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Entropy Production in Radiation-Affected Boundary Layers

Ahmet Selamet and Vedat S. Arpaci,
Department of Mechanical
Engineering and Applied Mechanics,
The University of Michigan, Ann
Arbor, Michigan

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AHMET SELAMET and VEDAT S. ARPACI

Department of Mechanical Engineering and Applied Mechanics

The University of Michigan, Ann Arbor, Michigan

ABSTRACT

A one-dimensional divergence of the monochromatic heat flux for wall-affected attenuating thingas,

$$\frac{dq_\nu^R}{dy} = 4\kappa_\nu \left\{ \left[E_{b\nu} - E_{b\nu}(\infty)\right] - \frac{\epsilon_\nu}{2}\left[E_{b\nu}(-0) - E_{b\nu}(\infty)\right] E_2(\tau_\nu) \right\} ,$$

is developed from general considerations. Here $E_{b\nu}$, κ_ν, ϵ_ν and τ_ν respectively denote the monochromatic values of the emissive power, absorption coefficient of gas, wall emissivity and optical thickness; E_2 is the second exponential integral, y the coordinate normal to boundary. The model applies to semi-infinite geometry.

For the spectral average of the heat flux divergence (needed for radiation-affected thermal transport), new definitions are introduced (including the wall and attenuation effects) for the absorption coefficient of gas and the wall emissivity. This heat flux is applied to thermal boundary layer over a horizontal flat plate. An explicit expression for the local Nusselt number involving both conduction and radiation is shown to be

$$\frac{Nu_x}{Nu_x^K} = \frac{(-d\theta/dy|_w)}{(-d\theta/dy|_w)^K} + \frac{3}{4}\epsilon_w P \left(\frac{\tau_x}{Nu_x^K}\right)\left(1 - \frac{5}{4}\frac{\tau_x}{Nu_x^K}\right) ,$$

Nu_x^K being the Nusselt number for pure conduction, $(-d\theta/dy|_w)^K$ the wall gradient of temperature for pure conduction, ϵ_w the wall emissivity, P the ratio of emission to conduction, and τ_x the local optical thickness.

The effect of radiation on the thermal part of entropy production is demonstrated in terms of the forced convection over a flat plate.

1. INTRODUCTION

The inherent complexity of radiation affected thermal energy transport has forced researchers in the past to development of models for the radiative heat flux valid either for small or large values of the optical thickness. The prime concern of these models has been the incorporation of boundary effects into the well-known astrophysical models for thingas and thick (Rosseland) gas.

In a study involving the effect of radiation on boundary layers in buoyancy driven flows, Arpaci[1] developed a thick gas model which includes boundaries. Although obtained in connection with a particular problem, the model was expected to be of general nature which in fact was later shown to be the case by Arpaci and Larsen.[2] The same model was used by Lord and Arpaci[3] in studying the radiation effect on forced convection boundary layers. Another boundary affected thick gas model was proposed by Viskanta[4] and Anderson and Viskanta[5] and the model was compared with Arpaci model by Viskanta and Anderson.[6]

For the other end of optical thickness, and in connection with both forced and natural convection boundary layers, Cess[7,8] developed models for attenuating thingas far from boundaries and for nonattenuating thingas. Also, in another forced convection boundary layer study, Lord and Arpaci[3] developed an attenuating thingas model. In spite of these efforts , the development of a thingas model strictly from general considerations, including especially spectral effects and the definition of a wall affected absorption coefficient apparently remained untreated. One of the prime motivations of the present study is the development of this model. The other is to study the entropy production in radiation-affected boundary layers in terms of the model.

The study consists of 7 sections: following this introduction, Section 2 develops the thingas model, Section 3 applies the model to the forced convection boundary layer over a horizontal flat plate. Section 4 considers two solution methods for the problem, Section 5 develops the transport aspects of local entropy production and applies them to the present problem, Section 6 deals with the heat transfer from the wall and relates the wall entropy production to the local Nusselt number, and Section 7 concludes the study with some final remarks.

2. A THINGAS MODEL

The one-dimensional heat flux associated with monochromatic radiation in semi-infinite domain is available in the literature (see, for example, Cess,[7] Arpaci and Larsen[2] for semi-infinite domain and References [9–13] for finite domain). This flux for diffuse radiation with negligible scattering is

$$q_\nu^R(\tau_\nu) = 2\, B_\nu\, E_3(\tau_\nu) + 2\int_0^{\tau_\nu} E_{b\nu} E_2(\tau_\nu - \tau_\nu')\,d\tau_\nu' - 2\int_{\tau_\nu}^{\infty} E_{b\nu} E_2(\tau_\nu' - \tau_\nu)\,d\tau_\nu' \ . \tag{1}$$

where B_ν being the monochromatic surface radiosity, $E_{b\nu}$ the monochromatic emissive power, E_2 and E_3 the second and third exponential integrals, respectively, τ_ν the monochromatic optical thickness, and τ_ν' a dummy variable. The wall value of this flux for $\tau_\nu = 0$, noting $E_3(0) = 1/2$, is

$$q_\nu^R(0) = B_\nu - 2\int_0^\infty E_{b\nu} E_2(\tau_\nu')\,d\tau_\nu' \ . \tag{2}$$

