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OPTIMAL TRAJECTORY COORDINATE-MULTIPLIER SYSTEMS WITH CONSTANT OF THE MOTION COMPONENTS

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Abstract

A relatively well-known property of continuously thrusting optimal trajectories is that there exists a vector constant of the motion which is completely analogous to the angular momentum integral of the three-body problem. In this analysis the range of applicability of the integral is extended in two ways. First, it is shown that there exists a large class of coordinate systems such that a conjugate Lagrange multiplier is a constant of the motion, and a method for generating systems with this property is presented. Such a method is applicable to the problem of defining nonstandard state variables for which a conjugate multiplier is a constant of the motion. Second, a nontrivial canonical transformation is used to generate a new system of canonical variables such that three of the variables are strictly functions of the components of the vector integral. Thus, the three variables are constants of the motion for the optimal trajectory problem. In addition, the canonical transformation is effected in such a way that all the new canonical variables are constants of the motion for the coast-arc problem.

I. Introduction

An attractive property of continuously thrusting optimal trajectory problems is that there exists a vector constant of the motion[1] identical in form to the angular momentum integral of the three-body problem. In the planar problem the integral reduces to a scalar which is actually the Lagrange multiplier conjugate to the range angle in a polar coordinate formulation. One can generalize this fact to show that there exists a constant Lagrange multiplier in nonplanar problems if the coordinate system is cylindrical or spherical.

The purpose of this paper is to extend the range of applicability of the vector integral in the following ways: (1) it will be shown that there exists a large class of nonplanar coordinate systems such that a conjugate Lagrange multiplier is a constant of the motion and a method for developing such systems will be presented; and (2) a canonical transformation will be effected in such a way that the new canonical system possesses three components which are strictly functions of the three components of the vector integral, i.e., three of the new variables are constants of the motion.

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II. Basic Theory

In this section the basic elements of canonical transformation theory necessary for the developments in the following sections are reviewed. It is assumed that the optimal trajectory problem under consideration is nonsingular and that the resultant Hamiltonian formulation of the problem has been transformed into a new Hamiltonian system which is isomorphic to the systems of classical mechanics[2], i.e., $H = \sum_{i=1}^{n} \lambda_i f_i(t, x, \lambda)$ does not contain the control variables.

DEFINITION II.1: Let $\{X(x, \lambda, t), \Lambda(x, \lambda, t)\} \in C^2$ be a nonsingular transformation. If for "every" Hamiltonian $H(x, \lambda, t)$ there exists a Hamiltonian $K(X, \Lambda, t)$, then the transformation is said to be canonical.

Note that the word "every" is emphasized in the above definition. The definition does not say that each transformation which preserves Hamiltonian form is canonical, but only those which preserve Hamiltonian form and are independent of the Hamiltonian function. Also, Definition II.1 is not a good "working" definition, i.e., one cannot check every Hamiltonian function. However, this definition leads to the following workable conditions for checking and generating canonical transformations.

PROPERTY II.1: (i) (Poisson brackets) The transformation $\{X(x, \lambda, t), \Lambda(x, \lambda, t)\}$ is canonical if and only if there exists a nonzero scalar constant μ such that $\{X_i, X_j\} = 0$, $\{\Lambda_i, \Lambda_j\} = 0$, and $\{\Lambda_i, X_j\} = \mu\delta_{ij}$ for each $i, j = 1, \ldots, n$, where $\{A, B\} \equiv \sum_{k=1}^{n} \left(\frac{\partial A}{\partial \lambda_k} \frac{\partial B}{\partial x_k} - \frac{\partial A}{\partial x_k} \frac{\partial B}{\partial \lambda_k} \right)$. (ii) (Generating functions) If there exists a scalar function S such that

$$\sum_{i=1}^{n} \lambda_i \dot{x}_i - H(x, \lambda, t) = \sum_{i=1}^{n} \Lambda_i \dot{X}_i - K(X, \Lambda, t) + \frac{dS}{dt} , \qquad (1)$$

then the transformation $\{X(x, \lambda, t), \Lambda(x, \lambda, t)\}$ is canonical.

With time as the independent variable, Eq. (1) can be expressed equivalently as

$$\delta S = \sum_{i=1}^{n} (\lambda_i \delta x_i - \Lambda_i \delta X_i) \qquad (2)$$

$$K = \frac{\partial S}{\partial t} + H \, . \qquad (3)$$

If the new Hamiltonian, K, is identically zero, then Eq. (3) becomes the Hamilton-Jacobi equation and the new variables X, Λ are constants of the motion.

Equations (2) and (3) are useful for defining the following class of canonical transformations.

