial w_k} &= \frac{q_l^{R}(m^{R}(\boldsymbol{p}^{M}, \boldsymbol{w}))q_k^{R}(m^{R}(\boldsymbol{p}^{M}, \boldsymbol{w}))}{\mu_2}, l \in S^R, l \neq k,\\
\frac{\partial q_k^{R}(m^{R}(\boldsymbol{p}^{M}, \boldsymbol{w}))}{\partial w_k} &= -\frac{q_k^{R}(m^{R}(\boldsymbol{p}^{M}, \boldsymbol{w}))[1 - q_k^{R}(m^{R}(\boldsymbol{p}^{M}, \boldsymbol{w}))]}{\mu_2}, k \in S^R.
\end{aligned}
$$

Using the derivatives above and some algebra one could verify that:

$$\frac{\partial \Pi^M(\boldsymbol{p}^M, \boldsymbol{w}, m^R(\boldsymbol{p}^M, \boldsymbol{w}))}{\partial p_k^M} = \frac{q_k^M(\boldsymbol{p}^M)\tau^M(\boldsymbol{p}^M, m^R(\boldsymbol{p}^M, \boldsymbol{w}))}{\mu_1}$$
$$\times \left[\begin{array}{l} \frac{\mu_2}{\mu_1}\left[\sum_{j\in S^M}(p_j^M - c_j)q_j^M(\boldsymbol{p}^M) - (p_k^M - c_k) + \mu_2\right] \\ - \sum_{j\in S^M}(p_j^M - c_j)q_j^M(\boldsymbol{p}^M)\left[1 - \tau^M(\boldsymbol{p}^M, m^R(\boldsymbol{p}^M, \boldsymbol{w}))\left[\begin{array}{l} 1 + W(\Omega(\boldsymbol{p}^M, \boldsymbol{w})) \\ - \tau^R(\boldsymbol{p}^M, m^R(\boldsymbol{p}^M, \boldsymbol{w}))\end{array}\right]\right] \\ + \sum_{j\in S^R}(w_j - c_j)\theta_j q_j^R(m^R(\boldsymbol{p}^M, \boldsymbol{w}))\tau^R(\boldsymbol{p}^M, m^R(\boldsymbol{p}^M, \boldsymbol{w}))[1 - \tau^R(\boldsymbol{p}^M, m^R(\boldsymbol{p}^M, \boldsymbol{w}))] \end{array} \right], \tag{C.14}$$

and

$$\frac{\partial \Pi^M(\boldsymbol{p}^M, \boldsymbol{w}, m^R(\boldsymbol{p}^M, \boldsymbol{w}))}{\partial w_k} = \frac{q_k^M(\boldsymbol{p}^M)\tau^R(\boldsymbol{p}^M, m^R(\boldsymbol{p}^M, \boldsymbol{w}))}{\mu_1}$$
$$\times \left[\begin{array}{l} \sum_{j\in S^M}(p_j^M - c_j)q_j^M(\boldsymbol{p}^M)\tau^M(\boldsymbol{p}^M, m^R(\boldsymbol{p}^M, \boldsymbol{w}))[1 - \tau^R(\boldsymbol{p}^M, m^R(\boldsymbol{p}^M, \boldsymbol{w}))] \\ - \sum_{j\in S^R}(w_j - c_j)\theta_j q_j^R(m^R(\boldsymbol{p}^M, \boldsymbol{w}))[1 - \tau^R(\boldsymbol{p}^M, m^R(\boldsymbol{p}^M, \boldsymbol{w}))]^2 \\ - \frac{\mu_1}{\mu_2}\left[\sum_{j\in S^R}(w_j - c_j)\theta_j q_j^R(m^R(\boldsymbol{p}^M, \boldsymbol{w})) - (w_k - c_k - \mu_2)\theta_k\right] \end{array} \right] \tag{C.15}$$

It can be shown that (C.14) is negative (positive) when p_k^M is very large (very small), i.e.,

$$\lim_{p_k^M \to \infty} \frac{\partial \Pi^M(\boldsymbol{p}^M, \boldsymbol{w}, m^R(\boldsymbol{p}^M, \boldsymbol{w}))}{\partial p_k^M} < 0 \text{ and } \lim_{p_k^M \to -\infty} \frac{\partial \Pi^M(\boldsymbol{p}^M, \boldsymbol{w}, m^R(\boldsymbol{p}^M, \boldsymbol{w}))}{\partial p_k^M} > 0.$$

Hence, it is never optimal to set one of the wholesale prices at a boundary, and the optimal $\boldsymbol{p}^M$ must satisfy the FOC. Similarly, it can be shown that (C.15) is negative (positive) when w_k is very large (very small), i.e.,

$$\lim_{w_k \to \infty} \frac{\partial \Pi^M(\boldsymbol{p}^{\scriptscriptstyle M}, \boldsymbol{w}, m^{\scriptscriptstyle R}(\boldsymbol{p}^{\scriptscriptstyle M}, \boldsymbol{w}))}{\partial w_k} < 0 \text{ and } \lim_{w_k \to -\infty} \frac{\partial \Pi^M(\boldsymbol{p}^{\scriptscriptstyle M}, \boldsymbol{w}, m^{\scriptscriptstyle R}(\boldsymbol{p}^{\scriptscriptstyle M}, \boldsymbol{w}))}{\partial w_k} > 0.$$

Hence, it is never optimal to set one of the prices for the indirect channel at a boundary, and the optimal $\boldsymbol{w}$ must satisfy the FOC.

To conclude the proof of Proposition 18 (a), we next prove that any $\boldsymbol{p}^{\scriptscriptstyle M}$ that satisfy FOC of $\Pi^M(\boldsymbol{p}^{\scriptscriptstyle M}, \boldsymbol{w}, m^{\scriptscriptstyle R}(\boldsymbol{p}^{\scriptscriptstyle M}, \boldsymbol{w}))$ with respect to $\boldsymbol{p}^{\scriptscriptstyle M}$ must also satisfy the property that $p_k^{\scriptscriptstyle M} - c_k$ is the same for all $k \in S^M$. Setting (C.14) equal to zero and rearranging terms yields the following:

$$\begin{aligned} p_k^{\scriptscriptstyle M} - c_k = & \sum_{j \in S^M} (p_j^{\scriptscriptstyle M} - c_j) q_j^{\scriptscriptstyle M}(\boldsymbol{p}^{\scriptscriptstyle M}) + \mu_2 \\ & - \frac{\mu_1}{\mu_2} \sum_{j \in S^M} (p_j^{\scriptscriptstyle M} - c_j) q_j^{\scriptscriptstyle M}(\boldsymbol{p}^{\scriptscriptstyle M}) \begin{bmatrix} 1 - \tau^{\scriptscriptstyle M}(\boldsymbol{p}^{\scriptscriptstyle M}, m^{\scriptscriptstyle R}(\boldsymbol{p}^{\scriptscriptstyle M}, \boldsymbol{w}))[1 + W(\Omega(\boldsymbol{p}^{\scriptscriptstyle M}, \boldsymbol{w}))] \\ - \tau^{\scriptscriptstyle R}(\boldsymbol{p}^{\scriptscriptstyle M}, m^{\scriptscriptstyle R}(\boldsymbol{p}^{\scriptscriptstyle M}, \boldsymbol{w}))\tau^{\scriptscriptstyle R}(\boldsymbol{p}^{\scriptscriptstyle M}, m^{\scriptscriptstyle R}(\boldsymbol{p}^{\scriptscriptstyle M}, \boldsymbol{w})) \end{bmatrix} \\ & + \frac{\mu_1}{\mu_2} \sum_{j \in S^R} (w_j - c_j) \theta_j q_j^{\scriptscriptstyle R}(m^{\scriptscriptstyle R}(\boldsymbol{p}^{\scriptscriptstyle M}, \boldsymbol{w})) \tau^{\scriptscriptstyle R}(\boldsymbol{p}^{\scriptscriptstyle M}, m^{\scriptscriptstyle R}(\boldsymbol{p}^{\scriptscriptstyle M}, \boldsymbol{w}))[1 - \tau^{\scriptscriptstyle R}(\boldsymbol{p}^{\scriptscriptstyle M}, m^{\scriptscriptstyle R}(\boldsymbol{p}^{\scriptscriptstyle M}, \boldsymbol{w}))]. \end{aligned}$$

Observe that the right-hand side (RHS) of the above equation is the same for all $k \in S^M$. Therefore, $p_k^{\scriptscriptstyle M} - c_k$ is the same for all $k \in S^M$. This concludes the proof of Proposition 18 (a). Likewise, setting (C.15) equal to zero and rearranging terms yields the following:

$$\begin{aligned} (w_k - c_k - \mu_2)\theta_k = & \frac{\mu_2}{\mu_1} \sum_{j \in S^M} (p_j^{\scriptscriptstyle M} - c_j) q_j^{\scriptscriptstyle M}(\boldsymbol{p}^{\scriptscriptstyle M}) \tau^{\scriptscriptstyle M}(\boldsymbol{p}^{\scriptscriptstyle M}, m^{\scriptscriptstyle R}(\boldsymbol{p}^{\scriptscriptstyle M}, \boldsymbol{w}))[1 - \tau^{\scriptscriptstyle R}(\boldsymbol{p}^{\scriptscriptstyle M}, m^{\scriptscriptstyle R}(\boldsymbol{p}^{\scriptscriptstyle M}, \boldsymbol{w}))] \\ & - \frac{\mu_2}{\mu_1} \sum_{j \in S^R} (w_j - c_j) \theta_j q_j^{\scriptscriptstyle R}(m^{\scriptscriptstyle R}(\boldsymbol{p}^{\scriptscriptstyle M}, \boldsymbol{w}))[1 - \tau^{\scriptscriptstyle R}(\boldsymbol{p}^{\scriptscriptstyle M}, m^{\scriptscriptstyle R}(\boldsymbol{p}^{\scriptscriptstyle M}, \boldsymbol{w}))]^2 \\ & - \sum_{j \in S^R} (w_j - c_j) \theta_j q_j^{\scriptscriptstyle R}(m^{\scriptscriptstyle R}(\boldsymbol{p}^{\scriptscriptstyle M}, \boldsymbol{w})). \end{aligned}$$

Observe that the RHS of the above equation is the same for all $k \in S^R$. This implies that $(w_k - c_k - \mu_2)\theta_k$ is the same for all $k \in S^R$. Therefore, the optimization over prices reduces to optimizing over two margins, $m^{\scriptscriptstyle M}$ and $\overline{w}$, where $m^{\scriptscriptstyle M} := p_k^{\scriptscriptstyle M} - c_k$ for all $k \in S^M$, and $\overline{w} := (w_k - c_k - \mu_2)\theta_k$ for all $k \in S^R$.

Proof of Proposition 19: Recall the definition of θ_k provided in (4.11), i.e. $\theta_k := \sigma_k\left(z_k + I_N(z_k)\right) + 1$. We now replace $I_N(z_k)$ in θ_k by its definition, $I_N(z_k) := \phi_N(z_k) - z_k(1 - \Phi_N(z_k))$, where $\phi_N(\cdot)$ and $\Phi_N(\cdot)$ are the standard normal density and distribution functions, respectively. After replacing $I_N(z_k)$ and with some algebraic manipulation we obtain

$$\theta_k := \sigma_k\left(\phi_N(z_k) + z_k\Phi_N(z_k)\right) + 1.$$

We note that $\phi_N(z_k) + z_k\Phi_N(z_k) > 0$, because $\lim_{z_k \to -\infty}\left[\phi_N(z_k) + z_k\Phi_N(z_k)\right] = 0$ and $\phi_N(z_k) + z_k\Phi_N(z_k)$ increases in z_k. Therefore, observe from the above equality that θ_k is increasing in σ_k. This fact will be useful in the rest of the proof.

Now recall from Proposition 18 that $\overline{w} = (w_k - c_k - \mu_2)\theta_k$ for all $k \in S^R$. Given that the proposition assumes that all variants are the same in all respects but σ_k, notice from the expression that variants with higher θ_k will have lower w_k. Recall that θ_k is increasing in σ_k. Hence, higher σ_k implies lower w_k.