Also, by definition,

$$q_\nu^R(0) = B_\nu - G_\nu \tag{3}$$

where G_ν is the monochromatic radiation incident on the wall. From Eqs. (2) and (3),

$$G_\nu = 2\int_0^\infty E_{b\nu} E_2(\tau_\nu')\,d\tau_\nu' \ . \tag{4}$$

Again, by definition,

$$B_\nu = \epsilon_\nu E_{b\nu}(-0) + \rho_\nu G_\nu \tag{5}$$

where $E_{b\nu}(-0)$, ϵ_ν *and* ρ_ν *denoting the monochromatic values* of wall emissive power, wall emissivity and reflectivity, respectively. Elimination of G_ν between Eqs. (4) and (5) gives

$$B_\nu = \epsilon_\nu E_{b\nu}(-0) + 2\rho_\nu \int_0^\infty E_{b\nu} E_2(\tau_\nu')\,d\tau_\nu' \ , \tag{6}$$

and in terms of Eq. (6), Eq. (1) *becomes*

$$q_\nu^R(\tau_\nu) = 2\,\epsilon_\nu\, E_{b\nu}(-0) E_3(\tau_\nu) + 4\,\rho_\nu\, E_3(\tau_\nu) \int_0^\infty E_{b\nu} E_2(\tau_\nu')\,d\tau_\nu' + 2\int_0^{\tau_\nu} E_{b\nu} E_2(\tau_\nu - \tau_\nu')\,d\tau_\nu' - 2\int_{\tau_\nu}^\infty E_{b\nu} E_2(\tau_\nu' - \tau_\nu)\,d\tau_\nu' \ . \tag{7}$$

Here the first term on the right is the wall emission being attenuated up to the generic point τ_ν in the gas, the second is the integrated effect of monochromatic gas emission incident on the wall, being reflected from wall and *attenuated up to* τ_ν, *the* third is the integrated effect of the monochromatic emission of gas over $(0, \tau_\nu)$, and finally the fourth is the integrated effect of the monochromatic emission of gas over (τ_ν, ∞).

Some arrangement of the above equation yields

$$q_\nu^R(\tau_\nu) = 4\,\Big\{ E_3(\tau_\nu) \int_0^\infty E_{b\nu} E_2(\tau_\nu')\,d\tau_\nu' + \frac{1}{2}\left[\int_0^{\tau_\nu} E_{b\nu} E_2\,(\tau_\nu - \tau_\nu')\,d\tau_\nu' - \int_{\tau_\nu}^\infty E_{b\nu} E_2(\tau_\nu' - \tau_\nu)\,d\tau_\nu'\right] + \frac{\epsilon_\nu}{2}\left[E_{b\nu}(-0) - 2\int_0^\infty E_{b\nu} E_2(\tau_\nu')\,d\tau_\nu'\right] E_3(\tau_\nu) \Big\} \ . \tag{8}$$

On boundaries, Eq. (8) reduces to

$$q_\nu^R(0) = \epsilon_\nu \left[E_{b\nu}(-0) - 2\int_0^\infty E_{b\nu} E_2(\tau_\nu)\,d\tau_\nu\right] . \tag{9}$$

Also, useful for thingas studies is the divergence of Eq. (8)

$$\frac{dq_\nu^R}{d\tau_\nu} = 4\Big\{ E_{b\nu} - E_2(\tau_\nu) \int_0^\infty E_{b\nu} E_2(\tau_\nu')\,d\tau_\nu' - \frac{1}{2}\int_0^\infty E_{b\nu} E_1\big(|\tau_\nu - \tau_\nu'|\big)\,d\tau_\nu' - \frac{\epsilon_\nu}{2}\,\Big\{\Big[E_{b\nu}(-0) - 2\int_0^\infty E_{b\nu} E_2(\tau_\nu')\,d\tau_\nu'\Big] E_2(\tau_\nu)\Big\}\Big\} \tag{10}$$

where E_1 being the first exponential integral. Note that both Eqs. (8) and (10) are general, and apply for any optical thickness. To simplify these for thingas, assume the variation of $E_{b\nu}$ over a semi-infinite domain from $E_{b\nu}(-0)$ to $E_{b\nu}(\infty)$ to have negligible effect on the integrals involved with Eqs. (8) and (10). Thus

$$\int_0^\infty E_{b\nu} E_2(\tau_\nu')\,d\tau_\nu' \simeq - E_{b\nu}(\infty) E_3(\tau_\nu)\Big|_0^\infty = +\frac{1}{2} E_{b\nu}(\infty)$$

$$\int_0^\infty E_{b\nu} E_1\big(|\tau_\nu - \tau_\nu'|\big)\,d\tau_\nu' \simeq \big[2 - E_2(\tau_\nu)\big]\, E_{b\nu}(\infty)$$

Then, Eq. (10) readily gives

$$\frac{dq_\nu^R}{dy} = 4\kappa_\nu \Big\{ \big[E_{b\nu} - E_{b\nu}(\infty)\big] - \frac{\epsilon_\nu}{2}\big[E_{b\nu}(-0) - E_{b\nu}(\infty)\big] E_2(\tau_\nu) \Big\} . \tag{11}$$

where y is the coordinate normal to wall.

