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Vorticity Capturing

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Introduction

THIS paper is intended to draw attention to a crisis in algorithm development. This crisis will already be familiar, in various forms, to some readers, and some of them may feel that "crisis" is too strong a word, but I want to point out connections between difficulties being experienced across a rather broad front. For several years now, we have had computers that are large enough to tackle three-dimensional problems, but we have available only algorithms that were designed using mostly one-dimensional analysis. I believe that the discrepancy shows up most vividly when the flows of interest involve strong vorticity, as indeed many three-dimensional flows do.

To set the stage for a discussion of the numerics, let us briefly review the reasons why vorticity is so important in three dimensions. The *vorticity transport equation* for $\omega = curl\ \mathbf{u}$ is

$$\frac{\partial \omega}{\partial t} + \mathbf{u} \cdot \nabla \omega = \omega \cdot \nabla \mathbf{u} + \frac{1}{\rho^2} \nabla \rho \times \nabla p + \nu \nabla^2 \omega$$

The left-hand side shows that vorticity is transported with the fluid, except as it is modified by the right-hand side. Clearly a good advection scheme is important, but unlikely to be sufficient.

The third term on the right is the diffusion of vorticity by viscosity. Viscosity also creates vorticity at solid surfaces. This process of creation is usually dealt with by making a sufficiently fine mesh close to the surface. The diffusion thereafter is then rather feeble, certainly at Reynolds numbers of aeronautical interest. If this weak diffusion is confined to a close neighborhood of the surface where the vorticity was generated (in other words, within the boundary layer) use of a sufficiently fine mesh may be adequate to treat it accurately. However, once the vorticity escapes unpredictably from the surface, following separation, then the choice is between an expensive enlargement of the refined region, or allowing the delicate physical dissipation to be overwhelmed by numerical error.

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The second term on the right is the so-called barotropic mechanism, arising from misalignment of the pressure and density gradients. If there is a density gradient then the center of gravity of a fluid element does not lie at its geometric center. Applying a pressure gradient will then create a force that may not pass through the center of gravity, and therefore may cause the element to spin. This mechanism can be called into play even in inviscid flow, provided that the functional dependence $p = p(\rho)$ has been broken, for instance by the generation of entropy. Again, the mechanism is rather weak, and it will be argued below that it too can be overwhelmed by numerical error.

The first term on the right deals with the important stretching effect, driven by changes of velocity in the direction of the vortex lines. It is this term that is usually held to be responsible for the evolution of small-scale structure. For example, vorticity arising from instabilities such as Tollmein-Schlichting waves is stretched by secondary instabilities, but since the circulation around a vortex line is preserved in an inviscid flow (and this is an inviscid mechanism) we must find that the new velocities are increased but found at smaller radii. Thus, vortical structures tend to become less well resolved as time passes. This problem is to some extent alleviated by the fact that the stretching and concentration of vorticity will accelerate its dissipation by the third term.

The particular importance of vorticity *"Vorticity is the muscle and sinews of fluid mechanics": Dietrich Küchemann* arises from the fact that it is hard to generate (except at solid surfaces) but also, once generated, hard to destroy. This makes vortical structures very long-lived, as evidenced by the contrails stretching for miles behind a high-flying airliner. This longevity can be useful. For example, many military aircraft use vorticity generated by strakes or on a canard control surface to induce low pressures over the main wing and thereby augment lift. The leading-edge vortices on a delta wing serve a similar function. The longevity can also be a nuisance. Helicopters in descent pass through their own helical trailing vortices, and the ensuing aerodynamic noise is a large disincentive to inner-city heliports. The very delicacy of the mechanisms that make vorticity important, also make it particularly vulnerable to numerical error.

From this discussion it seems that errors in computing strongly vortical flows might either arise in the inaccurate generation of vorticity, or in its inaccurate transport. Both of these phenomena might be thought simply to strengthen the case for more accurate (that is, higher order) methods. Indeed that is one perfectly sensible reaction. Tang and Baeder[1] have derived high-order compact versions of Godunov-type schemes and report remarkable improvements in some problems of vortex convection. Drikakis and Smolarkiewicz[2] have computed some unstable two-dimensional shear layers and find that second-order schemes can produce spurious features that are absent in higher-order simulations.

There are, however, other possibilities, and some of these will be listed in the next section.

