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Abstract

The exact equations of motion are derived for a point mass spinning in the orbital plane at the end of a linear elastic, massless spring. The spring is attached to a heavy point mass which moves in an undisturbed Keplerian orbit. These nonlinear equations are linearized and solved on a computer. The linear equations are shown to be ordinary, forced differential equations with time variant coefficients. Results for a one hour spin period and both circular and elliptic orbits are presented for radial and tangential motion of both the linearized and nonlinear equations. These results indicate the validity of the linearization. Stability of the linearized system of equations is shown for specific orbits and spin periods of one hour and 1/10 hour by numerically applying Floquet theory modified to handle forced equations. A matrix squaring technique is shown to adequately bound the stability multipliers without the necessity of calculating them. Preliminary results are presented for the above analysis applied to a 3 mass model which has two lumped masses which approximate the mass of the connecting cable. Both stable and unstable motions are found.

Nomenclature

Symbols

G	Gravitational constant times Earth's mass
k	Spring constant for single mass model
k_{1,2,3}	Spring constant for three mass model
l	Length of spring in single mass model
l_{1,2,3}	Lengths of springs in three mass model
l₀	Unstretched spring length of spring
l_s	Free space equilibrium length of spring
M	Mass for single mass model
m_{1,2,3}	Masses for three mass model
n	Number of equations
q₁	Deviation from l _s
q₂	Deviation from ϕ ₀
R	Orbital radius
T	Kinetic energy
V	Potential energy
W_{1,2,3}	Weight of three masses in three mass model
z	Dummy variable
Δ()	Implies deviation of enclosed variable from free space value
θ	Orbital angle
λ₁	Stability multipliers which are the roots of matrix [M]
ϕ	Relative angle
ϕ₁	Initial relative angle
ϕ₀	Free space reference angle

Matrices

[A(t)]	Matrix of coefficients of system of differential equations
{b(t)}	Matrix of forcing terms

[F(t)]	Fundamental matrix
[I]	Identity matrix
[M]	Monodromy matrix

Introduction

Interest in masses connected by cables or wires and spinning in a gravity field stemmed from a desire to provide an artificial gravity for astronauts by spinning their capsule about their booster vehicle at the end of a cable. Several papers resulted.¹⁻⁵ All but the paper by Austin⁴ considered the cable to have mass and assumed the tension to be constant or slightly perturbed.⁵ These papers deal with a continuous cable whose analysis yields partial differential equations. Assumptions such as uncoupling of radial and transverse motion and constant tension are made which enable solutions to be obtained. Garber⁶ did a preliminary investigation of the related problem of a long flexible wire in orbit by lumping masses and assuming rigid rods as connectors while Austin⁵ has assumed two point masses connected by a linear massless spring spinning in free space.

Current interest in building a large, orbiting, slowly spinning antenna for radio astronomy observation has pressed the need for analysis of wire connected systems where the tension is not constant and the motions are not uncoupled. For a system such as the Kilometer Wave Orbiting Telescope⁷ (KWOT) shown in Figure 1 an analysis capable of predicting both the motion and stability of the system in orbit is necessary.

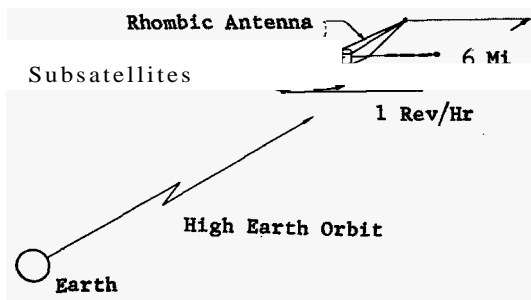


Figure 1. KWOT System in Orbit

Analysis

Simple Model

The gross passive motion and stability of a subsatellite in orbit can be obtained by studying the model shown in Figure 2. Namely, two masses connected by massless, linear elastic, undamped springs.

