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-- NOTES --

SWITCHING CONDITIONS AND A SYNTHESIS TECHNIQUE FOR THE SINGULAR SATURN GUIDANCE PROBLEM†

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Abstract

A singular optimal guidance problem which was motivated by difficulties encountered in the Saturn V SA-502 flight has been studied. It is shown that if the guidance equations are based upon a singular version of the flat-earth problem, then the control must be discontinuous at a junction of singular and nonsingular subarcs for almost all cases. A good suboptimal guidance scheme based upon a nonsingular approximation of the singular problem is presented. The resultant suboptimal control is continuous, which is more desirable than a discontinuous control, and causes only a noise-level difference in payload.

I. Introduction

In the second flight of the Saturn V vehicle (SA-502), two engines shut down early in the S-II stage. The measurements received by the on-board guidance scheme, the Iterative Guidance Mode (IGM)[1], indicated that only one engine was out. This resulted in a steep planar steering program in the S-IV stage which caused the time rate of change of the steering angle to reach its limiting value for a large portion of the S-IVB flight. Since the IGM is based on unconstrained variational theory, the resultant trajectory did not reach the desired terminal orbit.

In the flight mentioned above a large disturbance caused the guidance law to determine a steering angle rate of change which was too large. Thus, it would be desirable to design the guidance logic in such a way that the time rate of change of the steering angle is a bounded control variable, say u with $|u| \leq K$, such that the steering angle is a state variable since it cannot change rapidly (because of physical and reliability constraints). However, the resultant optimal control problem is a singular problem, and the variational and computational theory for such problems is far from satisfactory.

As is well-known, the variational and numerical theory for totally nonsingular optimal control problems is well developed, and recently McDanell and Powers[2], Speyer and Jacobson[3], and Goh[4] developed new necessary conditions and sufficient conditions for totally singular problems. Thus, the main problems to be resolved are: (1) The determination of necessary and sufficient conditions for optimal trajectories which possess both singular and nonsingular subarcs (which is the case in the Saturn guidance problem); and (2) The development of a computational scheme for the generation of optimal trajectories which possess both singular and nonsingular subarcs. Partial results in this direction have been obtained by Jacobson, Gershwin, and Lele[5] and Pagurek and Woodside[6].

In the following analysis, recently developed necessary conditions for composite optimal trajectories (i.e., trajectories which contain both singular and nonsingular subarcs) are used to characterize the local switching behavior of the singular Saturn guidance problem. Since the resultant behavior is not physically desirable, the problem is transformed into a good suboptimal nonsingular representation of the problem which could be incorporated easily into a recently proposed guidance scheme for Saturn class vehicles[7,8].

II. Singular Optimal Control Theory

In this section, properties from singular optimal control theory which we shall apply later will be summarized.

Consider the problem of minimizing

$$J = G(t_f, x_f) + \int_{t_0}^{t_f} L(t, x)\,dt \qquad (2.1)$$

subject to the following conditions

$$\dot{x} = f(t, x, u) \equiv f^0(t, x) + f_u(t, x)u \qquad (2.2)$$

$$x(t_0) = x_0 \qquad (2.3)$$

$$\psi(t_f, x_f) = 0 \qquad (2.4)$$

$$|u| \leq K. \qquad (2.5)$$

The state x is n-dimensional, u is a scalar control variable, and ψ is a p-dimensional vector function which defines the terminal surface, $p \leq n + 1$.

Along an optimal trajectory the following necessary conditions hold:

$$\dot{\lambda} = -H_x(t, x, \lambda, u) \qquad (2.6)$$

$$\lambda^T(t_f) = G^T_{x_f}(t_f, x_f) + \nu^T \psi_{x_f}(t_f, x_f) \qquad (2.7)$$

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70-965

$$H(t_f, x(t_f), \lambda(t_f), u(t_f)) + G_{t_f}(t_f, x_f) + \nu^T \psi_{t_f}(t_f, x_f) = 0 \qquad (2.8)$$

$$H(t,x,\lambda,u) = \min_{|v| \leq K} H(t,x,\lambda,v) \qquad (2.9)$$

where $\lambda(t)$ and ν are Lagrange multipliers and

$$H(t,x,\lambda,u) \equiv L(t,x) + \lambda^T f(t,x,u). \qquad (2.10)$$

This, of course, is the familiar Pontryagin maximum principle in a minimum form[9], i.e., the Hamiltonian is minimized with respect to the control if the performance index is to be minimized.

