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A LIE BRACKET SOLUTION OF THE OPTIMAL THRUST MAGNITUDE ON A SINGULAR ARC IN ATMOSPHERIC FLIGHT

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Abstract

Singular arcs form possible sub-arcs in various flight path optimization problems whenever we assume a constant ejection velocity type propulsion system, the thrust magnitude being the singular control. However, the actual evaluation of the thrust magnitude on these arcs is very cumbersome, especially for problems with an atmospheric flight segment. We do this using a recent extension of the Lie Bracket solution of singular controls on partially singular arcs by the authors. The lift and the bank controls are assumed to be interior while the thrust direction is assumed to be along the velocity vector. The Lie Bracket solution is shown to be much easier to compute and to preserve any symmetry properties in the problem. The solution is presented in vectorial form which allows for a compact and coordinate independent solution. An example canonical transformation illustrates how the results can be transformed to any set of state variables. Some interesting sub-cases such as flight in a vertical plane and flight in a circular orbit with no lift are studied.

Introduction

In atmospheric flight, there are five controls in general, two specifying the thrust direction, one thrust magnitude and the lift and the bank making up the aerodynamic controls. For a constant ejection velocity type propulsion system the optimal thrust magnitude is either a maximum, zero or intermediate, the last one resulting in a singular extremal arc. In order to find the thrust magnitude on a singular arc, we must differentiate the switching function an even number of times (two in our case). The conventional method of finding these derivatives is to express the lift and the bank controls as functions of the state and the adjoint variables (assuming that both the controls are interior) and then use the canonical equations for the latter. See Vinh and Medepalli[1] for such a solution.

In this paper, we derive the solution using the Lie Brackets. In literature, Lie bracket solution for singular controls is given only for the case of totally singular arcs with one control variable[2,3]. This has been extended recently by the authors[4] to include the case of partially singular arcs where both singular and non-singular controls are present. These results are summarized in the Appendix. Ross and Melton[5] have given a solution for the optimal thrust magnitude only for the case of a non-lifting vehicle. Thus the singular arc is still total in their case.

Here we give the complete solution of the optimal thrust magnitude on a singular arc for a thrusting and lifting spacecraft in atmospheric flight where the thrust vector is assumed to be aligned with the velocity vector. Due to our assumption on the thrust direction, we have only three controls. Clearly, the lift and the bank controls make our problem partially singular and the results of Ref.(4) are applicable in this case. It will be shown that the results of Ref. (5) are a special case of the results of this paper.

An additional feature of this paper is the vectorial representation of the problem and the results to make them independent of the choice of the variables. As shown in Ref.(1), using an appropriate canonical transformation, all the results can be expressed in terms of the desired set of variables. An example of such a canonical transformation is given in the Appendix.

In the following, the optimal control problem is stated along with all the assumptions. The optimal lift and bank controls and the explicit solution for the thrust magnitude on a singular arc are given next. A brief discussion will conclude the paper.

Note Throughout this paper, no distinction has been

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made between the symbol representing a vector and the corresponding column matrix. Both are represented by a symbol with a bar on top (or a hat for a unit vector). Thus, for example, $\bar{r}$ represents the radius vector as well as the column matrix representing the radius vector, depending on the context and r represents its magnitude.

Equations of Motion

The equations of motion of a spacecraft, considered as a point mass, flying in a stationary atmosphere are given by[6],

$$\begin{aligned} \dot{\bar{r}} &= \bar{v} \\ \dot{\bar{v}} &= \frac{\bar{T}}{m} + \frac{\bar{A}}{m} + \bar{g} \\ \dot{m} &= -\frac{T}{c} \end{aligned} \tag{1}$$

where, $\bar{r}$ is the radius vector, $\bar{v}$ the velocity, m the mass, $\bar{T}$ the thrust vector $(0 \le T \le T_{max})$, $\bar{A}$ the aerodynamic force and $\bar{g}$ is gravity. Dot represents differentiation with respect to time.

We make the following assumptions regarding the nature of the various forces. The thrust direction is aligned with the velocity vector, the gravitational field is central and inverse-square, atmospheric density is a function of radial distance only and the lift and the drag coefficients are related by a parabolic drag polar. Thus, we have

$$\begin{aligned} \bar{T} &= T\hat{e}_v \\ \bar{g} &= -g(r)\hat{e}_r \\ \bar{A} &= \bar{D} + \bar{L} \\ \bar{D} &= -\frac{1}{2}\rho(r)SC_D v\bar{v} \\ \bar{L} &= \frac{1}{2}\rho(r)Sv^2\hat{e}_l \\ C_D &= C_{D0} + KC_L{}^2 \end{aligned} \tag{2}$$

where, $\bar{D}$ is the drag, $\bar{L}$ is the lift, ρ is the density of the atmosphere, S is the effective surface area of the vehicle, C_D and C_L are the drag and the lift coefficients respectively and $\hat{e}_r$, $\hat{e}_v$ and $\hat{e}_l$ are the unit vectors along the radius, the velocity and the lift directions respectively.

We assume that the spacecraft has a plane of symmetry and using the bank angle, which is the angle between the orbital plane and the spacecraft symmetry plane, we can write,

$$\hat{e}_l = \cos\sigma\hat{e}_c + \sin\sigma\hat{e}_s \tag{3}$$

where $\hat{e}_c$ and $\hat{e}_s$ are two orthogonal unit vectors specifying the lift vector direction and each in turn is orthogonal to $\hat{e}_v$ and σ is the bank angle.

To simplify computations we introduce the following transformation of the variables.

$$\begin{aligned} q &= -c\log(\frac{m}{m_0}) \\ \tau &= \frac{T}{m} \\ \lambda_c &= \frac{C_L}{C_L{}^*}\cos\sigma \\ \lambda_s &= \frac{C_L}{C_L{}^*}\sin\sigma \end{aligned} \tag{4}$$

where, $C_D{}^* = 2C_{D_0}$ and $E^* = \frac{C_L{}^*}{C_D{}^*}$ correspond to the maximum lift to drag ratio conditions. Substituting (4) in (1), we get

$$\begin{aligned} \dot{\bar{r}} &= \bar{v} \\ \dot{\bar{v}} &= \tau\hat{e}_v + \bar{B} \\ \dot{q} &= \tau \end{aligned} \tag{5}$$

where

$$\begin{aligned} \bar{B} = & -\frac{\rho SC_{D0}}{2}\frac{e^{q/c}}{m_0}v^2(1+\lambda_c{}^2+\lambda_s{}^2)\hat{e}_v \\ & +\rho SC_{D0}\frac{e^{q/c}}{m_0}E^*v^2(\lambda_c\hat{e}_c+\lambda_s\hat{e}_s) - g\hat{e}_r \end{aligned} \tag{6}$$

The Optimal Control Problem

It is desired to find the controls λ_c, λ_s and τ, from a compact control set, that maximize a given function of the states at the final time while satisfying certain initial and final conditions. Thus we have a Mayer type problem. Introducing the adjoint vector $\bar{p} = (\bar{p}_r \;\; \bar{p}_v \;\; p_q)^T$, we form the Hamiltonian

Hamiltonian

$$H = \bar{p}_r^T\bar{v} + \bar{p}_v^T\bar{B} + \tau\left(\bar{p}_v^T\frac{\bar{v}}{v} + p_q\right) \tag{7}$$