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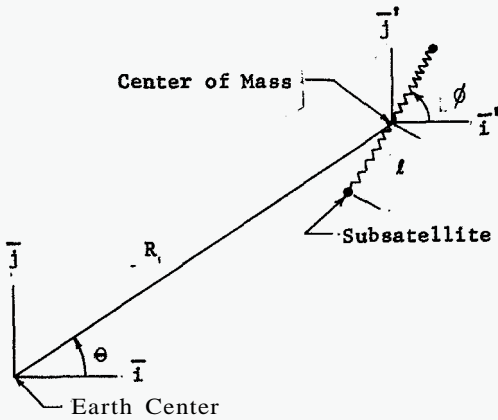


Figure 2. Spring-Mass Model of Subsattelites and Wire

The equations of motion for the simple model can be derived using the Lagrangian method. The gravity and elastic potential are defined in the usual manner. The contributions to the Lagrangian are as follows:

Kinetic energy

$$T = M [\dot{R}^2 + R^2 \dot{\theta}^2 + \dot{l}^2 + l^2 \dot{\phi}^2]$$

The first two terms are for the center of mass and the second two for the relative motion.

Potential energy

$$V = -GM \left\{ [R^2 + l^2 + 2Rl \cos(\theta - \phi)]^{-1/2} + [R^2 + l^2 - 2Rl \cos(\theta - \phi)]^{-1/2} \right\} + k(l - l_0)^2$$

Orbital radius

$$\ddot{R} = R\dot{\theta}^2 - \frac{G}{2} \left\{ \frac{R + l \cos(\theta - \phi)}{A} + \frac{R - l \cos(\theta - \phi)}{B} \right\} \quad (1)$$

Orbital angle

$$\ddot{\theta} = -\frac{2R\dot{\theta}}{R} + \frac{Gl}{2R} \sin(\theta - \phi) \left\{ \frac{1}{A} - \frac{1}{B} \right\} \quad (2)$$

Relative angle

$$\ddot{\phi} = -\frac{2l\dot{\phi}}{l} - \frac{GR}{2l} \sin(\theta - \phi) \left\{ \frac{1}{A} - \frac{1}{B} \right\} \quad (3)$$

Spring half length

$$\ddot{l} = l\dot{\phi}^2 - \frac{G}{2} \left\{ \frac{l + R \cos(\theta - \phi)}{A} + \frac{l - R \cos(\theta - \phi)}{B} \right\} - \frac{k}{M} (l - l_0)$$

where

$$A = (R^2 + l^2 + 2Rl \cos(\theta - \phi))^{3/2}$$

and

$$B = (R^2 + l^2 - 2Rl \cos(\theta - \phi))^{3/2}$$

G = gravitational constant times mass of Earth

k = spring constant

and

l_0 = unstretched spring length

Following the development of Goldstein⁸ for a point mass in orbit one can show for a spinning finite body that the orbit of the center of mass is perturbed but stable. In other words, since the relative and orbital motion are coupled as can be seen from equations (1)-(4), it could happen that the relative motion would gain energy while the orbital motion loses energy, this occurrence however is shown to be bounded.

To show it, one uses conservation of energy and angular momentum. If the conservation of angular momentum equation is solved for $\dot{\theta}^2$ and this put into the total energy equation and all terms which are positive definite collected into a single term, the remaining so called "fictitious potential" is a function of R , l , and $(\theta - \phi)$. This can be expanded in terms of the ratio l/R which is quite small for a satellite in orbit. If this fictitious potential is plotted on a three dimensional plot versus R and $(\theta - \phi)$ one finds that the surface is a well similar to the one for a point mass. However, the additional axis, $(\theta - \phi)$, gives the surface a "ripple" as one increases $(\theta - \phi)$. There are an infinite number of local wells corresponding to the gravity gradient stabilized, or "captured" configuration. If the system is spinning, it in effect moves through this rippled trough with its spin rate changing. The orbit is still limited by the total energy which differs from that of an equivalent point mass by only the relative kinetic energy and elastic potential energy both of which are small compared to the energy of the orbit. From this it can be concluded that little error will be introduced if the center of mass is assumed to move in a Keplerian orbit.