In general, the optimal trajectory for this problem consists of some combination of singular arcs and nonsingular (bang-bang) arcs. A singular arc is one along which

$$H_u(t,x,\lambda) \equiv 0 \qquad (2.11)$$

on a nonzero time interval. A nonsingular arc is one along which $H_u(t,x,\lambda) \neq 0$ except possibly at a countable number of points $\{t_1, t_2, \ldots\} \subset [t_0, t_f]$. On a nonsingular arc (2.9) implies $u = -\text{sgn}\, H_u$, which indicates the bang-bang character of nonsingular arcs.

The defining feature of a singular arc is that Eq. (2.9) of the minimum principle is satisfied trivially, and, thus, it cannot be used to distinguish between maxima and minima. In 1964, Kelley[10] developed a new necessary condition for singular arcs which allows one to distinguish between maxima and minima.

Kelley Condition
(Generalized Legendre-Clebsch Condition)

Let $x(t)$ be a weak, relative minimum for Eq. (2.1). Then,

$$(-1)^q \frac{\partial}{\partial u}\left[\frac{d^{2q}}{dt^{2q}} H_u\right] \geq 0, \qquad (2.12)$$

where $2q$ is the lowest order time derivative of H_u in which u appears explicitly. (Note: if $q = 0$, then the arc is nonsingular; if $q \neq 0$, then q is called the order of the singular arc.)

The Kelley condition is a pointwise or local necessary condition. Recently, a new arcwise necessary condition (a generalization of the classical Jacobi condition) was developed for totally singular problems[2,3,4]. A strengthened form of this condition (along with the strengthened Kelley condition and Eqs. (2.6)-(2.10)) leads to a sufficient condition for a totally singular arc. However, a useful sufficient condition for composite singular problems is still lacking.

In Reference 11, Kelley, Kopp, and Moyer used Taylor series expansions in the neighborhood of a singular-nonsingular junction along with the maximum principle to obtain necessary conditions at the junction. Recently these results have been generalized[12] as follows:

THEOREM 1: Let $x(t)$ be an optimal trajectory which contains both singular and nonsingular subarcs, and let the singular subarcs be of q^{th} order, i.e.,

$$H_u^{(2q)} \equiv \frac{d^{2q} H_u}{dt^{2q}} \equiv A(t,x,\lambda) + B(t,x,\lambda)u. \qquad (2.13)$$

Suppose the optimal control is piecewise analytic in a neighborhood of a junction (this is not always the case as is shown in Reference 11), and $B(t,x,\lambda) \neq 0$ at the junction. If $u^{(r)} \equiv d^r u/dt^r$ $(r \geq 0$, where $u^{(0)} \equiv u)$ is the lowest order derivative of u which is discontinuous at the junction, then $q + r$ must be an odd integer.

The main consequence of this theorem is that if a Taylor series expansion is valid in the neighborhood of a junction and the control is discontinuous at the junction, then the singular subarc must be of odd order. The singular Saturn guidance problem contains odd order $(q = 1)$ singular subarcs. Note that the theorem does not imply that the optimal control must jump if q is odd. Also, there exist well known cases of q-even problems with discontinuous controls but the controls are not piecewise analytic (e.g., an infinite number of switches between $u = \pm K$ on the nonsingular side of the junction in a finite time interval).

With the analyticity assumption removed, the following result can be obtained:

THEOREM 2: Let $x(t)$ be an optimal trajectory which contains both singular and nonsingular subarcs, where the singular subarcs are of q^{th} order. Then,

(i) $H_u^{(2q)} \neq 0$ on the nonsingular side of the junction implies the control must be discontinuous;

(ii) $H_u^{(2q)} = 0$ on the nonsingular side of the junction and $B \neq 0$ (where $H_u^{(2q)} \equiv A + Bu$) at the junction imply the control must be continuous.

Theorem 2 has the desirable quality of determining if the control "jumps" or is continuous at a junction without an analyticity assumption. However, the conditions are more difficult to verify than those of Theorem 1 if analyticity is a valid assumption. In some cases the theorems can be used together to indicate what one cannot assume, e.g., $H_u^{(2q)} \neq 0$ on the nonsingular side of the junction and q even imply that u is discontinuous by Theorem 2, but by Theorem 1, u must be continuous if it is piecewise analytic. Thus, one may conclude that if a junction occurs it is a nonanalytic junction.

Finally, another useful property[12] is:

Property 1: If $A(t,x,\lambda) = 0$ at a junction and

$|u| \leq K \neq 0$, then the optimal control is discontinuous at a singular-nonsingular (or nonsingular-singular) junction.

III. Simple Singular Guidance Example

In this section we shall present a simple example to demonstrate some of the features of a singular optimal guidance problem. Consider the problem of moving a boat from $(0,0)$ to $(x_f, y_f) \neq (x_f, 0)$ in minimum time. The boat has a constant speed, V, and steering angle, α (see Figure 1). If α could change discontinuously, then

$$\tan\alpha = \frac{y_f}{x_f} \qquad (3.1)$$

is the best steering program.