DEFINITION II. 2: A canonical transformation in which $\frac{\partial S}{\partial t} = 0$ and $\delta S = 0$ is called a homogeneous canonical transformation. If, in addition, n independent relations between $\{x_1, \ldots, x_n\}$ and $\{X_1, \ldots, X_n\}$ are specified, then the transformation is called an extended point-transformation.

The importance of extended point-transformations in the analysis of optimal control problems is demonstrated by the following property.

PROPERTY II. 2: Let $x = \phi(X)$ be a nonsingular transformation between the coordinates of two Hamiltonian systems defined by $H = \sum_{i=1}^{n} \lambda_i f_i(t, x, \lambda)$ and $K = \sum_{i=1}^{n} \Lambda_i F_i(t, X, \Lambda)$. Then, the time independent Lagrange multiplier transformation between the two systems is defined by the n-equations

$$\Lambda_i = \sum_{j=1}^{n} \lambda_j \frac{\partial \phi_j}{\partial X_i} \, . \quad (i = 1, \ldots, n) \qquad (4)$$

Property II. 2 has a number of important consequences. First of all, it tells us how to determine very simply the Lagrange multiplier transformation between any two coordinate formulations of the same optimal trajectory problem. Secondly, it tells us that these transformations are linear with respect to the Lagrange multipliers. This fact is important since the components of the vector integral for the optimal trajectory problem are linear in the multipliers. In Reference 3, Whittaker presents a method for performing a canonical transformation which transforms a scalar integral linear in the multipliers into a new multiplier variable. Thus, this method has an immediate application in trajectory analysis.

III. Coordinate Systems With One Constant Multiplier

In the previous section it was noted that the relationship between the Lagrange multipliers for two sets of state variables is linear. This fact motivates the following question: "Does there exist a set of state variables such that three of the conjugate multipliers are equal to three independent linear functions of the known constants of the motion?" This question is answered in the following theorem.

THEOREM III. 1: Let $A_1 \bar{i} + A_2 \bar{j} + A_3 \bar{k}$ be the known vector integral of the optimal trajectory problem expressed in cartesian coordinates. There does not exist a canonical transformation such that two (or three) of the new canonical variables are independent linear combinations of the A_i's.

This theorem is easily proved by applying the Poisson bracket conditions to all possible linear combinations of the A_i's. Also, the theorem tells us that if a canonical system of variables with three components dependent only upon the A_i's exists, then at least two of the three components must be nonlinear functions of the A_i's. This means that such a system will be a hybrid system in the sense that no n of the 2n new variables are natural state variables. The development of such a hybrid system will be discussed in Section IV.

Even though no two linear combinations of the A_i's can be transformed into new canonical variables, the method of Whittaker gives us the means for generating a large class of sets of state variables such that a conjugate multiplier in each set is a constant of the motion. The method for generating the new canonical systems is described below.

Consider a Hamiltonian system which possesses an integral linear and homogeneous in the Lagrange multipliers, say

$$g_1(x)\lambda_1 + \cdots + g_n(x)\lambda_n = \text{constant}. \qquad (5)$$

Without loss of generality, let Eq. (5) be Λ_n in the new $\{X, \Lambda\}$-system which is to be defined by an extended point-transformation $x = \phi(X)$. Then, by Eq. (4):

$$\Lambda_i = \sum_{j=1}^{n} \lambda_j \frac{\partial \phi_j}{\partial X_i} \, . \quad (i = 1, \ldots, n) \qquad (6)$$

In order that Eqs. (5) and (6) be consistent, there must exist n independent functions $\phi_1(X), \ldots, \phi_n(X)$ such that

$$g_j(x) = \frac{\partial \phi_j}{\partial X_n} \, . \quad (j = 1, \ldots, n) \qquad (7)$$

The existence of these functions is guaranteed by first noting that

$$dx_i = \sum_{j=1}^{n} \frac{\partial \phi_i}{\partial X_j} dX_j \equiv \sum_{j=1}^{n} \phi_{ij} dX_j \, , \qquad (8)$$

and then applying the classic integrability theorem for a system of total differential equations[4]

To determine the functions $\phi_1, \ldots, \phi_n$, we first observe that

$$dx_i = g_i(x)\, dX_n \; (i = 1, \ldots, n) \qquad (9)$$

which implies

$$\frac{dx_1}{g_1} = \frac{dx_2}{g_2} = \cdots = \frac{dx_n}{g_n} = dX_n \, . \qquad (10)$$

Note that the only restriction on the functions $\phi_i(x)$ is that $\frac{\partial \phi_i}{\partial X_n} = g_i \; (i = 1, \ldots, n)$. Thus, there exist many point-transformations which satisfy this criterion, and such a transformation can be defined by the following procedure:

(i) Determine n - 1 integrals of the system (10),

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and denote these integrals by $X_1, \ldots, X_{n-1}$. Thus,

$$X_i \equiv \psi_i(x_1, \ldots, x_n) = \text{constant}. \; (i = 1, \ldots, n-1) \qquad (11)$$