Since the boundary heat flux is controlled by the gas behavior near boundaries, its simplification for thingas requires a treatment different than the one employed for the divergence of heat flux given by Eq. (10). Let the integral involved with Eq. (9) be split into two, one over the interval $[0, \tau_{\nu\Delta}]$ and the other over the interval $[\tau_{\nu\Delta}, \infty]$, $\tau_{\nu\Delta}$ denoting the thickness of the conduction boundary layer. Over this boundary layer, $E_{b\nu}$ satisfies apparently the wall emissive power, $E_{b\nu}(-0)$, and

$$dE_{b\nu}(\tau_{\nu\Delta})/d\tau_\nu \simeq 0 \ , \quad E_{b\nu}(\tau_{\nu\Delta}) \simeq E_{b\nu}(\infty) \ , \tag{12}$$

and, in view of boundary thermal energy balance,

$$d^2E_{b\nu}(-0)/d\tau_\nu^2 \neq 0 \tag{13}$$

which, for the limit of weak radiation,

$$d^2E_{b\nu}(-0)/d\tau_\nu^2 \to 0 \ . \tag{14}$$

A polynomial approximation satisfying Eqs. (12) and (14) is

$$\frac{E_{b\nu} - E_{b\nu}(-0)}{E_{b\nu}(-0) - E_{b\infty}} = \frac{1}{2}\left(\frac{\tau_\nu}{\tau_{\nu\Delta}}\right)^3 - \frac{3}{2}\left(\frac{\tau_\nu}{\tau_{\nu\Delta}}\right) , \tag{15}$$

or, satisfying Eqs. (12) and (13) is

$$\frac{E_{b\nu} - E_{b\nu}(-0)}{E_{b\nu}(-0) - E_{b\infty}} = \left(\frac{\tau_\nu}{\tau_{\nu\Delta}}\right)^2 - 2\left(\frac{\tau_\nu}{\tau_{\nu\Delta}}\right) . \tag{16}$$

In terms of these profiles, the wall heat flux is found to be

$$q_\nu^R(0) = \epsilon_\nu [E_{b\nu}(-0) - E_{b\nu}(\infty)] (1 - \beta\tau_{\nu\Delta}) \qquad (17)$$

with

$$2/3 \le \beta \le 3/4 \ ,$$

upper limit corresponding to the diminishing effect of radiation. Clearly, Eq. (17) shows the separate enclosure and thingas effects.

Now, introducing a boundary-affected Planck mean absorption coefficient,

$$\kappa_P = \frac{\displaystyle\int_0^\infty \kappa_\nu \left\{ \left[E_{b\nu} - E_{b\nu}(\infty)\right] - \frac{\epsilon_\nu}{2}\left[E_{b\nu}(-0) - E_{b\nu}(\infty)\right] E_2(\tau_\nu) \right\} d\nu}{\left[(E_b - E_{b\infty}) - \frac{\epsilon_w}{2}(E_{bw} - E_{b\infty})\, E_2(\tau)\right]} , \qquad (18)$$

together with a thingas-affected wall emissivity,

$$\epsilon_w = \frac{\displaystyle\int_0^\infty \epsilon_\nu \left[E_{b\nu}(-0) - E_{b\nu}(\infty)\right] E_2(\tau_\nu) d\nu}{(E_{bw} - E_{b\infty})\, E_2(\tau)} \ , \quad \tau_\nu \neq 0 \qquad (19)$$

the divergence of monochromatic thingas flux of Eq. (11) is reduced to a spectrally-weighted thingas model as

$$\frac{dq_y^R}{dy} = 4\,\kappa_P \left[(E_b - E_{b\infty}) - \frac{\epsilon_w}{2}(E_{bw} - E_{b\infty})\, E_2(\tau)\right] \ , \qquad (20)$$

where the first difference between emissive powers shows the gas effect and the second difference denotes the attenuating wall effect.

Introducing the emissivity

$$\epsilon_w = \frac{\displaystyle\int_0^\infty \epsilon_\nu \left[E_{b\nu}(-0) - E_{b\nu}(\infty)\right] (1 - \beta\tau_{\nu\Delta}) d\nu}{(E_{bw} - E_{b\infty})(1 - \beta\tau_\Delta)} \ , \quad \tau_\nu = 0 \ , \qquad (21)$$

the monochromatic thingas flux given by Eq. (17) is reduced to a spectrally-weighted thingas flux as

$$q^R(0) = \epsilon_w (E_{bw} - E_{b\infty}) (1 - \beta\tau_{\nu\Delta}) \ . \qquad (22)$$

The different definitions for the wall emissivity given by Eqs. (19) and (21) should be noted.

The development from strictly general considerations of a monochromatic and wall-affected heat flux divergence for thingas given by Eq. (11) and the definitions of an absorption coefficient and a wall emissivity which include attenuating thingas and boundary effects and given by Eqs. (18) and (19) appears so far to remain untreated. In connection with radiation-affected boundary layer studies, Cess[7,8] considers the special case of Eq. (20) for $E_2(\tau) \simeq 1$ within boundary layers and for $E_b \simeq E_{b\infty}$ outside boundary layers, and Lord and Arpaci[3] in connection with a forced convection boundary layer study develop a model similar to Eq. (20).

Foregoing general considerations are applied to a boundary layer problem in the following section.

3. RADIATION AFFECTED FORCED CONVECTION

Consider the effect of radiation on the forced convection boundary layer over a horizontal flat plate. In a low speed flow, provided the difference between the temperature of the free stream and that of the wall is not too great (so that the density is sensibly constant) the momentum equation is decoupled from the thermal energy and may be solved separately. Furthermore, for heat transfer studies, rather than utilizing the velocity profiles, a good approximation of these profiles near boundaries is needed. This approach, in the absence of radiation, is well-known and has been studied extensively. The outstanding works are Fage and Falkner,[14] Lighthill,[15] Spalding[16] and Liepmann[17] (summarized in the monograph by Curle[18]). Also, the extension of the approach to the limiting cases of $Pr \ll 1$ and $Pr \gg 1$ are discussed in Arpaci and Larsen.[19] Since the case of $Pr \ll 1$ is for opaque fluids and has no application to radiation-affected problems, and the case of $Pr \gg 1$ is known to approximate for all fluids with $Pr \ge 1$, here only the latter case is considered.