Proofs for Section 4.5

For the purposes of this section, the retailer's assortment, S^R, is fixed and hence we drop S^R from the argument list of the functions. We first provide and prove three lemmas that will be used in the proofs of the propositions in Section 4.5.

Lemma 1. *The manufacturer's profit,* $\Pi^M(\boldsymbol{p}^{\scriptscriptstyle M}, \boldsymbol{w}, \boldsymbol{p}^{\scriptscriptstyle R})$ *in* (4.10)*, can be written as:*

$$\Pi^M(m^{\scriptscriptstyle M}, \overline{w}, m^{\scriptscriptstyle R}(m^{\scriptscriptstyle M}, \overline{w})) =$$

$$\tau^{\scriptscriptstyle M}(m^{\scriptscriptstyle M}, m^{\scriptscriptstyle R}(m^{\scriptscriptstyle M}, \overline{w}))m^{\scriptscriptstyle M} + \tau^{\scriptscriptstyle R}(m^{\scriptscriptstyle M}, m^{\scriptscriptstyle R}(m^{\scriptscriptstyle M}, \overline{w})) \sum_{k \in S^R} (\overline{w} + \theta_k \mu_2) q_k^{\scriptscriptstyle R}(m^{\scriptscriptstyle R}(m^{\scriptscriptstyle M}, \overline{w})),$$

where

$$v_k^{\scriptscriptstyle M}(m^{\scriptscriptstyle M}) = \exp([\alpha_k - c_k - m^{\scriptscriptstyle M}]/\mu_2) \text{ for } k \in S^M, \tag{C.16}$$

$$v_k^{\scriptscriptstyle R}(m^{\scriptscriptstyle R}(m^{\scriptscriptstyle M}, \overline{w})) = \exp([\alpha_k - \overline{w}/\theta_k - c_k - \mu_2 - \gamma_k - m^{\scriptscriptstyle R}(m^{\scriptscriptstyle M}, \overline{w})]/\mu_2) \text{ for } k \in S^R, \tag{C.17}$$

$$\tau^{\scriptscriptstyle R}(m^{\scriptscriptstyle M}, m^{\scriptscriptstyle R}(m^{\scriptscriptstyle M}, \overline{w})) = \frac{\left[\sum_{k \in S^R} v_k^{\scriptscriptstyle R}(m^{\scriptscriptstyle R}(m^{\scriptscriptstyle M}, \overline{w}))\right]^{\mu_2/\mu_1}}{\exp(\alpha_0) + \left[\sum_{k \in S^M} v_k^{\scriptscriptstyle M}(m^{\scriptscriptstyle M})\right]^{\mu_2/\mu_1} + \left[\sum_{k \in S^R} v_k^{\scriptscriptstyle R}(m^{\scriptscriptstyle R}(m^{\scriptscriptstyle M}, \overline{w}))\right]^{\mu_2/\mu_1}}, \tag{C.18}$$

$$\tau^{\scriptscriptstyle M}(m^{\scriptscriptstyle M}, m^{\scriptscriptstyle R}(m^{\scriptscriptstyle M}, \overline{w})) = \frac{\left[\sum_{k \in S^M} v_k^{\scriptscriptstyle M}(m^{\scriptscriptstyle M})\right]^{\mu_2/\mu_1}}{\exp(\alpha_0) + \left[\sum_{k \in S^M} v_k^{\scriptscriptstyle M}(m^{\scriptscriptstyle M})\right]^{\mu_2/\mu_1} + \left[\sum_{k \in S^R} v_k^{\scriptscriptstyle R}(m^{\scriptscriptstyle R}(m^{\scriptscriptstyle M}, \overline{w}))\right]^{\mu_2/\mu_1}} \tag{C.19}$$

$$\text{and } q_k^{\scriptscriptstyle R}(m^{\scriptscriptstyle R}(m^{\scriptscriptstyle M}, \overline{w})) = \frac{v_k^{\scriptscriptstyle R}(m^{\scriptscriptstyle R}(m^{\scriptscriptstyle M}, \overline{w}))}{\sum_{j \in S^R} v_j^{\scriptscriptstyle R}(m^{\scriptscriptstyle R}(m^{\scriptscriptstyle M}, \overline{w}))}. \tag{C.20}$$

Proof of Lemma 1: Let $v_k^{\scriptscriptstyle M}(m^{\scriptscriptstyle M})$ and $v_k^{\scriptscriptstyle R}(m^{\scriptscriptstyle R}(m^{\scriptscriptstyle M}, \overline{w}))$ be as defined in the statement of the lemma. Using Proposition 18 (a) it can be verified that

$$v_k^{\scriptscriptstyle M}(m^{\scriptscriptstyle M}) = v_k(p_k^{\scriptscriptstyle M}), \tag{C.21}$$

where $v_k(p_k^n)$ is as defined in (4.2). Similarly, using Proposition 18 (b) it can be verified that

$$v_k^{\scriptscriptstyle R}(m^{\scriptscriptstyle R}(m^{\scriptscriptstyle M}, \overline{w})) = v_k(p_k^{\scriptscriptstyle R}). \tag{C.22}$$

Using the equalities in (C.21) and (C.22) we will make the following three observations:

Observation 1: Recall the expression for $\tau^{R}(\boldsymbol{p}^{M}, \boldsymbol{p}^{R})$ given in (4.4), i.e.

$$\tau^{R}(\boldsymbol{p}^{M}, \boldsymbol{p}^{R}) = \frac{\left[\sum_{k \in S^R} v_k(p_k^{R})\right]^{\mu_2/\mu_1}}{\exp(\alpha_0) + \left[\sum_{k \in S^R} v_k(p_k^{R})\right]^{\mu_2/\mu_1} + \left[\sum_{k \in S^M} v_k(p_k^{M})\right]^{\mu_2/\mu_1}}.$$

Using the equalities in (C.21) and (C.22), observe that

$$\tau^{R}(\boldsymbol{p}^{M}, \boldsymbol{p}^{R}) = \tau^{R}(m^{M}, m^{R}(m^{M}, \overline{w})),$$

where $\tau^{R}(m^{M}, m^{R}(m^{M}, \overline{w}))$ is as defined in the statement of the lemma.

Observation 2: Recall the expression for $\tau^{M}(\boldsymbol{p}^{M}, \boldsymbol{p}^{R})$ given in (4.4), i.e.

$$\tau^{M}(\boldsymbol{p}^{M}, \boldsymbol{p}^{R}) = \frac{\left[\sum_{k \in S^M} v_k(p_k^{M})\right]^{\mu_2/\mu_1}}{\exp(\alpha_0) + \left[\sum_{k \in S^R} v_k(p_k^{R})\right]^{\mu_2/\mu_1} + \left[\sum_{k \in S^M} v_k(p_k^{M})\right]^{\mu_2/\mu_1}}.$$

Observe that $\tau^{M}(\boldsymbol{p}^{M}, \boldsymbol{p}^{R}) = \tau^{M}(m^{M}, m^{R}(m^{M}, \overline{w}))$, where $\tau^{M}(m^{M}, m^{R}(m^{M}, \overline{w}))$ is as defined in the statement of the lemma.

Observation 3: Recall the expression for $q_k^{R}(\boldsymbol{p}^{R})$ in (4.1), i.e.

$$q_k^{R}(\boldsymbol{p}^{R}) = \frac{v_k(p_k^{R})}{\sum_{k \in S^R} v_k(p_k^{R})} \text{ for all } k \in S^R.$$

Observe that $q_k^{R}(\boldsymbol{p}^{R}) = q_k^{R}(m^{R}(m^{M}, \overline{w}))$, where $q_k^{R}(m^{R}(m^{M}, \overline{w}))$ is as defined in the statement of the lemma.

Using Proposition 18, we make the following two observations.

Observation 4: From Proposition 18 (a), we have that $p_k^{M} = m^{M} + c_k$. Hence, observe that we can rewrite the term $\sum_{k \in S^M}(p_k^{M} - c_k) q_k^{M}(\boldsymbol{p}^{M})$ in (4.10) as just m^{M} (this is because $\sum_{k \in S^M} q_k^{M}(\boldsymbol{p}^{M}) = 1$).

Observation 5: From Proposition 18 (b), we have that $w_k = \overline{w}/\theta_k + c_k + \mu_2$.

Hence, observe that we can write the term $\sum_{k\in S^R}(w_k - c_k)\theta_k q_k^{R}(\boldsymbol{p}^{R})$ in (4.10) as $\sum_{k\in S^R}(\overline{w} + \theta_k\mu_2)q_k^{R}(\boldsymbol{p}^{R})$.

Recall the definition of $\Pi^M(\boldsymbol{p}^{M}, \boldsymbol{w}, \boldsymbol{p}^{R})$ in (4.10), i.e.

$$\Pi^M(\boldsymbol{p}^{M}, \boldsymbol{w}, \boldsymbol{p}^{R}) =$$
$$\tau^{M}(\boldsymbol{p}^{M}, \boldsymbol{p}^{R}) \sum_{k\in S^M}(p_k^{M} - c_k)q_k^{M}(\boldsymbol{p}^{M}) + \tau^{R}(\boldsymbol{p}^{M}, \boldsymbol{p}^{R}) \sum_{k\in S^R}(w_k - c_k)\theta_k q_k^{R}(\boldsymbol{p}^{R}).$$

We now use observations 1 through 5 and substitute in the above equation: $\tau^{M}(m^{M}, m^{R}(m^{M}, \overline{w}))$ for $\tau^{M}(\boldsymbol{p}^{M}, \boldsymbol{p}^{R})$, $\tau^{R}(m^{M}, m^{R}(m^{M}, \overline{w}))$ for $\tau^{R}(\boldsymbol{p}^{M}, \boldsymbol{p}^{R})$, $q_k^{R}(m^{R}(m^{M}, \overline{w}))$ for $q_k^{R}(\boldsymbol{p}^{R})$, $\sum_{k\in S^R}(\overline{w} + \theta_k\mu_2)q_k^{R}(m^{R}(m^{M}, \overline{w}))$ for $\sum_{k\in S^R}(w_k - c_k)\theta_k q_k^{R}(\boldsymbol{p}^{R})$ and m^{M} for $\sum_{k\in S^M}(p_k^{M} - c_k)q_k^{M}(\boldsymbol{p}^{M})$. This yields

$$\Pi^M(m^{M}, \overline{w}, m^{R}(m^{M}, \overline{w})) = \tau^{M}(m^{M}, m^{R}(m^{M}, \overline{w}))m^{M} + \tau^{R}(m^{M}, m^{R}(m^{M}, \overline{w})) \sum_{k\in S^R}(\overline{w} + \theta_k\mu_2)q_k^{R}(m^{R}(m^{M}, \overline{w})). \tag{C.23}$$

Lemma 2. *The manufacturer's profit function,* $\Pi^M(m^{M}, \overline{w}, m^{R}(m^{M}, \overline{w}))$*, given in Lemma 1, can be written as:*

$$\Pi^M(m^{M}, \overline{w}) = \widehat{\tau}^{M}(m^{M}, \overline{w})m^{M} + \widehat{\tau}^{R}(m^{M}, \overline{w}) \sum_{k\in S^R}(\overline{w} + \theta_k\mu_2)\widehat{q}_k^{R}(m^{M}, \overline{w}),$$

where

$$\Omega(m^{M},\overline{w}) = \frac{\exp(-1)\left[\sum_{j\in S^R}\exp([\alpha_k - c_k - \gamma_k - \mu_2 - \overline{w}/\theta_j]/\mu_2)\right]^{\mu_2/\mu_1}}{\exp\left(\alpha_0\right) + \left[\sum_{k\in S^M} v_k^{M}(m^{M})\right]^{\mu_2/\mu_1}},$$

$$v_k^{M}(m^{M}) = \exp([\alpha_k - c_k - m^{M}]/\mu_2) \text{ for } k \in S^M,$$

$$\widehat{v}_k^{R}(m^{M},\overline{w}) = \exp([\alpha_k - \overline{w}/\theta_k - c_k - \mu_2 - \gamma_k - \mu_1[1 + W(\Omega(m^{M},\overline{w}))]]/\mu_2),$$

$$\widehat{\tau}^{R}(m^{M},\overline{w}) = \frac{W(\Omega(m^{M},\overline{w}))}{1 + W(\Omega(m^{M},\overline{w}))},$$

$$\widehat{\tau}^{M}(m^{M},\overline{w}) = \frac{\left[\sum_{k\in S^M} v_k^{M}(m^{M})\right]^{\mu_2/\mu_1}}{\left[\exp(\alpha_0) + \left[\sum_{k\in S^M} v_k^{M}(m^{M})\right]^{\mu_2/\mu_1}\right]\left[1 + W(\Omega(m^{M},\overline{w}))\right]},$$

$$\text{and } \widehat{q}_k^{R}(m^{M},\overline{w}) = \frac{\widehat{v}_k^{R}(m^{M},\overline{w})}{\sum_{j\in S^R}\widehat{v}_j^{R}(m^{M},\overline{w})}.$$