Suppose that we take into account the facts that $\alpha(t_0)$ is probably not equal to $\tan^{-1}(y_f/x_f)$ and $\dot\alpha$ is finite, say $|\dot\alpha| \leq K < \infty$. For this example, let

$$\alpha(t_0) = 0. \qquad (3.2)$$

Since $\dot\alpha$ is finite, α is continuous and, thus, not a natural control variable. Therefore, let $\dot\alpha = u$ be the control. We now have the following variational problem, assuming $t_0 = 0$: minimize

$$J = t_f$$

subject to:

$$\begin{aligned} \dot x &= V\cos\alpha, & x(0) &= 0, & x(t_f) &= x_f \\ \dot y &= V\sin\alpha, & y(0) &= 0, & y(t_f) &= y_f \neq 0 \\ \dot\alpha &= u, & \alpha(0) &= 0, & |u| &\leq K \end{aligned} \qquad (3.3)$$

The Hamiltonian is:

$$H = \lambda_1 V\cos\alpha + \lambda_2 V\sin\alpha + \lambda_3 u, \qquad (3.4)$$

and

$$\dot\lambda_1 = -H_x = 0, \qquad \dot\lambda_2 = -H_y = 0 \qquad (3.5)$$

$$\dot\lambda_3 = -H_\alpha = \lambda_1 V\sin\alpha - \lambda_2 V\cos\alpha \qquad (3.6)$$

$$H_u = \lambda_3 \qquad (3.7)$$

If the optimal trajectory contains a singular subarc, then

$$H_u = \lambda_3 \equiv 0 \qquad (3.8)$$

on some nonzero time interval. Consider the Kelley condition.

$$\dot H_u = \dot\lambda_3 = \lambda_1 V\sin\alpha - \lambda_2 V\cos\alpha \qquad (3.9)$$

$$\ddot H_u = (\lambda_1 V\cos\alpha + \lambda_2 V\sin\alpha)u \equiv Bu \qquad (3.10)$$

$$-\frac{\partial}{\partial u}\ddot H_u = -B \geq 0 \Rightarrow B \leq 0 \qquad (3.11)$$

Since u appears explicitly in H_u, the problem is a $q = 1$ order singular problem. By Property 1 of the previous section, $A = 0$ in (3.10) implies that u must be discontinuous at a junction of singular and nonsingular arcs. On a nonsingular subarc the optimal control is $u = \pm K$, and on a singular subarc the optimal control is $u = 0$ if $B \neq 0$ (as implied by (3.10) since $\ddot H_u \equiv 0$ on a singular arc).

Because of the simplicity of this problem, the optimal control may be determined by inspection. At t_0, the steering angle is $\alpha(t_0) = 0$. If α could change to any other value instantaneously, then the optimal trajectory would be a straight line connecting (x_0, y_0) and (x_f, y_f), and the initial steering angle would be $\alpha(t_0^+) = \tan^{-1}(y_f/x_f)$, i.e., the velocity vector would swing to the dashed arrow at the origin in Figure 1. Since the best steering angle at any state is the angle between the x-axis and the line connecting the state with (x_f, y_f), the optimal control $u \equiv \dot\alpha$ is the control which will cause $\dot\alpha$ to approach the "best" value of α as fast as possible. Thus, on the subarc $[t_0, t_1)$, $\dot\alpha = +K$ is the optimal control, which is nonsingular. At $t = t_1$, $\alpha(t_1) = 0 + K(t_1 - 0) = \tan^{-1}(y_f - y_1)/(x_f - x_1))$, i.e., the true steering angle at (x_1, y_1) and the "best" steering angle for (x_1, y_1) coincide. This means that the vehicle may now be steered by $\alpha = \tan^{-1}((y_f - y_1)/(x_f - x_1))$ = constant for the remainder of the trajectory, and, thus, $u = \dot\alpha = 0$, which is a singular control. Note that the switch point is mainly dependent upon $\alpha(t_0)$, K, and the terminal conditions. For example, if $y_f = 0$, then the optimal control is totally singular; if K is very small, then the optimal trajectory will possess a longer nonsingular arc than the same problem with a larger value of K; if K is very large, then the nonsingular arc should be relatively short, e.g., $K \to \infty$ implies that the nonsingular arc disappears completely. The main consequence of these statements is that the joining of a singular subarc to a nonsingular subarc is a function of "nonlocal" information. This complicates considerably the procedure for synthesizing optimal singular guidance laws on-board a vehicle.