If the equations for only one mass assumed to be attached by a spring to a point moving in a Keplerian orbit are derived and the potential energy expanded in terms of l/R , the following equations result when only terms of order l/R are retained:

Orbit

$$\ddot{R} = R\dot{\theta}^2 - \frac{G}{R^2} \quad (5) \quad \text{and} \quad \ddot{\theta} = -\frac{2R\dot{\theta}}{R} \quad (6)$$

Relative motion

$$\ddot{l} = l\dot{\phi}^2 + \frac{Gl}{2R^3} [1 + 3 \cos 2(\phi - \theta)] - \frac{k}{M} (l - l_0) \quad (7)$$

and

$$\ddot{\phi} = -\frac{2l\dot{\phi}}{l} - \frac{3G}{2 \times 3} \sin 2(\phi - \theta) \quad (8)$$

It is interesting to note that these are exactly the same equations one would obtain if

equations (1)-(4) were expanded and the same order terms retained. The nonlinear equations may be linearized, however, there is no stationary point about which to linearize because the system is spinning in a gravity gradient. If the free space stationary point is used as the reference motion,

$$\text{namely } l_s = \frac{k}{M} l_o \text{ and } \dot{\phi}_o = \text{const.}$$

$$\left(\frac{k}{M} - \dot{\phi}_o^2 \right) \phi_o = \dot{\phi}_o t + \phi_1$$

$$l = l_s + q_1 \text{ and } \phi = \phi_o + q_2$$

The resulting linear equations for the deviations from free space motion of l and ϕ are

$$\ddot{q}_1 = - \left[\frac{k}{M} - \dot{\phi}_o^2 - \frac{G}{2R^3} - \frac{3G}{2R^3} \cos 2(\phi_o - \theta) \right] q_1$$

$$+ 2l_s \dot{\phi}_o \dot{q}_2 = \frac{3Gl_s}{R^3} \sin 2(\phi_o - \theta) q_2$$

$$+ \frac{Gl_s}{2R^3} \left[1 + 3 \cos 2(\phi_o - \theta) \right] \quad (9)$$

and

$$\ddot{q}_2 = - \frac{3G}{R^3} \cos 2(\phi_o - \theta) q_2 - \frac{2\dot{\phi}_o}{l_s} \dot{q}_1$$

$$- \frac{3G}{2R^3} \sin 2(\phi_o - \theta) \quad (10)$$

respectively.

The following observations can be made:

1. The equations are of the form

$$\dot{q} = [A(t)] \{q\} + \{b(t)\} \quad (11)$$

where q_1 and q_2 are defined as two new variables thereby changing the system of equations (9) and (10) into four first order equations. $[A(t)]$ is periodic for circular orbits where R and θ are constants and for specific elliptic orbits where the orbital period is an integer multiple of the relative spin period.

2. The natural frequency of the system is

$$\sqrt{\frac{k}{M} + 3\dot{\phi}_o^2} \text{ for } R = \infty$$

as noted by Pittman and Hall⁸ and hence the rotational rate "stiffens" the system.

3. The trace of $[A(t)]$ is zero for all t

$$\left(\text{i.e. } \sum_{i=1}^n a_{ii} = 0 \right).$$

4. Since the periodic terms are all multiplied by $1/R^3$, a perturbation technique could be applied in the same manner that Targoff⁷ did for a rigid mass-rod in orbit. However, this was not done in the present study.

5. Floquet theory as reported by Casari¹⁰ may be used to determine stability of equations (11) when $\{A(t)\}$ and $\{b(t)\}$ have a common largest period. This method is applied in the remainder of the paper.

Stability Analysis

Equations (11) are put in proper form so that Floquet theory may be applied by defining an artificial variable z with $\dot{z} = 0$ so that equations (11) become

$$\begin{Bmatrix} \dot{q}_1 \\ \vdots \\ \dot{q}_n \\ \dot{z} \end{Bmatrix} = \begin{bmatrix} [A(t)] & b(t) \\ [0] & 0 \end{bmatrix} \begin{Bmatrix} q_1 \\ \vdots \\ q_n \\ z \end{Bmatrix} \quad (12)$$

Briefly, the fundamental matrix of (12) at time t is related to itself at time $t + \tau$ where τ is the largest period of $\{A(t)\}$ and $\{b(t)\}$ by the equation

$$[F(t + \tau)] = [F(t)] [M]$$

where $[M]$ is the so called monodromy matrix.¹¹ If $\{F(0)\} = [I]$ then $[M]$ is found to be

$$[F(\tau)] = [I] [M].$$

The roots of $[M]$, $A_i s$, called multipliers, determine stability in the following way.

$$|\lambda_i| > 1.0 \text{ asymptotically unstable}$$

$$|\lambda_i| < 1.0 \text{ asymptotically stable}$$

$$|\lambda_i| = 1.0 \text{ and not repeated neutrally stable}$$

$$= 1.0 \text{ and repeated}$$

either neutrally stable or polynomial type growth with time, depending on the Jordan normal form of $[M]$.