(ii) Use Eqs. (11) to express n - 1 elements of the set $\{x_1, \ldots, x_n\}$ as functions of the X_i's and the remaining element of the set, say x_k. Then,

$$x_i \equiv \phi_i(X_1, \ldots, X_{n-1}, x_k). \; \begin{cases} i = 1, \ldots, n \\ i \neq k \end{cases} \qquad (12)$$

(iii) Substitute Eqs. (12) into $g_k(x)$ so that the function $g_k(x_k; X_1, \ldots, X_{n-1})$ is defined. Then, by Eqs. (10)

$$X_n = \int \frac{dx_k}{g_k(x_k; X_1, \ldots, X_{n-1})} \equiv \psi_n(x_k, X_1, \ldots, X_{n-1}) \;, \qquad (13)$$

where $X_1, \ldots, X_{n-1}$ are constants for the system defined by Eqs. (10). Then, solving for x_k in Eq. (13) the function $x_k = \phi_k(X_1, \ldots, X_n)$ is determined. This function and Eqs. (12) define the desired point-transformation.

After this method has been applied a new Hamiltonian system $\{X, \Lambda\}$ with Hamiltonian

$$K(X, \Lambda, t) \equiv H[x(X), \lambda(X, \Lambda), t] \qquad (14)$$

is defined. In this system $\frac{\partial K}{\partial X_n} = -\dot{\Lambda}_n = 0$ so X_n does not appear in $K(X, \Lambda, t)$. Thus, X_n will not appear in any of the Hamilton's equations and one need not even integrate the X_n-equation if the time-history of X_n is not a necessary part of the problem.

The method described above can be used to generate the cylindrical and spherical systems since they possess a conjugate constant of the motion Lagrange multiplier. However, the main reason for presenting the method is that it may prove useful in the generation of nonstandard coordinate systems (e.g., new orbital parameter systems) which possess a conjugate constant of the motion Lagrange multiplier.

IV. Application of Poisson Brackets in the Determination of the Total Canonical Transformation

In the previous section we found that there exist many canonical transformations which cause one of the new multipliers to be a constant of the motion. If the original system is denoted by $\{x_1, \ldots, x_n, \lambda_1, \ldots, \lambda_n\}$ and the new system by $\{X_1, \ldots, X_n, \Lambda_1, \ldots, \Lambda_n\}$, then the total Hamiltonian is, for example,

$$H(t, X_2, X_3, \ldots, X_n, \Lambda_1, \ldots, \Lambda_n), \qquad (15)$$

where the three constants of the motion are of the form

$$A_1 = \Lambda_1$$

$$A_2 = \sum_{i=1}^{n} G_{i_2}(X_1, \ldots, X_n)\Lambda_i \qquad (16)$$

$$A_3 = \sum_{i=1}^{n} G_{i_3}(X_1, \ldots, X_n)\Lambda_i \,.$$

Now we wish to perform a canonical transformation in which Λ_1 is invariant and two of the remaining new variables depend only upon A_1, A_2, and A_3 (at most).

The following theorem gives us a great deal of information concerning such a transformation.

THEOREM IV.1: Let $Q(X, \Lambda, t)$, $P(X, \Lambda, t)$ be a canonical transformation such that $P_1 = \Lambda_1$. Then,

(i) $Q_2, \ldots, Q_n, P_2, \ldots, P_n$ cannot depend upon X_1, and

(ii) Q_1 depends upon X_1 linearly, i.e., $Q_1 = cX_1 + f(X_2, \ldots, X_n, \Lambda_1, \ldots, \Lambda_n)$, where c is a nonzero constant.

Proof: (i) By Property II.1. (i), i.e., the Poisson bracket condition, it is necessary that $\{P_1, Q_j\} = \{P_1, P_j\} = 0$ for each j = 2, ..., n. But,

$$\{P_1, Q_j\} = \{\Lambda_1, Q_j\} = \frac{\partial Q_j}{\partial X_1} = 0$$

$$\{P_1, P_j\} = \{\Lambda_1, P_j\} = \frac{\partial P_j}{\partial X_1} = 0. \qquad (j = 2, \ldots, n)$$

Thus, $Q_2, \ldots, Q_n, P_2, \ldots, P_n$ cannot depend upon X_1. (ii) Again by the Poisson bracket theorem it is necessary that

$$\{P_1, Q_1\} = c \,,$$

where c is a nonzero scalar constant. Thus,

$$\{P_1, Q_1\} = \{\Lambda_1, Q_1\} = \frac{\partial Q_1}{\partial X_1} = c \,,$$

which implies

$$Q_1 = cX_1 + f(X_2, \ldots, X_n, \Lambda_1, \ldots, \Lambda_n).$$

With regard to a canonical transformation such that $P_1 = \Lambda_1$ and two of the remaining variables depend only upon A_1, A_2, and A_3 (at most), this theorem and Theorem III.1 imply the following:

(1) If P_k, P_ℓ, or P_k, Q_ℓ ($k \neq \ell$; $k, \ell \in \{2, 3, \ldots, n\}$) are the two desired canonical variables, then the nonlinear combinations of $A_1(\Lambda_1)$, $A_2(X, \Lambda)$, $A_3(X, \Lambda)$ which form them cannot depend upon X_1. Thus, if A_2 and/or A_3 depend upon X_1, then the nonlinear combination must be formed in such a way that X_1 is eliminated.