IV. Saturn Guidance: A Singular Flat-Earth Problem

In this section, an analysis of the switching procedure for a flat-earth representation of circular orbit insertion will be presented. This is directly applicable to the guidance of the Saturn V vehicle since the IGM is based upon the flat-earth approximation. It will be shown that except for an exceptional case, the optimal control is discontinuous at a junction of singular and nonsingular subarcs. Thus, the optimal control does not "ride" onto or off of the control boundary.

The planar equations of motion and boundary conditions for the singular flat-earth problem are (see Figure 2):

$$\begin{aligned} \dot x &= p & x(t_0) &= x_0 \\ \dot y &= q & y(t_0) &= y_0,\ y(t_f) = r_c \\ \dot p &= \frac{F}{m}\cos\alpha & p(t_0) &= p_0,\ p(t_f) = v_c \end{aligned} \qquad (4.1)$$

70-965

$$\dot{q} = \frac{F}{m}\sin\alpha - g \qquad q(t_0) = q_0,\ q(t_f) = 0$$

$$\dot{\alpha} = u \qquad \alpha(t_0) = \alpha_0,\ |u| \leq K$$

$$m(t) = m_0 + \dot{m}_0(t - t_0)$$

where r_c = radius of the circular orbit, v_c = circular velocity at r_c. It is desired to transfer the vehicle from the given initial conditions into the given circular orbit in minimum time.

The Hamiltonian is

$$H = \lambda_1 p + \lambda_2 q + \lambda_3 \frac{F}{m}\cos\alpha + \lambda_4\left(\frac{F}{m}\sin\alpha - g\right) + \lambda_5 u, \qquad (4.2)$$

which implies, by Eqs. (2.6)-(2.9):

$$\begin{aligned} &\lambda_1(t) \equiv 0 \qquad \lambda_3(t) \equiv c_3 \\ &\lambda_2(t) \equiv c_2 \qquad \lambda_4(t) = c_4 - c_2 t \\ &\dot{\lambda}_5 = \frac{F}{m}(\lambda_3\sin\alpha - \lambda_4\cos\alpha),\ \lambda_5(t_f) = 0 \end{aligned} \qquad (4.3)$$

If an optimal singular subarc exists, then by the *Kelley condition:*

$$\dot{H}_u = \frac{F}{m}(\lambda_3 \sin\alpha - \lambda_4\cos\alpha) \qquad (4.4)$$

$$\ddot{H}_u = \frac{F}{m}(\lambda_2\cos\alpha - \lambda_1\sin\alpha) + \frac{F}{m}(\lambda_3\cos\alpha + \lambda_4\sin\alpha)u \qquad (4.5)$$

$$\equiv A(t,x,\lambda) + B(t,x,\lambda)u$$

implies

$$B(t,x,\lambda) \leq 0. \quad \text{(on singular arc)} \qquad (4.6)$$

By Eq. (4.4), i.e., $\dot{H}_u \equiv 0$ on a singular subarc,

$$\tan\alpha = \frac{\lambda_4}{\lambda_3} \Rightarrow \begin{cases} \cos\alpha = \pm\lambda_3(\lambda_3^2 + \lambda_4^2)^{-\frac{1}{2}} \\ \sin\alpha = \pm\lambda_4(\lambda_3^2 + \lambda_4^2)^{-\frac{1}{2}} \end{cases} \qquad (4.7)$$

and by Eq. (4.6):

$$B(t,x,\lambda) = \pm\frac{F}{m}\sqrt{\lambda_3^2 + \lambda_4^2} \leq 0, \qquad (4.8)$$

which implies that the minus sign should be chosen in Eq. (4.7) for the minimum time problem. Upon substitution of Eqs. (4.3) and (4.7) into Eq. (4.5):

$$\ddot{H}_u = -\frac{F}{m}c_2 c_3 (c_3^2 + \lambda_4^2)^{-\frac{1}{2}} - \frac{F}{m}(c_3^2 + \lambda_4^2)^{\frac{1}{2}}u. \quad \text{(on singular arc only)} \qquad (4.9)$$

We shall now consider under what conditions a saturation junction is possible, i.e., the control rides on or off of the boundary (or, is continuous at the junction).

If the control is continuous and well-behaved at the junction, then Theorem 1 of Section 2 is applicable, i.e., the control is piecewise analytic in a neighborhood of the junction. Since q = 1 for this problem, if the control is assumed to be continuous at the junction, then by Theorem 1, $r \geq 2$ (i.e., if $r \neq 0$, then $r \neq 1$ since q + r must be odd). Thus, if u is continuous, then $\dot{u} = \ddot{\alpha}$ is continuous, also.