Adjoint Equations

The adjoint equations are given by

$$\begin{aligned} \dot{\bar{p}}_r &= -\frac{\partial H}{\partial \bar{r}} = -\left(\frac{\partial \bar{B}}{\partial \bar{r}}\right)^T \bar{p}_v \\ \dot{\bar{p}}_v &= -\frac{\partial H}{\partial \bar{v}} \\ &= -\bar{p}_r - \tau\left(\frac{\partial \hat{e}_v}{\partial \bar{v}}\right)^T\bar{p}_v - \left(\frac{\partial \bar{B}}{\partial \bar{v}}\right)^T\bar{p}_v \\ \dot{p}_q &= -\frac{\partial H}{\partial q} = -\left(\frac{\partial \bar{B}}{\partial q}\right)^T\bar{p}_v \end{aligned} \tag{8}$$

Integrals of Motion

The following integrals of motion exist[6] due to the problem being autonomous and due to the assumed spherical symmetry.

$$\begin{aligned} H &= c_0 \\ (\bar{r} \times \bar{p}_r) + (\bar{v} \times \bar{p}_v) &= \bar{c} \end{aligned} \quad (9)$$

Lift and Bank Controls

Assuming that λ_c and λ_s are interior optimizers, the Pontryagin's maximum principle gives the following first and second order necessary conditions.

$$\begin{aligned} \frac{\partial H}{\partial \lambda_c} &= 0 \implies \lambda_c = \frac{E^* v(\bar{p}_v^T \hat{e}_c)}{(\bar{p}_v^T \bar{v})} \\ \frac{\partial H}{\partial \lambda_s} &= 0 \implies \lambda_s = \frac{E^* v(\bar{p}_v^T \hat{e}_s)}{(\bar{p}_v^T \bar{v})} \\ \frac{\partial^2 H}{\partial {\lambda_c}^2} &= \frac{\partial^2 H}{\partial {\lambda_s}^2} \\ &= -\rho S C_{D0} \frac{e^{q/c}}{m_0} v(\bar{p}_v^T \bar{v}) < 0 \implies (\bar{p}_v^T \bar{v}) > 0 \end{aligned} \quad (10)$$

Optimal Thrust Magnitude

Let us define the switching function

$$\Phi = \bar{p}_v^T \hat{e}_v + p_q \quad (11)$$

The optimal thrust magnitude is $\tau = \tau_{max}$ if $\Phi > 0$, $\tau = 0$ if $\Phi < 0$ and τ has an intermediate value if $\Phi = 0$. If the final case is true for a finite interval of time, we have a singular arc and the switching function must be differentiated with respect to time in order to obtain τ explicitly. We express these derivatives in terms of Lie Brackets by the method described in the Appendix. All the required tensors and Jacobian matrices are also derived in the Appendix. The results are as follows.

$$\dot{\Phi} = -\bar{p}_r^T \hat{e}_v + \bar{p}_v^T \bar{C} = 0 \quad (12)$$

where,

$$\begin{aligned} \bar{C} &= \frac{\rho S C_{D0}}{2} \frac{e^{q/c}}{m_0} (1 + {\lambda_c}^2 + {\lambda_s}^2) \left(2 + \frac{v}{c}\right) \bar{v} \\ &\quad -\rho S C_{D0} \frac{e^{q/c}}{m_0} E^* v \left(1 + \frac{v}{c}\right) (\lambda_c \hat{e}_c + \lambda_s \hat{e}_s) \\ &\quad -\frac{g}{vr} \bar{r} + g \frac{(\bar{v}^T \bar{r})}{v^3 r} \bar{v} \end{aligned} \quad (13)$$

and

$$\ddot{\Phi} = \mathcal{A} + \tau \mathcal{B} = 0 \quad (14)$$

where,

$$\begin{aligned} \mathcal{A} &= \bar{p}_r^T \bar{Y} + \bar{p}_v^T \bar{Z} \\ &\quad + \begin{bmatrix} \bar{p}_v^T \bar{M}_c & \bar{p}_v^T \bar{M}_s \end{bmatrix} \begin{bmatrix} \frac{\partial^2 H}{\partial {\lambda_c}^2} & 0 \\ 0 & \frac{\partial^2 H}{\partial {\lambda_s}^2} \end{bmatrix}^{-1} \begin{bmatrix} -\bar{p}_r^T \left(\frac{\partial \bar{B}}{\partial \lambda_c}\right) + \bar{p}_v^T \bar{F}_c \\ -\bar{p}_r^T \left(\frac{\partial \bar{B}}{\partial \lambda_s}\right) + \bar{p}_v^T \bar{F}_s \end{bmatrix} \end{aligned} \quad (15)$$

$$\begin{aligned} \mathcal{B} &= \bar{p}_v^T \bar{W} \\ &\quad + \begin{bmatrix} \bar{p}_v^T \bar{M}_c & \bar{p}_v^T \bar{M}_s \end{bmatrix} \begin{bmatrix} \frac{\partial^2 H}{\partial {\lambda_c}^2} & 0 \\ 0 & \frac{\partial^2 H}{\partial {\lambda_s}^2} \end{bmatrix}^{-1} \begin{bmatrix} \bar{p}_v^T \bar{M}_c \\ \bar{p}_v^T \bar{M}_s \end{bmatrix} \end{aligned} \quad (16)$$

Various vector quantities in the above equations are given below.

$$\frac{\partial \bar{B}}{\partial \lambda_c} = \rho S C_{D0} \frac{e^{q/c}}{m_0} (-\lambda_c v \bar{v} + E^* v^2 \hat{e}_c) \quad (17)$$

$$\frac{\partial \bar{B}}{\partial \lambda_s} = \rho S C_{D0} \frac{e^{q/c}}{m_0} (-\lambda_s v \bar{v} + E^* v^2 \hat{e}_s) \quad (18)$$

$$\bar{M}_c = \rho S C_{D0} \frac{e^{q/c}}{m_0} \left[-\lambda_c \left(2 + \frac{v}{c}\right) \bar{v} + E^* v \left(1 + \frac{v}{c}\right) \hat{e}_c\right]$$

$$\bar{M}_s = \rho S C_{D0} \frac{e^{q/c}}{m_0} \left[-\lambda_s \left(2 + \frac{v}{c}\right) \bar{v} + E^* v \left(1 + \frac{v}{c}\right) \hat{e}_s\right]$$

$$\begin{aligned} \bar{F}_c &= \rho_r S C_{D0} \frac{e^{q/c}}{m_0} \frac{v}{r} (\bar{v}^T \bar{r}) [-\lambda_c \bar{v} + E^* v \hat{e}_c] \\ &\quad -(\rho S C_{D0})^2 \left(\frac{e^{q/c}}{m_0}\right)^2 (1 + {\lambda_c}^2 + {\lambda_s}^2) \frac{E^*}{2} v^3 \hat{e}_c \\ &\quad +(\rho S C_{D0})^2 \left(\frac{e^{q/c}}{m_0}\right)^2 \lambda_c E^* v^3 (\lambda_c \hat{e}_c + \lambda_s \hat{e}_s) \\ &\quad +\rho S C_{D0} \frac{e^{q/c}}{m_0} g \lambda_c \left(\frac{v}{r} \bar{r} + \frac{(\bar{v}^T \bar{r})}{vr} \bar{v}\right) \\ &\quad -\rho S C_{D0} \frac{e^{q/c}}{m_0} g E^* \left[2 \frac{(\bar{v}^T \bar{r})}{r} \hat{e}_c - \frac{(\bar{r}^T \hat{e}_c)}{r} \bar{v}\right] \end{aligned} \quad (19)$$

$$\begin{aligned} \bar{F}_s &= \rho_r S C_{D0} \frac{e^{q/c}}{m_0} \frac{v}{r} (\bar{v}^T \bar{r}) [-\lambda_s \bar{v} + E^* v \hat{e}_s] \\ &\quad -(\rho S C_{D0})^2 \left(\frac{e^{q/c}}{m_0}\right)^2 (1 + {\lambda_c}^2 + {\lambda_s}^2) \frac{E^*}{2} v^3 \hat{e}_s \\ &\quad +(\rho S C_{D0})^2 \left(\frac{e^{q/c}}{m_0}\right)^2 \lambda_s E^* v^3 (\lambda_c \hat{e}_c + \lambda_s \hat{e}_s) \\ &\quad +\rho S C_{D0} \frac{e^{q/c}}{m_0} g \lambda_s \left(\frac{v}{r} \bar{r} + \frac{(\bar{v}^T \bar{r})}{vr} \bar{v}\right) \\ &\quad -\rho S C_{D0} \frac{e^{q/c}}{m_0} g E^* \left[2 \frac{(\bar{v}^T \bar{r})}{r} \hat{e}_s - \frac{(\bar{r}^T \hat{e}_s)}{r} \bar{v}\right] \end{aligned} \quad (20)$$

where ρ_r is the partial derivative of ρ with respect to r.