(2) If Q_1, P_ℓ or Q_1, Q_ℓ (ℓ = 2, 3, ..., or n), where $Q_1 = cX_1 + f(X_2, \ldots, X_n, \Lambda_1, \ldots, \Lambda_n)$, are the two desired canonical variables, then the nonlinear combination of the A_i's which forms P_ℓ or Q_ℓ ($\ell \neq 1$) cannot depend upon X_1, and Q_1 must be formed from the A_i's in such a way that X_1 appears linearly.

Properties (1) and (2) mentioned above restrict considerably the possible choices for three of the new canonical variables (i.e., P_1 and two other variables). In the next section we shall see that the obvious choices for nonlinear combinations of the A_i's which satisfy (1) and (2) lead to the desired canonical transformation.

V. Canonical Systems with Three Constant of the Motion Components

In this section a new canonical system, which contains three constant of the motion components, will be determined for the optimal trajectory problem. From Theorem III.1 we know that such a system must contain at least two nonlinear combinations of A_1, A_2, and A_3 (where $A_1\bar{i} + A_2\bar{j} + A_3\bar{k}$ is the known vector constant of the motion).

Later in this section it will be shown that the A_i's can be expressed in spherical coordinates as

$$\begin{aligned} A_1 &= A\cos\phi - B\sin\phi \\ A_2 &= A\sin\phi + B\cos\phi \\ A_3 &= \lambda_6 , \end{aligned} \tag{17}$$

where ϕ is the coordinate canonically conjugate to the constant multiplier λ_6 and neither A nor B depends upon ϕ. (With respect to Section IV, λ_6 corresponds to Λ_1 and ϕ corresponds to X_1.) By Theorem IV. 1 at least one of the nonlinear combinations of the A_i's, which is to be a new canonical variable, cannot depend upon ϕ. Inspection of Eqs. (17) suggests two basic functional forms: $f(A_1^2 + A_2^2)$ or $f(A_1^2 + A_2^2 + A_3^2)$. The simplest of these forms are $A_1^2 + A_2^2$, $\sqrt{A_1^2 + A_2^2}$, $A_1^2 + A_2^2 + A_3^2$, and $\sqrt{A_1^2 + A_1^2 + A_3^2}$. With the canonical transformation technique of this paper (to be discussed later), $\sqrt{A_1^2 + A_2^2}$ does not allow the desired transformation whereas $\sqrt{A_1^2 + A_2^2 + A_3^2}$ does. Thus, $\sqrt{A_1^2 + A_2^2 + A_3^2}$ will be one of our new momenta variables. (Note that by performing a simple canonical transformation $\sqrt{A_1^2 + A_2^2 + A_3^2}$ could alternatively be a new generalized coordinate.)

Finally, consider the possibilities for the third new constant of the motion canonical variable. Since there does not exist another functionally independent, with respect to $f(A_1^2 + A_2^2 + A_3^2)$, nonlinear combination of the Ai's which does not contain ϕ, then the only possibility for the third variable is a linear function ϕ and it must be canonically conjugate to $\lambda_6 = A_3$ (by Theorem IV.1). At first glance it does not appear that a nonlinear combination of the A_i's can form a linear function of ϕ. However, $\tan^{-1}(A_1/-A_2)$ is such a function, and is indeed the desired third canonical variable. Since the Hamilton-Jacobi theory is used to determine the transformation, the new generalized coordinates result from simple differentiations. Thus, it appears that $\tan^{-1}(A_1/-A_2)$ would simply "fall out" and, would not be useful in generating the transformation. This is not the case since the requirement that the generalized coordinate conjugate to $\lambda_6 = A_1$ is also a constant of the motion is the means by which one chooses the proper functional form $f(A_1^2 + A_2^2 + A_3^2)$. (That is, it was found that $f(A_1^2 + A_2^2)$ does not produce a constant of the motion conjugate to λ_6, whereas $f(A_1^2 + A_2^2 + A_3^2)$ does.)

