LOCALIZED VIBRATIONS OF DISORDERED MULTI-SPAN BEAMS: THEORY AND EXPERIMENT

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ABSTRACT

The localization of the free modes of vibration of disordered multi-span beams is investigated, both theoretically and experimentally. It is shown that small deviations of the span lengths from an ideal value may have drastic effects on the dynamics of the system, by inhibiting the propagation of vibrations in the structure. Emphasis is placed on the development of a perturbation method that allows one to obtain the localized modes of vibration of the disordered system without a global eigenvalue analysis of the entire system. Such a perturbation analysis is cost effective and accurate. More importantly, it provides physical insight into the localization phenomenon, and allows one to formulate a criterion that predicts the occurence of localized modes. Also, an experiment is described, which has been carried out to verify the existence of localized modes for disordered two-span beams. Theoretical and experimental results are compared in detail and excellent agreement is found, thus confirming the existence of localized modes.

INTRODUCTION

The presence of irregularities in nominally periodic structures may inhibit the propagation of vibration within the structure. Depending on the magnitude of disorder and on the strength of internal coupling for the system, the irregularities localize the vibration modes and confine the vibrational energy to a region close to the source. This phenomenon is referred to as normal mode localization. The localization phenomenon was first predicted by Anderson in 1958 in the field of solid state physics. In a famous study[1], Anderson showed that the electron eigenstates in a disordered solid may become localized. This implies that metallic conduction may be very limited. Similarly, for structural elastic systems with localized modes of vibration, there is no long range propagation of vibration.

To date, both the analysis of the localization phenomenon and its potential applications have not received much attention in the field of structural dynamics. To the authors' knowledge, one of the few significant contributions was made by Hodges[2,3], who demonstrated the similarities between the propagation of vibration in an elastic system and the conduction of electrons in a solid, thus suggesting that localization can occur for some disordered elastic systems. Hodges also discussed several applications of the localization phenomenon in an acoustical context. Moreover, in a recent paper[3], Hodges and Woodhouse describe an experiment carried out to demonstrate localization, and find satisfactory agreement with the theoretical predictions. The system used in the experiment is a stretched string with masses attached to it, disorder being introduced in the structure by considering irregular spacing of the masses. Recent studies by Valero and Bendiksen[4,5] demonstrate the existence of localized free modes of vibration for shrouded blades of jet engine rotors and suggest that localization may have a stabilizing effect on the system. Finally, a few research papers[6,7] mention the high sensitivity of nearly periodic, weakly coupled systems to irregularities.

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Recently, a theoretical investigation of the mode localization phenomenon was conducted by the first and third authors[8,9] through the study of the free vibration modes of a few simple systems. The study determined that there are two categories of nearly periodic structural systems susceptible to localization:

- Systems consisting of coupled, similar but slightly **disordered** subsystems. Typical examples include chains of coupled pendula and jet engine rotors, for which the physical properties vary slightly from pendulum to pendulum and from blade to blade, respectively. It was shown that localization occurs when the **coupling** between subsystems is **small** and that localization becomes more pronounced as the coupling decreases.
- Structures with irregularly spaced constraints. Examples include a vibrating string with irregularly spaced masses attached and a beam or plate constrained at irregular intervals.

It is important to point out that when localization occurs, **small** irregularities result in **drastic** changes in the dynamics of the system. For this reason, neglecting these irregularities may lead to completely erroneous results. Thus it is particularly important to establish criteria capable of predicting the occurence of localization. For mistuned bladed disks, such localized vibrations may be damaging because the stresses remain localized, leading to blade fatigue. On the other hand, for other applications, such as large space structures, it may be desirable to use localization as a means of confining vibrations to a region close to the source of disturbance. Note that this application of localization leads to passive control of vibrations by irregularities.

In this paper, the localization of the free modes of vibration of multi-span beams is investigated, both theoretically and experimentally. Beams constrained at supposedly regular intervals are frequently encountered in structural analysis. Among numerous applications, aircraft fuselages and wings can be modeled by periodic beams. Other examples are building frames and bridges. These "periodic" structures are usually investigated by assuming ideal regularity , even though small deviations of the span lengths from an ideal value may have important effects on the free and forced response of the system.

The free modes of vibration of beams simply supported at regular intervals have been studied extensively in the research literature[10-13]. Particularly, one of the first and most important contributions was made by Miles in a well known paper[10]. Lin and Yang[14] investigated the effect of random deviations of the span lengths on the free modes of a beam simply supported at slightly irregular intervals. Nevertheless, their work was not concerned with the study of localization.

Miles[10] showed that the natural frequencies of periodic multi-span beams are clustered in an infinite number of groups, or bands, with n frequencies in each band, where n is the number of spans. If, in addition to a zero deflection, torsional springs exert restoring moments at the $n-1$ intermediate constraint locations, then the width of the frequency bands diminishes as the spring constant increases. In the limit as the spring constant c goes to infinity, the beam becomes clamped at the constraint locations, and the width of the frequency band goes to zero.

The inverse of the torsional spring constant may be viewed as a coupling parameter between spans. As $1/c \to 0$, the different spans of the beam are "decoupled" because no moment can be transmitted from one span to another. For $c = 0$, the beam is simply supported at the constraint locations, and the spans are strongly "coupled", since no restoring moment is exerted. Hence, depending on the value of the spring constant, c, the n-span beam can be regarded as a strongly or weakly coupled set of n, infinite number of degrees of freedom (DOF) oscillators. Moreover, the multi-span beam is an ordered system if all the spans have the same length. It can be rendered disordered by considering slightly irregular constraint locations.

Hence for large values of the spring constant and irregular spacing between supports, a multi-span beam can be regarded as a **disordered chain of weakly coupled subsystems**, where each subsystem is a span. From the theory of the mode localization phenomenon developed in References 8 and 9, the free modes of vibration of such a system are susceptible to becoming localized. Also, as mentioned above, the natural frequencies of multi-span beams are in bands of small width if the spring constant is large. Moreover, disorder in the length of the different spans introduces a spread in the individual natural frequencies of the spans. A general criterion was proposed in Reference 9, stating that localization may occur if the width of the frequency band of the ordered system is of the order of, or smaller than the spread in individual natural frequencies of the disordered component systems. Clearly, these conditions can be met by disordered multi-span beams, provided the spring constant c is large enough. Hence, by analogy with the systems studied in Reference 9, it is expected that the free modes of vibration of nearly periodic multi-span beams may become localized.

In the first part of the paper, the free modes of transverse vibration of a disordered two-span beam are investigated theoretically. It is shown that, under certain conditions, the modes are localized. The degree of localization is dependent upon two parameters: the deviation of the constraint location from the middle of the beam, and the value of the stiffness constant of the spring which exerts a restoring moment at the constraint location. The free vibration modes are determined by using a Rayleigh-Ritz formulation with the constraints conditions enforced by means of Lagrange multipliers[15]. The method is described in Section I.1. In Section I.2, classical and modified **perturbation methods** are developed for the analysis of nonlocalized and localized modes, respectively. These perturbation methods provide physical insight into the mechanisms of the mode localization phenomenon. In Section I.3, numerical results are presented and discussed. In particular, the theory of the localization phenomenon developed in Reference 9 is applied successfully to the two-span beam. These results can be readily extended to an n-span beam.

The second part of the paper presents an experiment which has been carried out to verify the existence of localized modes for disordered two-span beams. The experimental set-up is described and justified in detail in Section II.1. Section II.2 presents the corresponding experimental results, along with a detailed comparison with theoretical results derived in the first part of the paper. Excellent agreement between theory and experiment is observed.

PART I : THEORY

I.1. Free Vibration of a Disordered Two-Span Beam

Consider the uniform two-span beam of length l shown in Fig. 1. E, I, and m are respectively its Young's modulus, area moment of inertia, and mass per unit length. The beam is simply supported at $x = 0$ and $x = l$, and is constrained to have zero deflection at $x = x_1$. Moreover, a torsional spring of stiffness constant c exerts a restoring moment at $x = x_1$. If $x_1 = l/2$, the beam is said to be tuned, or ordered; otherwise, it is mistuned, or disordered.