$$\begin{aligned} \bar{W} &= \frac{\rho S C_{D0}}{2} \frac{e^{q/c}}{m_0} (1 + {\lambda_c}^2 + {\lambda_s}^2) \left(\frac{2}{v} + \frac{4}{c} + \frac{v}{c^2}\right) \bar{v} \\ &\quad -\rho S C_{D0} \frac{e^{q/c}}{m_0} E^* \frac{v}{c} \left(2 + \frac{v}{c}\right) (\lambda_c \hat{e}_c + \lambda_s \hat{e}_s) \\ &\quad +\frac{2g}{v^2 r} \bar{r} - 2g \frac{(\bar{v}^T \bar{r})}{v^4 r} \bar{v} \end{aligned} \quad (21)$$

$$\bar{Y} = -\frac{\rho S C_{D0}}{2}\frac{e^{q/c}}{m_0}(1+\lambda_c^2+\lambda_s^2)\left(2+\frac{v}{c}\right)\bar{v} + \rho S C_{D0}\frac{e^{q/c}}{m_0}E^*\frac{v^2}{c}(\lambda_c\hat{e}_c+\lambda_s\hat{e}_s) + \frac{2g}{vr}\bar{r} - 2g\frac{(\bar{v}^T\bar{r})}{v^3 r}\bar{v} \tag{22}$$

and finally,

$$\bar{Z} = Z_1\bar{v} + Z_2\bar{r} + Z_3(\lambda_c\hat{e}_c+\lambda_s\hat{e}_s) \tag{23}$$

where,

$$\begin{aligned} Z_1 = {} & \rho_r\frac{S C_{D0}}{2}\frac{e^{q/c}}{m_0}(1+\lambda_c^2+\lambda_s^2)\left(1+\frac{v}{c}\right)\frac{(\bar{v}^T\bar{r})}{r} \\ & +2\left(\frac{\rho S C_{D0}}{2}\right)^2\left(\frac{e^{q/c}}{m_0}\right)^2(1+\lambda_c^2+\lambda_s^2)^2 v \\ & +\frac{\rho S C_{D0}}{2}\frac{e^{q/c}}{m_0}(1+\lambda_c^2+\lambda_s^2)g\left(2-\frac{v}{c}\right)\frac{(\bar{v}^T\bar{r})}{v^2 r} \\ & -\rho S C_{D0}\frac{e^{q/c}}{m_0}E^*\frac{g}{vr}\left(1+\frac{v}{c}\right)\left[\lambda_c(\bar{r}^T\hat{e}_c)+\lambda_s(\bar{r}^T\hat{e}_s)\right] \\ & -\frac{g^2}{v^3}+3g^2\frac{\left((\bar{v}^T\bar{r})\right)^2}{v^5 r^2}-\frac{g}{vr}+\frac{\left((\bar{v}^T\bar{r})\right)^2}{v^3 r^2}\left(-\frac{g}{r}+g_r\right) \end{aligned} \tag{24}$$

$$Z_2 = -\frac{\rho S C_{D0}}{2}\frac{e^{q/c}}{m_0}(1+\lambda_c^2+\lambda_s^2)\frac{g}{r}\left(4+\frac{v}{c}\right) - 2g^2\frac{(\bar{v}^T\bar{r})}{v^3 r^2} + 2\frac{(\bar{v}^T\bar{r})}{v r^2}\left(\frac{g}{r}-g_r\right) \tag{25}$$

$$\begin{aligned} Z_3 = {} & -\rho_r S C_{D0}\frac{e^{q/c}}{m_0}E^*\frac{v^2}{c}\frac{(\bar{v}^T\bar{r})}{r} \\ & -(\rho S C_{D0})^2\left(\frac{e^{q/c}}{m_0}\right)^2(1+\lambda_c^2+\lambda_s^2)E^* v^2 \\ & +2\rho S C_{D0}\frac{e^{q/c}}{m_0}E^* g\left(1+\frac{v}{c}\right)\frac{(\bar{v}^T\bar{r})}{vr} \end{aligned} \tag{26}$$

Special Case 1 : Flight in the Vertical Plane

The vertical plane is the plane containing $\bar{r}$ and $\bar{v}$ and by definition of the bank angle and the unit vectors $\hat{e}_c$ and $\hat{e}_s$, the former unit vector is in the vertical plane while the latter is orthogonal to it. Thus, in the expressions for $\bar{B}$, $\bar{C}$, $\bar{F}_c$, $\bar{M}_c$, $\bar{W}$, $\bar{Y}$ and $\bar{Z}$ above, we set $\lambda_s \equiv 0$ and drop all the terms containing $\hat{e}_s$. Similarly we drop the vectors $\bar{F}_s$ and $\bar{M}_s$ since they correspond to the time derivative of λ_s. This gives us the expression for the thrust magnitude on a singular arc for flight in the vertical plane.

A note of caution is due here. We cannot obtain the case of horizontal flight by simply setting terms corresponding to $\lambda_c \equiv 0$ similar to the vertical flight case above. The main reason for the lack of symmetry is the expression for gravity. We have two types of symmetry in the problem. The equations of motion and the gravity term have a spherical symmetry about the radius vector where as the atmospheric force has a spherical symmetry about the velocity vector. As long as these two are uncoupled, as in the case of general three dimensional flight and flight in the vertical plane, we can use the symmetry properties. But to obtain the case of horizontal flight (at constant altitude), we cannot simply set $\lambda_c \equiv 0$ because then the lift force in the vertical direction is zero and any difference between the gravity and the centrifugal force causes the spacecraft to change its altitude. Instead we must impose a constraint that the sum of the forces in the vertical direction is zero. This gives a relation between the lift and the bank controls and the general expression we derived for the thrust magnitude is no longer applicable for this constrained optimization problem. Only if we ignore gravity, as is sometimes done for flight at low altitude in dense atmosphere, can we set $\lambda_c \equiv 0$ to obtain the horizontal flight case.

Special Case 2 : Circular Orbit with Zero Lift

Historically, an interesting question regarding the singular arc solution is that whether the cruise solution, obtained by setting the thrust equal to the drag, coincides with the singular one. Since it is very difficult to verify whether the $\ddot{\Phi}$ equation for the general three dimensional flight and for flight in a vertical plane yields the solution that the thrust is equal to the drag (though it seems highly unlikely on a quick inspection), we consider yet another simple case, namely, the flight of the spacecraft in a circular orbit with zero lift. Here we compare the circular solution to the Singular arc solution and see if they coincide. First of all we set $\lambda_c = \lambda_s = 0$ in the equations of motion as well as all the derivatives of the switching function derived above.