Before the desired canonical transformation is effected, let us consider the possible ways in which we can generate the transformation. First of all note that since the constants of the motion depend upon all of the original variables, i.e., $A_1(x,\lambda)$, $A_2(x,\lambda)$, $A_3(x,\lambda)$, then the possibility of performing simple transformations and using independence arguments[2] is not applicable, i.e., the system $\{x,\lambda\}$ cannot be transformed into an intermediate system $\{X,\Lambda\}$ such that $\delta S = \sum_{i=1}^{n}(\Lambda_i\delta X_i - P_i\delta Q_i(X))$ can be used to define the transformation by independence arguments.

Another approach which is possible but will not be pursued here is the following. We know by Definition II.1 that a canonical transformation is independent of the Hamiltonian function, and by Eq. (3) that

$$\frac{\partial S}{\partial t} + H(x, \frac{\partial S}{\partial x}, t) = 0 \tag{18}$$

implies a canonical transformation which results in the equilibrium solution for the system defined by $H(x, \frac{\partial S}{\partial t}, t)$. Thus, the desired transformation can be obtained by solving the partial differential equation (18) for any Hamiltonian such that $\sqrt{A_1^2 + A_2^2 + A_3^2}$ and A_3 are constants of the motion for the corresponding Hamiltonian system. Since $A_1^2 + A_2^2$ does not depend upon ϕ, then such a Hamiltonian is

$$H = A_1^2(\tilde{x},\lambda) + A_2^2(\tilde{x},\lambda) + \lambda_6^2 , \tag{19}$$

where $\tilde{x}$ does not depend upon ϕ. Since this H does not depend upon time it must be a constant of the motion (say α_1^2) and since ϕ does not appear explicitly, then λ_6 must be a constant of the motion (say α_2). To complete the canonical transformation, one must determine a complete solution $S(t, x_1, \ldots, x_n, \alpha_1, \ldots, \alpha_n)$ of the Hamilton-Jacobi equation

$$-\alpha_1^2 + A_1^2(\tilde{x}, \frac{\partial S}{\partial \tilde{x}}) + A_2^2(\tilde{x}, \frac{\partial S}{\partial \tilde{x}}) + \alpha_2^2 = 0. \tag{20}$$

The new momenta variables are $\alpha_1, \ldots, \alpha_n$ and the new generalized coordinates are $\beta_i = \frac{\partial S}{\partial x_i} (i = 1, \ldots, n)$.

The approach mentioned above is undesirable because the Hamiltonian of Eq. (19) is not physically motivated and the Hamilton-Jacobi equation of Eq. (20) is not easy to solve. Therefore, since we must solve a nontrivial partial differential equation, we should try to obtain as much physical knowledge about the problem as possible. Since $\sqrt{A_1^2 + A_2^2 + A_3^2}$ and A_3 are also constants of the motion for the optimal trajectory problem with zero-thrust (i.e., the coast-arc problem), then a logical choice for the Hamiltonian is the zero-thrust Hamiltonian. In this case the new canonical system $\{\alpha, \beta\}$ will have two desirable properties: (1) two of the new momenta variables and one of the new generalized coordinates will be constants of the motion for the total optimal trajectory problem; and (2) all of the variables $\alpha_1, \ldots, \alpha_n$, $\beta_1, \ldots, \beta_n$ are constants of the motion for the coast-arc problem. Thus, for low-thrust missions the set $\{\alpha, \beta\}$ should be a slowly varying set of variables.

The equations of motion for a continuously thrusting rocket in an inverse square gravitational force field are

$$
\begin{aligned}
\dot{u} &= \frac{v^2}{r^3} + \frac{w^2}{r^3 \cos^2\theta} - \frac{k}{r^2} \\
&\quad + \frac{F}{m}[\cos\theta \cos\tau \cos\chi^* + \sin\tau \sin\theta] \\
\dot{v} &= -\frac{w^2}{r^2}\sec^2\theta \tan\theta \\
&\quad + r\frac{F}{m}[\sin\tau \cos\theta - \cos\tau \sin\theta \cos\chi^*] \qquad (21) \\
\dot{w} &= r\frac{F}{m}\cos\tau \cos\theta \sin\chi^* \\
\dot{r} &= u \\
\dot{\theta} &= \frac{v}{r^2} \\
\dot{\phi} &= \frac{w}{r^2 \cos^2\theta} \\
\dot{m} &= -\sigma
\end{aligned}
$$

where the variables r, θ, ϕ, τ, and χ are defined in Figure 1 and $\chi^* \equiv \chi - \phi$. The thrust magnitude, F, and mass flow rate parameter, σ, are assumed constant. Upon application of the maximum principle (assuming that a scalar quantity $g(t_f, x_f)$ is to be minimized), the control variables are determined as functions of state variables and Lagrange multipliers, and the following generalized Hamiltonian describes the problem:

$$
\begin{aligned}
H &= \lambda_1\left(\frac{v^2}{r^3} + \frac{w^2}{r^3}\sec^2\theta - \frac{k}{r^2}\right) \\
&\quad - \lambda_2 \frac{w^2}{r^2}\sec^2\theta \tan\theta \\
&\quad + \lambda_4 u + \lambda_5 \frac{v}{r^2} + \lambda_6 \frac{w}{r^2}\sec^2\theta - \lambda_7\sigma \qquad (22) \\
&\quad + \frac{F}{m}\sqrt{\lambda_1^2 + r^2\lambda_2^2 + r^2\lambda_3^2\cos^2\theta} \; .
\end{aligned}
$$