Replacing the longitudinal velocity by its tangent on the wall and using this velocity in the conservation of mass to determine the transversal velocity, and including the radiation effect in terms of Eq. (20), the thermal energy balance gives

$$\rho c_p \left[y \left(\frac{\tau_w}{\mu}\right) \frac{\partial T}{\partial x} - \frac{1}{2} y^2 \frac{d}{dx}\left(\frac{\tau_w}{\mu}\right) \frac{\partial T}{\partial y} \right] = k \frac{\partial^2 T}{\partial y^2} - \frac{\partial q_y^R}{\partial y} \qquad (23)$$

subject to

$$\frac{\partial q_y^R}{\partial y} = 4\,\kappa_P \left[(E_b - E_{b\infty}) - \frac{\epsilon_w}{2}(E_{bw} - E_{b\infty})\, E_2(\tau)\right] \ , \qquad (20)$$

where τ_w denotes the wall shear stress, μ the dynamic viscosity, ρ the density, c_p the specific heat. The boundary conditions to be satisfied are

$$T(0, y) = T_\infty \ , \quad T(x, 0) = T_w \ , \quad T(x, \infty) = T_\infty \ , \qquad (24)$$

The next section deals with two solutions of the foregoing formulation.

4. TWO SOLUTIONS

Case I: Complete Solution for Pr ≫ 1

This case deals with the domain

$$0 \le y \le \Delta$$

and includes the effect of conduction as well as viscosity. A formulation in terms of a similarity variable including both conduction and radiation is not feasible because of intrinsic lack of similarity between conduction and radiation. However, the effect of thingas radiation on conduction is small. This fact suggests the use of the similarity variable for conduction by which the radiation effect can be treated locally similar.

Introducing

$$\eta = y/g(x)$$

(see, for example, Arpaci and Larsen[19]), into Eq. (23) leads to the equation satisfied g(x),

$$\left(\frac{\tau_w}{\mu}\right)\frac{dg^3}{dx} + \frac{3}{2}g^3\frac{d}{dx}\left(\frac{\tau_w}{\mu}\right) = \alpha$$

which readily gives

$$g(x) = \frac{\left[\alpha \int_0^x (\tau_w/\mu)^{1/2}\,dx\right]^{1/3}}{(\tau_w/\mu)^{1/2}}$$

and

$$\eta = \frac{(\tau_w/\mu)^{1/2}\,y}{\left[\alpha \int_0^x (\tau_w/\mu)^{1/2}\,dx\right]^{1/3}} \,. \tag{25}$$

In terms of Eq. (25) and the approximation $E_2 \simeq \exp(-\sqrt{3}\tau)$, Eqs. (20) and (23) are reduced to

$$\frac{d^2\theta}{d\eta^2} + \frac{1}{3}\eta^2\frac{d\theta}{d\eta} = \chi P \gamma^2 x \left(\Theta^4 - \frac{\epsilon_w}{2}e^{-\gamma x^{1/2}\eta}\right) \tag{26}$$

subject to $\theta(0)=1$ and $\theta(\infty)=0$. Here, $\chi=(\kappa_P/\kappa_R)^{1/2}$ is the weighted nongrayness, κ_R the Rosseland mean absorption coefficient and

$$\theta = \frac{T - T_\infty}{T_w - T_\infty} \;,\quad \Theta^4 = \frac{T^4 - T_\infty^4}{T_w^4 - T_\infty^4} \;,\quad \gamma = \sqrt{3}\,\kappa_M G$$

$$g = G\,x^{1/2} \;,\quad G = \left[\frac{4\alpha/3}{0.332\,U_\infty(U_\infty/\nu)^{1/2}}\right]^{1/3}$$

$$P = \frac{4}{3}\frac{\sigma(T_w^4 - T_\infty^4)}{k(T_w - T_\infty)\kappa_M} \simeq \frac{\text{Emission}}{\text{Conduction over } \kappa_M^{-1}} \tag{27}$$

$\kappa_M=(\kappa_P\kappa_R)^{1/2}$ being the mean absorption coefficient. As $P \to 0$, the effect of radiation diminishes and Eq. (26) reduces to the case of pure conduction, as expected.

A solution following asymptotic matching of an inner solution based on conductive boundary layer and an outer solution based on radiative boundary layer is somewhat involved because of the transcendental nature of the latter. A local similarity approach [integrating Eq. (26) for a fixed x] is straightforward and is pursued here. Equation (26) was solved first as a boundary-value problem by using the finite difference code PASVA3 developed by Lentini and Pereyra.[20,21] Results for pure conduction agree to five decimals with those obtained from the well-known (integral) solution evaluated by using a 15-point Gauss-Legendre quadrature.[22] Equation (26) was solved also as an initial-value problem[23] depending on the wall gradient of temperature obtained from PASVA3. The single step code DVERK based on a fifth and sixth order Runge Kutta – Verner approximation developed by *Hull et al.*[24] *was utilized.* The results obtained separately from PASVA3 and DVERK are found to agree to five decimals. Figure 1 shows the variation of θ against η, for pure conduction which can be obtained by letting the right hand side of Eq. (26) equal to zero, and combination of conduction and radiation as expressed by Eq. (26). The present study utilizes air properties at the film temperature and assumes $U_\infty = 2$m/s.