The equations of free bending motion are derived from Hamilton's principle, and a Rayleigh-Ritz procedure with the con-

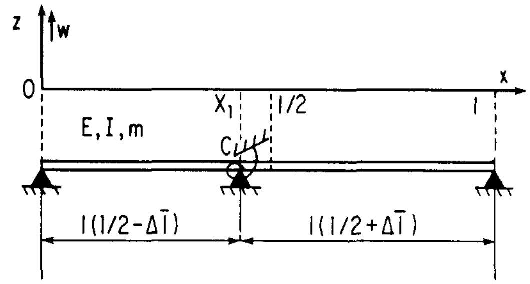

Figure 1. Geometry of disordered two-span beam.

straint conditions enforced by means of Lagrange multipliers is chosen[15]. The transverse deflection is expanded on the free modes of a single-span beam of length l pinned at both ends. Note that these component modes are comparison functions for the two-span beam, since the boundary conditions remain unchanged. The transverse deflection $w(x,t)$ of the two-span beam is expanded as

$$w(x,t) = \sum_{i=1}^{NM} a_i(t)\, \phi_i(x) \tag{1}$$

where the a_i's are the generalized coordinates, NM is the number of modes used in the Rayleigh-Ritz analysis, and $\phi_i(x) = \sqrt{\frac{2}{ml}} sin(\frac{i\pi x}{l})$ are the normalized natural modes of the single-span beam, whose corresponding natural frequencies are given by

$$\omega_i = (i\pi)^2 \sqrt{\frac{EI}{ml^4}} \tag{2}$$

The strain energy of the two-span beam is

$$U = \frac{1}{2}\sum_{i=1}^{NM} \omega_i^2 a_i^2 + \frac{1}{2} c[w'(x_1)]^2 \tag{3}$$

where $'$ denotes a derivative with respect to x. Its kinetic energy is:

$$T = \frac{1}{2}\sum_{i=1}^{NM} \dot{a}_i^2 \tag{4}$$

where $\dot{}$ denotes a derivative with respect to time. In addition, the beam is constrained at $x = x_1$, and the two constraint equations are given by:

$$f_1 = \sum_{i=1}^{NM} a_i(t)\phi_i(x_1) = 0 \tag{5}$$

$$f_2 = w'(x_1) - \sum_{i=1}^{NM} a_i(t)\phi_i'(x_1) = 0 \tag{6}$$

Thus the Lagrangian of the system is

$$L = U - T + \beta_1 f_1 + \beta_2 f_2 \tag{7}$$

where β_1 and β_2 are the two Lagrange multipliers corresponding to the constraints (5-6).

Applying Hamilton's principle, the equations of free motion are found to be:

$$\begin{cases} \ddot{a}_i + \omega_i^2 a_i^2 - \beta_1\phi_i(x_1) + \beta_2\phi_i'(x_1) = 0 \quad i = 1, ..., NM & (8) \\ \beta_2 = cw'(x_1) & (9) \\ f_1 = 0 \quad ; \quad f_2 = 0 & (10) \end{cases}$$

There are $NM + 3$ equations for $NM + 3$ unknowns a_i, β_1, β_2 and $w'(x_1)$. Assuming simple harmonic motion of natural frequency, Ω, one has:

$$\begin{cases} a_i = \bar{a}_i e^{j\Omega t} \quad i = 1, ..., NM & (11) \\ \beta_k = \bar{\beta}_k e^{j\Omega t} \quad k = 1, 2 & (12) \end{cases}$$

For $\Omega \neq \omega_i$, Eq. (8) may be written as:

$$\bar{a}_i = \frac{1}{\omega_i^2 - \Omega^2}[\bar{\beta}_1 \phi_i(x_1) - \bar{\beta}_2 \phi_i'(x_1)] \tag{13}$$

Substituting the above expression of $\bar{a}_i$ into Eqs. (9-10) yields, for $\Omega \neq \omega_i$:

$$\left\{ \begin{array}{l} \bar{\beta}_1 \Big[\sum_{i=1}^{NM} \frac{\phi_i^2(x_1)}{\omega_i^2 - \Omega^2}\Big] - \bar{\beta}_2 \Big[\sum_{i=1}^{NM} \frac{\phi_i(x_1)\phi_i'(x_1)}{\omega_i^2 - \Omega^2}\Big] = 0 \quad (14) \\ \bar{\beta}_1 \Big[\sum_{i=1}^{NM} \frac{\phi_i(x_1)\phi_i'(x_1)}{\omega_i^2 - \Omega^2}\Big] - \bar{\beta}_2 \Big[\frac{1}{c} + \sum_{i=1}^{NM} \frac{\phi_i'^2(x_1)}{\omega_i^2 - \Omega^2}\Big] = 0 \quad (15) \end{array} \right.$$

The latter equations constitute an eigenvalue problem in Ω. Non-zero solutions are obtained for $\bar{\beta}_1$ and $\bar{\beta}_2$ if and only if the determinant of the system (14-15) is equal to zero, yielding:

$$\Big[\sum_{i=1}^{NM} \frac{sin^2(i\pi\bar{x}_1)}{\bar{\omega}_i^2 - \bar{\Omega}^2}\Big]\Big[\frac{1}{\bar{c}} + \sum_{i=1}^{NM} \frac{(i\pi)^2 cos^2(i\pi\bar{x}_1)}{\bar{\omega}_i^2 - \bar{\Omega}^2}\Big] - \Big[\sum_{i=1}^{NM} \frac{(i\pi) sin(i\pi\bar{x}_1) cos(i\pi\bar{x}_1)}{\bar{\omega}_i^2 - \bar{\Omega}^2}\Big]^2 = 0 \tag{16}$$

where $\bar{\omega}_i$ and $\bar{\Omega}$ are dimensionless frequencies defined by

$$\bar{\omega}_i = \omega_i / \sqrt{\frac{EI}{ml^4}} = (i\pi)^2 \ ; \ \bar{\Omega} = \Omega / \sqrt{\frac{EI}{ml^4}} \tag{17}$$

$\bar{c} = c2l/EI$ is the dimensionless spring constant, and $\bar{x}_1 = x_1/l = \frac{1}{2} - \overline{\Delta l}$ is the dimensionless location of the intermediate support, where $\overline{\Delta l} = \Delta l / l$ is the dimensionless deviation from the middle of the beam.

Recall that Eq. (16) presupposes $\bar{\Omega} \neq \bar{\omega}_i$. It is an eigenvalue equation whose solutions are the free vibration natural frequencies $\bar{\Omega}$ of the two-span beam. For each value of $\bar{\Omega}$ solution of Eq. (16), the corresponding ratio $\bar{\beta}_1/\bar{\beta}_2$ is obtained from either Eq. (14) or Eq. (15), and the generalized coordinates amplitudes $\bar{a}_i$ are given by Eq. (13), from which the expression of the spatial mode shape $\bar{w}$ is readily obtained:

$$\bar{w}(\bar{x}) = \sum_{i=1}^{NM} \bar{a}_i sin(i\pi\bar{x}) \tag{18}$$

where $\bar{x} = x/l$.

TUNED BEAM

In this case $\overline{\Delta l} = 0$ (or $\bar{x}_1 = l/2$). First consider a beam **simply supported** at its middle.

• $\bar{c} = 0$

Then the eigenvalue equation (16) reduces to:

$$\sum_{\substack{i=1 \\ i \ odd}}^{NM} \frac{1}{\bar{\omega}_i^2 - \bar{\Omega}^2} = 0 \tag{19}$$

Since the summation is only on odd values of i, solving Eq. (19) provides only half of the natural frequencies $\bar{\Omega}$. It is well known[10] that these frequencies are the natural frequencies of a hinged-clamped beam of length $l/2$. The remaining half of the natural frequencies is given by $\bar{\Omega} = \bar{\omega}_i$, for i even, as can be readily deduced from the original eigenvalue problem (8-10), the corresponding mode shape being such that the only nonzero generalized coordinate is a_i. Hence the natural frequencies of a tuned beam simply supported at its middle are

$$\left\{ \begin{array}{ll} \bar{\Omega}_k = \bar{\omega}_{k+1} = [(k+1)\pi]^2 & k \ odd \quad (20) \\ \bar{\Omega}_k \simeq [\frac{2k+1}{2}\pi]^2 & k \ even \quad (21) \end{array} \right.$$

As mentioned in the introduction, these natural frequencies have a pass-band character, and are placed in groups of two along the frequency axis.

• $\bar{c} \neq 0$

Then $1/\bar{c}$ is defined, and Eq. (16) reduces to:

$$\sum_{\substack{i=1 \\ i \ odd}}^{NM} \frac{1}{\bar{\omega}_i^2 - \bar{\Omega}^2} = 0 \tag{22}$$

$$\frac{1}{\bar{c}} + \sum_{\substack{i=1 \\ i \ even}}^{NM} \frac{(i\pi)^2}{\bar{\omega}_i^2 - \bar{\Omega}^2} = 0 \tag{23}$$

Each of Eqs. (22) and (23) provides half of the natural frequencies $\bar{\Omega}$. Note that Eqs. (22) and (19) are identical, hence the $\bar{\Omega}_k$'s for k even given by Eq. (21) are natural frequencies of the tuned beam for any value of the constant $\bar{c}$. As $\bar{c}$ increases, the distance between the two natural frequencies of a same group diminishes: the first frequency of the group increases, while the second remains unchanged. This behavior is represented in Fig. 2 for the first group of modes. Higher groups reveal a similar pattern. It is observed that, as $\bar{c}$ goes to infinity, the first frequency tends to the second one. In the limit $1/\bar{c} = 0$, corresponding to a beam clamped in the middle, the two natural frequencies of each group are equal, leading to two-fold multiple eigenvalues. Hence, as $\bar{c}$ increases, the width of the frequency bands decreases and goes to zero as $\bar{c}$ goes to infinity. As will be shown later, this bandwidth is one of two key parameters in determining the occurence of localized modes.