By Eq. (4.7), the expression for $\ddot{\alpha}$ on the singular arc may be determined.

$$\dot{u} = \ddot{\alpha} = -\frac{2c_2^2 c_3 \lambda_4}{(c_3^2 + \lambda_4^2)^2} \quad \text{(on singular arc).} \qquad (4.10)$$

Since $\dot{\alpha} = \pm K$ on the nonsingular arc, it follows that

$$\dot{u} = \ddot{\alpha} = 0 \quad \text{(on nonsingular arc).} \qquad (4.11)$$

Therefore, the continuity of $\dot{u}$ at the junction requires $\dot{u} = 0$ at the junction, which implies:

$$c_2 = 0 \text{ or } c_3 = 0 \text{ or } \lambda_4 = 0. \qquad (4.12)$$

If $c_2 = 0$ or $c_3 = 0$, then the first term in Eq. (4.9) *is zero, which implies u is discontinuous* at the junction by Property 1 of Section 2. Thus, $c_2 \neq 0$, $c_3 \neq 0$ if the junction is continuous, and Eq. (4.12) *then implies:*

$$\lambda_4 = 0, \text{ (at a continuous junction)} \qquad (4.13)$$

or by Eq. (4.7):

$$\alpha = 0°, 180°, \text{ (at a continuous junction)} \qquad (4.14)$$

i.e., if $\alpha \neq 0°$, $180°$ at a junction, then the junction must be discontinuous. Since the steering angles $\alpha = 0°$, $180°$ do not appear to possess special prop*erties, further analysis would probably eliminate* the possibility of a continuous junction at these points, also.

If indeed smooth junctions are possible when $\alpha = 0°$, $180°$, then one can easily show that only one smooth junction is possible on the trajectory since λ_4 is a linear function of time and, thus, can go through zero only once. Also, if one considers an inverse-square gravity field and it is assumed that a continuous junction is possible, then necessary conditions for such a junction can be derived in the same way as Eq. (4.13) was derived for the flat-earth problem above.

V. Synthesis of Guidance Laws for Singular Problems

In the previous section it was shown that a junction of singular and nonsingular subarcs in the singular flat-earth problem requires a jump in the control (except possibly for the zero-probability cases when $\alpha = 0°$, $180°$). In Section 3 it was shown that the time t · ump is mainly a function of non-local information, and, thus, a formidable synthesis problem arises. In this section a suboptimal synthesis procedure to be used in conjunction with the guidance scheme of References 7 and 8 is suggested.

In References 7 and 8, a guidance scheme based upon the on-board solution of a nonsingular two-point boundary-value problem is proposed. Such a scheme is possible for Saturn class vehicles since it has a relatively large on-board computer. In this section a nonsingular approximation of the singular Saturn guidance problem will be developed, and the flat-earth approximation will be relaxed. The resultant optimal time rate of change of the steering angle (i.e., the optimal control) has the desirable property of continuity.

The planar equations of motion and boundary conditions for the singular inverse-square problem in polar coordinates are (see Figure 3):

$$\begin{aligned} &\dot{r} = v_r && r(t_0) = r_0,\ r(t_f) = r_c \\ &\dot{\theta} = v_\theta/r && \theta(t_0) = \theta_0 \\ &\dot{v}_r = v_\theta^2/r - \mu/r^2 + \frac{F}{m}\sin\gamma && v_r(t_0) = v_{r_0},\ v_r(t_f) = 0 \\ &\dot{v}_\theta = -v_r v_\theta/r + \frac{F}{m}\cos\gamma && v_\theta(t_0) = v_{\theta_0},\ v_\theta(t_f) = v_c \\ &\dot{\gamma} = u && \gamma(t_0) = \gamma_0,\ |u| \leq K \end{aligned} \qquad (5.1)$$

$$m(t) = m_0 + \dot{m}_0(t - t_0),$$

where

$$J_1 = t_f \qquad (5.2)$$

is to be minimized.

In Reference 5, the following method for computing singular control problems is suggested: adjoin $\epsilon\int_{t_0}^{t_f} u^2\,dt$ to the performance index, i.e., Eq. (5.2), and solve the resultant nonsingular boundary-*value problem for successively smaller values of* ϵ. As $\epsilon \to 0$, it is argued that the solutions approach the optimal singular solution. Another *computational scheme is also suggested in Refer*ence 5 since numerical stability problems may result for small values of ϵ. However, the main effect of the second scheme is to *sharpen the con*trol history while only a slight improvement in the performance index is noted. Data from the Saturn SA-502 flight will be used to show that merely adding the $\epsilon\int_{t_0}^{t_f} u^2\,dt$ term to the performance index results in a good suboptimal control for a relatively large value of ϵ.