Solution Along a Circular Orbit

On a circular orbit, the radius and the velocity are constant. Thus,

$$\begin{aligned} (\bar{r}^T\bar{r}) &= r^2 = constant \\ (\bar{v}^T\bar{v}) &= v^2 = constant \end{aligned} \tag{27}$$

Differentiating the first relation, we get the following relations among the states and controls:

$$\begin{aligned} (\bar{v}^T\bar{r}) &= 0 \\ (\dot{\bar{v}}^T\bar{r}) + (\bar{v}^T\bar{v}) &= 0 \end{aligned} \tag{28}$$

Equation (28), expressed in scalar form gives

$$\begin{aligned} \sin\gamma &\equiv 0 \\ V^2 - rg &= 0 \end{aligned} \qquad (29)$$

the second one being the force balance equation along the radial direction. Upon differentiation, the second relation in (27) gives

$$\tau = \frac{\rho S C_{D0}}{2}\frac{e^{q/c}}{m_0}v^2 \qquad (30)$$

which says that the thrust is equal to the drag thus balancing the forces along the velocity.

Solution Along a Singular Arc

Now we suppose that the state variables on the circular arc and the singular arc coincide at some point. Without loss of generality, we take that as our initial point. Note that if the singular arc yields a solution for the thrust which is equal to the drag at this point, the derivatives of $\bar{r}$ and $\bar{v}$ remain zero and we still remain on the circular orbit. Thus as long as the thrust magnitude, which is now dependent on the adjoint variables, does not change, the singular arc and the circular arc remain coincident. Thus we get a condition on the adjoint variables under which this happens. Note that the mass of the spacecraft continues to decrease since we are thrusting continuously.

In the singular arc solution, we set

$$\begin{aligned} (\bar{v}^T\bar{r}) &= 0 \\ r &= constant \\ v &= constant \\ v^2 &= rg \end{aligned} \qquad (31)$$

Thus we get

$$\begin{aligned} H^* = C_0 &= (\bar{p}_r^T\bar{v}) - \frac{\rho S C_{D0}}{2}\frac{e^{q/c}}{m_0}v(\bar{p}_v^T\bar{v}) - \frac{g}{r}(\bar{p}_v^T\bar{r}) \\ \Phi = 0 &= (\bar{p}_v^T\hat{e}_v) + p_q \qquad (32) \\ \dot{\Phi} = 0 &= -(\bar{p}_r^T\hat{e}_v) + \frac{\rho S C_{D0}}{2}\frac{e^{q/c}}{m_0}\left(2+\frac{v}{c}\right)(\bar{p}_v^T\bar{v}) \\ &\quad -\frac{g}{vr}(\bar{p}_v^T\bar{r}) \\ \ddot{\Phi} = 0 &= \mathcal{A} + \tau\mathcal{B} \end{aligned}$$

where

$$\begin{aligned} \mathcal{A} &= -\frac{\rho S C_{D0}}{2}\frac{e^{q/c}}{m_0}\left(2+\frac{v}{c}\right)(\bar{p}_r^T\bar{v}) + \frac{2g}{vr}(\bar{p}_r^T\bar{r}) \\ &\quad +(\bar{p}_v^T\bar{v})\left[2\left(\frac{\rho S C_{D0}}{2}\right)^2\left(\frac{e^{q/c}}{m_0}\right)^2 v - \frac{g}{vr} - \frac{g^2}{v^3}\right] \\ &\quad -\frac{\rho S C_{D0}}{2}\frac{e^{q/c}}{m_0}\left(4+\frac{v}{c}\right)\frac{g}{r}(\bar{p}_v^T\bar{r}) \end{aligned} \qquad (33)$$

and

$$\mathcal{B} = \frac{\rho S C_{D0}}{2}\frac{e^{q/c}}{m_0}\left(\frac{2}{v}+\frac{4}{c}+\frac{v}{c^2}\right)(\bar{p}_v^T\bar{v}) + \frac{2g}{v^2 r}(\bar{p}_v^T\bar{r}) \qquad (34)$$

Note that the first equation in (32) is the Hamiltonian integral due to the equations of motion being autonomous.

After some algebraic manipulation, we can express $(\bar{p}_v^T\bar{r})$ and $(\bar{p}_r^T\bar{v})$ as functions of $(\bar{p}_v^T\bar{v})$ using (32) as given below.

$$\begin{aligned} \frac{g}{r}(\bar{p}_v^T\bar{r}) &= \frac{1}{2}\frac{\rho S C_{D0}}{2}\frac{e^{q/c}}{m_0}v\left(1+\frac{v}{c}\right)(\bar{p}_v^T\bar{v}) - \frac{c_0}{2} \qquad (35) \\ (\bar{p}_r^T\bar{v}) &= \frac{c_0}{2} + \frac{1}{2}\frac{\rho S C_{D0}}{2}\frac{e^{q/c}}{m_0}v\left(3+\frac{v}{c}\right)(\bar{p}_v^T\bar{v}) \qquad (36) \end{aligned}$$

Substituting (35) and (36) in (33) and (34), we get

$$\begin{aligned} \mathcal{A} &= \frac{2g}{vr}(\bar{p}_r^T\bar{r}) - \left(\frac{g}{vr}+\frac{g^2}{v^3}\right)(\bar{p}_v^T\bar{v}) \\ &\quad +\frac{\rho S C_{D0}}{2}\frac{e^{q/c}}{m_0}c_0 \\ &\quad -\left(\frac{\rho S C_{D0}}{2}\right)^2\left(\frac{e^{q/c}}{m_0}\right)^2 v\left(3+\frac{5v}{c}+\frac{v^2}{c^2}\right)(\bar{p}_v^T\bar{v}) \end{aligned} \qquad (37)$$

and

$$\mathcal{B} = -\frac{c_0}{v^2} + \frac{\rho S C_{D0}}{2}\frac{e^{q/c}}{m_0}\left(3+\frac{5v}{c}+\frac{v^2}{c^2}\right)\frac{(\bar{p}_v^T\bar{v})}{v} \qquad (38)$$

Substituting $v^2 = rg$ in the second term of (37), we can write the second derivative of the switching function as

$$\begin{aligned} \ddot{\Phi} &= \mathcal{A} + \tau\mathcal{B} \\ &= \frac{2g}{vr}\left[(\bar{p}_r^T\bar{r}) - (\bar{p}_v^T\bar{v})\right] \\ &\quad +\left[\frac{\rho S C_{D0}}{2}\frac{e^{q/c}}{m_0}\left(3+\frac{5v}{c}+\frac{v^2}{c^2}\right)\frac{(\bar{p}_v^T\bar{v})}{v} - \frac{c_0}{v^2}\right] \\ &\quad \left(\tau - \frac{\rho S C_{D0}}{2}\frac{e^{q/c}}{m_0}v^2\right) \end{aligned} \qquad (39)$$

Clearly $\tau = \frac{\rho S C_{D0}}{2}\frac{e^{q/c}}{m_0}v^2$ is a solution of the above equation if and only if

$$(\bar{p}_r^T\bar{r}) - (\bar{p}_v^T\bar{v}) \equiv 0 \quad or \quad g \equiv 0 \qquad (40)$$