Proof of Lemma 2: Let us first define

$$\Omega(m^{M},\overline{w}) := \frac{\exp(-1)\left[\sum_{j\in S^R}\exp([\alpha_k - c_k - \gamma_k - \mu_2 - \overline{w}/\theta_j]/\mu_2)\right]^{\mu_2/\mu_1}}{\exp\left(\alpha_0\right) + \left[\sum_{k\in S^M} v_k^{M}(m^{M})\right]^{\mu_2/\mu_1}}. \tag{C.24}$$

Recall from Proposition 17 that

$$m^{R}(m^{M},\overline{w}) = \frac{\mu_1}{1 - \tau^{R}(m^{M}, m^{R}(m^{M},\overline{w}))}.$$

Replacing $\tau^{R}(m^{M}, m^{R}(m^{M},\overline{w}))$ in this expression with its definition in Lemma 1 we obtain:

$$\begin{aligned}\frac{m^{R}(m^{M},\overline{w})}{\mu_1} &= \frac{\exp\left(\alpha_0\right) + \left[\sum_{k\in S^M} v_k^{M}(m^{M})\right]^{\mu_2/\mu_1} + \left[\sum_{k\in S^R} v_k^{R}(m^{R}(m^{M},\overline{w}))\right]^{\mu_2/\mu_1}}{\exp\left(\alpha_0\right) + \left[\sum_{k\in S^M} v_k^{M}(m^{M})\right]^{\mu_2/\mu_1}} \\ &= 1 + \frac{\left[\sum_{j\in S^R}\exp([\alpha_k - c_k - \gamma_k - \mu_2 - \overline{w}/\theta_j]/\mu_2)\right]^{\mu_2/\mu_1}\exp(-m^{R}/\mu_1)}{\exp\left(\alpha_0\right) + \left[\sum_{k\in S^M} v_k^{M}(m^{M})\right]^{\mu_2/\mu_1}}.\end{aligned}$$

With some algebraic manipulation we now obtain:

$$\left[\frac{m^{R}(m^{M},\overline{w})}{\mu_1}-1\right]\exp\left(\frac{m^{R}(m^{M},\overline{w})}{\mu_1}-1\right)=$$
$$\frac{\exp(-1)\left[\sum_{j\in S^R}\exp([\alpha_k-c_k-\gamma_k-\mu_2-\overline{w}/\theta_j]/\mu_2)\right]^{\mu_2/\mu_1}}{\exp(\alpha_0)+\left[\sum_{k\in S^M}v_k^{M}(m^{M})\right]^{\mu_2/\mu_1}}.$$

Observe that the right-hand side of the expression above is the same as $\Omega(m^{M},\overline{w})$. Using the Lambert-W function provided in the preamble of Appendix A, it could be shown that the expression above can be written as:

$$\frac{m^{R}(m^{M},\overline{w})}{\mu_1}-1=W(\Omega(m^{M},\overline{w})). \tag{C.25}$$

We now use the expression above written in the form of

$$m^{R}(m^{M},\overline{w})=\mu_1[1+W(\Omega(m^{M},\overline{w}))] \tag{C.26}$$

to make some observations.

Observation 1: Recall from Proposition 17 that $\tau^{R}(m^{M},m^{R}(m^{M},\overline{w}))=1-\frac{\mu_1}{m^R(m^M,\overline{w})}$. Replacing $m^{R}(m^{M},\overline{w})$ in this expression by the definition of $m^{R}(m^{M},\overline{w})$ in (C.26), we can observe that:

$$\widehat{\tau}^{R}(m^{M},\overline{w})=\tau^{R}(m^{M},m^{R}(m^{M},\overline{w})),$$

where $\widehat{\tau}^{R}(m^{M},\overline{w})$ is defined in the statement of the lemma.

Observation 2: Recall the expression for $v_k^{R}(m^{R}(m^{M},\overline{w}))$ defined in Lemma 1, i.e.

$$v_k^{R}(m^{R}(m^{M},\overline{w}))=\exp([\alpha_k-\overline{w}/\theta_k-c_k-\mu_2-\gamma_k-m^{R}(m^{M},\overline{w})]/\mu_2).$$

Substituting above for $m^R(m^M, \overline{w})$ with $\mu_1[1 + W(\Omega(m^M, \overline{w}))]$ we obtain:

$$v_k^R(m^R(m^M, \overline{w})) = \exp([\alpha_k - \overline{w}/\theta_k - c_k - \mu_2 - \gamma_k - \mu_1[1 + W(\Omega(m^M, \overline{w}))]]/\mu_2).$$

Observe from above that $v_k^R(m^R(m^M, \overline{w})) = \widehat{v}_k^R(m^M, \overline{w})$ where $\widehat{v}_k^R(m^M, \overline{w})$ is as defined in the lemma.

Observation 3: Recall from Proposition 18 that $p_k^M = m^M + c_k$. Replacing p_k^M with $m^M + c_k$ in the expression for $v_k(p_k^M)$ provided in (4.2), we conclude that $v_k(p_k^M) = v_k^M(m^M)$. Using Proposition 18 we recall also that $w_k = \overline{w}_k/\theta_k + c_k + \mu_2$. Replacing w_k with $\overline{w}_k/\theta_k + c_k + \mu_2$ and $v_k(p_k^M)$ with $v_k^M(m^M)$ in the expression for $\Omega(\boldsymbol{p}^M, \boldsymbol{w})$ provided in (C.8) we conclude that $\Omega(\boldsymbol{p}^M, \boldsymbol{w}) = \Omega(m^M, \overline{w})$, where $\Omega(m^M, \overline{w})$ is provided in (C.24).

Recall the expression for $\tau^M(\boldsymbol{p}^M, m^R(\boldsymbol{p}^M, \boldsymbol{w}))$ provided in (C.11), i.e.

$$\tau^M(\boldsymbol{p}^M, m^R(\boldsymbol{p}^M, \boldsymbol{w})) = \frac{\left[\sum_{k \in S^M} v_k(p_k^M)\right]^{\mu_2/\mu_1}}{\left[\exp(\alpha_0) + \left[\sum_{k \in S^M} v_k(p_k^M)\right]^{\mu_2/\mu_1}\right][1 + W(\Omega(\boldsymbol{p}^M, \boldsymbol{w}))]}.$$

Notice that replacing $v_k(p_k^M)$ with $v_k^M(m^M)$ and $\Omega(\boldsymbol{p}^M, \boldsymbol{w})$ with $\Omega(m^M, \overline{w})$ in the expression above we can conclude that

$$\tau^M(m^M, m^R(m^M, \overline{w})) = \widehat{\tau}^M(m^M, \overline{w}).$$

Observation 4: Recall also the expression for $q_k^R(m^R)$ in Lemma 1, i.e.

$$q_k^R(m^R(m^M, \overline{w})) = \frac{v_k^R(m^R(m^M, \overline{w}))}{\sum_{j \in S^R} v_j^R(m^R(m^M, \overline{w}))}.$$

Using Observation 2, we can substitute $\widehat{v}_k^R(m^M, \overline{w})$ for $v_k^R(m^R(m^M, \overline{w}))$, which will allow us to observe that

$$q_k^R(m^R(m^M, \overline{w})) = \widehat{q}_k^R(m^M, \overline{w}).$$

Using observations 1 through 5, we now replace $\tau^{M}(m^{M}, m^{R})$ with $\widehat{\tau}^{M}(m^{M}, \overline{w})$, $\tau^{R}(m^{M}, m^{R})$ with $\widehat{\tau}^{R}(m^{M}, \overline{w})$ and $q_k^{R}(m^{R})$ with $\widehat{q}_k^{R}(\Omega(m^{M}, \overline{w}))$ in the manufacturer's profit function $\Pi^M(m^{M}, \overline{w}, m^{R})$ in (C.23). This step yields

$$\Pi^M(m^{M}, \overline{w}) = \widehat{\tau}^{M}(m^{M}, \overline{w})m^{M} + \widehat{\tau}^{R}(m^{M}, \overline{w}) \sum_{k \in S^R} (\overline{w} + \theta_k \mu_2)\widehat{q}_k^{R}(m^{M}, \overline{w}). \tag{C.27}$$

Lemma 3. *Consider a manufacturer selling through the direct channel only. Let $m_{k,direct}^{M} := p_k^{M} - c_k$ denote the gross margin for variant $k \in S^M$. Then, any price vector that is optimal for the manufacturer is such that all variants have the same gross margin, i.e. $m_{k,direct}^{M} = m_{direct}^{M}$ for all $k \in S^M$. Furthermore, the manufacturer's optimal direct channel margin is the unique value of m_{direct}^{M} that satisfies*

$$m_{direct}^{M} = \frac{\mu_1}{1 - \tau_{direct}^{M}(m_{direct}^{M})}, \tag{C.28}$$

where

$$\tau_{direct}^{M}(m_{direct}^{M}) = \frac{\left[\sum_{k \in S^M} v_k^{M}(m_{direct}^{M})\right]^{\mu_2/\mu_1}}{\exp{(\alpha_0)} + \left[\sum_{k \in S^M} v_k^{M}(m_{direct}^{M})\right]^{\mu_2/\mu_1}}, \tag{C.29}$$

and $v_k^{M}(m_{direct}^{M}) = \exp([\alpha_k - c_k - m_{direct}^{M}]/\mu_2)$.

Proof of Lemma 3: For a manufacturer selling only through a direct channel, the probability that a customer purchases from the direct channel, denoted with $\tau_{direct}^{M}(\boldsymbol{p}^{M})$, will be given by:

$$\tau_{direct}^{M}(\boldsymbol{p}^{M}) = \frac{\left[\sum_{k \in S^M} v_k(p_k^{M})\right]^{\mu_2/\mu_1}}{\exp{(\alpha_0)} + \left[\sum_{k \in S^M} v_k(p_k^{M})\right]^{\mu_2/\mu_1}}. \tag{C.30}$$

Using this probability, the manufacturer's expected profit will then be given by:

$$\Pi_{direct}^{M}(\boldsymbol{p}^{M}) = \tau_{direct}^{M}(\boldsymbol{p}^{M}) \sum_{k \in S^M} (p_k^{M} - c_k)q_k^{M}(\boldsymbol{p}^{M}). \tag{C.31}$$

Taking the derivative of (C.31) yields,

$$\frac{\partial \Pi^M_{direct}(\boldsymbol{p}^M)}{\partial p^M_k} = \frac{\tau^M_{direct}(\boldsymbol{p}^M) q^M_k(\boldsymbol{p}^M)}{\mu_2} \left[\begin{array}{l} -\sum_{j \in S^M} (p^M_j - c_j) q^M_j(\boldsymbol{p}^M) \left(\frac{\mu_2[1 - \tau^M_{direct}(\boldsymbol{p}^M)]}{\mu_1} - 1 \right) \\ - p^M_k + c_k - 1 \end{array} \right]. \tag{C.32}$$