Case II: Pure Radiative Solution

For the asymptotic case corresponding to $Pr \to \infty$, the viscous boundary layer δ is much thicker than the conductive boundary layer Δ. In the region bounded by these layers,

$$\Delta \le y < \delta \,,$$

the effect of conduction is negligible. Considering the approximations for u and v employed in the preceding case, and treating the derivatives of temperature in the manner that led to Eq. (26), it can be shown that

$$v\frac{\partial T}{\partial y} = -\frac{1}{2}u\frac{\partial T}{\partial x} \,.$$

Thus, the thermal energy, Eq. (23), may be reduced to

$$\frac{1}{2}\rho c_p\,y\left(\frac{\tau_w}{\mu}\right)\frac{\partial T}{\partial x} = -\frac{\partial q_y^R}{\partial y} \,. \tag{28}$$

or, in terms of Eq. (20), to

$$\frac{1}{2}\rho c_p\,y\left(\frac{\tau_w}{\mu}\right)\frac{\partial T}{\partial x} = 4\,\kappa_P\left[\frac{\epsilon_w}{2}(E_{bw} - E_{b\infty})E_2(\tau) - (E_b - E_{b\infty})\right] . \tag{29}$$

Employing

$$\tau_w/\mu = 0.332\,U_\infty\,(U_\infty/\nu x)^{1/2} \;,$$

and the Stefan-Boltzmann law,

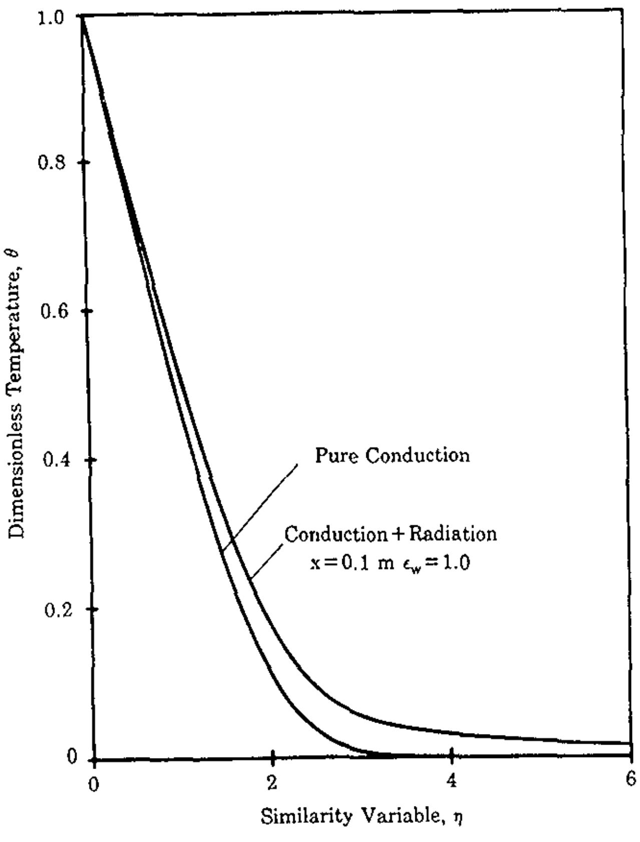

Figure 1

$$E_b = \sigma T^4 \ ,$$

Eq. (29) may be rearranged as

$$\frac{\partial T}{\partial x} = \frac{24\ \sigma\ \kappa_P\ x^{1/2}}{\rho\ c_p\ U_\infty (U_\infty/\nu)^{1/2}\ y} \left[\frac{\epsilon_w}{2}(T_w^4 - T_\infty^4)E_2(\tau) - (T^4 - T_\infty^4)\right] \quad (30)$$

or, in terms of an effective temperature, say

$$T_e(\tau) = \left[\frac{\epsilon_w}{2}(T_w^4 - T_\infty^4)E_2(\tau) \ + \ T_\infty^4\right]^{1/4} , \quad (31)$$

and the dimensionless temperature

$$\theta^* = T/T_e \ ,$$

as

$$\frac{\partial \theta^*}{\partial x} = \frac{24\ \kappa_P\ \sigma\ T_e^3\ x^{1/2}}{\rho\ c_p\ U_\infty (U_\infty/\nu)^{1/2}\ y}(1 - \theta^{*4}) \ . \quad (32)$$

Separation of variables, integration, and the use of $\theta^*(x,\infty) = \theta^*_\infty$ for the integration constant readily gives the transcendental closed form solution

$$\ell n \left[\left(\frac{1+\theta^*}{1+\theta^*_\infty}\right)\left(\frac{1-\theta^*_\infty}{1-\theta^*}\right)\right] + 2[\tan^{-1}\theta^* - \tan^{-1}\theta^*_\infty] = \frac{16\chi B_e {\tau_x}^{3/2}}{R^{1/2}\tau_y} \quad (33)$$

where τ_x and τ_y being the optical thicknesses based on κ_M in longitudinal and transversal directions, respectively. B_e is the Boltzmann number based on T_e, and R is the Reynolds number defined as follows

$$B_e = 4\sigma T_e^4/\rho c_p U_\infty T_e \quad , \quad R = U_\infty/\nu\kappa_M \ .$$

The next section first develops an expression for local entropy production in radiatively participating media and then applies the result to the present forced convective boundary layer problem.

5. LOCAL ENTROPY PRODUCTION

The interpretation of the contemporary problems of thermomechanics in terms of entropy production is lately receiving increased attention. Because of its size no attempt will be made here to survey the literature (see, for example, Bejan[25,26] for applications involving heat transfer and Arpaci[27,28] and Arpaci and Selamet[29-31] for applications involving radiation and flames). The following brief review on the local entropy production is for later convenience.