By differentiating the first condition and substituting from the state and the adjoint equations we get

$$\begin{aligned} \frac{d}{dt}\left[(\bar{p}_r^T\bar{r}) - (\bar{p}_v^T\bar{v})\right] &= (\bar{p}_v^T\bar{v})\frac{e^{q/c}}{m_0}\frac{S C_{D0}}{2}v\,(\rho_r r + \rho) \\ &\quad + g_r(\bar{p}_v^T\bar{r}) - c_0 \end{aligned} \qquad (41)$$

which is non zero on a circular orbit. Thus the first condition does not yield a solution of thrust being equal to the drag. On the other hand, the second condition does yield this solution and the corresponding singular arc is optimal if

$$\frac{\rho S C_{D0}}{2}\frac{e^{q/c}}{m_0}\left(3+\frac{5v}{c}+\frac{v^2}{c^2}\right)\frac{(\bar{p}_v^T\bar{v})}{v} > \frac{c_0}{v^2} \tag{42}$$

Discussion

The most striking feature of the Lie Bracket solution, is the systematic procedure that is involved in applying it. Thus one can even use a symbolic manipulator to evaluate all the necessary brackets. The contributions of the non-singular solutions can be easily identified in the resulting expressions. Thus one can easily obtain sub-cases where one or more of the non-singular solutions is not present, as we have done above to obtain the special cases of vertical and non-lifting flights. The term containing the singular solution (linearly) in the second order derivative (represented as $\mathcal{B}$) appears separately from the remaining terms ($\mathcal{A}$). Thus if one only wants to check the Generalized Legendre-Clebsch condition given by $\mathcal{B} > 0$ for first order singular arcs, it is much easier to use the Lie Bracket solution than the conventional method where one needs to group all the terms involved by inspection.

$\bar{F}_s$, $\bar{M}_c$ and $\bar{M}_s$ are symmetric with respect to λ_c and λ_s. Thus we only need to derive one of each pair of vectors and by simply interchanging the subscripts c and s, we get the other vector. This is facilitated by the way we set up the equations of motion and it saves us a lot of computational effort. It can be shown that this symmetry is lost in the conventional derivation. The solution obtained here has been verified and validated by the conventional method[7].

Even though we gave the complete solution for the thrust magnitude, one should note that it contains both the state and the adjoint variables. Since we have seven adjoint variables but only six relations among them as given by Eqs. (9), (11) and (12), the problem is not completely integrable.

The vectorial formulation used here is for the sake of brevity and generality. One can start with the equations of motion for a given set of variables and derive the expression directly. The solution given in the Appendix is quite general and is applicable whenever the equations of motion have the form specified there.

Appendix

Extended Lie Bracket Solution[4]

Let the equations of motion be of the form

$$\dot{\mathbf{x}} = \mathbf{f}_0(\mathbf{x},\mathbf{v}) + u\,\mathbf{f}_1(\mathbf{x}) \tag{43}$$

where $\mathbf{x},\mathbf{f}_0,\mathbf{f}_1 \in R^n$, $\mathbf{v} \in R^m$ and $u \in R^1$. Then the singular solution for u is given as follows.

$$\Phi = \mathbf{p}^T\mathbf{f}_1 \equiv 0 \tag{44}$$

$$\dot{\Phi} = \mathbf{p}^T(\mathbf{f}_0,\mathbf{f}_1) = 0 \tag{45}$$

Where, the Lie bracket $(\mathbf{f}_0,\mathbf{f}_1)$ is defined as

$$(\mathbf{f}_0,\mathbf{f}_1) = \left(\frac{\partial \mathbf{f}_1}{\partial \mathbf{x}}\right)\mathbf{f}_0 - \left(\frac{\partial \mathbf{f}_0}{\partial \mathbf{x}}\right)\mathbf{f}_1 \tag{46}$$

and $\left(\frac{\partial \mathbf{f}_0}{\partial \mathbf{x}}\right)$ and $\left(\frac{\partial \mathbf{f}_1}{\partial \mathbf{x}}\right)$ are the $(n\times n)$ Jacobian matrices.

$$\ddot{\Phi} = \mathcal{A} + u\mathcal{B} = 0 \tag{47}$$

where,

$$\mathcal{A} = \mathbf{p}^T(\mathbf{f}_0,(\mathbf{f}_0,\mathbf{f}_1)) + \left[\mathbf{p}^T\left(\mathbf{f}_1,\frac{\partial \mathbf{f}_0}{\partial \mathbf{v}}\right)\right]^T \left(\frac{\partial^2 H}{\partial \mathbf{v}^2}\right)^{-1}\left[\mathbf{p}^T\left(\mathbf{f}_0,\frac{\partial \mathbf{f}_0}{\partial \mathbf{v}}\right)\right]$$

$$\mathcal{B} = \mathbf{p}^T(\mathbf{f}_1,(\mathbf{f}_0,\mathbf{f}_1)) + \left[\mathbf{p}^T\left(\mathbf{f}_1,\frac{\partial \mathbf{f}_0}{\partial \mathbf{v}}\right)\right]^T \left(\frac{\partial^2 H}{\partial \mathbf{v}^2}\right)^{-1}\left[\mathbf{p}^T\left(\mathbf{f}_1,\frac{\partial \mathbf{f}_0}{\partial \mathbf{v}}\right)\right]$$

where we define $\mathbf{p}^T\left(\mathbf{f}_0,\frac{\partial \mathbf{f}_0}{\partial \mathbf{v}}\right)$ as a $(m\times 1)$ column vector such that

$$\mathbf{p}^T\left(\mathbf{f}_0,\frac{\partial \mathbf{f}_0}{\partial \mathbf{v}}\right)_i = \mathbf{p}^T\left(\mathbf{f}_0,\frac{\partial \mathbf{f}_0}{\partial v_i}\right) \tag{48}$$

Evaluation of the Various Lie Brackets

By comparing (5) with (43), we note that

$$\mathbf{x} = (\bar{r},\bar{v},q)^T \tag{49}$$

$$\mathbf{p} = (\bar{p}_r,\bar{p}_v,p_q)^T \tag{50}$$

$$\mathbf{v} = \begin{bmatrix}\lambda_c\\ \lambda_s\end{bmatrix} \tag{51}$$

$$u = \tau \tag{52}$$

$$\mathbf{f}_0 = \begin{bmatrix}\bar{v}\\ \bar{B}\\ 0\end{bmatrix} \tag{53}$$

and

$$\mathbf{f}_1 = \begin{bmatrix}0\\ \hat{e}_v\\ 1\end{bmatrix} \tag{54}$$

The various Jacobians and Lie Brackets are evaluated as given below.

$$\frac{\partial \mathbf{f}_0}{\partial \mathbf{x}} = \begin{bmatrix} \mathbf{O}_3 & \mathbf{I}_3 & Oc_3 \\ \frac{\partial \bar{B}}{\partial \bar{r}} & \frac{\partial \bar{B}}{\partial \bar{v}} & \frac{\partial \bar{B}}{\partial q} \\ Or_3 & Or_3 & 0 \end{bmatrix} \tag{55}$$

and

$$\frac{\partial \mathbf{f}_1}{\partial \mathbf{x}} = \begin{bmatrix} \mathbf{O}_3 & \mathbf{O}_3 & Oc_3 \\ \mathbf{O}_3 & \frac{\partial \hat{e}_v}{\partial \bar{v}} & Oc_3 \\ Or_3 & Or_3 & 0 \end{bmatrix} \tag{56}$$

where $\mathbf{O}_3$ is the 3×3 null matrix, $\mathbf{I}_3$ is the 3×3 identity matrix, Oc_3 is the 3×1 null matrix (column vector) and Or_3 is the 1×3 null matrix (row vector). Thus we have

$$(\mathbf{f}_0, \mathbf{f}_1) = \begin{bmatrix} -\hat{e}_v \\ \bar{C} \\ 0 \end{bmatrix} \tag{57}$$

where

$$\bar{C} = \left(\frac{\partial \hat{e}_v}{\partial \bar{v}}\right) \bar{B} - \left(\frac{\partial \bar{B}}{\partial \bar{v}}\right) \hat{e}_v - \left(\frac{\partial \bar{B}}{\partial q}\right) \tag{58}$$