It can be shown that:
$\lim_{p^M_k \to -\infty} \frac{\partial \Pi^M_{direct}(\boldsymbol{p}^M)}{\partial p^M_k} > 0$ and $\lim_{p^M_k \to \infty} \frac{\partial \Pi^M_{direct}(\boldsymbol{p}^M)}{\partial p^M_k} < 0$.
Hence, $\boldsymbol{p}^M$ must satisfy FOC, where the FOC are obtained by setting (C.32) equal to zero yielding:

$$p^M_k - c_k = \sum_{j \in S^M} (p^M_j - c_j) q^M_j(\boldsymbol{p}^M) \left(1 - \frac{\mu_2[1 - \tau^M_{direct}(\boldsymbol{p}^M)]}{\mu_1} \right) - 1. \tag{C.33}$$

Observe from above that the right-hand side is the same for all $k \in S^M$. Let m^M_{direct} be the profit margin for the manufacturer, i.e. $m^M_{direct} := p^M_k - c_k$. Using this margin definition we now write the manufacturer's purchasing probability and expected profit as,

$$\tau^M_{direct}(m^M_{direct}) = \frac{\left[\sum_{k \in S^M} v^M_k(m^M_{direct})\right]^{\mu_2/\mu_1}}{\exp(\alpha_0) + \left[\sum_{k \in S^M} v^M_k(m^M_{direct})\right]^{\mu_2/\mu_1}}, \tag{C.34}$$

where $v_k(m^M_{direct}) := \exp([\alpha_k - c_k - m^M_{direct}]/\mu_2)$, and

$$\Pi^M_{direct}(m^M_{direct}) = \tau^M_{direct}(m^M_{direct}) m^M_{direct}. \tag{C.35}$$

Recall that m^M_{direct} must satisfy FOC. Taking the derivative of (C.35) and setting it equal to zero yields the following FOC for the profit margin:

$$m^M_{direct} = \frac{\mu_1}{1 - \tau^M_{direct}(m^M_{direct})}. \tag{C.36}$$

Proof of Proposition 20: Let m^{M}_{direct} be the manufacturer's optimal direct channel margin when selling through the direct channel only, and let m^{M}_{dual} be the manufacturer's optimal direct channel margin when selling through dual channels, as required by the proposition. We first derive conditions that m^{M}_{dual} must satisfy for the dual-channel setting and provide an expression for the m^{M}_{dual} that satisfies those conditions.

In particular, we show that the optimal m^{M} for the dual-channel is such that:

$$
\begin{aligned}
m^{M}_{dual} =& \mu_1 \left[1 + \frac{\left[\sum_{k\in S^M} v^{M}_{k}(m^{M}_{dual})\right]^{\mu_2/\mu_1}}{\exp(\alpha_0)} \right] \\
& \times \left[1 + \sum_{j\in S^R} \theta_j \widehat{q}^{R}_{j}(m^{M}_{dual}, \overline{w}) W(\Omega(m^{M}_{dual}, \overline{w})) \right]. \qquad \text{(C.37)}
\end{aligned}
$$

Recall from the proof of Proposition 18 that any optimal price vector for the manufacturer must satisfy FOCs. Therefore, m^{M} and $\overline{w}$ will also satisfy FOCs. We next use the FOCs to derive a simpler condition that the optimal m^{M} must satisfy. This proof uses the functions defined in the statement of Lemma 2.

Provided next are the derivatives of the functions $\Omega(m^{M}, \overline{w})$, $\widehat{\tau}^{R}(m^{M}, \overline{w})$, $\widehat{\tau}^{M}(m^{M}, \overline{w})$ and $\widehat{q}^{R}_{k}(m^{M}, \overline{w})$, all defined in Lemma 2, with respect to m^{M} and $\overline{w}$:

$$
\begin{aligned}
\frac{\partial \Omega(m^{M}, \overline{w})}{\partial m^{M}} &= \frac{\Omega(m^{M}, \overline{w})}{\mu_1} \widehat{\tau}^{M}(m^{M}, \overline{w})[1 + W(\Omega(m^{M}, \overline{w}))], \\
\frac{\partial \widehat{\tau}^{R}(m^{M}, \overline{w})}{\partial m^{M}} &= \frac{\widehat{\tau}^{M}(m^{M}, \overline{w})\widehat{\tau}^{R}(m^{M}, \overline{w})[1 - \widehat{\tau}^{R}(m^{M}, \overline{w})]}{\mu_1}, \\
\frac{\partial \widehat{\tau}^{M}(m^{M}, \overline{w})}{\partial m^{M}} &= \frac{-\widehat{\tau}^{M}(m^{M}, \overline{w})}{\mu_1} \left[\begin{array}{l} 1 - \widehat{\tau}^{M}(m^{M}, \overline{w})[1 + W(\Omega(m^{M}, \overline{w}))] \\ + \widehat{\tau}^{M}(m^{M}, \overline{w})\widehat{\tau}^{R}(m^{M}, \overline{w}) \end{array} \right], \\
\frac{\partial \Omega(m^{M}, \overline{w})}{\partial \overline{w}} &= -\frac{\Omega(m^{M}, \overline{w})}{\mu_1} \sum_{j\in S^R} \frac{\widehat{q}^{R}_{j}(m^{M}, \overline{w})}{\theta_j}, \\
\frac{\partial \widehat{\tau}^{R}(m^{M}, \overline{w})}{\partial \overline{w}} &= \frac{-\widehat{\tau}^{R}(m^{M}, \overline{w})}{\mu_1}[1 - \widehat{\tau}^{R}(m^{M}, \overline{w})]^2 \sum_{j\in S^R} \frac{\widehat{q}^{R}_{j}(m^{M}, \overline{w})}{\theta_j}, \\
\frac{\partial \widehat{\tau}^{M}(m^{M}, \overline{w})}{\partial \overline{w}} &= \frac{\widehat{\tau}^{M}(m^{M}, \overline{w})\widehat{\tau}^{R}(m^{M}, \overline{w})[1 - \widehat{\tau}^{R}(m^{M}, \overline{w})]}{\mu_1} \sum_{j\in S^R} \frac{\widehat{q}^{R}_{j}(m^{M}, \overline{w})}{\theta_j},
\end{aligned}
$$

$$\frac{\partial \widehat{q}_k^{R}(m^{M},\overline{w})}{\partial \overline{w}} = -\frac{\widehat{q}_k^{R}(m^{M},\overline{w})}{\mu_2}\left[\frac{1}{\theta_k} - \sum_{j\in S^R}\frac{\widehat{q}_j^{R}(m^{M},\overline{w})}{\theta_j}\right].$$

Using the derivatives above, it can be shown that the derivatives of $\Pi^M(m^M,\overline{w})$, given in Lemma 2 are:

$$\begin{aligned}\frac{\partial \Pi^{M}(m^{M},\overline{w})}{\partial m^{M}} &= \widehat{\tau}^{M}(m^{M},\overline{w})\\ &\times\left[\begin{array}{l}1-\dfrac{m^{M}}{\mu_1}\left[1+\widehat{\tau}^{M}(m^{M},\overline{w})\left(-1-W(\Omega(m^{M},\overline{w}))+\widehat{\tau}^{R}(m^{M},\overline{w})\right)\right]\\ +\displaystyle\sum_{j\in S^R}(\overline{w}+\theta_j\mu_2)\widehat{q}_j^{R}(m^{M},\overline{w})\frac{W(\Omega(m^{M},\overline{w}))}{\mu_1[1+W(\Omega(m^{M},\overline{w}))]^2}\end{array}\right],\end{aligned} \tag{C.38}$$

and

$$\begin{aligned}\frac{\partial \Pi^{M}(m^{M},\overline{w})}{\partial \overline{w}} &= \sum_{j\in S^R}\frac{\widehat{q}_j^{R}(m^{M},\overline{w})}{\theta_j}\widehat{\tau}^{R}(m^{M},\overline{w})\\ &\times\left[\begin{array}{l}\left[m^{M}\widehat{\tau}^{M}(m^{M},\overline{w})-\dfrac{\overline{w}}{1+W(\Omega(m^{M},\overline{w}))}\right]\dfrac{1}{\mu_1\left[1+W(\Omega(m^{M},\overline{w}))\right]}\\ +\displaystyle\sum_{j\in S^R}\theta_j\widehat{q}_j^{R}(m^{M},\overline{w})\left[1-\frac{\mu_2}{\mu_1[1+W(\Omega(m^{M},\overline{w}))]^2}\right]\end{array}\right].\end{aligned} \tag{C.39}$$

To obtain the FOC for the manufacturer, we set (C.38) and (C.39) equal to zero and rearrange terms which yields,

$$\begin{aligned}&\frac{1}{W(\Omega(m^{M},\overline{w}))}\left[1-\frac{m^{M}}{\mu_1}\left[1-\widehat{\tau}^{M}(m^{M},\overline{w})\left(1+W(\Omega(m^{M},\overline{w}))\right)\right]\right]\\ &=\frac{1}{\mu_1[1+W(\Omega(m^{M},\overline{w}))]}\left[m^{M}\widehat{\tau}^{M}(m^{M},\overline{w})-\frac{\sum_{j\in S^R}(\overline{w}+\theta_j\mu_2)\widehat{q}_j^{R}(m^{M},\overline{w})}{[1+W(\Omega(m^{M},\overline{w}))]}\right],\end{aligned} \tag{C.40}$$

and

$$\frac{1}{\mu_1[1+W(\Omega(m^{M},\overline{w}))]}\left[m^{M}\widehat{\tau}^{M}(m^{M},\overline{w})-\frac{\sum_{j\in S^R}(\overline{w}+\theta_j\mu_2)\widehat{q}^{R}_j(m^{M},\overline{w})}{[1+W(\Omega(m^{M},\overline{w}))]}\right]$$
$$=-\sum_{j\in S^R}\theta_j\widehat{q}^{R}_j(m^{M},\overline{w}). \tag{C.41}$$

Notice from the above two expressions that the RHS of (C.40) and the LHS of (C.41) are the same. Hence, to simplify the comparison between models, we replace the LHS of (C.40) with the RHS of (C.41) to obtain the following condition for m^{M}:

$$m^{M}=\frac{\mu_1\left[1+\sum_{j\in S^R}\theta_j\widehat{q}^{R}_j(m^{M},\overline{w})W(\Omega(m^{M},\overline{w}))\right]}{[1-\widehat{\tau}^{M}(m^{M},\overline{w})\,[1+W(\Omega(m^{M},\overline{w}))]]}. \tag{C.42}$$

For exposition purposes, we introduce the subscript *dual* to the functions and variables for the problem where the manufacturer sells through two channels. Let m^{M}_{dual} be the direct channel margin that satisfies the FOCs for the dual channel problem. Replacing $\widehat{\tau}^{M}(m^{M},\overline{w})$ in (C.42) with its definition in Lemma 2, we can write

$$m^{M}_{dual}=\mu_1\left[1+\frac{\left[\sum_{k\in S^M}v^{M}_k(m^{M}_{dual})\right]^{\mu_2/\mu_1}}{\exp(\alpha_0)}\right]$$
$$\times\left[1+\sum_{j\in S^R}\theta_j\widehat{q}^{R}_j(m^{M}_{dual},\overline{w})W(\Omega(m^{M}_{dual},\overline{w}))\right].$$

Through algebra, we can rewrite the above condition as follows:

$$\frac{m^{M}_{dual}}{\mu_1\left[1+\frac{\left[\sum_{k\in S^M}v^{M}_k(m^{M}_{dual})\right]^{\mu_2/\mu_1}}{\exp(\alpha_0)}\right]}-1=\sum_{j\in S^R}\theta_j\widehat{q}^{R}_j(m^{M}_{dual},\overline{w})W(\Omega(m^{M}_{dual},\overline{w})). \tag{C.43}$$

Next, using the condition above we compare m^{M}_{dual} with m^{M}_{direct}.

Recall from the proof of Lemma 3 that m^{M}_{direct} must satisfy the expression in (C.36). Substituting in (C.36) for $\tau^{M}_{direct}(m^{M}_{direct})$ with its definition given in (C.34) and with some algebraic manipulation, yields the following condition:

$$\frac{m^M_{direct}}{\mu_1\left[1+\frac{\left[\sum_{k\in S^M} v^M_k(m^M_{direct})\right]^{\mu_2/\mu_1}}{\exp(\alpha_0)}\right]}-1=0. \tag{C.44}$$

Recall that $v^M_k(m^M)$ is decreasing in m^M. Given that (i) the left-hand side of the expressions in (C.43) and (C.44) are increasing in m^M, and (ii) the expression $\sum_{j\in S^R}\theta_j\widehat{q}^R_j(m^M_{dual},\overline{w})W(\Omega(m^M_{dual},\overline{w}))$ is always positive, we can now conclude that the m^M_{direct} that satisfies (C.44) is smaller than the m^M_{dual} that satisfies (C.43).