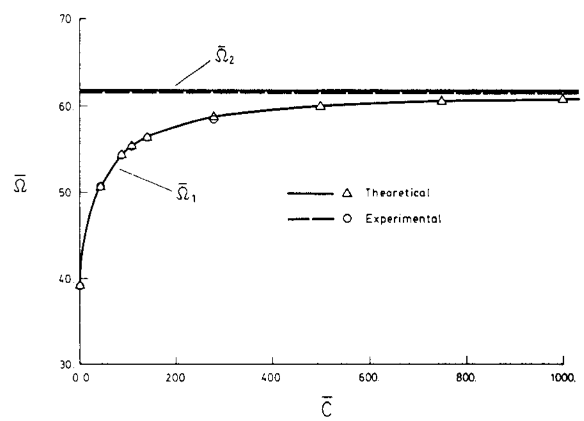

Figure 2. Natural frequencies of the first group of modes versus $\bar{c}$, for a tuned two-span beam. Theoretical and experimental results.

MISTUNED BEAM

If, for a given value of $\bar{c}$, a mistuning $\overline{\Delta l}$ is introduced, the two frequencies of a group move apart: the width of the frequency band increases with $\overline{\Delta l}$. This behavior is shown in Fig. 3, which represents the first and second natural frequencies (first group of modes) in terms of $\overline{\Delta l}$ for various values of $\bar{c}$. It should be noted that, for relatively large values of $\overline{\Delta l}$ such as .07, the band character of the natural frequencies is lost.

CONVERGENCE

The natural frequencies are calculated by solving the non-linear algebraic equation (16), which involves summations over

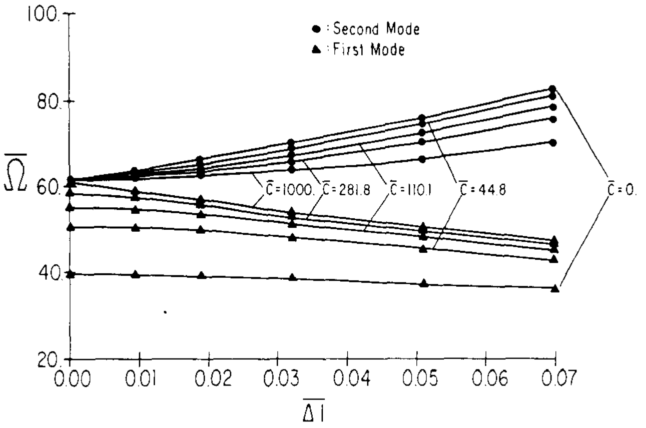

Figure 3. Natural frequencies of the first group of modes versus mistuning $\overline{\Delta l}$, for various values of $\bar{c}$.

the number of component modes NM. The convergence of the Rayleigh-Ritz procedure with NM has been checked by considering a mistuned beam clamped at $\bar{x}_1$, hence defined by $\overline{\Delta l} \neq 0$ and $\bar{c} \to \infty$. In this case the first two modes consist of the first modes of the two hinged-clamped spans of lengths $\frac{1}{2} - \overline{\Delta l}$ and $\frac{1}{2} + \overline{\Delta l}$. Since the beam is mistuned, the first two natural frequencies are distinct. Note that the exact mode shapes ought to have exactly a zero deflection over one of the spans, since the beam is clamped at $\bar{x}_1$ and mistuned to avoid repeated eigenvalues. It was found that a large number of component modes must be considered in order to achieve good convergence. Typically, the mode shapes were almost perfectly flat over one of the spans if $NM \geq 1000$. In the subsequent calculations, 1000 component modes were used. Convergence was also checked for higher modes: until the 20-th mode at least, zero deflection in one of the spans was obtained if 1000 or more component modes were used.

This rather slow convergence can be explained by noticing that the eigenvalue equation (16) contains summations over i of terms such as $1/(\bar{\omega}_i^2 - \bar{\Omega}^2)$ and $i^2/(\bar{\omega}_i^2 - \bar{\Omega}^2)$. Considering the expression (2) of $\bar{\omega}_i$, the series $\sum_{i=1}^{NM}(1/i^4)$ and $\sum_{i=1}^{NM}(1/i^2)$ are involved in the eigenvalue equation. It is well kwown that, although the former series converges quickly, the latter is slowly convergent, thus requiring the use of a large number of terms NM to approach its sum satisfactorily. It follows that the Rayleigh-Ritz expansion adopted here converges slowly to the exact solution. However, since this is a linear calculation, the computer cost remains reasonably low. Moreover, if the number of component modes is large enough ($\geq$ 1000), very accurate results are obtained, even for higher modes.

I.2. Perturbation Analyses

As shown in the previous section, it is much easier to calculate the free modes of vibration of a tuned two-span beam than of a mistuned one. For the modes of an ordered beam are obtained by solving two simple eigenvalue equations (22-23), whereas the modes of a disordered beam are calculated from the more complicated equation (16). This is characteristic of nearly periodic structures with small irregularities: when the structure is disordered, its periodicity properties are lost, and investigating its modes of vibration requires a computational effort much greater than for the associated periodic system. Hence the idea, for small irregularities, of performing a perturbation analysis. Since the unperturbed system is the periodic structure whose modes are easily obtained, this procedure allows one to avoid solving the eigenvalue problem for the mistuned system.

Here, the two-span beam is mistuned by the dimensionless deviation $\overline{\Delta l}$ of the support from its middle location. Small values of $\overline{\Delta l}$ are considered. The perturbation analyses presented below are very similar to the classical and modified perturbation methods developed in Reference 9 for a chain of coupled pendula.

CLASSICAL PERTURBATION ANALYSIS

The unperturbed system consists of the tuned beam. It is perturbed by moving the constraint by a distance Δl. A perturbation analysis can be readily defined from Eq. (16) by expanding the terms $sin(i\pi\bar{x}_1)$ and $cos(i\pi\bar{x}_1)$ in terms of $\overline{\Delta l}$ to the first or second order, hence obtaining the corresponding natural frequency perturbations $\delta\bar{\Omega}$ and $\delta^2\bar{\Omega}$. This perturbation analysis is straightforward and will not be presented in detail. Note that this approach presupposes that the term $1/\bar{c}$ in Eq. (16) is not small, but has a finite or large value. The modes of vibration of the mistuned system are perturbations of (hence are very similar to) the modes of the tuned system, and are certainly not localized. Thus this case is not of great interest to the present study.

MODIFIED PERTURBATION ANALYSIS

Here small values of $1/\bar{c}$ are considered. If, when the system is mistuned, **only** $\overline{\Delta l}$ is considered as a perturbation, then one can expect qualitatively erroneous results: **all** the small parameters (not only $\overline{\Delta l}$, but also $1/\bar{c}$) must be treated as perturbations. Since this case is very similar to the one of mistuned, weakly coupled pendula encountered in Reference 9, one may anticipate drastic changes in the modes when the system is mistuned. As a matter of fact, it is shown in the next paragraph that the modes become localized.

Considering $1/\bar{c}$ as a perturbation, the unperturbed system would be characterized by $\overline{\Delta l} = 0$ and $1/\bar{c} = 0$, defining a beam clamped at its middle. The perturbations would consist of $\overline{\Delta l}$ and $1/\bar{c}$, leading to a mistuned, "almost" clamped beam (since $1/\bar{c}$ is small, of the order of, or smaller than $\overline{\Delta l}$). The natural frequencies of the unperturbed system are then repeated, with one two-fold multiple eigenvalue in each group. The corresponding mode shapes are defined by any linear combination of a left-span hinged-clamped mode and of a right-span clamped-hinged mode, since the eigenfunctions associated with each double natural frequency span a space of dimension two. In order to perform a perturbation analysis, one must first determine the unique set of two unperturbed mode shapes from which the modes of the perturbed system are continuously obtained. It can be shown that this is equivalent to solving the eigenvalue problem for the modes of interest, hence rendering this perturbation procedure ineffective. The conclusion is that one must avoid multiple eigenvalues for the unperturbed system.

This is achieved by introducing some mistuning in the unperturbed system. The unperturbed state is then defined by $1/\bar{c} = 0$ and $\bar{x}_1 = \frac{1}{2} - \overline{\Delta l}$. It consists of a mistuned two-span beam clamped at the constraint location. Since the unperturbed beam is already mistuned, the only perturbation parameter is $1/\bar{c}$. This perturbation method is referred to as Modified Perturbation Method (MPM), and it is similar to the one developed in Reference 9 for a disordered chain of weakly coupled pendula. Since the unperturbed beam is mistuned, its eigenvalues are simple. Also, since it is clamped at $\bar{x} = \bar{x}_1$, its natural modes are the ones of hinged-clamped beams of respective lengths $\frac{1}{2} - \overline{\Delta l}$ and $\frac{1}{2} + \overline{\Delta l}$. Note that these unperturbed modes are **decoupled**, that is, they have a zero deflection over one of the two spans, depending on the mode number. Also, they are very easily determined analytically, without having to use the Rayleigh-Ritz procedure. When the system is perturbed by $1/\bar{c}$, the modes cease to be decoupled in the left or right spans to become collective, that is, they have nonzero deflection in both spans. However, since $1/\bar{c}$ is small, they are perturbations of the decoupled modes, hence are characterized by a deflection which is much larger in one span than in the other one: the modes are **localized**. It is remarkable that one is able to predict whether the modes are localized or not, just by considering perturbations of the eigenvalue equation (16).