To apply the Floquet theory, modified to handle forced equations, to equations (12), they must be integrated $n + 1$ times with initial conditions so that $\{F(0)\} = [I]$.

This was carried out on an IBM 7090 using a fourth order Runge Kutta technique. Experience has shown that about 20 steps per shortest cycle will give adequate accuracy from the standpoint of both truncation and round off errors. The monodromy matrix which resulted for all cases run had repeated roots of value 1.0. The Jordan normal form of $[M]$ indicated a linear growth with time of a solution. This is however a "trivial" instability. It only means that the system can be given an initial angular momentum different from the reference value and a linear growth in the angular position from the nominal position results. For spin periods of one hour and one tenth hour, a range of elastic constants and most orbital radii, the spring-mass model was neutrally stable except for the trivial instability mentioned. The only non trivial instability resulted when the orbital radius was too low so that the subsatellite was captured by the gravity field and merely oscillated about the vertical direction.

In most cases, the roots were not actually calculated. They were merely bounded using three facts

$$1. \quad |\lambda_{\max}| \leq \sum_{i=1}^n \max_j |m_{ij}| \quad j = 1, \dots, n$$

2. The roots of $[M]^n$ are λ_1^n

and

3. The bound in (1) becomes better as the power of $[M]$ increases.

This matrix squaring technique proved highly satisfactory from the standpoint of accuracy and computer time required. For example, 15 squarings of a 5×5 matrix reduced the upper bound from 1.75×10^4 to 1.0007 in 2.0 seconds, including printing all intermediate bounds. It should be pointed out that 15 squarings gives $[M]$ to the 32,768th power. To apply this technique to large matrices, some type of repeated scaling will have to be carried out to prevent over and underflows.

The method used to determine stability may be applied to any number of masses. The limiting factor will probably be computer time required to generate the monodromy matrix. The time required, increases as $(n+1)^2$ for a forced system since $n+1$ equations must be integrated over one period, $n+1$ times. There is a further restriction that enters as masses are added for a better approximation, the shorter spring segments and smaller mass points cause the highest frequency (which dictates step size) to increase. Hence there is a trade off between the number of lumped masses used for each wire and round off errors due to a large number of steps.

Motion of Spring-Mass System

The motion was investigated in two ways; first the linearized equations were solved on an analog computer for ease of solution, secondly, the non-linear equations were numerically integrated, again using a fourth order Runge Kutta technique.

Figure 3 shows a plot of the axial stretching due to gravity gradient vs. effective spring stiffness for various circular orbits and a spring length of 1.64×10^4 ft.

An empirical formula given below may be used to obtain the axial deflection for a given set of conditions:

$$\Delta l = (115) \frac{G}{R^3} \left(\frac{l_0}{\frac{k}{m} - \dot{\phi}_0^2 - \frac{g^2}{2}} \right)$$

This equation has been verified by comparison with Figure 3 but its validity for spin rates other than one revolution per hour has not been demonstrated.

Figure 4 gives the deviation in spin rate for various orbits. The effective spring stiffness had no noticeable effect on these deviations (at least for the range of stiffnesses noted in Figure 3). The spin rate deviation changed as $1/R^3$ for various orbits. This is to be expected since the coefficients of the forcing term of the linear equations also change as $1/R^3$.