Proof of Proposition 21: Let $\Pi^M_{direct}(m^M_{direct})$ be the manufacturer's optimal expected profit when selling through the direct channel only and $\Pi^M_{dual}(m^M_{dual},\overline{w})$ be the manufacturer's optimal expected profit when selling through dual channels.

Recall the expression for m^M_{direct} provided in Lemma 3. Rearranging its terms, we obtain that $\tau^M_{direct}(m^M_{direct}) = 1-\frac{\mu_1}{m^M_{direct}}$. We now replace $\tau^M_{direct}(m^M_{direct})$ with $1-\frac{\mu_1}{m^M_{direct}}$ in $\Pi^M_{direct}(m^M_{direct})$, (C.35), to obtain the manufacturer's optimal expected profit, i.e.

$$\Pi^M_{direct}(m^M_{direct}) = m^M_{direct}-\mu_1. \tag{C.45}$$

This last expression for $\Pi^M_{direct}(m^M_{direct})$ is going to be used later in this proof to compare it with the manufacturer's optimal profit for the dual channel setting.

Recall the expression for $\Pi^M_{dual}(m^M_{dual},\overline{w})$ given in Lemma 2. To obtain an expression for the optimal expected profit for the manufacturer when selling through dual channels, we first rewrite the FOC of the manufacturer with respect to m^M_{dual}, given in (C.40). Observe we can rewrite (C.40) with some algebra to obtain:

$$m^M_{dual}-\mu_1+W(\Omega(m^M_{dual},\overline{w}))\left[m^M_{dual}[1-\widehat{\tau}^M(m^M_{dual},\overline{w})(1+W(\Omega(m^M_{dual},\overline{w})))]-\mu_1\right]=$$
$$\widehat{\tau}^M(m^M_{dual},\overline{w})m^M_{dual}+\widehat{\tau}^R(m^M_{dual},\overline{w})\sum_{k\in S^R}(\overline{w}+\theta_k\mu_2)\widehat{q}^R_k(m^M_{dual},\overline{w}).$$

Notice that the right-hand side (RHS) of the equation above corresponds to the definition of $\Pi^M_{dual}(m^M_{dual}, \overline{w})$ in (C.27). We can now replace the terms that correspond to the RHS of the equation above with the function $\Pi^M_{dual}(m^M_{dual}, \overline{w})$, to obtain,

$$\begin{aligned}\Pi^M_{dual}(m^M_{dual}, \overline{w}) &= m^M_{dual} - \mu_1 + W(\Omega(m^M_{dual}, \overline{w})) \\ &\times \left[m^M_{dual}[1 - \widehat{\tau}^M(m^M_{dual}, \overline{w})(1 + W(\Omega(m^M_{dual}, \overline{w})))] - \mu_1\right]. \end{aligned} \tag{C.46}$$

Given the fact that $m^M_{direct} \leq m^M_{dual}$, as stated on Proposition 20, what is left to show in order to prove that the expression in (C.46) is greater than or equal to the expression in (C.45), is that

$$m^M_{dual}(1 - \widehat{\tau}^M(m^M_{dual}, \overline{w})[1 + W(\Omega(m^M_{dual}, \overline{w}))]) - \mu_1 > 0.$$

Observe from (C.42) that m^M_{dual} satisfies $m^M_{dual}(1 - \widehat{\tau}^M(m^M_{dual}, \overline{w})(1 + W(\Omega(m^M_{dual}, \overline{w})))) > \mu_1$. This concludes the proof of the profit comparisons between the dual channel and direct channel strategies.

Proofs of Section 4.6

We first provide a lemma that will be used in the proofs of the propositions.

Lemma 4. *The optimal margins, m^M and $\overline{w}$, for $\Pi^M(S^R, m^M, \overline{w})$ in* (C.27) *must satisfy $F_1 = 0$ and $F_2 = 0$ where*

$$\begin{aligned}F_1 &= m^M\left[1 - \widehat{\tau}^M(S^R, m^M, \overline{w})\left[1 + W(\Omega(S^R, m^M, \overline{w}))\right]\right] \\ &- \mu_1\left[1 + \sum_{j\in S^R} \theta_j \widehat{q}^R_j(S^R, m^M, \overline{w}) W(\Omega(S^R, m^M, \overline{w}))\right],\end{aligned}$$

and

$$F_2 = \sum_{j \in S^R} \left[\overline{w} - \mu_2 \theta_j \widehat{q}_j^{R}(S^R, m^{M}, \overline{w}) \right] \left[1 - \widehat{\tau}^{R}(S^R, m^{M}, \overline{w}) \right]$$

$$- \sum_{j \in S^R} \theta_j \widehat{q}_j^{R}(S^R, m^{M}, \overline{w}) \mu_1 [1 + W(\Omega(S^R, m^{M}, \overline{w}))] - m^{M} \widehat{\tau}^{M}(S^R, m^{M}, \overline{w}).$$

Proof of Lemma 4: Recall from the proof of Proposition 18 that any optimal price vector for the manufacturer, $\boldsymbol{p}^{M}$ and $\boldsymbol{w}$, must satisfy FOCs. Therefore, m^{M} and $\overline{w}$ will also satisfy FOCs. We next derive the functions F_1 and F_2 using as our starting point the FOCs of $\Pi^M(S^R, m^{M}, \overline{w})$ as defined in Lemma 2.

Recall the FOCs of $\Pi^M(S^R, m^{M}, \overline{w})$ with respect to m^{M} and $\overline{w}$ provided in (C.40) and (C.41). Notice from two expressions that the RHS of (C.40) and the LHS of (C.41) are the same. Hence, to derive the function F_1 we replace the LHS of (C.40) with the RHS of (C.41) to obtain that:

$$m^{M} \left[1 - \widehat{\tau}^{M}(S^R, m^{M}, \overline{w}) \left[1 + W(\Omega(S^R, m^{M}, \overline{w})) \right] \right]$$

$$- \mu_1 \left[1 + \sum_{j \in S^R} \theta_j \widehat{q}_j^{R}(S^R, m^{M}, \overline{w}) W(\Omega(S^R, m^{M}, \overline{w})) \right] = 0.$$

Observe that the left-hand side of the expression above is the same as F_1 as defined in the lemma.

As for F_2, it follows directly from (C.41).

Proof of Proposition 22: Let m^{M*} and $\overline{w}^*$ be the margins that satisfy the conditions $F_1 = 0$ and $F_2 = 0$ (provided in Lemma 4) for a given assortment. This proof is done in two parts: In Part 1, we show that the manufacturer's expected profit,

$\Pi^M(S^R, m^{M*}, \overline{w}^*)$ in (C.27), increases in α_k for $k \in S^R$. In Part 2, we show that

$$\lim_{\alpha_z \to -\infty} \Pi^M(S^R \cup \{z\}, m^{M*}, \overline{w}^*) = \Pi^M(S^R, m^{M*}, \overline{w}^*).$$

Part 1: We note below the derivatives of $\Omega(S^R, m^{\scriptscriptstyle M}, \overline{w})$, $\widehat{\tau}^{\scriptscriptstyle R}(S^R, m^{\scriptscriptstyle M}, \overline{w}))$, and $\widehat{\tau}^{\scriptscriptstyle M}(S^R, m^{\scriptscriptstyle M}, \overline{w})$, all provided in Lemma 2 with respect to α_k.

$$\begin{aligned}
\frac{\partial \Omega(S^R, m^{\scriptscriptstyle M}, \overline{w})}{\partial \alpha_k} &= \Omega(S^R, m^{\scriptscriptstyle M}, \overline{w}) \frac{\widehat{q}_k^{\scriptscriptstyle R}(S^R, m^{\scriptscriptstyle M}, \overline{w})}{\mu_1}, \\
\frac{\partial \widehat{\tau}^{\scriptscriptstyle R}(S^R, m^{\scriptscriptstyle M}, \overline{w}))}{\partial \alpha_k} &= \widehat{\tau}^{\scriptscriptstyle R}(S^R, m^{\scriptscriptstyle M}, \overline{w})[1 - \widehat{\tau}^{\scriptscriptstyle R}(S^R, m^{\scriptscriptstyle M}, \overline{w})]^2 \frac{\widehat{q}_k^{\scriptscriptstyle R}(S^R, m^{\scriptscriptstyle M}, \overline{w})}{\mu_1}, \\
\frac{\partial \widehat{\tau}^{\scriptscriptstyle M}(S^R, m^{\scriptscriptstyle M}, \overline{w})}{\partial \alpha_k} &= \widehat{\tau}^{\scriptscriptstyle M}(S^R, m^{\scriptscriptstyle M}, \overline{w}) \widehat{\tau}^{\scriptscriptstyle R}(S^R, m^{\scriptscriptstyle M}, \overline{w}) \\
&\quad \times [1 - \widehat{\tau}^{\scriptscriptstyle R}(S^R, m^{\scriptscriptstyle M}, \overline{w})]^2 \frac{\widehat{q}_k^{\scriptscriptstyle R}(S^R, m^{\scriptscriptstyle M}, \overline{w})}{\mu_1}, \\
\frac{\partial \widehat{q}_k^{\scriptscriptstyle R}(S^R, m^{\scriptscriptstyle M}, \overline{w})}{\partial \alpha_k} &= \frac{\widehat{q}_k^{\scriptscriptstyle R}(S^R, m^{\scriptscriptstyle M}, \overline{w})}{\mu_2} \left[1 - \widehat{q}_k^{\scriptscriptstyle R}(S^R, m^{\scriptscriptstyle M}, \overline{w})\right], k \in S^R, \\
\frac{\partial \widehat{q}_j^{\scriptscriptstyle R}(S^R, m^{\scriptscriptstyle M}, \overline{w})}{\partial \alpha_k} &= -\widehat{q}_j^{\scriptscriptstyle R}(S^R, m^{\scriptscriptstyle M}, \overline{w}) \frac{\widehat{q}_k^{\scriptscriptstyle R}(S^R, m^{\scriptscriptstyle M}, \overline{w})}{\mu_2}, k \in S^R, k \neq j.
\end{aligned}$$