In this formulation the three components of the vector integral become

$$
\begin{aligned}
A_1 &\equiv A\cos\phi - B\sin\phi \\
A_2 &\equiv A\sin\phi + B\cos\phi \qquad (23) \\
A_3 &= \lambda_6 \, ,
\end{aligned}
$$

where

$$
\begin{aligned}
A &\equiv \lambda_3 v + \lambda_6 \tan\theta - \lambda_2 w \sec^2\theta \\
B &\equiv \lambda_5 - \lambda_3 w \tan\theta \, . \qquad (24)
\end{aligned}
$$

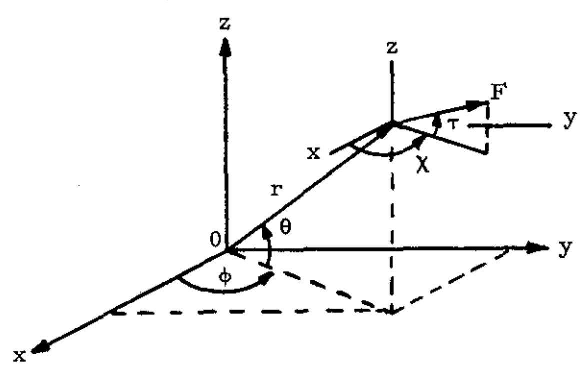

Figure 1. Geometry and Control Angle Definition

In References 2 and 5, the application of simple canonical transformations to the solution of Hamilton-Jacobi equations is discussed. Based on those results the following simple canonical transformation is defined for the problem considered here:

$$
\begin{aligned}
&Q_1 = \lambda_1, Q_2 = \lambda_2, Q_3 = w, Q_4 = r, Q_5 = \theta, Q_6 = \phi, Q_7 = m \\
&\qquad (25) \\
&P_1 = -u, P_2 = -v, P_3 = \lambda_3, P_4 = \lambda_4, P_5 = \lambda_5, P_6 = \lambda_6, P_7 = \lambda_7 \, ,
\end{aligned}
$$

where the Q_i's and P_i's represent new generalized coordinates and momenta, respectively. The coast-arc Hamilton-Jacobi equation (i.e., the Hamilton-Jacobi equation for the Hamiltonian of Eq. (22) with F = 0) is then:

$$
\begin{aligned}
&\frac{\partial S}{\partial t} + Q_1\left[\frac{S_2^2 + Q_3^2\sec^2 Q_5}{Q_4^3} - \frac{k}{Q_4}\right] \\
&- \frac{Q_2 Q_3^2}{Q_4^2}\sec^2 Q_5 \tan Q_5 \qquad (26) \\
&-S_4 S_1 - \frac{S_5 S_2}{Q_4^2} + \frac{S_6 Q_3 \sec^2 Q_5}{Q_4^2} - \sigma S_7 = 0 \, ,
\end{aligned}
$$

where $S_i \equiv \frac{\partial S}{\partial Q_i}$. A complete solution of this equation will now be effected, and two of the new momenta variables, α_3 and α_7, will be strictly functions of A_1, A_2, and A_3.

Since neither t, Q_7, nor Q_6 appear explicitly in Eq. (26), then $\partial S/\partial t$, S_7, and S_6 must be constants of the motion, say α_1, α_2/σ, and α_3. From knowledge of the two-body problem (which describes the state on the coast-arc) two more constant relationships are known:

$$
\begin{aligned}
\alpha_4^2 &= v^2 + w^2\sec^2\theta \\
&= S_2^2 + Q_3^2\sec^2 Q_5 \quad \text{(angular momentum)} \qquad (27) \\
-\alpha_5 &= u^2 + \frac{\alpha_4^2}{r^2} - \frac{2k}{r} \\
&= S_1^2 + \frac{\alpha_4^2}{Q_4^2} - \frac{2k}{Q_4} \, . \quad \text{(energy)}
\end{aligned}
$$