Methods Tuned to Vortical Flows

Vortex Methods

The mathematical basis of these is the fact that a vector field (here the fluid velocity) can be recovered from knowing the distribution in space of its divergence and curl, together with appropriate boundary conditions on solid surfaces and at infinity. In general we have to solve the Cauchy-Riemann equations with a given right-hand side, but if we can assume that the vorticity is localized into some finite set of *vortex lines* and that the flow is incompressible, then a direct recovery of the velocity follows from the *Biot-Savart* law;

$$\mathbf{u}(\mathbf{x}) = \sum_V \frac{\Gamma_V}{4\pi} \int \frac{(\mathbf{x}-\mathbf{s}) \times d\mathbf{s}}{|\mathbf{x}-\mathbf{s}|^{3/2}}. \qquad (1)$$

where the sum is over a certain number of vortex lines V, each of which carries a constant circulation Γ_V, and either forms a closed loop, or extends to infinity, or terminates at a surface. Along each such line, the position vector is s. The integral is replaced numerically by a sum over short segments of the vortex line. In each timestep, every segment is moved following the velocities "induced" by the other segments. In practice, the Biot-Savart law is "desingularized" by assigning small finite radius δ to the vortex cores. A nice survey and the principle references are given in Almgren, Buttke and Colella.[3] To quote from these authors *Vortex methods are especially useful for flows which are dominated by localized vorticity distribution; e.g., shear flows, wakes, and jets. In these flows most of the vorticity is confined to a very small portion of the flow, and then a method based on following the vorticity can be very economical.*

Weaknesses of the vortex method are the limitations on its applicability to situations to which the words "dominated" and "localized" truly apply, the apparent restriction to incompressible flow, and the fact that vorticity generation must be handled by separate, and somewhat empirical, procedures. Nevertheless, when the restrictions are met, this is a powerful and economical tool.

Hybrid Methods

A Navier-Stokes solver can be used to predict the near-field (generation) part of the flow, with vorticity transport modelled in the far field by a pure vortex method. in[4] an intermediate regime, model-led by potential flow, is also included. In a later paper given to the American Helicopter Society in Virginia Beach, in 1997, generally good agreement with experimental data is reported, although there seem to be a few anomalies. Details of the hybridization probably need to be tailored to the specific application of concern.

Vorticity Confinement

This is a very interesting attempt to address the problem of vortex dissipation in some generality. It originates with Steinhoff,[5] who adds artificial terms to the momentum equation of the Euler equations.

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \frac{1}{\rho} grad\, p = \mu\nabla^2\mathbf{u} + \epsilon\, \mathbf{n} \times \omega$$

Here μ is an artificial diffusion coefficient, added for reasons of stability, but it is the last term that is interesting. The vector **n** is a unit vector, chosen to lie in the direction of the vorticity gradient. For structures that resemble potential vortices with a finite core, such as the Lamb vortex, this vector is perpendicular to the axis of vorticity. The product $\mathbf{n} \times \omega$ is also perpendicular to this axis, and points toward it, irrespective of the sense of rotation. By taking the curl of this equation, one finds that the artificial terms appear in the vorticity transport equation, and that they have the effect of convecting the vorticity toward the core axis.

Because there will be a paper devoted to this topic in a later session, I will not say much more about it, but it does strike me as an excellent device for transporting the vorticity once it has been generated. Essentially, the intention is to obtain Euler solutions in the high Reynolds number limit.

Spurious structures; the carbuncle

Under some circumstances, Euler codes produce anomalous solutions, unlike the flows that would be anticipated in an experiment. Frequently, perhaps even invariably, the undesirable solution contains vortical structures. Two of the best-known anomalies are the 'carbuncle' that sometimes appears ahead of blunt bodies in supersonic flow computations[1], and which has recently been surveyed in.[6] Another example is the "Quirk phenomenon",[7] where an instability can

[1] I have also seen this behavior, in unpublished work, when a cylindrically expanding shock is computed on a square grid. It tends to arise where the shock crosses an axis of symmetry of the grid.

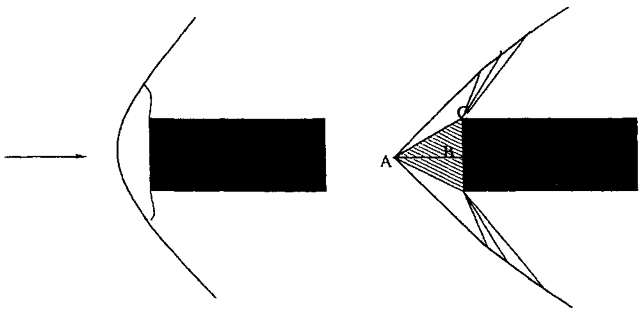

Fig. 1 Two valid solutions for the flow past a flat-faced obstacle.