Let $\bar{\Omega}_0$ be a natural frequency of the unperturbed system. Recall that, although $\bar{\Omega}_0$ can be obtained by solving Eq. (16) for $1/\bar{c} = 0$, it can be calculated analytically more easily, along with the corresponding mode shape. The system is perturbed by replacing the clamped condition by a spring of high stiffness $\bar{c}$, and the natural frequencies $\bar{\Omega}$ of the perturbed system are such that:

$$\bar{\Omega} = \bar{\Omega}_0 + \delta\bar{\Omega} + \mathcal{O}(\frac{1}{\bar{c}^2}) \tag{24}$$

where $\mathcal{O}$ is the Landau notation "of the order of", and $\delta\bar{\Omega}$ is a first order perturbation in $1/\bar{c}$. This first order expansion is substituted into Eq. (16), which is expanded to the first order. Since $\bar{\Omega}_0$ is solution of the unperturbed problem (such that $1/\bar{c} = 0$), the zeroth order term cancels out, and the first order perturbation $\delta\bar{\Omega}$ can be easily shown to be:

$$\delta\bar{\Omega} = \frac{1}{\bar{c}}\,\frac{-1}{2\bar{\Omega}_0\alpha_{NM}}\sum_{i=1}^{NM}\frac{sin^2(i\pi\bar{x}_1)}{\bar{\omega}_i^2 - \bar{\Omega}_0^2} \tag{25.a}$$

where

$$\alpha_{NM} = \sum_{i=1}^{NM}\frac{sin^2(i\pi\bar{x}_1)}{\bar{\omega}_i^2 - \bar{\Omega}_0^2}\sum_{i=1}^{NM}\frac{(i\pi)^2cos^2(i\pi\bar{x}_1)}{[\bar{\omega}_i^2 - \bar{\Omega}_0^2]^2}$$
$$+\sum_{i=1}^{NM}\frac{sin^2(i\pi\bar{x}_1)}{[\bar{\omega}_i^2 - \bar{\Omega}_0^2]^2}\sum_{i=1}^{NM}\frac{(i\pi)^2cos^2(i\pi\bar{x}_1)}{\bar{\omega}_i^2 - \bar{\Omega}_0^2}$$
$$-2\sum_{i=1}^{NM}\frac{(i\pi)sin(i\pi\bar{x}_1)cos(i\pi\bar{x}_1)}{\bar{\omega}_i^2 - \bar{\Omega}_0^2}\sum_{i=1}^{NM}\frac{(i\pi)sin(i\pi\bar{x}_1)cos(i\pi\bar{x}_1)}{[\bar{\omega}_i^2 - \bar{\Omega}_0^2]^2} \tag{25.b}$$

The corresponding perturbed mode shape is readily calculated by substituting the perturbed value $\bar{\Omega}$ into Eqs. (13-15,18). Note that a second order perturbation analysis can also be easily developed.

It is worth mentioning that the perturbation analysis described above provides analytical relationships only for the perturbations of the natural frequencies, and not for the mode shape perturbations, although such relations could possibly be developed. For the system of weakly coupled pendula studied in Reference 9, closed form relations for the eigenvector perturbations were used to show that the degree of localization depends only on the coupling to disorder ratio of the system, and to determine quantitatively the degree of localization for a given value of this ratio. Comparable information cannot be deduced from the perturbation theory presented here. Nevertheless, two important characteristics of perturbation methods are retained by the present analysis:

- The method is cost effective.
- Localized modes are predicted for small values of $1/\bar{c}$ if the beam is mistuned: perturbation methods provide physical insight into the localization phenomenon.

I.3. Results and Discussion

I.3.1 Results

The resolution of the eigenvalue equation (16) has been implemented on a digital computer. For given values of $\bar{x}_1$ and $\bar{c}$, this nonlinear equation is solved by a standard bisection technique. The bisection process converges rapidly. Typically, 20 to 35 iterations are necessary to obtain natural frequencies converged up to the 10-th decimal place. This kind of accuracy is required because very small variations in the natural frequencies may result in significant variations in the mode shapes, since a large number of component modes are considered. The accuracy of the Rayleigh-Ritz procedure and of the bisection process was checked against well known results, and in all cases excellent agreement was observed. Even though the number of component modes considered is very large, the CPU time necessary was not excessive, partly because the bisection process converges rapidly. Unless otherwise stated, the following results were obtained by solving directly Eq. (16), and not by a modified perturbation analysis.

Fig. 4 shows the lower two modes of a tuned beam ($\overline{\Delta l} = 0$) such that $\bar{c} = 1000$. These modes constitute the first group of modes. One observes that the modes are collective, as opposed to localized: the magnitude of the deflection is the same in each span. Fig. 5 displays the lower two modes of a mistuned beam such that $\overline{\Delta l} = .01$, for the same $\bar{c} = 1000$. One clearly sees that the peak deflection is much larger in one span that in the other one: the modes are localized. In this case, slight mistuning is sufficient to localize strongly the natural modes. The localized modes are perturbations of the "decoupled" modes corresponding to $1/\bar{c} = 0$. Recall that the term "decoupled" refers to a mode of zero deflection over one of the two spans. Since the system is **mistuned**, these decoupled modes correspond to **simple** eigenvalues. On the other hand, the modes of the **tuned** system such that $1/\bar{c} = 0$ correspond to **twofold multiple** eigenvalues, and perturbed modes for small $1/\bar{c}$ do not vary continuously from individual decoupled modes, giving rise to collective modes.

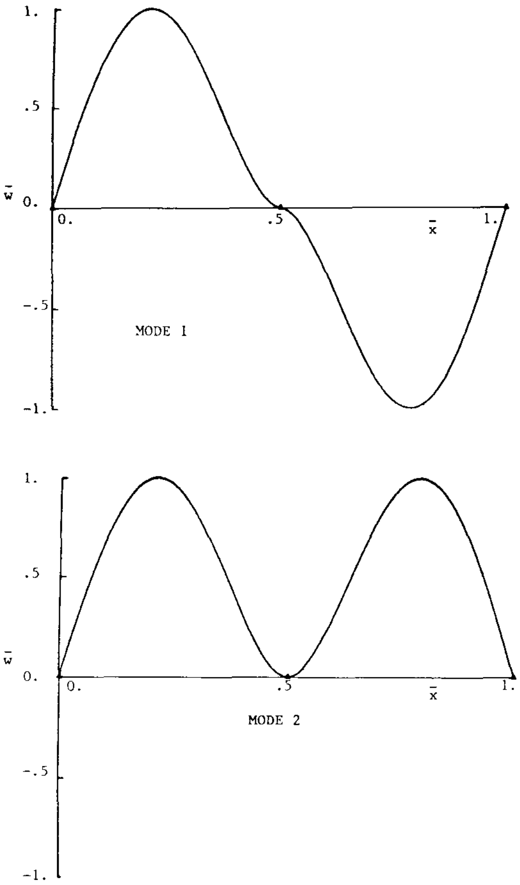

Figure 4. Lower two mode shapes for a tuned two-span beam ($\overline{\Delta l} = 0$) for $\bar{c} = 1000$.

The results shown in Fig. 5 were obtained by both exact method and Modified Perturbation Method. The agreement is observed to be excellent, confirming the fact that the MPM is suitable for the analysis of localized modes.

If the spring constant $\bar{c}$ increases, the mode shapes become even more strongly localized. This is observed in Fig. 6, which displays the lower two modes for $\bar{c} = 5000$, and for the same mistuning parameter $\overline{\Delta l} = .01$. In the limit $\bar{c} \to \infty$, the modes of the mistuned beam tend to become decoupled. On the other hand, for larger values of $1/\bar{c}$, the modes are only partially localized, and to the limit $\bar{c} \to 0$, the modes of the (simply supported) mistuned beam are not localized. For $\overline{\Delta l} = .01$ and $\bar{c} = 0$, it was found that there is only a slight difference between the peak deflections in each span. Due to lack of space, these modes are not displayed in this paper. In this case, the mode shapes are no longer perturbations of decoupled modes, but are perturbations of the collective modes of the tuned beam, which are shown in Fig. 4. Hence the Classical Perturbation Method (defined for large values of $1/\bar{c}$) would be suitable for this analysis.