Upon substitution of $\partial S/\partial t = \alpha_1$, $S_7 = \alpha_2/\sigma$, $S_6 = \alpha_3$, $S_1(\alpha, Q)$, and $S_2(\alpha, Q)$ into Eq. (26) another separation of variables can be performed so that a sixth constant α_6 may be defined such that both of the following equations hold:

$$
\begin{aligned}
\alpha_6\alpha_4 &= (\alpha_1 - \alpha_2)Q_4^2 + Q_1\left(\frac{\alpha_4^2}{Q_4} - k\right) \\
&\quad - Q_4 S_4\sqrt{2kQ_4 - \alpha_5 Q_4^2 - \alpha_4^2} \qquad (28)
\end{aligned}
$$

$$
\begin{aligned}
\alpha_6\alpha_4 &= Q_2 Q_3^2\sec^2 Q_5 \tan Q_5 - \alpha_3 Q_3 \sec^2 Q_5 \\
&\quad - S_5\sqrt{\alpha_4^2 - Q_3^2\sec^2 Q_5} \, . \qquad (29)
\end{aligned}
$$

A more detailed discussion of the solution technique outlined above may be found in Reference 6. Also, it should be noted that ± signs have been omitted in the determination of S_1 and S_2 from Eqs. (27). The consequences of this will be discussed later.

For a complete solution of Eq. (26), seven constants are required and so far we have obtained only six. However, we have not made use of the $\sqrt{A_1^2 + A_2^2 + A_3^2}$ constant. With the constants that we already know we can use the solution technique of References 5 and 6 to form an incomplete solution of Eq. (26) of the form

$$S^*(t, Q_1, \ldots, Q_7, \alpha_1, \ldots, \alpha_6). \tag{30}$$

To form the complete solution we note that Eq. (26) does not contain an $S_3 \equiv \partial S/\partial Q_3$-term (this is a result of the base Hamiltonian not containing λ_3). Thus, the addition of any function of Q_3 to S^* will not affect the Hamilton-Jacobi equation, i.e., $S^* + f(Q_3, \alpha)$ is also a solution of Eq. (26). Instead of picking an arbitrary function of Q_3, we shall choose the one which results from defining $\alpha_7^2 = A_1^2 + A_2^2 + A_3^2$.

Upon evaluation of A_1^2, A_2^2, and A_3^2 in terms of S_3, Q_i's and α_i's a quadratic in S_3 is formed, and then

$$S_3 = \frac{Q_3 g_2 \tan Q_5 - g_1 S_2(Q_3, Q_5, \alpha_4)}{(\alpha_4^2 - Q_3^2)} \pm \frac{\sqrt{(\alpha_7^2 - \alpha_3^2)(\alpha_4^2 - Q_3^2) - (\alpha_3 Q_3 + \alpha_4 \alpha_6)^2}}{(\alpha_4^2 - Q_3^2)} \tag{31}$$

where

$$\begin{aligned} g_1 &\equiv \alpha_3 \tan Q_5 - Q_2 Q_3 \sec^2 Q_5 \\ g_2 &\equiv S_5 = [Q_2 Q_3^2 \sec^2 Q_5 \tan Q_5 \\ &\quad -\alpha_3 Q_3 \sec^2 Q_5 - \alpha_4 \alpha_6]/S_2(Q_3, Q_5, \alpha_4) \end{aligned} \tag{32}$$

and use has been made of the relationship

$$g_1 Q_3 \tan Q_5 + g_2 S_2(Q_3, Q_5, \alpha_4) = -(\alpha_3 Q_3 + \alpha_4 \alpha_6). \tag{33}$$

Note that Eq. (31) has the following functional form

$$S_3 = f_1(\alpha;\ Q_2, Q_3, Q_5) \pm f_2(\alpha, Q_3). \tag{34}$$

If one differentiates S^* of Eq. (30) with respect to Q_3, then $f_1(\alpha; Q_2, Q_3, Q_5)$ will be the result. Thus, the function of Q_3 which we wish to adjoin to S^* is just the indefinite integral of $f_2(\alpha, Q_3)$. Evaluating this integral and writing out the S^*-function, we obtain the following complete solution to the Hamilton-Jacobi equation (26):