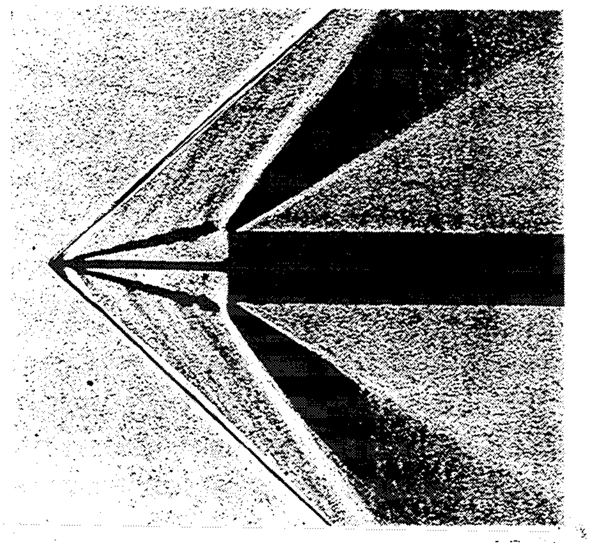

Fig. 2 Schlieren photograph of alternative solution.

be triggered by a tiny irregularity in the mesh, during the propagation of a one-dimensional shock through an otherwise uniform mesh. Both of these anomalies have usually been treated as purely numerical artifacts. What is especially disturbing is that both tend to turn up in what are, by other criteria, the most accurate codes.

Let us begin by considering a rather simple geometry, the two-dimensional flow past a flat-faced obstacle. Figure shows two possible solutions to the Euler equations under these conditions. The expected solution is of course a smooth bow shock, detached ahead of the obstacle, producing a mix of sub- and supersonic flow. However, we can construct a second solution by supposing a triangular region of "dead air" ahead of the body, past which the flow turns just as it would if the obstacle were shaped like a wedge. There seems to be no mathematical objection to this second-solution. It is assembled out of standard solutions to the Euler equations, and the discontinuities that it contains are stable and admissible.

Moreover, this second solution is experimentally realizable, by setting a thin plate along the line of symmetry. Figure 2 shows plate 272 from.[8] What appears to happen physically is that when the wind tunnel is started up, vorticity is created in the boundary layer on the plate. It is created faster than than it can escape by advection, and therefore accumulates in the region ahead of the body, which acquires an increasing circulation. Eventually a steady state is reached[2] in which circulation around the contour ABC can be accounted for by vorticity that has been produced in the past, but which by then is no longer being produced. If this explanation is correct (and there is no direct evidence) then no critical Reynolds number is involved. Repeating the experiment in imagination with less viscosity should just mean that it takes longer to build up the circulation. The flow is "inviscid" in the sense that a flow satisfying the Kutta condition is inviscid. Some viscosity is needed, but the amount can be arbitrarily small.

The axisymmetric version of this flow was studied by Maull.[9] Figure shows a plate from his paper, and Figure 3 shows a computation by Pandolfi and d'Ambrosiano[6] of a two-dimensional cylinder suffering from a severe carbuncle. Despite the differences in geometry and test conditions these seem to be examples of the same flow pattern. An explanation for the carbuncle phenomenon could therefore be that vorticity can be produced in a computation by truncation error, and that this can accumulate to produce significant circulation.

In a very interesting recent paper, Robinet *et al*[10] argue that the "Quirk phenomenon" is closely related. They show an exception to the usual text-book argument that a plane shock propagating through a uniform flow does so stably. They consider the usual case of a plane shock with a small sinusoidal perturbation along its length, and find that there is in fact an unstable mode that arises from a resonance between acoustic and advective modes. The eigenfunction of this mode has vorticity generated at the shock and convected downstream from it. Moreover, this eigenfunction resembles closely the behavior encountered numerically in Quirk's phenomenon. It seems that the mechanism for generating vorticity here is the baroclinic one associated with entropy variation behind a curved shock. Velocities induced by this vorticity then feed back into the shock through acoustic signals. The mode is actually unstable, though, only for one isolated flow condition, corresponding to a particular Mach number of the unperturbed shock.

Therefore, both the carbuncle and Quirk phenomena are "almost legitimate" flow patterns. Apparently both can be found in nature, but not unless they are triggered somehow. We might almost regard them as

[2]Not in all cases; some of the experiments cited gave rise to flow patterns that oscillated between two kinds of solution, often in an asymmetric manner. Generally the phenomenon was not repeatable enough to be useful as part of any aerodynamic design.