Define

$$J_2 = t_f + \epsilon\int_{t_0}^{t_f} u^2\,dt, \qquad (5.3)$$

where ϵ is a given constant. The Hamiltonian is

$$H = \lambda_1 v_r + \lambda_2 v_\theta/r + \lambda_3(v_\theta^2/r - \mu/r^2 + (F/m)\sin\gamma) + \lambda_4(-v_r v_\theta/r + (F/m)\cos\gamma) + \lambda_5 u + \epsilon u^2, \qquad (5.4)$$

which defines a nonsingular optimization problem. The minimum principle states that the Hamiltonian must be minimized with respect to the control. This implies the following ordinary minimization problem: minimize

$$\tilde{h} = \lambda_5 u + \epsilon u^2 \qquad (5.5)$$

subject to the inequality constraint

$$|u| \leq K. \qquad (5.6)$$

Eq. (5.6) can be transformed into an equality constraint by introducing a slack variable, z, i.e.,

$$z^2 = K - u^2 \geq 0 \qquad (5.7)$$

is an equality constraint which enforces the desired inequality constraint. By defining the augmented function

$$h(u,z) = \lambda_5 u + \epsilon u^2 + \Lambda(z^2 + u^2 - K), \qquad (5.8)$$

and forming

$$\frac{\partial h}{\partial u} = 0, \quad \frac{\partial h}{\partial z} = 0, \qquad (5.9)$$

and then checking the second-order sufficient condition for ordinary minimization problems, the following optimal control is determined:

$$u = \begin{cases} -K & \text{if } \lambda_5 \geq 2\epsilon K \\ -\lambda_5/2\epsilon & \text{if } -2\epsilon K \leq \lambda_5 \leq 2\epsilon K \\ +K & \text{if } \lambda_5 \leq 2\epsilon K. \end{cases} \qquad (5.10)$$

Note that the control is continuous at the junction points $\lambda_5 = \pm 2\epsilon K$ since λ_5 must be continuous by the Weierstrass-Erdmann corner conditions.

The usual Euler-Lagrange equations hold for the multipliers. The only other new condition of interest is the transversality condition for $\lambda_5(t_f)$. *Since* $\gamma(t_f)$ *is unspecified,* then

$$\lambda_5(t_f) = 0. \qquad (5.11)$$

which implies that $-2\epsilon K < \lambda_5(t_f) < 2\epsilon K$, or

$$u(t_f) = -\lambda_5(t_f)/2\epsilon = 0 \qquad (5.12)$$

This states that the control must have an interior segment in a neighborhood of t_f. However, in some *numerical studies for* ϵ *small this terminal interior* arc was very short, e.g., 0.1 seconds of a 160 second trajectory.

Since the guidance scheme of References 7 and 8 involves an iteration scheme which uses ini*tial Lagrange multiplier estimates, a similar* scheme was used to converge the optimal trajectories of this study. (Since a sufficient condition for composite singular problems does not exist, we can only use physical reasoning to argue that the resultant singular extremals are indeed optimal.)

In Figure 4, the "best" steering angle history from the initial position and velocity of the vehicle (see Appendix A for the numerical values used in this study) is shown. The initial position and velocity represent a point on the SIVB stage trajectory of the Saturn SA-502 flight. Note that the desired steering angle at the given initial position and velocity is $\gamma = -29.5°$. However, at that instant the steering angle was approximately 46.6° away from the desired angle, i.e., $\gamma(t_0) = +17.1°$. Since the steering rate on the Saturn is constrained to approximately one degree per second, one cannot assume that the steering angle can change instantaneously to the desired value.

In Figure 5, the suspected optimal time rate of change of the steering angle is presented. The optimal control is nonsingular on the interval [0,55.4) and singular on the interval [55.4, 157.305]. Note that the control is discontinuous at the junction, which is expected since the flat-earth problem is an excellent approximation of the problem of this section. Note that the difference in the initial value of γ causes the constrained trajectory to be approximately 6.5 seconds longer than the trajectory of Figure 4, which results in a 3500 pound fuel loss.

The optimal control for a nonsingular approximation of the given singular problem is presented in Figure 5, also. The value ϵ = 100,000 was found to give good results with respect to optimality and ease of convergence. Note that the suboptimal control is continuous, and in some sense approximates the optimal singular control. The final time of the suboptimal trajectory is 157.392, which represents a fuel penalty of only 46.5 pounds.

A puzzling trend was encountered when the ϵ-method was used for converging the suboptimal trajectories of this study. It was found that lower values of the original performance index, i.e., Eq. (5.2), were obtained as ϵ increased instead of as ϵ decreased, which at first glance seems contrary to intuition. Of course this trend may be due to the fact that an initial multiplier guessing scheme was used to converge the trajectories instead of a function space method. However, the Pontryagin minimum principle was satisfied numerically in each case.