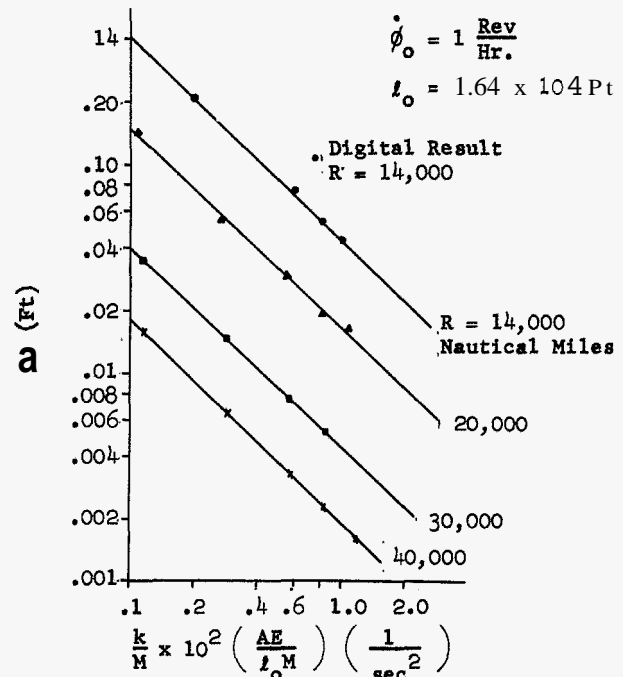


Figure 3. Axial Deviation vs. Effective Spring Stiffness

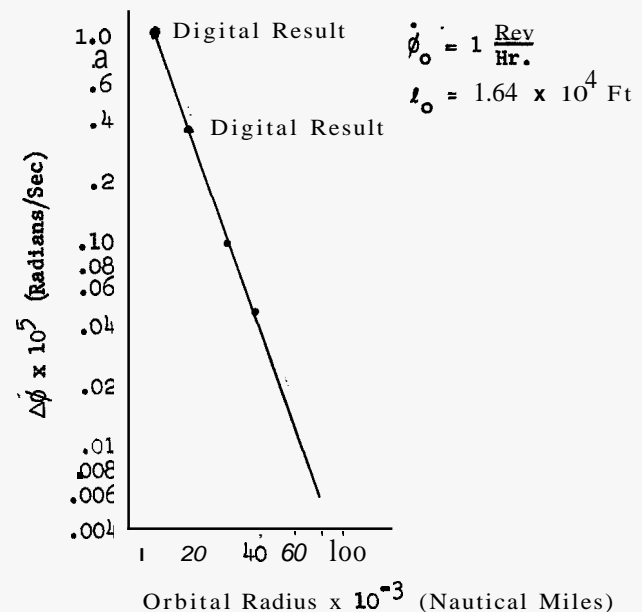


Figure 4. Spin Rate Deviation vs. Orbital Radius

The angular deviation converted to feet and called the transverse deviation is shown in Figure 5. This deviation also changes as $1/R^3$ and is periodic down to an orbit which captures it. When this happens, damping will cause it to become gravity gradient stabilized in one of the local potential wells mentioned above.

Note in Figure 3-5 that the "Digital-nonlinear Results" are in very good agreement with the analog-linearized equation results.

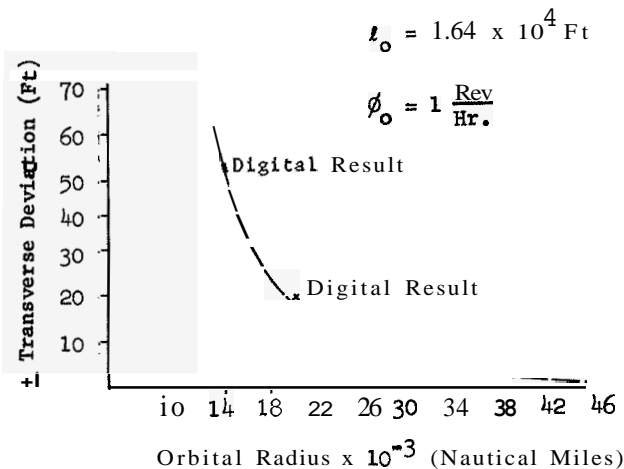


Figure 5. Transverse Deviation vs. Orbital Radius

The results shown in Figure 4 and the angular deviation data which was used to generate Figure 5 can be shown to be independent of the length of the springs. If a new non-dimensional variable is defined as

$$m_1(t) = \frac{q_1(t)}{l_s}$$

then equations (9) and (10) can be made to be independent of the length of the spring.