The development of the entropy production in moving media requires the consideration of the momentum balance as well as the energy balance. For the Stokesean fluid, the momentum balance in terms of the usual nomenclature is

$$\rho\frac{Dv_i}{Dt} = -\frac{\partial p}{\partial x_i} + \frac{\partial \tau_{ij}}{\partial x_j} + \rho f_i \ . \quad (34)$$

The entropy balance (the Second Law balanced by the local entropy production) is

$$\rho\frac{Ds}{Dt} = -\frac{\partial}{\partial x_i}\left(\frac{q_i}{T}\right) + s''' \ , \quad (35)$$

where s''' denotes the local entropy production. Also, the conservation of total (thermomechanical) energy (or the First Law) including the heat flux expressed in terms of the entropy flux,

$$\frac{\partial q_i}{\partial x_i} \equiv \frac{\partial}{\partial x_i}\left[\left(\frac{q_i}{T}\right)T\right] = T\frac{\partial}{\partial x_i}\left(\frac{q_i}{T}\right) + \left(\frac{q_i}{T}\right)\frac{\partial T}{\partial x_i} , \quad (36)$$

is

$$\rho\frac{D}{Dt}\left(u + \frac{1}{2}v_i^2\right) = -\frac{\partial}{\partial x_i}\left[\left(\frac{q_i}{T}\right)T\right] - \frac{\partial}{\partial x_i}(pv_i) + \frac{\partial}{\partial x_j}(\tau_{ij}v_i) + \rho f_i v_i + u''' . \quad (37)$$

Now, the fundamental difference,

$$\text{Total energy} - (\text{Momentum})v_i - (\text{Entropy})T \quad (38)$$

in terms of Eqs. (34), (35), (37) and the conservation of mass,

$$\frac{D\rho}{Dt} + \rho\frac{\partial v_i}{\partial x_i} = 0 \ , \quad (39)$$

yields,

$$\rho\left(\frac{Du}{Dt} - T\frac{Ds}{Dt} + p\frac{Dv}{Dt}\right) = -\left(\frac{q_i}{T}\right)\frac{\partial T}{\partial x_i} + \tau_{ij}s_{ij} + u''' - Ts''' , \quad (40)$$

where s_{ij} is the rate of deformation. For a reversible process, all forms of dissipation vanish, and

$$\left(\frac{Du}{Dt} - T\frac{Ds}{Dt} + p\frac{Dv}{Dt}\right) = 0 \quad (41)$$

which is the Gibbs Thermodynamic relation. For an irreversible process, Eq. (41) continues to hold provided the process can be assumed in local equilibrium. Then, the local entropy production is found to be

$$s''' = \frac{1}{T}\left[-\left(\frac{q_i}{T}\right)\left(\frac{\partial T}{\partial x_i}\right) + \tau_{ij}s_{ij} + u'''\right] , \quad (42)$$

where the first term in brackets denotes the dissipation of thermal energy into entropy (lost heat), the second term denotes the dissipation of mechanical energy into heat (lost work), and the third term denotes the dissipation of any (except thermomechanical) energy into heat. When radiation is appreciable, q_i denotes the total flux involving the sum of the conductive flux and the radiative flux,

$$q_i = q_i^K + q_i^R \ . \quad (43)$$

In the present study, neglecting contribution due to viscous dissipation and noting that conductive and radiative heat fluxes are considered only in transversal direction, Eq. (42) may be rearranged as

$$s''' = -\frac{1}{T^2}\left(q_y^K + q_y^R\right)\left(\frac{\partial T}{\partial y}\right). \quad (44)$$

The conductive constitution, expressing T *in terms of* θ *from* Eq. (27)

$$q_y^K = -\frac{k}{g}\frac{d\theta}{d\eta}\left(T_w - T_\infty\right), \quad (45)$$

where η and g are defined by Eqs. (25) and (27), respectively. Inserting T, radiative heat flux approximated by its wall value neglecting the second order correction term in Eq. (22), and conductive heat flux expressed by Eq. (45), into Eq. (44), the volumetric local entropy production

$$s_\eta''' = \frac{\left(-\frac{d\theta}{d\eta}\right)}{g\left(\theta+\frac{T_\infty}{T_w-T_\infty}\right)^2}\left[-\frac{k}{g}\frac{d\theta}{d\eta} + \epsilon_w\sigma(T_w+T_\infty)(T_w^2+T_\infty^2)\right]. \quad (46)$$

Figure 2 shows the variation of nondimensionalized (with respect *to wall value) entropy production* s_η'''/s_w''' *against similarity* variable η for three wall temperatures; 400 K, 500 K, and 600 K where T_∞ is 300 K. For T_w=500 K, Fig. 3 depicts the variation of s_η''' against η, for pure conduction, conductive and total (conductive+radiative) components in combined conduction and radiation problems.

6. HEAT TRANSFER

Radiation problems are usually linearized before any *attempt for a solution. Thus the solution complexity due to* nonlinear radiation is eliminated at the expense of sacrificed quantitative physics.

The linearization about a mean temperature T_M yields

$$\Theta^4 \rightarrow \theta \;,\; P \rightarrow 4\mathcal{P}$$

where

$$\mathcal{P} = \frac{4\sigma T_M^4}{3kT_M\kappa_M} \simeq \frac{\text{Emission}}{\text{Conduction over } \kappa_M^{-1}} \quad (47)$$

is the Planck number, and

$$T_M = \left(\frac{\epsilon_w T_w^4 + T_\infty^4}{\epsilon_w + 1}\right)^{1/4} \quad (48)$$

is the mean temperature obtained from an energy balance on the transparent limit of thingas. With this linearization, Eq. (26) is reduced to

$$\frac{d^2\theta}{d\eta^2} + \frac{1}{3}\eta^2\frac{d\theta}{d\eta} = 4\,\mathcal{P}\,\chi\,\gamma^2 x\left(\theta - \frac{\epsilon_w}{2}e^{-\gamma x^{1/2}\eta}\right) \quad (49)$$

subject to the boundary conditions of the nonlinear problem.