A possible explanation of the above trend is simply that the augmented performance index of Eq. (5.3) does not converge to the minimum value of the original performance index (t_f) for this particular problem as $\epsilon \to 0$. The proof of convergence for the ϵ-algorithm in Reference 5 is for fixed t_f, and since t_f is not fixed in this problem, convergence cannot be assumed. Indeed, further analysis revealed that the augmented performance index decreased monotonically as ϵ decreased and appeared to be converging to a value considerably larger than the minimum value of the original performance index. In other words, as ϵ decreased, J_2 decreased, but t_f increased, indicating that $\lim_{\epsilon \to 0} J_2(\epsilon) \neq J_2(0)$.

To lend support to our contention that the ϵ-algorithm may not converge to the optimal singular solution for a minimum time problem, consider the following argument. From Eq. (5.3) the minimum value of J_2 for a particular value of ϵ can be written

$$J_2(\epsilon) = t_f(\epsilon) + \epsilon a(\epsilon) t_f(\epsilon) \tag{5.13}$$

where $a(\epsilon) > 0$ is the average value of the optimal $u^2(t)$ over the interval $[t_0, t_f]$ for the given value of ϵ, and for simplicity we have taken $t_0 = 0$. Differentiating Eq. (5.13) by the chain rule,

$$\frac{dJ_2}{d\epsilon} = at_f + (1 + \epsilon a)\frac{dt_f}{d\epsilon} + \epsilon t_f \frac{da}{d\epsilon} \tag{5.14}$$

where the terms containing ϵ are negligible for ϵ sufficiently small. Let $\epsilon_1 < \epsilon_0 << 1$. The series expansion for $J_2(\epsilon_1)$ to first order is

$$J_2(\epsilon_1) = J_2(\epsilon_0) + \frac{dJ}{d\epsilon}(\epsilon_1 - \epsilon_0)$$

$$= J_2(\epsilon_0) + \left(a(\epsilon_0)t_f(\epsilon_0) + \frac{dt_f}{d\epsilon}\right)(\epsilon_1 - \epsilon_0). \tag{5.15}$$

If $J_2(\epsilon_1) < J_2(\epsilon_0)$, then it is necessary that

$$\frac{dt_f}{d\epsilon} \geq -a(\epsilon_0)t_f(\epsilon_0). \tag{5.16}$$

On the other hand, if $t_f(\epsilon_1) < t_f(\epsilon_0)$ it is necessary that

$$\frac{dt_f}{d\epsilon} \geq 0. \tag{5.17}$$

Satisfaction of the inequality (5.16) does not imply satisfaction of (5.17). Therefore, it is to be expected that t_f may increase while J_2 decreases as $\epsilon \to 0$.

The above analysis is valid for sufficiently small ϵ. It is still somewhat surprising that a good suboptimal control would result from using ϵ = 100,000. In this regard, note that if ϵ is very small in the performance index of Eq. (5.3), and if a number of control histories give near-optimal performance (with respect to the original performance index), which is a common occurrence in singular problems, then nothing is acting to keep u interior in the neighborhood of the optimal singular subarc. On the other hand, if ϵ is large, then to minimize the second part of J_2, u should be as near to zero as possible, i.e., ϵ large helps to enforce an interior u in the neighborhood of the singular arc.

To demonstrate this argument, another example was considered. The only difference between this example and the given Saturn SA-502 data is that $\gamma_0 = -1.8°$ instead of $\gamma_0 = +17.1°$. Because of the smaller difference between γ_0 and $\gamma_{COV} = -29.5°$, we have more confidence that the singular extremal obtained for this problem is indeed optimal.

In Figure 6, three control histories are shown. The singular control results in $t_f = 151.66$ seconds, the ϵ = 100,000 approximation results in $t_f = 151.93$ seconds, and the ϵ = 100 approximation results in $t_f = 157.70$ seconds. As ϵ was decreased from ϵ = 100,000, the trajectories tended to become more bang-bang. In fact, with $\epsilon = 1$, the interior segments are approximately only 0.1 seconds long and, thus, are essentially bang-bang. Therefore, as $\epsilon \to 0$, the ϵ-approximate optimal controls actually go away from the desired singular control.

It should be emphasized that the feasibility of generating suboptimal controls with desirable properties has been demonstrated even though the convergence of the ϵ-method as $\epsilon \to 0$ for the free-final

-time problem is still an open question. In fact, the use of relatively large values of ϵ is desirable since this results in a problem which is numerically well-conditioned. Nonetheless, research on convergence is being continued, and alternate approaches which circumvent the convergence question are also being considered. One possible approach is to transform the minimum time problem into a fixed interval problem by treating v_θ as the independent variable with $v_\theta(t_0)$ and $v_\theta(t_f)$ specified, since v_θ is monotonic with respect to time.