As was mentioned in the preceding section, the simple spring-mass model was neutrally stable even for certain elliptic orbits as determined by Floquet theory. This can be shown also for the solution of the full, nonlinear equations. Figure 6 shows the deviation of the position angle ϕ from the free space, linearly increasing reference angle. There is a periodic deviation superimposed on a linear growth (the so-called trivial growth). The amplitude of the periodic motion varies like $1/R^3$, independent of the eccentricity of the orbit. The linear growth decreases with eccentricity because the radius at time of release increases.

$$\phi_0 = 1 \frac{\text{Rev}}{\text{Hr.}} \quad l_0 = 1.64 \times 10^4 \text{ Ft}$$

$$\bar{M} = 7.4 \times 10^{-3} \frac{1}{\text{sec}^2}$$

"P" Denotes Perigee

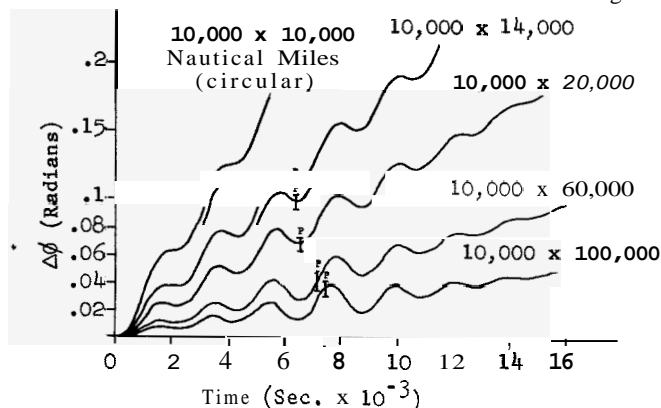


Figure 6. Angular Deviation from Free Space Motion vs. Time

Figure 7 shows the effects of a low perigee and a slow spin rate. In this case the orbit is highly eccentric (5,000 x 60,000 NM) and the spin rate is altered as much as 50% during perigee passing. It can be seen that the system can either gain or lose energy depending on the initial conditions, in this case, angular position. As pointed out previously, the equations which were solved are valid for either a dumbbell configuration (Figure 2) or a single mass constrained at the center of rotation. For a dumbbell, with its center of mass free to move, total energy is conserved and hence a gain or loss in spin rate must show up as a loss or gain in orbital semi-major axis measured to the center of mass. For a single mass with a constrained center of rotation, total energy is not conserved, nor is angular momentum.

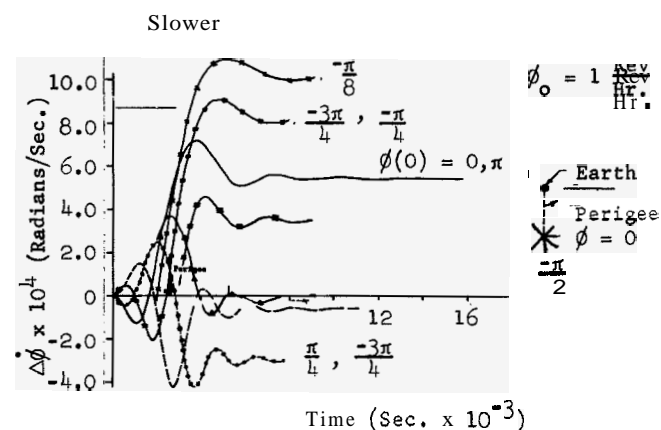


Figure 7. Angular Rate Deviation vs. Time (5,000 x 60,000 N.M.)

Equations (1)-(4) were integrated using double precision arithmetic on the 7090. The orbital radius and angular rate did indeed adjust as the relative motion gained or lost energy. However, the calculations for the relative motion (such as shown in Figure 7) where the center of rotation or center of mass were assumed to be constrained to a Keplerian orbit, agreed with solution of the exact equations where the center of mass was free to move, to at least 5 significant figures. This coupled with the fact that an indefinite transfer of energy cannot take place (as pointed out earlier) justifies neglecting the orbital perturbations.

Multi-mass Model

The method of analysis above can be applied to any number of masses. As discussed before, a lumped mass model can be used to study the motion and stability of a complex system. It can equally well be used to study the problems of thrusting, deployment, and spin-up or spin-down. The equations for the three mass model shown in Figure 8 were derived and linearized about a free space motion.