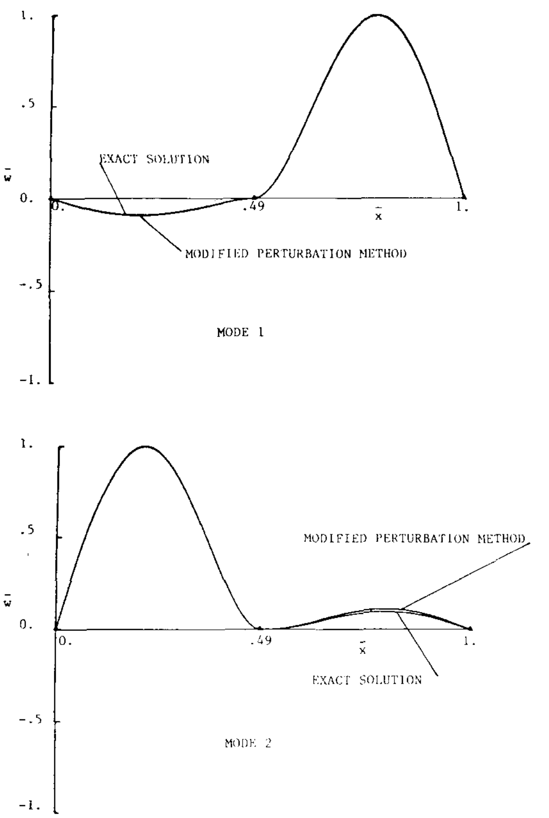

Figure 5. Lower two mode shapes for a mistuned two-span beam, for $\overline{\Delta l} = .01$ and $\bar{c} = 1000$, by exact method and Modified Perturbation Method.

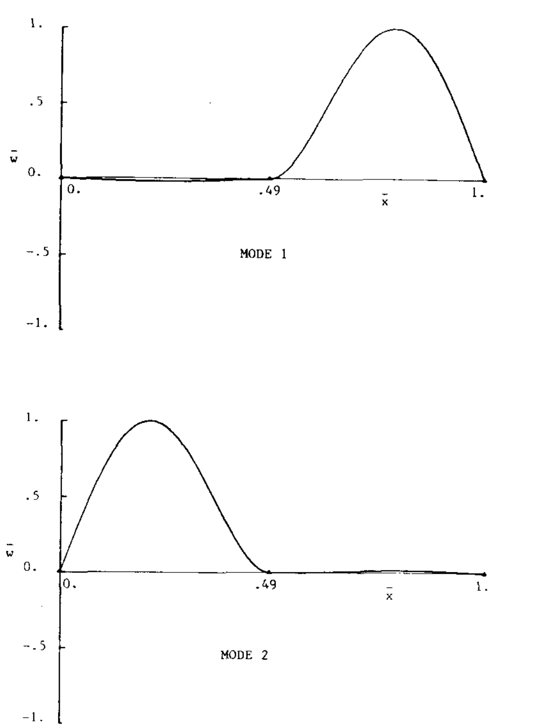

Figure 6. Lower two mode shapes for a mistuned two-span beam, for $\overline{\Delta l} = .01$ and $\bar{c} = 5000$.

In order to investigate systematically the effect of the mistuning parameter $\overline{\Delta l}$ and of the spring constant $\bar{c}$, it is suitable to adopt a compact representation of the modes. The degree of localization of a mode can be characterized by the ratio A of the peak deflection in one span to the peak deflection in the other span, such that the numerator of this ratio corresponds to the span with the smaller peak deflection:

$$A = \frac{A_s}{A_l} \tag{26}$$

where A_s and A_l are the peak deflections in each span, such that $A_s \leq A_l$. Note that the ratio A takes values ranging between -1 and $+1$. The smaller the absolute value of A, the more localized the corresponding mode. For decoupled modes, $A = 0$. For a tuned beam, $A = \pm 1$, depending on the mode number.

Fig. 7 displays values of $|A|$ in the $(\bar{c}, \overline{\Delta l})$-plane, for the first group of modes. To fix ideas, localization is said to occur if the absolute value of the peak ratio A is less than 10% (or .1). One notes that for a given $\bar{c}$, A decreases as $\overline{\Delta l}$ increases, hence the mode localization becomes more pronounced as the amount of mistuning is increased. In the limit $\bar{c} \to \infty$, the modes are localized for an arbitrarily small, but non zero, mistuning. Also, for a given mistuning $\overline{\Delta l}$, localization becomes more pronounced as $\bar{c}$ increases. The larger $\overline{\Delta l}$, the smaller the threshold value of $\bar{c}$ necessary to give rise to localized modes. However, localization does not occur for $\bar{c} < 110$, even for relatively large values of mistuning $\overline{\Delta l}$ such as .07. In particular, the lower modes of a beam simply supported at the constraint location do not become localized. Even if the value .07 seems to be small, the reader should bear in mind that this study is conducted within the context of small perturbations, and that $\overline{\Delta l} = .07$ corresponds to a 14% deviation of the length of the individual spans, a fairly large value.

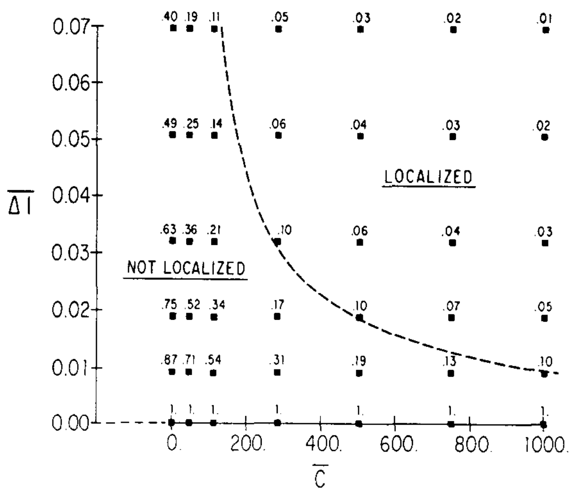

Figure 7. Values of $|A|$ in the $(\bar{c}, \overline{\Delta l})$-plane, for the first group of modes.

An approximate boundary of localization, corresponding to $|A| = 10\%$, is represented on Fig. 7 by a dotted line. From numerical results, one can show that, for various small values of $1/\bar{c}$ and $\overline{\Delta l}$ such that the product $\bar{c}\overline{\Delta l}$ is the same, the peak ratio A remains approximately constant. This can be seen in Fig. 7, as the localization boundary is similar to an hyperbola of equation $\bar{c} = constant/\overline{\Delta l}$. The product $\bar{c}\overline{\Delta l}$ is in fact a disorder to coupling ratio $\overline{\Delta l}/\frac{1}{\bar{c}}$, and the degree of localization seems to depend only on the value of this ratio. This result is similar to the one obtained in Reference 9 for a chain of coupled pendula. Nevertheless, in the latter case, this was shown analytically, whereas in the present study, one is reduced to investigate numerically the dependence of A upon $\overline{\Delta l}$ and $\bar{c}$.

I.3.2 Discussion

There is a strong analogy between the two-span beam and the system of two coupled pendula studied in Reference 9. The two pendulum system consists of two coupled single DOF oscillators, each of them being characterized by an individual natural frequency which is obtained by letting the coupling between the pendula go to zero. The amount of coupling is governed by the value of the spring constant, k, and mistuning is achieved by varying the length of the pendula, that is, by changing slightly their individual natural frequency. Similarly, for the two-span beam, the coupling between spans is determined by the inverse $1/\bar{c}$ of the torsional spring constant. If $1/\bar{c} = 0$ (clamped constraint), the spans are "decoupled", the same way the two pendula are decoupled for $k = 0$. The amount of "coupling" in the two-span beam increases with $1/\bar{c}$, its maximum value being reached for $1/\bar{c} \to \infty$, corresponding to a simply supported constraint. Each of the individual spans possesses an infinity of natural frequencies, which are for hinged-clamped boundary conditions. Hence if the beam is constrained at its middle location, the two spans have **identical** individual natural frequencies: the system is **ordered**, or **tuned**. On the other hand, if the location of the constraint is slightly off the middle of the beam, the two spans have **different** natural frequencies, and the system is **disordered**, or **mistuned**. It should also be noted that the two-span beam is a system of two coupled, infinite number of DOF oscillators, whereas the pendulum system is constituted of single DOF oscillators. However, recall that the natural frequencies of a two-span beam are distributed by groups of two, and each of these groups can be regarded as corresponding to a two pendulum system.

It has been shown in Reference 9 that the modes of the pendulum system are localized for small values of mistuning and coupling. Similarly, for the two-span beam, localization occurs for small values of $\overline{\Delta l}$ and $1/\bar{c}$. For the pendulum system, localized modes are perturbations of decoupled oscillations; for the two-span beam, they are perturbation of "decoupled" hinged-clamped modes. It has also been shown in Reference 9 that localization does not occur for strong coupling between pendula. Similarly, the free modes of a two-span beam are not localized for finite or large values of $1/\bar{c}$. In particular, localization does not occur for $\bar{c} = 0$, even for relatively large values of $\overline{\Delta l}$ such as .07.