We can also write

$$\frac{\partial \mathbf{f}_0}{\partial \mathbf{v}_0} = \begin{bmatrix} \frac{\partial \mathbf{f}_0}{\partial \lambda_c} & \frac{\partial \mathbf{f}_0}{\partial \lambda_s} \end{bmatrix} \tag{59}$$

where

$$\frac{\partial \mathbf{f}_0}{\partial \lambda_c} = \begin{bmatrix} Oc_3 \\ \frac{\partial \bar{B}}{\partial \lambda_c} \\ 0 \end{bmatrix} \tag{60}$$

and

$$\frac{\partial \mathbf{f}_0}{\partial \lambda_s} = \begin{bmatrix} Oc_3 \\ \frac{\partial \bar{B}}{\partial \lambda_s} \\ 0 \end{bmatrix} \tag{61}$$

We can now calculate the next set of Jacobians for the higher order Lie Brackets as follows.

$$\frac{\partial}{\partial \mathbf{x}}(\mathbf{f}_0, \mathbf{f}_1) = \begin{bmatrix} \mathbf{O}_3 & -\frac{\partial \hat{e}_v}{\partial \bar{v}} & Oc_3 \\ \frac{\partial \bar{C}}{\partial \bar{r}} & \frac{\partial \bar{C}}{\partial \bar{v}} & \frac{\partial \bar{C}}{\partial q} \\ Or_3 & Or_3 & 0 \end{bmatrix} \tag{62}$$

$$\frac{\partial}{\partial \mathbf{x}}\left(\frac{\partial \mathbf{f}_0}{\partial \lambda_c}\right) = \begin{bmatrix} \mathbf{O}_3 & \mathbf{O}_3 & Oc_3 \\ \frac{\partial}{\partial \bar{r}}\left(\frac{\partial \bar{B}}{\partial \lambda_c}\right) & \frac{\partial}{\partial \bar{v}}\left(\frac{\partial \bar{B}}{\partial \lambda_c}\right) & \frac{\partial}{\partial q}\left(\frac{\partial \bar{B}}{\partial \lambda_c}\right) \\ Or_3 & Or_3 & 0 \end{bmatrix}$$

and

$$\frac{\partial}{\partial \mathbf{x}}\left(\frac{\partial \mathbf{f}_0}{\partial \lambda_s}\right) = \begin{bmatrix} \mathbf{O}_3 & \mathbf{O}_3 & Oc_3 \\ \frac{\partial}{\partial \bar{r}}\left(\frac{\partial \bar{B}}{\partial \lambda_s}\right) & \frac{\partial}{\partial \bar{v}}\left(\frac{\partial \bar{B}}{\partial \lambda_s}\right) & \frac{\partial}{\partial q}\left(\frac{\partial \bar{B}}{\partial \lambda_s}\right) \\ Or_3 & Or_3 & 0 \end{bmatrix}$$

Thus we can calculate the higher order Lie Brackets as:

$$(\mathbf{f}_0, (\mathbf{f}_0, \mathbf{f}_1)) = \begin{bmatrix} \bar{Y} \\ \bar{Z} \\ 0 \end{bmatrix} \tag{63}$$

where,

$$\bar{Y} = -\left(\frac{\partial \hat{e}_v}{\partial \bar{v}}\right) \bar{B} - \bar{C} \tag{64}$$

and

$$\bar{Z} = \left(\frac{\partial \bar{C}}{\partial \bar{r}}\right) \bar{v} + \left(\frac{\partial \bar{C}}{\partial \bar{v}}\right) \bar{B} + \left(\frac{\partial \bar{B}}{\partial \bar{r}}\right) \hat{e}_v - \left(\frac{\partial \bar{B}}{\partial \bar{v}}\right) \bar{C} \tag{65}$$

Similarly, we have

$$(\mathbf{f}_1, (\mathbf{f}_0, \mathbf{f}_1)) = \begin{bmatrix} -\left(\frac{\partial \hat{e}_v}{\partial \bar{v}}\right) \hat{e}_v \\ \bar{W} \\ 0 \end{bmatrix}, \tag{66}$$

where,

$$\bar{W} = \left(\frac{\partial \bar{C}}{\partial \bar{v}}\right) \hat{e}_v + \left(\frac{\partial \bar{C}}{\partial q}\right) - \left(\frac{\partial \hat{e}_v}{\partial \bar{v}}\right) \bar{C}, \tag{67}$$

$$\left(\mathbf{f}_0, \frac{\partial \mathbf{f}_0}{\partial \lambda_c}\right) = \begin{bmatrix} -\left(\frac{\partial \bar{B}}{\partial \lambda_c}\right) \\ \bar{F}_c \\ 0 \end{bmatrix}, \tag{68}$$

where,

$$\bar{F}_c = \left[\frac{\partial}{\partial \bar{r}}\left(\frac{\partial \bar{B}}{\partial \lambda_c}\right)\right] \bar{v} + \left[\frac{\partial}{\partial \bar{v}}\left(\frac{\partial \bar{B}}{\partial \lambda_c}\right)\right] \bar{B} - \left(\frac{\partial \bar{B}}{\partial \bar{v}}\right)\left(\frac{\partial \bar{B}}{\partial \lambda_c}\right),$$

$$\left(\mathbf{f}_0, \frac{\partial \mathbf{f}_0}{\partial \lambda_s}\right) = \begin{bmatrix} -\left(\frac{\partial \bar{B}}{\partial \lambda_s}\right) \\ \bar{F}_s \\ 0 \end{bmatrix}, \tag{69}$$

where,

$$\bar{F}_s = \left[\frac{\partial}{\partial \bar{r}}\left(\frac{\partial \bar{B}}{\partial \lambda_s}\right)\right] \bar{v} + \left[\frac{\partial}{\partial \bar{v}}\left(\frac{\partial \bar{B}}{\partial \lambda_s}\right)\right] \bar{B} - \left(\frac{\partial \bar{B}}{\partial \bar{v}}\right)\left(\frac{\partial \bar{B}}{\partial \lambda_s}\right),$$

$$\left(\mathbf{f}_1, \frac{\partial \mathbf{f}_0}{\partial \lambda_c}\right) = \begin{bmatrix} Oc_3 \\ \bar{M}_c \\ 0 \end{bmatrix}, \tag{70}$$

where,

$$\bar{M}_c = \left[\frac{\partial}{\partial \bar{v}}\left(\frac{\partial \bar{B}}{\partial \lambda_c}\right)\right] \hat{e}_v + \frac{\partial}{\partial q}\left(\frac{\partial \bar{B}}{\partial \lambda_c}\right) - \left(\frac{\partial \hat{e}_v}{\partial \bar{v}}\right)\left(\frac{\partial \bar{B}}{\partial \lambda_c}\right),$$

and

$$\left(\mathbf{f}_1, \frac{\partial \mathbf{f}_0}{\partial \lambda_s}\right) = \begin{bmatrix} Oc_3 \\ \bar{M}_s \\ 0 \end{bmatrix}, \tag{71}$$

where,

$$\bar{M}_s = \left[\frac{\partial}{\partial \bar{v}}\left(\frac{\partial \bar{B}}{\partial \lambda_c}\right)\right] \hat{e}_v + \frac{\partial}{\partial q}\left(\frac{\partial \bar{B}}{\partial \lambda_c}\right) - \left(\frac{\partial \hat{e}_v}{\partial \bar{v}}\right)\left(\frac{\partial \bar{B}}{\partial \lambda_c}\right).$$