The masses are coupled only through the elastic potential terms. The effect of the gravity gradient on each mass must be approximately independent of its distance from the center of rotation as was found previously for the linear equations of a single mass. Hence, if no parametric excitation occurs one would expect no transverse oscillations to build up since each mass would speed up and slow down at the same rate as its neighbor. This was indeed the case. Both an application of Floquet

theory to the linear set of equations and the actual numerical integration of the nonlinear set showed transverse deflections of less than a few feet out of a length, l_3 , of 1.54×10^4 ft. for most cases.

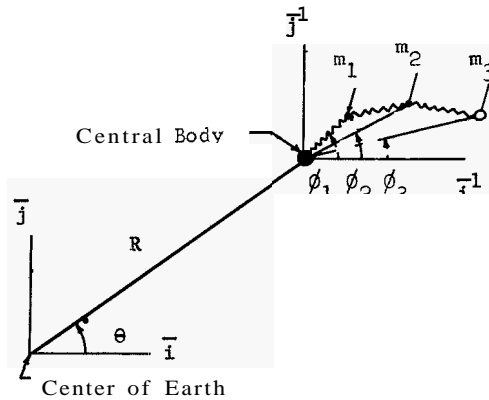


Figure 8. Three Mass Model in Orbit

One numerical integration, did reveal what appeared to be an instability, that is, the transverse deviation began to get large. This occurred when the axial frequency of the third mass (subsattelite) was close to the lowest transverse frequency of the two small masses. In this case the deflections became as high as 300 ft. Figure 9 shows the time history of the first mass' deviation from a line passing through the center of rotation and the end mass (l_3 in Figure 8).

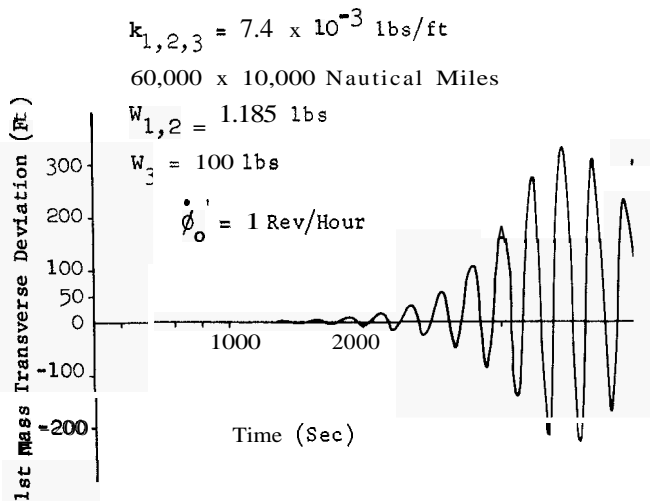


Figure 9. Transverse Deflection of m vs. Time

This motion suggests a parametric excitation of the Wire due to the oscillations of the end mass. Note that as the amplitude increases, nonlinearities entered in and limited the amplitude. The frequency and growth was not directly related to the gravity gradient but this disturbance probably helped the notion get started. The behavior of the stability multiplier of the linear equations for the same conditions has not been examined because of computer time limitations.

Conclusions

1. Lumping a continuous complex cable connected system in orbit is a convenient way to analyze the system because:
 - a. The exact equations of motion can be systematically derived using the Lagrangian method.
 - b. A fourth order Runge-Kutta can effectively be used to numerically integrate the nonlinear equations when the motion is desired,
 - c. Floquet theory can be applied to the linearized equations when the coefficients are periodic with commensurate periods.
 - d. Computer experiments can be conducted to examine spin up, deployment, etc.
2. The orbit of an elastic dumbbell is bounded and the center of mass may be assumed to follow a Keplerian orbit when calculating the relative motion of a large cable connected system in orbit.
3. Neutral stability was shown in all cases examined for the linear equations of motion for the simple spring-mass system as long as the orbit was high enough and/or the spin rate fast enough to avoid gravity gradient stabilization.

4. The three mass model showed stability for some cases. An instability (large but finite deflections) was found from numerical integration of the nonlinear equations. The motion suggests a parametric excitation of the transverse modes due to axial motion. It does not appear to be caused by gravity gradient effects alone.

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