Using the derivatives above and given the fact that

$$\left.\frac{\partial \Pi^M(S^R, m^{\scriptscriptstyle M}, \overline{w})}{\partial m^{\scriptscriptstyle M}}\right|_{m^M = m^{M*}} = 0 \text{ and } \left.\frac{\partial \Pi^M(S^R, m^{\scriptscriptstyle M}, \overline{w})}{\partial \overline{w}}\right|_{\overline{w} = \overline{w}^*} = 0,$$

we find that:

$$\begin{aligned}
\frac{\mathbf{d}\Pi^M(S^R, m^{M*}, \overline{w}^*)}{\mathbf{d}\alpha_k} &= \frac{\partial \Pi^M(S^R, m^{M*}, \overline{w}^*)}{\partial \alpha_k} \\
&= \widehat{\tau}^{\scriptscriptstyle R}(S^R, m^{M*}, \overline{w}^*)[1 - \widehat{\tau}^{\scriptscriptstyle R}(S^R, m^{M*}, \overline{w}^*)] \\
&\quad + \frac{\widehat{q}_k^{\scriptscriptstyle R}(S^R, m^{M*}, \overline{w}^*)}{\mu_1} \left[F_2 + \frac{\mu_1 \theta_k}{1 - \widehat{\tau}^{\scriptscriptstyle R}(S^R, m^{M*}, \overline{w}^*)}\right],
\end{aligned} \tag{C.47}$$

where F_2 is as defined in Lemma 4. Recall that the optimal margins for the manufacturer will satisfy the conditions $F_1 = 0$ and $F_2 = 0$; hence

$$\frac{\mathbf{d}\Pi^M(S^R, m^{M*}, \overline{w}^*)}{\mathbf{d}\alpha_k} > 0.$$

Part 2: For any given pair $\overline{w}$ and $m^{\scriptscriptstyle M}$, it can be shown that:

$$\lim_{\alpha_z \to -\infty} \widehat{\tau}^{\scriptscriptstyle M}(S^R \cup \{z\}, m^{\scriptscriptstyle M}, \overline{w}) = \widehat{\tau}^{\scriptscriptstyle M}(S^R, m^{\scriptscriptstyle M}, \overline{w}),$$

$$\lim_{\alpha_z \to -\infty} \Omega(S^R \cup \{z\}, m^{\scriptscriptstyle M}, \overline{w}) = \Omega(S^R, m^{\scriptscriptstyle M}, \overline{w}),$$

$$\lim_{\alpha_z \to -\infty} \widehat{\tau}^{\scriptscriptstyle R}(S^R \cup \{z\}, m^{\scriptscriptstyle M}, \overline{w}) = \widehat{\tau}^{\scriptscriptstyle R}(S^R, m^{\scriptscriptstyle M}, \overline{w}),$$

$$\lim_{\alpha_z \to -\infty} \sum_{\substack{j \in S^R \\ j \neq z}} \widehat{q}_j^{\scriptscriptstyle R}(S^R \cup \{z\}, m^{\scriptscriptstyle M}, \overline{w}) = \sum_{\substack{j \in S^R \\ j \neq z}} \widehat{q}_j^{\scriptscriptstyle R}(S^R, m^{\scriptscriptstyle M}, \overline{w}),$$

$$\lim_{\alpha_z \to -\infty} \widehat{q}_z^{\scriptscriptstyle R}(S^R \cup \{z\}, m^{\scriptscriptstyle M}, \overline{w}) = 0.$$

Using the above limits along with F_1 and F_2 as defined in Lemma 4, note that any pair of $m^{\scriptscriptstyle M}$ and $\overline{w}$ that satisfy F_1 and F_2 for the assortment S^R will also satisfy F_1 and F_2 for the assortment $S^R \cup \{z\}$ when α_z goes to negative infinity. Furthermore, using the same limits and the expression for $\Pi^M(m^{\scriptscriptstyle M}, \overline{w})$ as defined in Lemma 2, it follows that:

$$\lim_{\alpha_z \to -\infty} \Pi^M(S^R \cup \{z\}, m^{\scriptscriptstyle M}, \overline{w}) = \Pi^M(S^R, m^{\scriptscriptstyle M}, \overline{w}).$$

Putting together the facts that the manufacturer's profit function is increasing in α_z, and that at a very small α_z the manufacturer's profit function is at least as good as the assortment that does not includes variant z, we conclude the proof.

Proof of Proposition 23: From Proposition 17 we know that the optimal effective margin for the retailer is $m^{\scriptscriptstyle R}(S^R, \boldsymbol{p}^{\scriptscriptstyle M}, \boldsymbol{w}) = \frac{\mu_1}{1-\tau^R(S^R, \boldsymbol{p}^M, m^R(S^R, \boldsymbol{p}^M, \boldsymbol{w}))}$. Hence we can write the profit function for the retailer given in (4.16) as:

$$\Pi^R(S^R, \boldsymbol{p}^{\scriptscriptstyle M}, \boldsymbol{w}, m^{\scriptscriptstyle R}(S^R, \boldsymbol{p}^{\scriptscriptstyle M}, \boldsymbol{w})) = m^{\scriptscriptstyle R}(S^R, \boldsymbol{p}^{\scriptscriptstyle M}, \boldsymbol{w}) - \mu_1. \tag{C.48}$$

Observe from above that the right hand side is increasing in $m^{\scriptscriptstyle R}(S^R, \boldsymbol{p}^{\scriptscriptstyle M}, \boldsymbol{w})$. Hence, this proof will follow after showing that $m^{\scriptscriptstyle R}(S^R, \boldsymbol{p}^{\scriptscriptstyle M}, \boldsymbol{w})$ always increases with the addition of a new variant to the retailer's assortment.

It follows from (C.7) that:

$$\frac{m^{\scriptscriptstyle R}(S^R, \boldsymbol{p}^{\scriptscriptstyle M}, \boldsymbol{w})}{\mu_1} - 1 = W\left(\frac{\exp(-1)\left[\sum_{k\in S^R}\exp(\eta_k^{\scriptscriptstyle R}/\mu_2)\right]^{\mu_2/\mu_1}}{\exp(\alpha_0) + \left[\sum_{k\in S^M} v_k(p_k^{\scriptscriptstyle M})\right]^{\mu_2/\mu_1}}\right). \tag{C.49}$$

Notice that the right-hand side of the expression above is increasing in $m^{\scriptscriptstyle R}(S^R, \boldsymbol{p}^{\scriptscriptstyle M}, \boldsymbol{w})$ and the right-hand side depends only on problem parameters for the retailer's pricing problem. Hence, any change in parameters that will force the right-hand side to increase will in fact increase $m^{\scriptscriptstyle R}(S^R, \boldsymbol{p}^{\scriptscriptstyle M}, \boldsymbol{w})$. Let z be an available variant that can be added to the retailer's assortment, i.e. $z \in S^M$, $z \notin S^R$, and recall that the Lambert-W function is increasing in its arguments. Observe that introducing this variant into the retailer's assortment will produce the following relationship,

$$\begin{aligned}\frac{m^{\scriptscriptstyle R}(S^R, \boldsymbol{p}^{\scriptscriptstyle M}, \boldsymbol{w})}{\mu_1} - 1 &= W\left(\frac{\exp(-1)\left[\sum_{k\in S^R}\exp(\eta_k^{\scriptscriptstyle R}/\mu_2)\right]^{\mu_2/\mu_1}}{\exp(\alpha_0) + \left[\sum_{k\in S^M} v_k(p_k^{\scriptscriptstyle M})\right]^{\mu_2/\mu_1}}\right)\\ &< W\left(\frac{\exp(-1)\left[\sum_{k\in S^R}\exp(\eta_k^{\scriptscriptstyle R}/\mu_2) + \exp(\eta_z^{\scriptscriptstyle R}/\mu_2)\right]^{\mu_2/\mu_1}}{\exp(\alpha_0) + \left[\sum_{k\in S^M} v_k(p_k^{\scriptscriptstyle M})\right]^{\mu_2/\mu_1}}\right)\\ &= \frac{m^{\scriptscriptstyle R}(S^R \cup \{z\}, \boldsymbol{p}^{\scriptscriptstyle M}, \boldsymbol{w})}{\mu_1} - 1.\end{aligned}$$

Hence, $m^{\scriptscriptstyle R}(S^R, \boldsymbol{p}^{\scriptscriptstyle M}, \boldsymbol{w}) < m^{\scriptscriptstyle R}(S^R \cup \{z\}, \boldsymbol{p}^{\scriptscriptstyle M}, \boldsymbol{w})$ which implies

$$\Pi^R(S^R, \boldsymbol{p}^{\scriptscriptstyle M}, \boldsymbol{w}, m^{\scriptscriptstyle R}(S^R, \boldsymbol{p}^{\scriptscriptstyle M}, \boldsymbol{w})) < \Pi^R(S^R \cup \{z\}, \boldsymbol{p}^{\scriptscriptstyle M}, \boldsymbol{w}, m^{\scriptscriptstyle R}(S^R, \boldsymbol{p}^{\scriptscriptstyle M}, \boldsymbol{w})).$$

Proof of Proposition 24: Observe that the manufacturer's profit under Scenario 1 will always be at least as high as the manufacturer's profit under Scenario 2 (because under Scenario 1 the manufacturer chooses the retailer's assortment while the retailer makes the same decision under Scenario 2). Now, recall that the manufacturer picks its prices before the retailer. Hence, even under Scenario 2 the manufacturer will pick the same variant that it would pick under Scenario 1 and will choose the same wholesale and direct channel prices that it would choose under Scenario 1. The manufacturer will make all other variants unattractive for the retailer, which could be achieved by charging a very high wholesale price. Hence, under Scenario 2, the retailer will offer the variant that the manufacturer will offer under Scenario 1.

Proof of Proposition 25: For this proof, we set up the simplified versions of the retailer's and manufacturer's pricing problems. Consider $S^M = \{1, 2\}$. Suppose the retailer offers variant k, i.e. $S^R = \{k\}$. We will prove this result by showing that the manufacturer's and retailer's profits, evaluated at the equilibrium wholesale and direct channel prices, are increasing in γ_k.

Let k be the only variant offered at the retailer, i.e. $S^R = \{k\}$. Since this is a special case of the more general model, the two conditions described in Lemma 4 continue to hold. For exposition purposes, we are going to make a few observations taking advantage of the fact that $S^R = \{k\}$.

Observation 1: Let

$$\widetilde{\Omega}(S^R, m^{\scriptscriptstyle M}, w_k) = \frac{\exp(-1)\left[\exp([\alpha_k - \gamma_k - w_k]/\mu_2)\right]^{\mu_2/\mu_1}}{\exp\left(\alpha_0\right) + \left[\sum_{j \in S^M} v_j^{\scriptscriptstyle M}(m^{\scriptscriptstyle M})\right]^{\mu_2/\mu_1}}. \tag{C.50}$$

From Proposition 18 we use the pricing property that $\overline{w} = (w_k - c_k - \mu_2)\theta_k$ to observe that $\widetilde{\Omega}(S^R, m^{\scriptscriptstyle M}, w_k) = \Omega(S^R, m^{\scriptscriptstyle M}, \overline{w})$, where $\Omega(S^R, m^{\scriptscriptstyle M}, \overline{w})$ is defined in Lemma 2.

Using the same pricing property, we observe that:

$$\sum_{j \in S^R} [\overline{w} - \mu_2 \theta_j] \, \widehat{q}_j^{\scriptscriptstyle R}(S^R, m^{\scriptscriptstyle M}, \overline{w}) = (w_k - c_k)\theta_k.$$

Observation 2: Let

$$\widetilde{\tau}^{\scriptscriptstyle R}(S^R, m^{\scriptscriptstyle M}, w_k) = \frac{W(\widetilde{\Omega}(S^R, m^{\scriptscriptstyle M}, w_k))}{1 + W(\widetilde{\Omega}(S^R, m^{\scriptscriptstyle M}, w_k))}, \tag{C.51}$$

and

$$\widetilde{\tau}^{\scriptscriptstyle M}(S^R, m^{\scriptscriptstyle M}, w_k) = \frac{\left[\sum_{j \in S^M} v_j^{\scriptscriptstyle M}(m^{\scriptscriptstyle M})\right]^{\mu_2/\mu_1}}{\left[\exp(\alpha_0) + \left[\sum_{j \in S^M} v_j^{\scriptscriptstyle M}(m^{\scriptscriptstyle M})\right]^{\mu_2/\mu_1}\right] [1 + W(\widetilde{\Omega}(S^R, m^{\scriptscriptstyle M}, w_k))]}. \tag{C.52}$$

Using the expressions for $\tau^{\scriptscriptstyle R}(S^R, m^{\scriptscriptstyle M}, \overline{w})$ and $\tau^{\scriptscriptstyle M}(S^R, m^{\scriptscriptstyle M}, \overline{w})$ in Lemma 2 together with Observation 1, note that

$$\widetilde{\tau}^{\scriptscriptstyle R}(S^R, m^{\scriptscriptstyle M}, w_k) = \tau^{\scriptscriptstyle R}(S^R, m^{\scriptscriptstyle M}, \overline{w}) \text{ and } \widetilde{\tau}^{\scriptscriptstyle M}(S^R, m^{\scriptscriptstyle M}, w_k) = \tau^{\scriptscriptstyle M}(S^R, m^{\scriptscriptstyle M}, \overline{w}).$$

Using Observations 1 and 2, we can now write the manufacturer's profit function in Lemma 2 as:

$$\widetilde{\Pi}^{\scriptscriptstyle M}(S^R, m^{\scriptscriptstyle M}, w_k) = \widetilde{\tau}^{\scriptscriptstyle M}(S^R, m^{\scriptscriptstyle M}, w_k) m^{\scriptscriptstyle M} + \widetilde{\tau}^{\scriptscriptstyle R}(S^R, m^{\scriptscriptstyle M}, w_k)(w_k - c_k)\theta_k. \tag{C.53}$$