Here, it is necessary to pause to remind the reader that the concept of localization is defined within the context of small perturbations: under some conditions, **small** mistuning has a drastic effect on the free modes of vibration. In the case $\bar{c} = 0$, one observes from Fig. 7 that the peak ratio A of the first group of modes is .4 for $\overline{\Delta l} = .07$. This is a significant change in the mode shape from the tuned case. However, it cannot be called localization, because the value of mistuning for which it is obtained is not small: for it corresponds to a 14% change of the span length. Moreover, it is clearly seen in Fig. 7 that the effect of mistuning on the peak ratio is not drastic for $\bar{c} = 0$, but rather slowly increasing with $\overline{\Delta l}$. On the other hand, for larger values of $\bar{c}$, a rapid change of A in terms of $\overline{\Delta l}$ is observed: localization occurs. To conclude, the theory of the mode localization phenomenon is for **small** departure from ideal regularity. For larger values of mistuning, significant changes can also be observed. But although of interest and of potential importance to the designer, these cases are not relevant to the study of localization.

It should be mentioned that the analogy between the two pendulum system and the two-span beam can be readily generalized to an n pendulum system and an n-span beam, for any n, suggesting that localized vibrations also occur for multi-span beams.

In order to understand thoroughly the physical mechanisms of localization for multi-span beams, the general criterion formulated in Reference 9 is now considered. This criterion states that a nearly periodic system is susceptible of having localized modes if the natural frequencies of the corresponding periodic system are distributed in groups, and if the widths of these pass bands are **small** relatively to the values of the frequencies belonging to the pass bands. Localization may occur for such systems if there are some discrepancies (mistuning) in the individual natural frequencies of the component subsystems constituting the almost periodic system, and if a characteristic spread in these individual frequencies is small, and of the order of, or larger than the pass band width of the ordered system:

$$PBW \leq \mathcal{O}(SNF) \tag{27}$$

where PBW stands for the Pass Band Width of the ordered system, and SNF for the Spread in individual Natural Frequencies. Note that for a tuned, or periodic system, $SNF = 0$.

For the pendulum system, it has been shown in Reference 9 that PBW is proportional to the amount of coupling between pendula, and that SNF is proportional to a characteristic perturbation of pendulum length. Similar results can be shown for the two-span beam. Considering a tuned beam of spring constant $\bar{c}$, its natural frequencies are placed along the frequency axis by groups of two, and thus have a pass band character. Denoting the pass band width of the j-th group of modes, which contains the $(2j-1)$-th and $2j$-th modes, by PBW_j, one can write:

$$PBW_j(\bar{c}) = \sqrt{\frac{EI}{ml^4}}\left[\overline{\Omega}_{2j} - \overline{\Omega}_{2j-1}(\bar{c})\right] \tag{28}$$

where the natural frequency of the $2j$-th mode is given by Eq. (21). The value of the natural frequency $\overline{\Omega}_{2j-1}(\bar{c})$ is dependent upon the value of $\bar{c}$. As $\bar{c}$ increases, $\overline{\Omega}_{2j-1}(\bar{c})$ becomes closer to $\overline{\Omega}_{2j}$, that is, PBW_j diminishes. In the limit $1/\bar{c} \to 0$, PBW_j goes to zero. Hence small values of the "coupling" $1/\bar{c}$ mean small pass band width of the ordered system.

The other variable that needs to be defined is the Spread in Natural Frequencies (SNF). As previously stated, the beam is decoupled if $1/\bar{c} = 0$, its natural frequencies being the ones of the two individual hinged-clamped spans. The spread resulting from mistuning can be written as:

$$SNF_j(\overline{\Delta l}) = \sqrt{\frac{EI}{ml^4}}\,\overline{\Omega}_{2j}\left|1 - \frac{1}{(1 \pm 2\overline{\Delta l})^2}\right| \tag{29}$$

For small mistuning, a first order approximation is obtained:

$$SNF_j(\overline{\Delta l}) \simeq \sqrt{\frac{EI}{ml^4}}\,\overline{\Omega}_{2j}\,4|\overline{\Delta l}| \tag{30}$$

Hence, for small $\overline{\Delta l}$, SNF_j is proportional to the amount of mistuning.

It is convenient to nondimensionalize $PBW_j(\bar{c})$ and $SNF_j(\overline{\Delta l})$ by the natural frequency of the second mode of the j-th group, Ω_{2j}. Eqs. (28-29) become:

$$\overline{PBW}_j(\bar{c}) = 1 - \overline{\Omega}_{2j-1}(\bar{c})\big/\overline{\Omega}_{2j} \tag{31}$$

$$\overline{SNF}_j(\overline{\Delta l}) = \left|1 - \frac{1}{(1 \pm 2\overline{\Delta l})^2}\right| \simeq 4|\overline{\Delta l}| \tag{32}$$

The following discussion investigates the ability of the criterion (27) to predict localized modes. This paragraph is concerned with the first group of modes, corresponding to $j = 1$. Localization of higher modes is considered later. Fig. 8 displays the absolute value of the peak ratio, $|A|$, in the $(\overline{PBW}_1, \overline{SNF}_1)$-plane, for the first group of modes. It is observed that localization occurs when $\overline{SNF}_1$ and $\overline{PBW}_1$ are both small. Moreover, with the definition of localization $|A| \leq 10\%$, the modes are localized in the region approximately defined by $\overline{PBW}_1 \leq .42\,\overline{SNF}_1$, the localization boundary being given by $\overline{PBW}_1 \simeq .42\,\overline{SNF}_1$. Note that this boundary is dependent upon the definition chosen for localization: stronger or weaker requirements for localization to occur would result in a quantitatively different, but qualitatively similar boundary. It should also be noted that, from numerical results, the degree of localization $|A|$ seems to be only dependent upon the ratio $\overline{PBW}_1/\overline{SNF}_1$. Since the localization region shown in Fig. 8 is consistent with the criterion (27), the latter has the ability to predict the occurence of localization for the first group of modes.

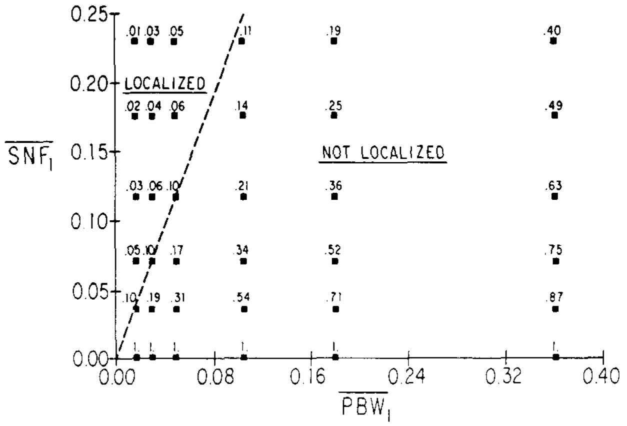

Figure 8. Values of $|A|$ in the $(\overline{PBW}_1, \overline{SNF}_1)$-plane, for the first group of modes.

Finally, it is of interest to investigate the localization of higher groups of modes. Consider the variation of $\overline{PBW}_j$ and $\overline{SNF}_j$ given by Eqs. (31-32) in terms of the group number j. It is clear that $\overline{SNF}_j$ remains constant when j increases. For a given $\overline{c}$, $\overline{PBW}_j$ decreases as j increases, and in the limit $j \to \infty$, $\overline{PBW}_j$ goes to zero. For instance, for $\overline{c} = 0$, $\overline{PBW}_j$ can be written as:

$$\overline{PBW}_j(\overline{c} = 0) = 1 - \frac{4j^2}{(4j+1)^2/4} \simeq \frac{1}{2j} \; for\ large\ j \qquad (33)$$

Thus for any given $\overline{c}$ (even small), and for a given $\overline{\Delta l}$, there exists a group number j^* such that $\overline{PBW}_j$ is smaller than $\overline{SNF}_j$, for any $j > j^*$. If the criterion (27) were valid for higher groups of modes, this would mean that, for any $\overline{c}$ and $\overline{\Delta l}$, no matter how small, there always exists a threshold value j^* such that higher groups of modes are localized. However, the results obtained do not seem to confirm this hypothesis.

Fig. 9 shows the variation of the peak ratio $|A|$ in terms of the mode number, for various values of $\overline{c}$ and $\overline{\Delta l}$. For $\overline{c} = 1000$ and $\overline{\Delta l} = .01$, localization occurs in the first group of modes. Higher modes are still localized, but no more strongly than the first two modes. As a matter of fact, the peak ratio remains almost constant when the mode number increases. For $\overline{c} = 110$ and $\overline{\Delta l} = .019$, the first group of modes is not localized, and the peak ratio decreases only slightly in the higher modes, from .33 for the first group to a plateau value of .26 for the sixth group. Finally, for $\overline{c} = 0$ and $\overline{\Delta l} = .03$, the peak ratio decreases significantly from .64 for the first group to .42 for the fifth group. However, the modes do not become localized. Moreover, after the tenth mode, $|A|$ increases to reach .93 in the eighth group of modes, and goes back to .49 for the twentieth mode. In this case, $\overline{PBW}_j$ given by Eq. (31) decreases monotonically and one can show that localization ought to occur in the eighth or ninth groups of modes. However, it does not.