$$\begin{aligned} S &= \alpha_1 t + \alpha_2 Q_7 + \alpha_3 \left[Q_6 - \sin^{-1} \frac{Q_3 \tan Q_5}{\sqrt{\alpha_4^2 - Q_3^2}} \right] \\ &- Q_1 \sqrt{\frac{2k}{Q_4} - \frac{\alpha_4^2}{Q_4^2} - \alpha_5} - Q_2 \sqrt{\alpha_4^2 - Q_3^2 \sec^2 Q_5} \\ &+ \alpha_6 \left[\cos^{-1} \frac{\alpha_4^2 - k Q_4}{Q_4 \sqrt{k^2 - \alpha_4^2 \alpha_5}} - \sin^{-1} \frac{\alpha_4 \sin Q_5}{\sqrt{\alpha_4^2 - Q_3^2}} \right] \\ &+ (\alpha_1 - \alpha_2) \left[\frac{Q_4}{\alpha_5} \sqrt{\frac{2k}{Q_4} - \frac{\alpha_4^2}{Q_4^2} - \alpha_5} - \frac{k}{\alpha_5^{3/2}} \cos^{-1} \frac{k - \alpha_5 Q_4}{\sqrt{k^2 - \alpha_4^2 \alpha_5}} \right] \\ &+ \left\{ \alpha_7 \sin^{-1} \frac{\alpha_7^2 Q_3 + \alpha_3 \alpha_4 \alpha_6}{\alpha_4 \sqrt{\alpha_7^2 - \alpha_3^2} \sqrt{\alpha_7^2 - \alpha_6^2}} \right. \\ &\quad + \alpha_3 \cos^{-1} \frac{\alpha_4 \alpha_6 + \alpha_3 Q_3}{\sqrt{\alpha_4^2 - Q_3^2} \sqrt{\alpha_7^2 - \alpha_3^2}} \\ &\quad \left. + \alpha_6 \cos^{-1} \frac{\alpha_3 \alpha_4 + \alpha_6 Q_3}{\sqrt{\alpha_4^2 - Q_3^2} \sqrt{\alpha_7^2 - \alpha_6^2}} \right\} \end{aligned} \tag{35}$$

where $\{\ldots\}$ is the $f_2(\alpha, Q_3)$ contribution. The seven remaining canonical variables may be obtained by applying Jacobi's Theorem, i.e., $\beta_i = \partial S/\partial \alpha_i$ $(i = 1, \ldots, 7)$. Since these new variables can be obtained by differentiation alone, we shall only develop $\beta_3 = \partial S/\partial \alpha_3$ to show that it is equal to $\tan^{-1}(A_1/-A_2)$.

Differentiation of Eq. (35) with respect to α_3 gives

$$\beta_3 = \frac{\partial S}{\partial \alpha_3} = Q_6 - \sin^{-1} \frac{Q_3 \tan Q_5}{\sqrt{\alpha_4^2 - Q_3^2}} + \cos^{-1} \frac{\alpha_4 \alpha_6 + \alpha_3 Q_3}{\sqrt{\alpha_4^2 - Q_3^2} \sqrt{\alpha_7^2 - \alpha_3^2}}. \tag{36}$$

If one combines the last two terms of Eq. (36) to form a single $\sin^{-1}$-function and also combines them to form a single $\cos^{-1}$-function, and then makes use of the following equalities:

$$\begin{aligned} &\alpha_4 \alpha_6 + \alpha_3 w = -w[\lambda_3 v - \lambda_2 w \sec^2\theta + \lambda_6 \tan\theta] \tan\theta \\ &v(\lambda_5 - \lambda_3 w \tan\theta) = A w \tan\theta + B v \\ &\alpha_7^2 - \alpha_3^2 = A^2 + B^2 \\ &\alpha_4^2 - w^2 = v^2 + w^2 \tan^2\theta \\ &(\alpha_4^2 - Q_3^2)(\alpha_7^2 - \alpha_3^2) - (\alpha_4 \alpha_6 + \alpha_3 w)^2 = (vA - wB\tan\theta)^2 \end{aligned} \tag{37}$$

then the following expressions are valid:

$$\sin(\beta_3 - \phi) = \frac{A}{\sqrt{A^2 + B^2}}, \quad \cos(\beta_3 - \phi) = \frac{-B}{\sqrt{A^2 + B^2}}. \tag{38}$$

From these equations and Eqs. (23) it follows that

$$\tan\beta_3 = \frac{A_1}{-A_2}. \tag{39}$$

Therefore, the system $\{\alpha, \beta = \partial S/\partial\beta\}$ represents a set of canonical variables which are canonic constants for the coast-arc problem and α_3, α_7, and β_3 are strictly functions of A_1, A_2, and A_3 so they are constants of the motion for the total problem.

VI. Concluding Remarks

The present study has sought to extend the applicability of the known vector integral for the optimal trajectory problem. It was shown that a classic theorem due to Whittaker can be used to define a large class of state variables such that a conjugate Lagrange multiplier is a component of the vector integral. Also, a canonical transformation was used to define a new canonical system in which three of the variables are constants of the motion. Although the resultant system is cumbersome, it demonstrates the existence of a canonical system with constant of the motion components for the coast-arc problem such that three of the components are constants for the total problem. Since the primary goal here was to generate such a transformation, the ± signs which result from solving quadratic equations throughout the analysis were dropped (thus the resultant solution is only valid for the positive case). However, now that it is known that such a transformation exists, the ± sign difficulty should be removable in a manner similar

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to the way that the Hamilton-Jacobi solution of Reference 2 removed the ± difficulty in the solutions of References 5 and 6.

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