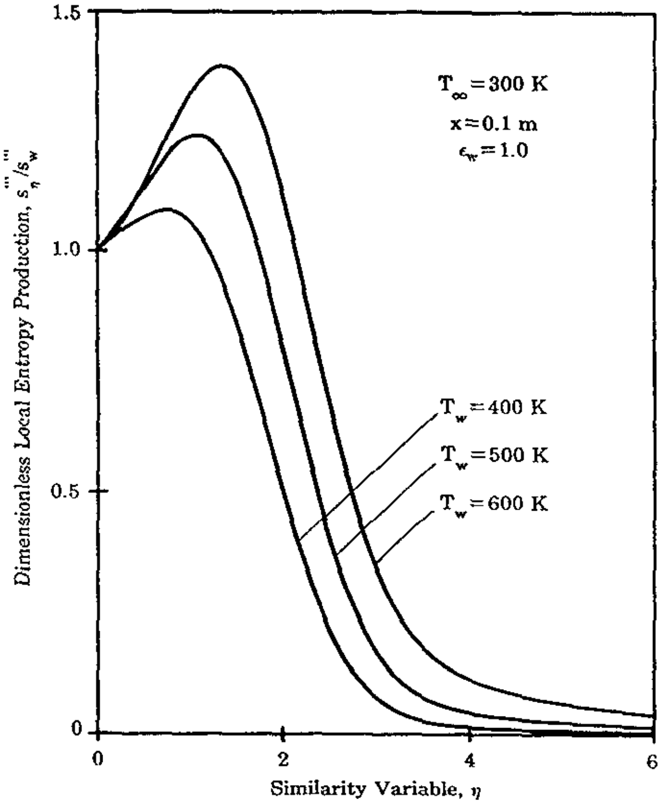

Figure 2

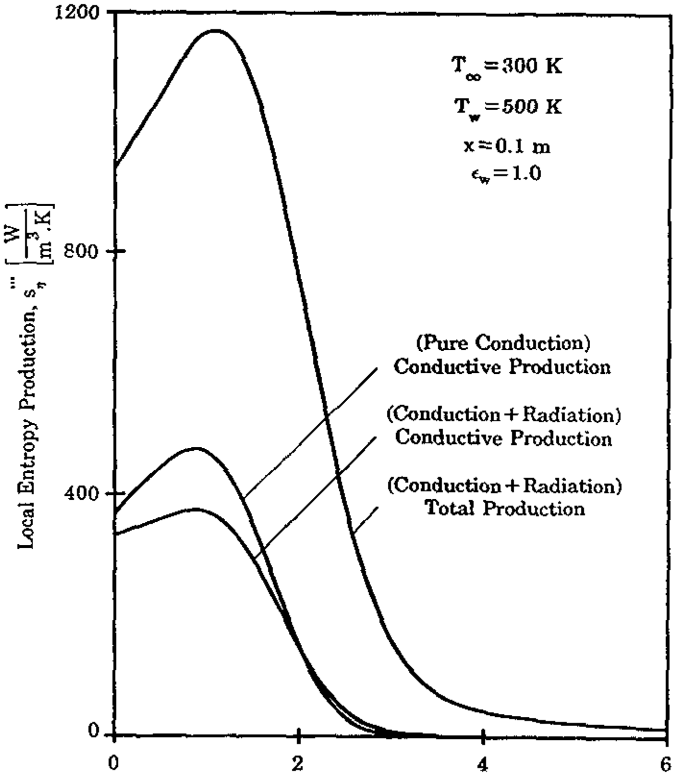

Figure 3

A thermal boundary layer study based on approximate velocity profiles is known to yield satisfactory results for heat transfer. Accordingly, the present study is well suited for evaluation of a total Nusselt number including the effects of both conduction and radiation.

The total heat flux on boundaries,

$$q_w = q_w^C + q_w^R \ , \tag{50}$$

q_w^C being available from a usual boundary approach and q_w^R being already evaluated from strict radiative considerations based on the spectral average of the monochromatic heat flux [leading to Eq. (22)]. However, since the temperature distribution is affected by both conduction and radiation, the radiative heat flux given by Eq. (22) should have an explicit effect of conduction. To demonstrate this effect, consider the spectral average of heat flux given by Eq. (9),

$$q_w^R = \epsilon_w \left[E_{bw} - 2\textstyle\int_0^\infty E_b E_2(\tau')d\tau'\right] \ . \tag{51a}$$

Split the interval into two domains: $[0, \tau_\Delta]$ and $[\tau_\Delta, \infty]$. Then the integration of Eq (51a) yields

$$q_w^R \simeq -2\,\epsilon_w \int_0^{\tau_\Delta} \frac{dE_b}{d\tau'} E_3(\tau')d\tau' \ . \tag{51b}$$

Assume a third order polynomial in τ for E_b,

$$E_b = a_o + a_1\tau + a_2\tau^2 + a_3\tau^3 \ . \tag{52}$$

First satisfy the apparent conditions,

$$E_b(0) = E_{bw} \ , \quad E_b(\tau_\Delta) \simeq E_{b\infty} \ \text{and} \ dE_b(\tau_\Delta)/d\tau \simeq 0 \ , \tag{53}$$

and, for the fourth, utilize the balance of the thermal energy,

$$k\left.\frac{d^2T}{dy^2}\right|_w = \left.\frac{dq_y^R}{dy}\right|_w \tag{54}$$

which in terms of Eq. (20) may be rearranged to give

$$k\left.\frac{d^2T}{dy^2}\right|_w = 4\,\kappa_P\,(1 - \frac{\epsilon_w}{2})\,(E_{bw} - E_{b\infty}) \ . \tag{55}$$

Also, from the (linearized) Stefan-Boltzmann law

$$\frac{d^2E_b}{dy^2} = 4\,\sigma\,T_M^3\,\frac{d^2T}{dy^2} \tag{56}$$

Without this linearization, an explicit fourth condition does not appear to be available. However, this linearization may be shown to have negligible effect.