VI. Conclusions

A singular optimal guidance problem which was motivated by difficulties encountered in the Saturn V SA-502 flight has been studied. It was shown that if the guidance system uses a singular version of the flat-earth problem, then the control must be discontinuous at a junction of singular and nonsingular subarcs for almost all cases. Since the junctions of singular and nonsingular subarcs are determined by nonlocal information, the situation described above is undesirable.

A suboptimal guidance scheme based upon a nonsingular approximation of the singular problem was suggested. Since it allows for rapid computation of a nonsingular two-point boundary-value problem, the scheme could be incorporated into the guidance scheme of References 7 and 8.

In addition to the use of the ϵ-method in an on-board iteration guidance scheme, since the ϵ-method leads to nonsingular representations of singular problems, it may also be useful for constructing suboptimal neighboring optimum guidance schemes for singular problems

Appendix A

Data from the Saturn SA-502 flight, 586.72 seconds into the mission (with approximately 160 seconds of flight remaining)

$x = 6.2133939 \times 10^6$ meters

$y = 2.1301780 \times 10^6$ meters

$\dot{x} = -2.0242460 \times 10^3$ meters/second

$\dot{y} = 6.4412899 \times 10^3$ meters/second

$F = 2.2790300 \times 10^5$ pounds

$W = 3.5280200 \times 10^5$ pounds

$Isp = 4.2476900 \times 10^2$ seconds

$\gamma = +17.1$ degrees

$r_c = 6.5633660 \times 10^6$ meters

$v_c = 7.7930430 \times 10^3$ meters/second

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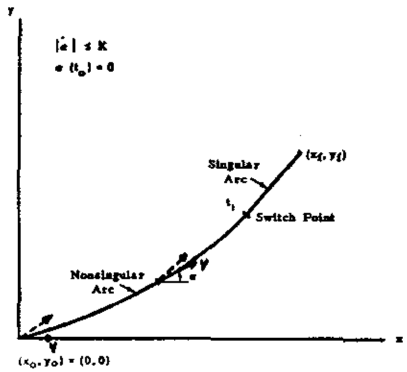


FIGURE 1. OPTIMAL TRAJECTORY FOR THE SIMPLE EXAMPLE.

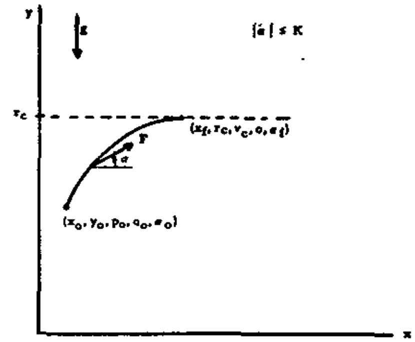


FIGURE 2. THE SINGULAR FLAT EARTH PROBLEM.

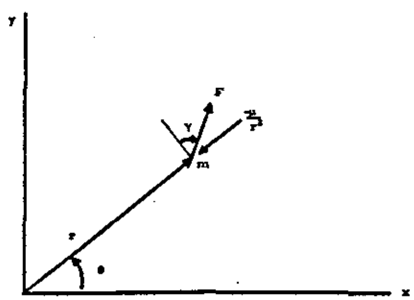
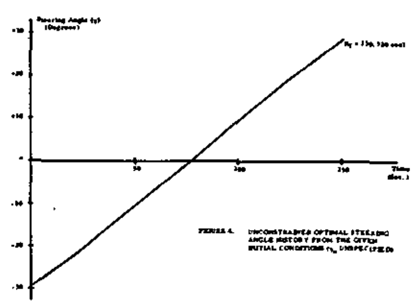
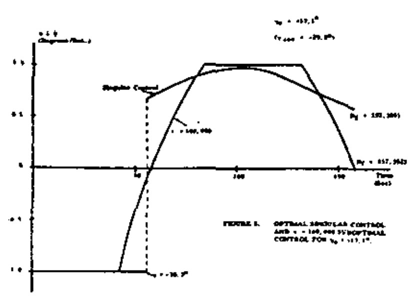
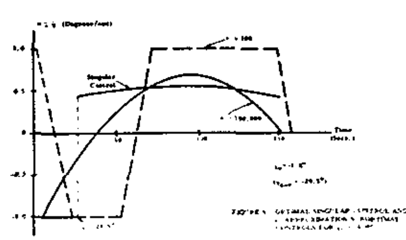


FIGURE 3. COORDINATE SYSTEM AND STEERING ANGLE FOR THE INVERSE-SQUARE GRAVITY PROBLEM.







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