A few hypotheses can be formulated from the study of these few representative cases:

(a) If the modes of the first group are not localized, it seems that localization will not occur for the modes of higher groups either.

(b) If the first two modes are localized, then higher modes are also localized, but no more strongly than in the first group.

This suggests that higher modes do not significantly affect the occurence of localization. (b) is a reassuring result, since it states that localization does not disappear in higher modes. (a) is, of course, disappointing. It seems paradoxical that even though the criterion (27) is satisfied for higher modes, localization does not occur. However, a tentative explanation is as follows.

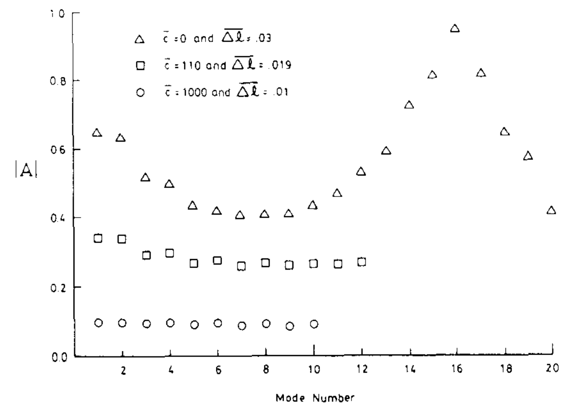

Figure 9. Variation of $|A|$ in terms of mode number.

Localization occurs for weakly coupled, weakly disordered systems. If $\overline{c}$ is small, the system is not weakly coupled in the first group of modes, neither is it in the higher modes. Hence, from a physical standpoint, if localization does not occur in the lower modes, it should not occur either in the higher ones. This shows that the criterion (27) **cannot** be used **independently** of the physical system to which it applies: the most important condition for localization to occur is to have a weakly coupled system, that is to have $1/\overline{c}$ **small**. If this requirement is met, then the criterion (27) can be applied effectively to determine the minimum value of mistuning $\overline{\Delta l}$ necessary to obtain localized modes.

Finally, one ought to mention that, even though the higher modes do not become localized for $\overline{c} = 0$, mistuning may have a significant effect, since for $\overline{\Delta l} = .03$ the peak ratio of the seventh mode is .40.

PART II : EXPERIMENT

II.1. Experimental Set-up

An experiment has been carried out to verify the existence of localized modes for disordered two-span beams. The vibration tests were performed on a spring steel beam resting on three supports. The beam was pinned at both ends. In addition, a third support with variable torsional stiffness was located near the middle of the beam, but could be moved to various locations. The experimental set-up is shown in Figs. 10.a and 10.b. The geometric dimensions of the specimen beam were 53 cm (length) and .0635x1.015 cm^2 (cross-section). The variable torsional stiffness of the intermediate support was created by a pinned-clamped beam, the distance between the pinned point and the clamped one being varied to adjust the torsional stiffness. The torsional beam was parallel to the specimen beam (see Fig. 10).

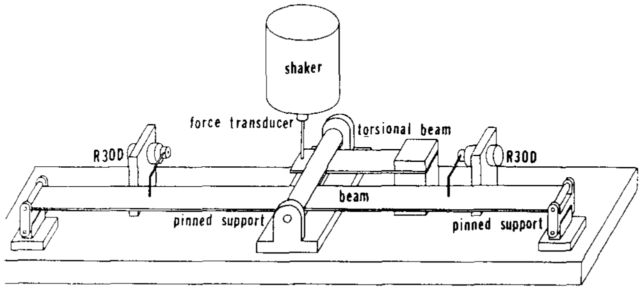

Figure 10.a. Experimental set-up: model.

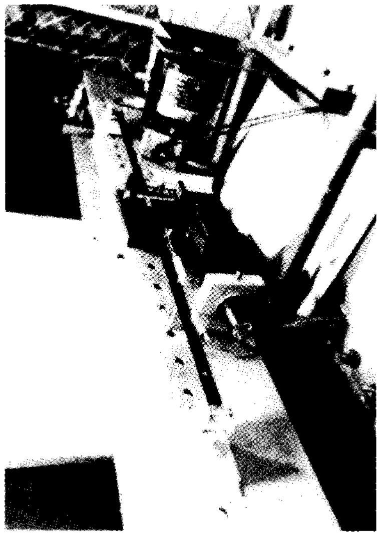

Figure 10.b. Experimental set-up: photograph.

The possibility of dynamic interaction between the two beams had to be considered. The frequency range of the first and second modes of the specimen beam was 19-40 Hz. Fig. 11 shows the measured dynamic torsional stiffness curve of the torsional beam near the pinned point, when the specimen beam was removed. It was found that the torsional stiffness remained essentially constant in the frequency range of interest (19-40 Hz). Since the fundamental natural frequency of the torsional beam was 2-3 times higher than the one of the specimen beam, the dynamic torsional stiffness did not vary significantly, and thus could be considered to be constant.

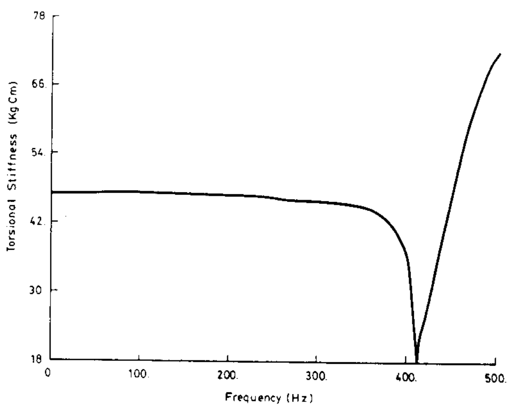

Figure 11. Dynamic torsional stiffness versus frequency.

The major equipment components for the vibration test were as follows:

- A sine generator provided a sweeping sinusoidal signal.
- A mini-shaker (B&K 4810) and a power amplifier (B&K 2706) were used to excite the beam. In order to reduce the effect of the additional mass of the joint components between the shaker and the beam on the vibration characteristics of the specimen beam, the driving point was located near the pinned end of the torsional beam (see Fig. 10). Thus a pure excitation torque was applied to the specimen beam near its intermediate support, without inducing any appreciable added mass effect. Because of the small mass of the specimen beam, this effect could have been potentially very important.
- A force transducer (B&K 8200) and two rotary-variable-differential transformers (R30D) were used to measure the exciting force and transverse beam displacement, respectively. The displacement transducers had only a small added mass.
- The charge amplifier was a portable conditioning amplifier (B&K 2635) which provided high voltage output sensitivity of the force. Digital voltage meters were used to record all signals and to analyze natural frequencies and mode shapes.

In addition to the effect of added mass, there were two other important considerations in the design of the experiment. The first one was concerned with minimizing the effect of the additional constraint due to jointing the R30D transformers with the specimen beam. In order to avoid additional stiffness constraint when large amplitude vibration occurs, the contact between the needle and the beam had to be sufficiently flexible. This requirement was met by using a flexible needle with a pinned end.

The second consideration concerned the design of the pinned end supports of the specimen beam. From a transient decay test, the critical damping ratio of the beam was found to be approximately .001. Thus, large amplitudes occured near the resonance frequencies. If the horizontal displacement of the end supports were constrained, the measured frequency was found to be dependent upon the level of the excitation torque, which is characteristic of a **nonlinear** system. Thus, in the experimental set-up, **two** degrees of freedom, namely rotation and horizontal displacement, were allowed at the pinned ends, in order to insure the **linearity** of the system. It was then found that the natural frequencies were independent of the excitation torque level, and that even when the response amplitude was very large, the system behaved in a linear fashion.

II.2. Experimental results. Comparison with Theory.

The displacement mobility concept (response displacement / excitation torque) was used to determine the natural frequencies. Fig. 12 shows a typical frequency response curve of displacement mobility at the measurement point $\bar{x} = .24$, for $\overline{\Delta l} = .05$ and $\bar{c} = 143$. The natural frequencies correspond to the peaks of the curve.

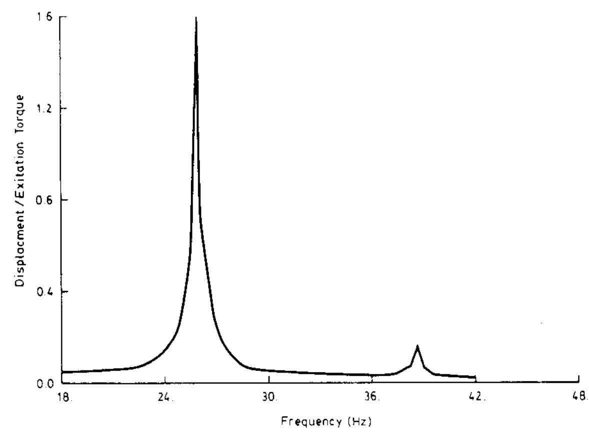

Figure 12. Displacement mobility versus frequency, for $\overline{\Delta l} = .05$ and $\bar{c} = 143$, measured at $\bar{x} = .24$.