The elimination of thermal curvature between Eqs. (55) and (56) gives

$$\left.\frac{d^2E_b}{d\tau^2}\right|_w = 12\,\chi\,\mathcal{P}\,(1 - \frac{\epsilon_w}{2})\,(E_{bw} - E_{b\infty}) \ . \tag{57}$$

Now, Eq. (52) subject to Eqs. (53) and (57) yields

$$\frac{E_b - E_{bw}}{E_{bw} - E_{b\infty}} = \frac{1}{2}\left[-\left(3+\frac{1}{2}\mathcal{P}_o\right)\frac{\tau}{\tau_\Delta} + \mathcal{P}_o\left(\frac{\tau}{\tau_\Delta}\right)^2 + \left(1-\frac{1}{2}\mathcal{P}_o\right)\left(\frac{\tau}{\tau_\Delta}\right)^3\right] \tag{58}$$

where

$$\mathcal{P}_o = 12\,\chi\,\mathcal{P}\,(1 - \frac{\epsilon_w}{2})\,\tau_\Delta^2$$

In terms of Eq. (58), Eq. (51b) results in

$$q_w^R \simeq \epsilon_w(E_{bw} - E_{b\infty})\left\{1 - \tau_\Delta\left[\frac{3}{4} - (1-\frac{\epsilon_w}{2})\,\tau_\Delta^2\,\chi\,\mathcal{P}\right]\right\} \tag{59}$$

which shows the explicit effect of conduction on the radiative heat flux. However, for the thingas radiation

$$\tau_\Delta\mathcal{P} \sim 1 \ , \quad \tau_\Delta \ll 1$$

and, to first order, the explicit effect of conduction on the radiation flux is negligible. Thus

$$q_w^R = \epsilon_w(E_{bw} - E_{b\infty})\,(1 - \frac{3}{4}\tau_\Delta) \tag{60}$$

which is the upper limit of the radiative flux obtained from strict radiative considerations.

Now, for the total heat transfer,

$$q_w = -k\left.\frac{\partial T}{\partial y}\right|_w + \epsilon_w(E_{bw} - E_{b\infty})\,(1 - \frac{3}{4}\tau_\Delta) \ , \tag{61}$$

where, after neglecting the effect of thingas radiation on the thermal boundary layer,

$$\tau_\Delta = \kappa_M\,\Delta = \kappa_M\,\delta\,/\,Pr^{1/3} \ . \tag{62}$$

From approximate studies on viscous boundary layers,

$$\delta \simeq 5.0\,x\,/\,Re_x^{1/2} \ , \tag{63}$$

and

$$\tau_\Delta = 5.0\,\tau_x\,/\,Re_x^{1/2}\,Pr^{1/3} \ . \tag{64}$$

Also, from thermal boundary layer studies,

$$Nu_x = 0.629(-d\theta/d\eta|_w)\,Re_x^{1/2}\,Pr^{1/3} \ , \tag{65}$$

which, for the pure conduction case

$$(-d\theta/d\eta|_w)^K = 0.538 \ ,$$

gives

$$Nu_x^K = 0.339\,Re_x^{1/2}\,Pr^{1/3} \ . \tag{66}$$

In terms of Eq. (66), Eq. (64) becomes

$$\tau_\Delta \simeq \frac{5}{3}\,\tau_x\,/\,Nu_x^K \ . \tag{67}$$

Thus

$$\frac{Nu_x}{Nu_x^K} \simeq \frac{(-d\theta/dy|_w)}{(-d\theta/dy|_w)^K} + \frac{3}{4}\,\epsilon_w\,P\left(\frac{\tau_x}{Nu_x^K}\right)\left(1 - \frac{5}{4}\frac{\tau_x}{Nu_x^K}\right) \ .$$

In terms of Eq. (44), the local thermal entropy production on the wall is

$$s_x''' = -\frac{1}{T_w^2}\,(q_w^K + q_w^R)\left.\left(\frac{\partial T}{\partial y}\right)\right|_w . \tag{68}$$

Introducing a wall local entropy production number,

$$\Pi_x = s_x''' x^2/k \tag{69}$$

Eq. (68) may be rearranged as

$$\Pi_x = \left(1 - \frac{T_\infty}{T_w}\right)^2 \left(1 + \frac{q_w^R}{q_w^K}\right) \left[\frac{(\partial T/\partial y)|_w}{(T_w - T_\infty)/x}\right]^2 \tag{70}$$

With the definition of local Nusselt number

$$Nu_x = \frac{q_x^C}{q_x^K} = \frac{q_w^K}{q_x^K} = \frac{(\partial T/\partial y)|_w}{(T_w - T_\infty)/x} \tag{71}$$

Eq. (70) may be finally expressed as

$$\Pi_x = \left(1 - \frac{T_\infty}{T_w}\right)^2 \left(1 + \frac{q_w^R}{q_w^K}\right) Nu_x^2 . \tag{72}$$

7. FINAL REMARKS

A boundary-affected and attenuating thingas model is developed. The radiation-affected forced convection over a flat plate is investigated in terms of this model.

The distribution of entropy production within and outside the radiation-affected thermal boundary layer is evaluated. The retained nonlinearity of temperature in the entropy production leads to an extremum in this production within the boundary layer rather than on the boundary.

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