Expressions for the Tensors

Partial of $\hat{e}_v$

$$\frac{\partial \hat{e}_v}{\partial \bar{v}} = \frac{\mathbf{I}_3}{v} - \frac{\bar{v}\bar{v}^T}{v^3} \tag{72}$$

Note that from (72), we get

$$\left(\frac{\partial \hat{e}_v}{\partial \bar{v}}\right)\hat{e}_v = 0 \tag{73}$$

Partials of $\hat{e}_c$ and $\hat{e}_s$ We have to find $\frac{\partial \hat{e}_c}{\partial \bar{v}}$ and $\frac{\partial \hat{e}_s}{\partial \bar{v}}$ since the unit vectors $\hat{e}_c$ and $\hat{e}_s$ are dependent on $\bar{v}$ through the implicit relations $\bar{v}^T\hat{e}_c = 0$ and $\bar{v}^T\hat{e}_s = 0$. It can be shown that[7]

$$\frac{\partial \hat{e}_c}{\partial \bar{v}} = -\frac{1}{v^2}\bar{v}\hat{e}_c^T \tag{74}$$

and

$$\frac{\partial \hat{e}_s}{\partial \bar{v}} = -\frac{1}{v^2}\bar{v}\hat{e}_s^T \tag{75}$$

Partials of $\bar{B}$

$$\begin{aligned}\frac{\partial \bar{B}}{\partial \bar{r}} &= -\left(\frac{\rho_r}{\rho}\right)\frac{\rho S C_{D0}}{2}\frac{e^{q/c}}{m_0}(1+{\lambda_c}^2+{\lambda_s}^2)\frac{v}{r}\bar{v}\bar{r}^T \\ &+\left(\frac{\rho_r}{\rho}\right)\rho S C_{D0}\frac{e^{q/c}}{m_0}E^*\frac{v^2}{r}(\lambda_c\hat{e}_c+\lambda_s\hat{e}_s)\bar{r}^T \\ &-\frac{g}{r}\mathbf{I}_3-\left(\frac{g_r}{g}\right)\frac{g}{r^2}\bar{r}\bar{r}^T+\frac{g}{r^3}\bar{r}\bar{r}^T\end{aligned} \tag{76}$$

$$\begin{aligned}\frac{\partial \bar{B}}{\partial \bar{v}} &= -\frac{\rho S C_{D0}}{2}\frac{e^{q/c}}{m_0}(1+{\lambda_c}^2+{\lambda_s}^2)\left(v\mathbf{I}_3+\frac{\bar{v}\bar{v}^T}{v}\right) \\ &+2\rho S C_{D0}\frac{e^{q/c}}{m_0}E^*(\lambda_c\hat{e}_c+\lambda_s\hat{e}_s)\bar{v}^T \\ &-\rho S C_{D0}\frac{e^{q/c}}{m_0}E^*\bar{v}(\lambda_c\hat{e}_c+\lambda_s\hat{e}_s)^T\end{aligned} \tag{77}$$

$$\begin{aligned}\frac{\partial \bar{B}}{\partial q} &= -\frac{\rho S C_{D0}}{2}\frac{e^{q/c}}{m_0}\frac{v}{c}(1+{\lambda_c}^2+{\lambda_s}^2)\bar{v} \\ &+\rho S C_{D0}\frac{e^{q/c}}{m_0}E^*\frac{v^2}{c}(\lambda_c\hat{e}_c+\lambda_s\hat{e}_s)\end{aligned} \tag{78}$$

$$\frac{\partial \bar{B}}{\partial \lambda_c} = \rho S C_{D0}\frac{e^{q/c}}{m_0}(-\lambda_c v\bar{v}+E^*v^2\hat{e}_c) \tag{79}$$

$$\frac{\partial \bar{B}}{\partial \lambda_s} = \rho S C_{D0}\frac{e^{q/c}}{m_0}(-\lambda_s v\bar{v}+E^*v^2\hat{e}_s) \tag{80}$$

Partials of $\frac{\partial \bar{B}}{\partial \lambda_c}$

$$\frac{\partial}{\partial \bar{r}}\left(\frac{\partial \bar{B}}{\partial \lambda_c}\right) = \rho_r S C_{D0}\frac{e^{q/c}}{m_0}\left(-\lambda_c\frac{v}{r}\bar{v}+E^*\frac{v^2}{r}\hat{e}_c\right)\bar{r}^T \tag{81}$$

$$\begin{aligned}\frac{\partial}{\partial \bar{v}}\left(\frac{\partial \bar{B}}{\partial \lambda_c}\right) &= -\rho S C_{D0}\frac{e^{q/c}}{m_0}\lambda_c\left(v\mathbf{I}_3+\frac{\bar{v}\bar{v}^T}{v^3}\right) \\ &+\rho S C_{D0}\frac{e^{q/c}}{m_0}E^*(2\hat{e}_c\bar{v}^T-\bar{v}\hat{e}_c^T)\end{aligned} \tag{82}$$

$$\frac{\partial}{\partial q}\left(\frac{\partial \bar{B}}{\partial \lambda_c}\right) = \rho S C_{D0}\frac{e^{q/c}}{m_0}\left(-\lambda_c\frac{v}{c}\bar{v}+E^*\frac{v^2}{c}\hat{e}_c\right) \tag{83}$$

Partials of $\frac{\partial \bar{B}}{\partial \lambda_s}$

$$\frac{\partial}{\partial \bar{r}}\left(\frac{\partial \bar{B}}{\partial \lambda_s}\right) = \rho_r S C_{D0}\frac{e^{q/c}}{m_0}\left(-\lambda_s\frac{v}{r}\bar{v}+E^*\frac{v^2}{r}\hat{e}_s\right)\bar{r}^T \tag{84}$$

$$\begin{aligned}\frac{\partial}{\partial \bar{v}}\left(\frac{\partial \bar{B}}{\partial \lambda_s}\right) &= -\rho S C_{D0}\frac{e^{q/c}}{m_0}\lambda_s\left(v\mathbf{I}_3+\frac{\bar{v}\bar{v}^T}{v^3}\right) \\ &+\rho S C_{D0}\frac{e^{q/c}}{m_0}E^*(2\hat{e}_s\bar{v}^T-\bar{v}\hat{e}_s^T)\end{aligned} \tag{85}$$

$$\frac{\partial}{\partial q}\left(\frac{\partial \bar{B}}{\partial \lambda_s}\right) = \rho S C_{D0}\frac{e^{q/c}}{m_0}\left(-\lambda_s\frac{v}{c}\bar{v}+E^*\frac{v^2}{c}\hat{e}_s\right) \tag{86}$$