Fig. 2 (see Part I) shows the dependence of the lower two natural frequencies upon torsional stiffness for a tuned beam ($\overline{\Delta l} = 0$), for both experimental and theoretical results. The agreement between theory and experiment is observed to be excellent. Note that the torsional stiffness was determined from the static stiffness measurement.

Fig. 13 displays the comparison between theoretical and experimental natural frequencies versus mistuning $\overline{\Delta l}$, for a coupling $\bar{c} = 281.8$. Again, the agreement is found to be excellent. Lack of space precludes the authors from presenting results obtained for other values of the coupling $\bar{c}$. Nevertheless, in all cases studied, the maximum discrepancy between theoretical and experimental results was always less than 2.5%.

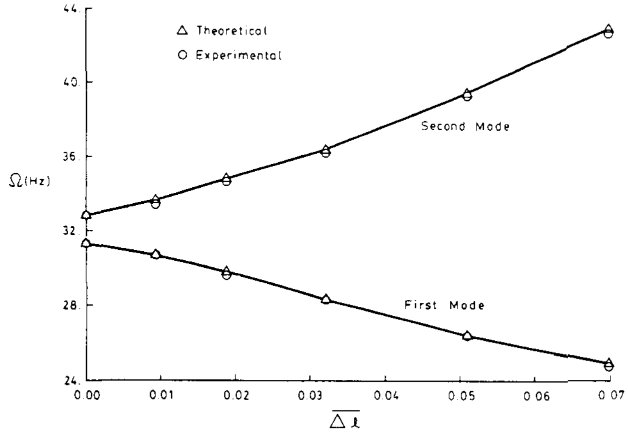

Figure 13. Comparison of experimental and theoretical natural frequencies of the first group of modes, for $\bar{c} = 281.8$.

Very good agreement was also found between theoretical and experimental results in terms of mode shapes. This can be observed on Figs. 14.a and 14.b, which display the peak ratio A

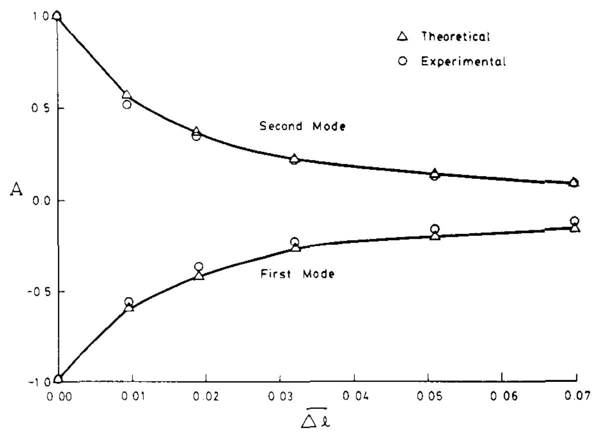

Figure 14.a

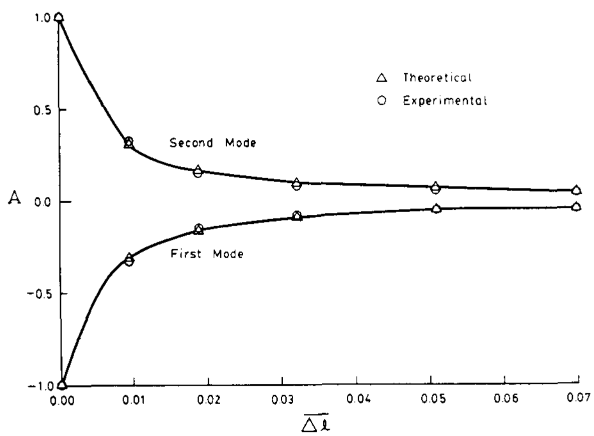

Figure 14.b

Figure 14. Comparison of experimental and theoretical peak ratio A for the first group of modes. (a): $\bar{c} = 90.4$; (b): $\bar{c} = 281.8$.

for the lower two modes in terms of mistuning $\overline{\Delta l}$, for values of the torsional spring constant $\bar{c} = 90.4$ and $\bar{c} = 281.8$, respectively. Peak ratios were also compared for other values of $\bar{c}$, but these results are not presented here. The maximum difference between theoretical and experimental data was always less than 15%. This error was mainly due to inaccuracies in the measurement of small response amplitudes, which were encountered for strongly localized modes ($\overline{\Delta l}$ large). For at very small amplitudes the ratio signal to noise of the transducers R30D becomes smaller.

Finally, Figs. 15 and 16 show the motion in the first mode, for a torsional spring constant $\bar{c} = 281.8$. Fig. 15 is for the tuned system, whereas Fig. 16 is for a slightly mistuned beam for which $\overline{\Delta l} = 2\%$. Figures marked (a) are obtained from displacement measurement. Figures marked (b) are stroboscopic photographs of the motion in the first mode, when the stroboscopic frequency coincides with the fundamental frequency of the beam. It is observed that for these values of $\overline{\Delta l}$ and $\bar{c}$, the first mode of the mistuned beam is strongly localized in the second span, whereas the one of the tuned beam is collective, that is the peak deflection is the same in both spans.

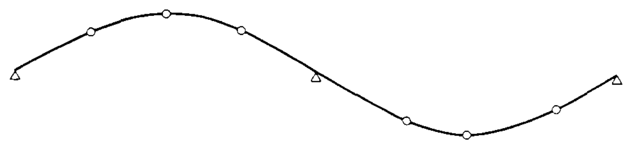

Figure 15.a

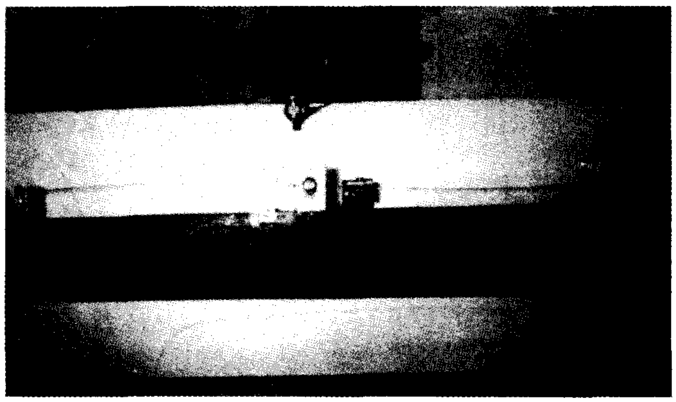

Figure 15.b

Figure 15. First mode shape of tuned two-span beam, for $\overline{\Delta l} = 0$ and $\bar{c} = 281.8$. (a): from measurements ; (b): stroboscopic photograph.

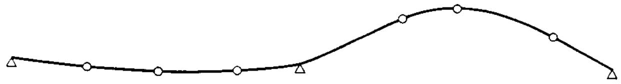

Figure 16.a

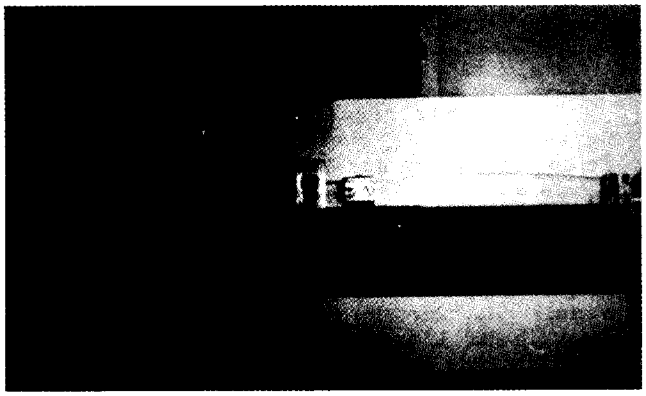

Figure 16.b

Figure 16. First mode shape of mistuned two-span beam, for $\overline{\Delta l} = .02$ and $\bar{c} = 281.8$. (a): from measurements ; (b): stroboscopic photograph.

CONCLUDING REMARKS

The free modes of vibration of disordered two-span beams subject to a torsional spring at the intermediate constraint location have been investigated, both theoretically and experimentally. Disorder is achieved by slightly moving the location of the constraint from the middle of the beam. The following conclusions can be drawn:

- For small mistuning and large values of the torsional spring constant, the free modes of vibration become localized in one of the two spans.
- A Modified Perturbation Method has been developed. It predicts localized modes accurately and is cost effective. Moreover, it provides physical insight into the localization phenomenon.
- For the **first group** of modes, localization occurs if the relative pass band width of the tuned beam is of the order of, or smaller than the relative spread in the frequencies of the individual spans, and if these two quantities are both **small**.
- From preliminary results, it is suspected that if localization does not occur in the lower two modes, then it does not occur in the higher ones either. On the other hand, if the first two modes are localized, then the higher ones are also localized.
- An experiment has been carried out to verify the existence of localized modes for disordered two-span beams. Excellent agreement has been found with theoretical results, thus confirming the existence of localized modes.
- An immediate generalization of the present study is to investigate the localization of vibrations for n-span beams, where $n > 2$. Future work is also in order concerning localized vibrations of **two-dimensional** structures and the behavior of disordered structures under forced excitation.

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