Furthermore, we can write the conditions $F_1 = 0$ and $F_2 = 0$, defined in Lemma 4 as:

$$\begin{aligned} \widetilde{F}_1 = m^{\scriptscriptstyle M} &\left[1 - \widetilde{\tau}^{\scriptscriptstyle M}(S^R, m^{\scriptscriptstyle M}, w_k) \left[1 + W(\widetilde{\Omega}(S^R, m^{\scriptscriptstyle M}, w_k))\right]\right] \\ &- \mu_1 [1 + \theta_k W(\widetilde{\Omega}(S^R, m^{\scriptscriptstyle M}, w_k)] = 0. \end{aligned} \tag{C.54}$$

and

$$\begin{aligned}\widetilde{F}_2 &= -m^{M}\widetilde{\tau}^{M}(S^R, m^{M}, w_k)\\ &+ (w_k - c_k)\theta_k\left[1 - \widetilde{\tau}^{R}(S^R, m^{M}, w_k)\right] - \frac{\mu_1\theta_k}{1 - \widetilde{\tau}^{R}(S^R, m^{M}, w_k)} = 0. \end{aligned} \tag{C.55}$$

Observation 3: It can be shown that the derivatives of $\widetilde{\tau}^{R}(S^R, m^{M}, w_k)$, given by (C.51), and $\widetilde{\tau}^{M}(S^R, m^{M}, w_k)$, given by (C.52), with respect to γ_k are:

$$\begin{aligned}\frac{\partial\widetilde{\tau}^{R}(S^R, m^{M}, w_k)}{\partial\gamma_k} &= \frac{-\widetilde{\tau}^{R}(S^R, m^{M}, w_k)[1 - \widetilde{\tau}^{R}(S^R, m^{M}, w_k)]^2}{\mu_1},\\ \frac{\partial\widetilde{\tau}^{M}(S^R, m^{M}, w_k)}{\partial\gamma_k} &= \frac{\widetilde{\tau}^{M}(S^R, m^{M}, w_k)\widetilde{\tau}^{R}(S^R, m^{M}, w_k)[1 - \widetilde{\tau}^{R}(S^R, m^{M}, w_k)]}{\mu_1}.\end{aligned}$$

Let m^{M*} and w^* denote the margin and wholesale price in equilibrium. Given that m^{M*} and w^* satisfy FOCs of $\widetilde{\Pi}^M(S^R, m^{M}, w)$, we use the derivatives above find that:

$$\frac{\mathbf{d}\widetilde{\Pi}^M(S^R, m^{M*}, w^*)}{\mathbf{d}\gamma_k} = \frac{\partial\widetilde{\Pi}^M(S^R, m^{M*}, w^*)}{\partial\gamma_k} =$$

$$\frac{\widetilde{\tau}^{R}(S^R, m^{M*}, w^*)[1 - \widetilde{\tau}^{R}(S^R, m^{M*}, w^*)]}{\mu_1}\left[\begin{aligned}&m^{M*}\widetilde{\tau}^{M}(S^R, m^{M*}, w^*)\\ &-(w^* - c_k)\theta_k[1 - \widetilde{\tau}^{R}(S^R, m^{M*}, w^*)]\end{aligned}\right].$$

Because m^{M*} and w^* satisfy (C.55), we have

$$(w^* - c_k)\theta_k[1 - \widetilde{\tau}^{R}(S^R, m^{M*}, w^*)] > m^{M*}\widetilde{\tau}^{M}(S^R, m^{M*}, w^*).$$

Therefore:

$$\frac{\mathbf{d}\widetilde{\Pi}^{M}(S^R, m^{M*}, w^*)}{\mathbf{d}\gamma_k} < 0.$$

Given that (i) co_k and cu_k play a role only in the function γ_k, (ii) γ_k is increasing in

co_k and cu_k, and (iii) $\frac{\mathbf{d}\widetilde{\Pi}^{M}(S^R, m^{M*}, w^*)}{\mathbf{d}\gamma_k} < 0$, we now conclude that $\widetilde{\Pi}^{M}(S^R, m^{M*}, w^*)$ is decreasing in co_k and cu_k.

Observation 4: Recall from (C.26) that $m^{R}(m^{M*}, \overline{w}^*) = \mu_1[1 + W(\Omega(m^{M*}, \overline{w}^*))]$. Given Observation 1, we can write

$$m^{R}(m^{M*}, w^*) = \mu_1[1 + W(\Omega(m^{M*}, w^*))].$$

Replacing $m^{R}(m^{M*}, w^*)$ with $\mu_1[1 + W(\Omega(m^{M*}, w^*))]$ in (C.48), we can write

$$\widetilde{\Pi}^R(S^R, m^{M*}, w^*) = W(\widetilde{\Omega}(S^R, m^{M*}, w^*))\mu_1 \tag{C.56}$$

Given that the function $W(\cdot)$ is increasing in its arguments and μ_1 is a problem parameter, observe that the expression for $\widetilde{\Pi}^R(S^R, m^{M*}, w^*)$ provided in (C.56) is strictly increasing in $\widetilde{\Omega}(S^R, m^{M*}, w^*)$. Then to show that $\widetilde{\Pi}^R(S^R, m^{M*}, w^*)$ is increasing in γ_k it suffices to show that:

$$\frac{\mathbf{d}\widetilde{\Omega}(S^R, m^{M*}, w^*)}{\mathbf{d}\gamma_k} < 0.$$

Observation 5: Taking the derivative of $\widetilde{\Omega}(S^R, m^{M*}, w^*)$, given by (C.50), with respect to γ_k provides:

$$\frac{\mathbf{d}\widetilde{\Omega}(S^R, m^{M*}, w^*)}{\mathbf{d}\gamma_k} = \frac{\partial\widetilde{\Omega}(S^R, m^{M}, w)}{\partial\gamma_k} + \left.\frac{\partial\widetilde{\Omega}(S^R, m^{M}, w)}{\partial m^{M}}\right|_{m^M = m^{M*}} \frac{\mathbf{d}m^{M*}}{\mathbf{d}\gamma_k}$$
$$+ \left.\frac{\partial\widetilde{\Omega}(S^R, m^{M*}, w^*)}{\partial w^*}\right|_{w = w^*} \frac{\mathbf{d}w}{\mathbf{d}\gamma_k}.$$

We will observe that the above expression is negative after showing that:

$$\frac{\partial\widetilde{\Omega}(S^R, m^{M*}, w^*)}{\partial\gamma_k} + \frac{\partial\widetilde{\Omega}(S^R, m^{M*}, w^*)}{\partial w_k}\frac{\mathbf{d}w^*}{\mathbf{d}\gamma_k} < 0 \text{ and } \frac{\partial\widetilde{\Omega}(S^R, m^{M*}, w^*)}{\partial m^{M*}}\frac{\mathbf{d}m^{M}}{\mathbf{d}\gamma_k} < 0.$$

Recall the two conditions that m^{M*} and w^* must satisfy for the manufacturer's problem, i.e. $\widetilde{F}_1 = 0$ and $\widetilde{F}_2 = 0$. Using implicit differentiation we have that:

$$\frac{\mathbf{d}m^{M*}}{\mathbf{d}\gamma_k} = \frac{-\frac{\partial \widetilde{F}_1}{\partial \gamma_k}\frac{\partial \widetilde{F}_2}{\partial w^*} + \frac{\partial \widetilde{F}_2}{\partial \gamma_k}\frac{\partial \widetilde{F}_1}{\partial w_k}}{\frac{\partial \widetilde{F}_1}{\partial m^{M*}}\frac{\partial \widetilde{F}_2}{\partial w^*} - \frac{\partial \widetilde{F}_2}{\partial m^{M}}\frac{\partial \widetilde{F}_1}{\partial w^*}}, \text{ and} \tag{C.57}$$

$$\frac{\mathbf{d}w^{*}}{\mathbf{d}\gamma_k} = \frac{-\frac{\partial \widetilde{F}_2}{\partial \gamma_k}\frac{\partial \widetilde{F}_1}{\partial m^{M*}} + \frac{\partial \widetilde{F}_1}{\partial \gamma_k}\frac{\partial \widetilde{F}_2}{\partial m^{M*}}}{\frac{\partial \widetilde{F}_1}{\partial m^{M*}}\frac{\partial \widetilde{F}_2}{\partial w^*} - \frac{\partial \widetilde{F}_2}{\partial m^{M*}}\frac{\partial \widetilde{F}_1}{\partial w^*}}. \tag{C.58}$$

It can be shown that:

$$\begin{aligned}
\frac{\partial \widetilde{\Omega}(S^R, m^{M}, w)}{\partial \gamma_k} &= \frac{-\widetilde{\Omega}(S^R, m^{M}, w)}{\mu_1}, \\
\frac{\partial \widetilde{\tau}^{R}(S^R, m^{M}, w)}{\partial \gamma_k} &= \frac{-\widetilde{\tau}^{R}(S^R, m^{M}, w)[1-\widetilde{\tau}^{R}(S^R, m^{M}, w)]^2}{\mu_1}, \\
\frac{\partial \widetilde{\tau}^{M}(S^R, m^{M}, w)}{\partial \gamma_k} &= \frac{-\widetilde{\tau}^{M}(S^R, m^{M}, w)\widetilde{\tau}^{R}(S^R, m^{M}, w)[1-\widetilde{\tau}^{R}(S^R, m^{M}, w)]}{\mu_1}, \\
\frac{\partial \widetilde{F}_1}{\partial \gamma_k} &= \theta_k \widetilde{\tau}^{R}(S^R, m^{M}, w), \\
\frac{\partial \widetilde{F}_2}{\partial \gamma_k} &= 2\theta_k \widetilde{\tau}^{R}(S^R, m^{M}, w) + \frac{\widetilde{\tau}^{R}(S^R, m^{M}, w)[1-\widetilde{\tau}^{R}(S^R, m^{M}, w)]}{\mu_1}\widetilde{F}_2, \\
\frac{\partial \widetilde{F}_1}{\partial w} &= \theta_k \widetilde{\tau}^{R}(S^R, m^{M}, w), \\
\frac{\partial \widetilde{F}_2}{\partial w} &= \theta_k[1+\widetilde{\tau}^{R}(S^R, m^{M}, w)] + \frac{\widetilde{\tau}^{R}(S^R, m^{M}, w)[1-\widetilde{\tau}^{R}(S^R, m^{M}, w)]}{\mu_1}\widetilde{F}_2, \\
\frac{\partial \widetilde{F}_1}{\partial m^{M}} &= \widetilde{F}_1 \widetilde{\tau}^{M}(S^R, m^{M}, w)[1+W(\widetilde{\Omega}(S^R, m^{M}, w)] \\
&\qquad + 1 + \widetilde{\tau}^{M}(S^R, m^{M}, w)\theta_k[W(\widetilde{\Omega}(S^R, m^{M}, w)]^2, \\
\frac{\partial \widetilde{F}_2}{\partial m^{M}} &= -\theta_k \widetilde{\tau}^{M}(S^R, m^{M}, w)W(\widetilde{\Omega}(m^{M}, w)) + \frac{\widetilde{\tau}^{M}(S^R, m^{M}, w)}{\mu_1} \\
&\quad \times \left[\widetilde{F}_1 - \widetilde{F}_2 \widetilde{\tau}^{R}(S^R, m^{M}, w)\right].
\end{aligned}$$

Using the derivatives above it will follow that $\frac{\mathbf{d}m^{M*}}{\mathbf{d}\gamma_k}$, given by (C.57), is negative. Observe now that:

$$\frac{\partial \widetilde{\Omega}(S^R, m^{M*}, w^*)}{\partial \gamma_k} + \frac{\partial \widetilde{\Omega}(S^R, m^{M*}, w^*)}{\partial w}\frac{\mathbf{d}w^*}{\mathbf{d}\gamma_k} = \frac{\partial \widetilde{\Omega}(S^R, m^{M*}, w^*)}{\partial \gamma_k}\left[1 + \frac{\mathbf{d}w^*}{\mathbf{d}\gamma_k}\right] < 0.$$

This observation follows after noting that that $1 + \frac{\mathbf{d}w^*}{\mathbf{d}\gamma_k} > 0$, and $\frac{\partial\widetilde{\Omega}(S^R, m^{M*}, w^*)}{\partial\gamma_k} < 0$, by using the derivatives of the conditions $\widetilde{F}_1$ and $\widetilde{F}_2$ listed previously.