Partials of $\bar{C}$

$$\begin{aligned}\frac{\partial \bar{C}}{\partial \bar{r}} &= \rho_r\frac{S C_{D0}}{2}\frac{e^{q/c}}{m_0}(1+{\lambda_c}^2+{\lambda_s}^2)\left(2+\frac{v}{c}\right)\frac{\bar{v}\bar{r}^T}{r} \\ &-\rho_r S C_{D0}\frac{e^{q/c}}{m_0}E^*\frac{v}{r}\left(1+\frac{v}{c}\right)(\lambda_c\hat{e}_c+\lambda_s\hat{e}_s)\bar{r}^T \\ &-\frac{g}{vr}\mathbf{I}_3+\frac{g}{vr^3}\bar{r}\bar{r}^T+\frac{g}{v^3r}\bar{v}\bar{v}^T \\ &-\frac{g(\bar{v}^T\bar{r})}{v^3r^3}\bar{v}\bar{r}^T-\frac{g_r}{vr^2}\bar{r}\bar{r}^T+\frac{g_r(\bar{v}^T\bar{r})}{v^3r^2}\bar{v}\bar{r}^T\end{aligned} \tag{87}$$

$$\begin{aligned}\frac{\partial \bar{C}}{\partial \bar{v}} \\ &= \frac{\rho S C_{D0}}{2}\frac{e^{q/c}}{m_0}(1+{\lambda_c}^2+{\lambda_s}^2)\left[\left(2+\frac{v}{c}\right)\mathbf{I}_3+\frac{\bar{v}\bar{v}^T}{cv}\right] \\ &+\rho S C_{D0}\frac{e^{q/c}}{m_0}E^*\frac{1}{v}\left(1+\frac{v}{c}\right)\bar{v}(\lambda_c\hat{e}_c+\lambda_s\hat{e}_s)^T \\ &-\rho S C_{D0}\frac{e^{q/c}}{m_0}E^*\frac{1}{v}\left(1+\frac{2v}{c}\right)(\lambda_c\hat{e}_c+\lambda_s\hat{e}_s)\bar{v}^T \\ &+\frac{g}{v^3r}\left[\bar{r}\bar{v}^T+(\bar{v}^T\bar{r})\mathbf{I}_3+\bar{v}\bar{r}^T-3\frac{(\bar{v}^T\bar{r})}{v^2}\bar{v}\bar{v}^T\right]\end{aligned} \tag{88}$$

$$\begin{aligned}\frac{\partial \bar{C}}{\partial q} &= \frac{\rho S C_{D0}}{2}\frac{e^{q/c}}{m_0}(1+{\lambda_c}^2+{\lambda_s}^2)\frac{1}{c}\left(2+\frac{v}{c}\right)\bar{v} \\ &-\rho S C_{D0}\frac{e^{q/c}}{m_0}E^*\frac{1}{v}\left(1+\frac{v}{c}\right)(\lambda_c\hat{e}_c+\lambda_s\hat{e}_s)\end{aligned}$$

Example Canonical Transformation[1,6]

For a point transformation between the variables (p,q) and (P,Q) to be canonical, we have the condition that

$$\bar{p}\cdot d\bar{q} = \bar{P}\cdot d\bar{Q} \tag{89}$$

Here we give an example of a canonical transformation between the vector quantities $\bar{r}$ and $\bar{v}$ and the flight path variables, r(radius), θ(longitude), ϕ(latitude), V(velocity), γ(flight path angle) and ψ(heading angle). See the figures below for the definition of the spherical unit vector triads $(\hat{e}_r, \hat{e}_\theta, \hat{e}_\phi)$ and $(\hat{e}_\gamma, \hat{e}_v, \hat{e}_\psi)$.

Noting that

$$\bar{r} = r\hat{e}_r \tag{90}$$

$$\bar{v} = V\hat{e}_v \tag{91}$$

we can write,

$$\begin{aligned} d\bar{r} &= dr\hat{e}_r + rd\hat{e}_r \\ &= dr\hat{e}_r + rd\bar{\omega} \times \hat{e}_r \\ &= dr\hat{e}_r + d\theta(\hat{k} \times \bar{r}) - d\phi(\hat{e}_\theta \times \bar{r}) \end{aligned} \tag{92}$$

and

$$\begin{aligned} d\bar{v} &= dV\hat{e}_v + Vd\hat{e}_v \\ &= dV\hat{e}_v + V(d\bar{\omega} + d\bar{\Omega}) \times \hat{e}_v \\ &= dV\hat{e}_v + d\theta(\hat{k} \times \bar{v}) - d\phi(\hat{e}_\theta \times \bar{v}) + d\psi(\hat{e}_r \times \bar{v}) \\ &\quad -d\gamma(\hat{e}_\psi \times \bar{v}) \end{aligned} \tag{93}$$

where,

$$\bar{\omega} = \dot{\theta}\hat{k} - \dot{\phi}\hat{e}_\theta \tag{94}$$

$$\bar{\Omega} = \dot{\psi}\hat{e}_r - \dot{\gamma}\hat{e}_\psi \tag{95}$$

are the angular velocities of the $(\hat{e}_r, \hat{e}_\theta, \hat{e}_\phi)$ and the $(\hat{e}_\gamma, \hat{e}_v, \hat{e}_\psi)$ coordinate systems respectively.

Now using (89), (92) and (93) and simplifying, we get

$$\begin{aligned} \bar{p}_r &= p_r\hat{e}_r \\ &+ \left[\frac{p_\theta}{r\cos\phi} + \frac{p_\gamma}{r\cos\phi}(\hat{k}\cdot\hat{e}_\psi) - \frac{p_\psi}{r\cos\phi}\frac{(\hat{k}\cdot\hat{e}_\gamma)}{\cos\gamma}\right]\hat{e}_\theta \\ &+ \left[\frac{p_\phi}{r} - \frac{p_\gamma}{r}(\hat{e}_\theta\cdot\hat{e}_\psi) + \frac{p_\psi}{r\cos\gamma}(\hat{e}_\theta\cdot\hat{e}_\gamma)\right]\hat{e}_\phi \end{aligned} \tag{96}$$

and

$$\bar{p}_v = \frac{p_\gamma}{V}\hat{e}_\gamma + p_V\hat{e}_v + \frac{p_\psi}{V\cos\gamma}\hat{e}_\psi \tag{97}$$

Similarly we can also write the inverse transformation as

$$\begin{aligned} p_r &= \bar{p}_r \cdot \frac{\bar{r}}{r} \\ p_V &= \bar{p}_v \cdot \frac{\bar{v}}{V} \\ p_\theta &= \hat{k}\cdot\bar{c} \\ p_\phi &= -\hat{e}_\theta\cdot\bar{c} \\ p_\psi &= \hat{e}_r\cdot\bar{c} \\ p_\gamma &= \frac{\bar{r}\cdot[\bar{v}\times(\bar{p}_v\times\bar{v})]}{rV\cos\gamma} \end{aligned} \tag{98}$$

where $\bar{c}$ is defined in (9).

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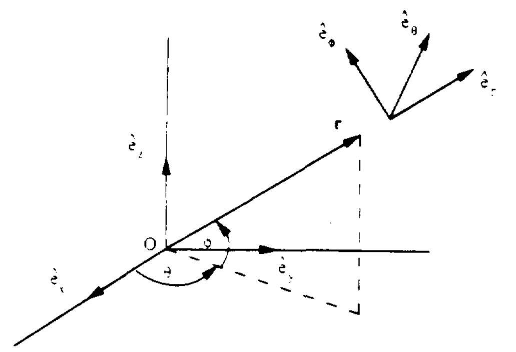

Fig. 1. Radius Vector

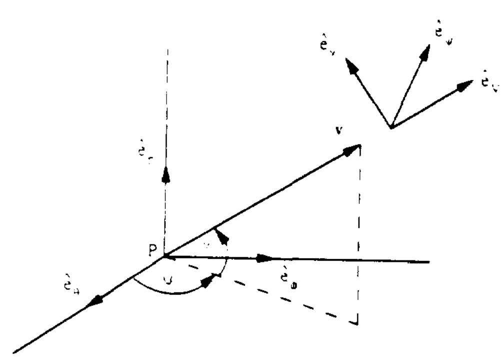

Fig. 2. Velocity Vector