Using Observations 4 and 5, we conclude

$$\frac{\mathbf{d}\widetilde{\Pi}^R(S^R, m^{M*}, w^*)}{\mathbf{d}\gamma_k} < 0.$$

Since γ_k is increasing in co_k and cu_k we conclude that $\widetilde{\Pi}^R(S^R, m^{M*}, w^*)$ is decreasing in co_k and cu_k.

Proof of Proposition 26: In this proposition we assume all variants are the same (which implies θ_k is the same for all $k \in S^R$). Hence, we have that w_k is the same for all $k \in S^R$. Given that all variants are the same, it is no longer the composition of the retailer's assortment, but the size of the retailer's assortment that influences profits. Let M denote the number of variants carried by the manufacturer and let N denote the number of variants carried by the retailer. For notational convenience let $w = w_k$, $\alpha = \alpha_k$, $c = c_k$, $\gamma = \gamma_k$ and $\theta = \theta_k$.

Let $\overline{\Pi}^M(N, m^M, w)$ be the manufacturer's profit and $\overline{\Pi}^R(N, m^M, w)$ be the retailer's profit for this simplified problem. The proof of this proposition will follow after showing that:

$$\frac{\mathbf{d}\overline{\Pi}^M(N, m^{M*}, w^*)}{\mathbf{d}N} > \frac{\mathbf{d}\overline{\Pi}^R(N, m^{M*}, w^*)}{\mathbf{d}N},$$

Observation 1: Let

$$\overline{\Omega}(N, m^M, w) = \frac{\exp(-1)\left[N \exp([\alpha - \gamma - w]/\mu_2)\right]^{\mu_2/\mu_1}}{\exp(\alpha_0) + \left[M \exp([\alpha - c - m^M]/\mu_2)\right]^{\mu_2/\mu_1}}. \quad \text{(C.59)}$$

Given that all variants are the same for the purposes of this proposition, then observe that $\overline{\Omega}(N, m^M, w) = \Omega(S^R, m^M, \overline{w})$, where $\Omega(S^R, m^M, \overline{w})$ is defined in Lemma 2.

Observation 2: Let

$$\overline{\tau}^{R}(N, m^{M}, w) = \frac{W(\overline{\Omega}(N, m^{M}, w))}{1 + W(\overline{\Omega}(N, m^{M}, w))}, \tag{C.60}$$

and

$$\overline{\tau}^{M}(N, m^{M}, w) = \frac{[M \exp([\alpha - c - m^{M}]/\mu_2)]^{\mu_2/\mu_1}}{\left[\exp(\alpha_0) + [M \exp([\alpha - c - m^{M}]/\mu_2)]^{\mu_2/\mu_1}\right] [1 + W(\overline{\Omega}(N, m^{M}, w))]}. \tag{C.61}$$

Using Observation 1, note that $\overline{\tau}^{R}(N, m^{M}, w) = \tau^{R}(S^{R}, m^{M}, \overline{w})$ and $\overline{\tau}^{M}(N, m^{M}, w) = \tau^{M}(S^{R}, m^{M}, \overline{w})$ where $\tau^{R}(S^{R}, m^{M}, \overline{w})$ and $\tau^{M}(S^{R}, m^{M}, \overline{w})$ are defined in Lemma 2.

Observation 3: Let

$$\overline{\Pi}^{R}(N, m^{M}, w) = \mu_1 W(\overline{\Omega}(N, m^{M}, w)),$$

and

$$\Pi^{M}(N, m^{M}, w) = \overline{\tau}^{M}(N, m^{M}, w) m^{M} + \overline{\tau}^{R}(N, m^{M}, w)(w - c)\theta, \tag{C.62}$$

Using Observation 2, note that

$$\overline{\Pi}^{R}(N, m^{M}, w) = \Pi^{R}(S^{R}, m^{M}, \overline{w}) \text{ and } \overline{\Pi}^{M}(N, m^{M}, w) = \Pi^{M}(S^{R}, m^{M}, \overline{w}).$$

Given that the problem in this proposition is a simplification of the general problem, the conditions in Lemma 4 continue to hold. Using Observations 1 through 3, conditions F_1 and F_2 in Lemma 4 can now be simplified to $\overline{F}_1$ and $\overline{F}_2$ as follows:

$$\overline{F}_1 = m^{M} \left[1 - \overline{\tau}^{M}(N, m^{M}, w) \left[1 + W(\overline{\Omega}(N, m^{M}, w))\right]\right] - \mu_1[1 + \theta W(\overline{\Omega}(N, m^{M}, w))],$$

and

$$\overline{F}_2 = \frac{\mu_1 \theta}{1 - \overline{\tau}^{R}(N, m^{M}, w)} + m^{M}\overline{\tau}^{M}(N, m^{M}, w) - (w - c)\theta[1 - \overline{\tau}^{R}(N, m^{M}, w)].$$

The conditions above will be used later to find the derivatives of the retailer's and manufacturer's profit functions.

We next provide some partial derivatives that will help with the derivations of the profit functions. The derivatives for $\overline{\Omega}(N, m^{M}, w)$ in (C.59), $\overline{\tau^{R}}(N, m^{M}, w)$ in (C.60), and $\overline{\tau^{M}}(N, m^{M}, w)$ in (C.61) with respect to N, m^{M} and w_k are:

$$\begin{aligned}
\frac{\partial \overline{\Omega}(N, m^{M}, w_k)}{\partial N} &= \frac{\mu_2}{\mu_1 N}\overline{\Omega}(N, m^{M}, w_k), \\
\frac{\partial \overline{\tau}^{R}(N, m^{M}, w_k)}{\partial N} &= \frac{\mu_2}{\mu_1 N}\overline{\tau}^{R}(N, m^{M}, w_k)[1 - \overline{\tau}^{R}(N, m^{M}, w_k)]^2, \\
\frac{\partial \overline{\tau}^{M}(N, m^{M}, w_k)}{\partial N} &= -\frac{\mu_2}{\mu_1 N}\overline{\tau}^{M}(N, m^{M}, w_k)\overline{\tau}^{R}(N, m^{M}, w_k)[1 - \overline{\tau}^{R}(N, m^{M}, w_k)], \\
\frac{\partial \overline{F}_1}{\partial N} &= -\frac{\mu_2}{N}\theta\overline{\tau}^{R}(N, m^{M}, w_k), \\
\frac{\partial \overline{F}_2}{\partial N} &= 2\frac{\mu_2}{N}\theta\overline{\tau}^{R}(N, m^{M}, w_k), \\
\frac{\partial \overline{\Omega}(N, m^{M}, w_k)}{\partial m^{M}} &= \frac{\overline{\Omega}(N, m^{M}, w_k)}{\mu_1}\overline{\tau}^{M}(N, m^{M}, w_k)[1 + W(\overline{\Omega}(N, m^{M}, w_k))], \\
\frac{\partial \overline{\tau}^{R}(N, m^{M}, w_k)}{\partial m^{M}} &= \frac{[1 - \overline{\tau}^{R}(N, m^{M}, w_k)]}{\mu_1}\overline{\tau}^{R}(N, m^{M}, w_k)\overline{\tau}^{M}(N, m^{M}, w_k), \\
\frac{\partial \overline{\tau}^{M}(N, m^{M}, w_k)}{\partial m^{M}} &= -\frac{\overline{\tau}^{M}(N, m^{M}, w_k)}{\mu_1}\left[1 + \overline{\tau}^{M}(N, m^{M}, w_k)[\overline{\tau}^{R}(N, m^{M}, w_k)]\right. \\
&\qquad \left. -1 - W(\overline{\Omega}(N, m^{M}, w_k))\right], \\
\frac{\partial \overline{F}_1}{\partial m^{M}} &= 1 + \overline{\tau}^{M}(N, m^{M}, w_k)\theta[W(\overline{\Omega}(N, m^{M}, w_k))]^2, \\
\frac{\partial \overline{F}_2}{\partial m^{M}} &= \overline{\tau}^{M}(N, m^{M}, w_k)\theta W(\overline{\Omega}(N, m^{M}, w_k)), \\
\frac{\partial \overline{\Omega}(N, m^{M}, w_k)}{\partial w_k} &= -\frac{\overline{\Omega}(N, m^{M}, w_k)}{\mu_1}, \\
\frac{\partial \overline{\tau}^{R}(N, m^{M}, w_k)}{\partial w_k} &= -\frac{\overline{\tau}^{R}(N, m^{M}, w_k)}{\mu_1}[1 - \overline{\tau}^{R}(N, m^{M}, w_k)]^2,
\end{aligned}$$

$$\frac{\partial \overline{\tau}^{M}(N, m^{M}, w_k)}{\partial w_k} = \frac{\overline{\tau}^{M}(N, m^{M}, w_k)\overline{\tau}^{R}(N, m^{M}, w_k)}{\mu_1}[1 - \overline{\tau}^{R}(N, m^{M}, w_k)],$$
$$\frac{\partial \overline{F}_1}{\partial w_k} = \theta \overline{\tau}^{R}(N, m^{M}, w_k),$$
$$\frac{\partial \overline{F}_2}{\partial w_k} = -\theta[1 + \overline{\tau}^{R}(N, m^{M}, w_k)].$$

Using implicit differentiation we find the following derivatives with respect to N for the manufacturer and retailer:

$$\frac{\mathbf{d}\Pi^{M}(N, m^{M*}, w^*)}{\mathbf{d}N} = \frac{\mu_2 \overline{\tau}^{R}(N, m^{M*}, w^*)\theta}{N}, \tag{C.63}$$
$$\frac{\mathbf{d}\Pi^{R}(N, m^{M*}, w^*)}{\mathbf{d}N} = \frac{\mu_2 \overline{\tau}^{R}(N, m^{M*}, w^*)\theta}{N} \times \frac{\overline{\tau}^{M}(N, m^{M*}, w^*)\theta\left[1 + W(\overline{\Omega}(N, m^{M*}, w^*))\right] + 1/W(\overline{\Omega}(N, m^{M*}, w^*))}{\left[\begin{array}{l} 2\overline{\tau}^{M}(N, m^{M*}, w^*)\theta W(\overline{\Omega}(N, m^{M*}, w^*)) + 1/W(\overline{\Omega}(N, m^{M*}, w^*)) \\ + 2[1 + \overline{\tau}^{M}(N, m^{M*}, w^*)\theta[W(\overline{\Omega}(N, m^{M*}, w^*))]^2] \end{array}\right]}. \tag{C.64}$$

Observe that (C.64) is smaller than (C.63) when

$$\frac{\overline{\tau}^{M}(N, m^{M*}, w^*)\theta\left[1 + W(\overline{\Omega}(N, m^{M*}, w^*))\right] + 1/W(\overline{\Omega}(N, m^{M*}, w^*))}{\left[\begin{array}{l} 2\overline{\tau}^{M}(N, m^{M*}, w^*)\theta W(\overline{\Omega}(N, m^{M*}, w^*)) + 1/W(\overline{\Omega}(N, m^{M*}, w^*)) \\ + 2[1 + \overline{\tau}^{M}(N, m^{M*}, w^*)\theta[W(\overline{\Omega}(N, m^{M*}, w^*))]^2] \end{array}\right]} < 1. \tag{C.65}$$

One can show that the above condition holds when

$$\overline{\tau}^{M}(N, m^{M*}, w^*)\left[1 - W(\overline{\Omega}(N, m^{M*}, w^*)) - 2W(\overline{\Omega}(N, m^{M*}, w^*))^2\right] < \frac{2}{\theta}.$$

Hence, we conclude that when $\theta < 2$ (C.64) is smaller than (C.63).

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