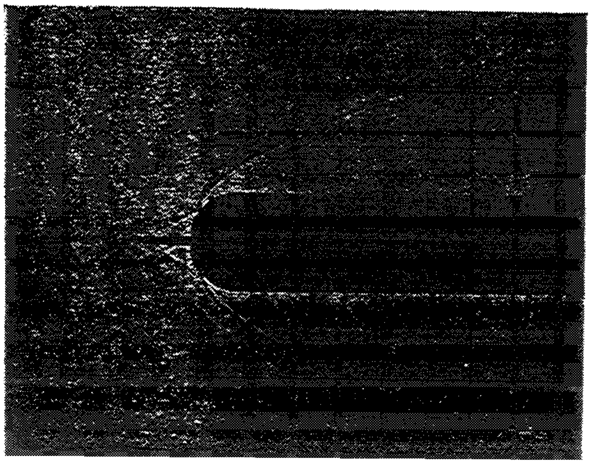

Fig. 3 Schlieren photograph of flow past a sphere with a needle ahead of it.

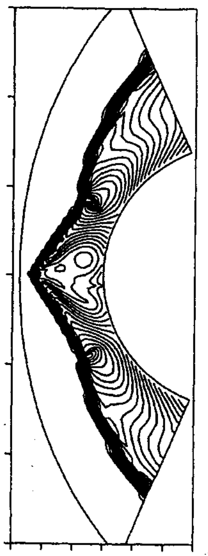

Fig. 4 A well-developed carbuncle, reproduced form Pandolfi and d'Ambrosiano.[6]

interesting discoveries that we have made about the Euler equations by doing computations. Nevertheless, they are unwelcome. In the context of this paper they can be thought of as "spurious vortical structures".

We now turn to the question why these phenomena are apparently characteristic of schemes that in other contexts give particularly good accuracy. In Pandolfi and d'Ambrosiano[6] the two-dimensional flow past a cylinder is computed using a variety of flux functions to solve the one-dimensional Riemann problem. The largest carbucles are generated by the exact Riemann solution and by the Roe linearization. These solvers both yield accurate results for isolated contact discontinuities, which is a crucial property for the accurate treatment of boundary layers.[11] Almost no carbuncle is generated by the HLL flux, which dispenses with the contact in the interest of speed and simplicity, or by simple flux-splitting schemes that are also inaccurate in shear layers. However, a large carbuncle results when the contact is reinserted into the HLL scheme (HLLC). Hybrid schemes, such as the different versions of AUSM, give intermediate results, although AUSM-M shows only a barely detectable glitch. Robinet *et al* state that the carbuncle could be removed from their computations by including the Navier-Stokes terms in the governing equations, but that this was not very effective. It was necessary to reduce the Reynolds number to a few hundred to make the phenomena disappear.

I now propose an hypothesis to account for the available evidence. *Spurious vortical structures can be introduced into a computation by discretization errors. It is not necessary that these errors be particularly large, since they can accumulate over time. The structures can be eliminated by introducing sufficient damping, either of a numerical or physical nature.* If true, this puts us between a rock and a hard place. Schemes with much damping will not generate the spurious structures, but they will excessively dissipate the genuine structures.

I admit that much of this conjecture, and that a high-order scheme might suffer from none of these problems, but I want to explore in the next section the idea that some special notions of accuracy may be useful here.

Dealing with spurious production

In the design of upwind codes, we have learned to work simultaneously with more than one form of the Euler equations. To ensure the correct propagation of nonlinear discontinuities, we work with the conservation form, but to obtain cleanly propagating waves we work with the characteristic form. Conservation form is based on ensuring that *some* properties of the exact solution are shared *precisely* by the discrete solution, so that there are certain kinds of error that just will not happen. I speculate that *something* like this may be possible for vorticity.

The sort of code I have in mind would still need to be in conservation form because we will not be dealing with flows that are totally dominated by vorticity. There will still be shocks that must be treated properly, but the way that the fluxes are calculated will acknowledge that vorticity is important, just as upwind codes acknowledge that characteristic information is important. It may not be necessary to incorporate the vorticity transport equation directly into the code, any more than an upwind code directly incorporates the Rankine-Hugoniot relations. Instead, the principle should probably be this. *From the discrete equations that are used in the code, it should be possible to obtain discrete equations for vorticity transport that are physically valid. In particular, vorticity should be generated only by terms that have physical meaning, and not by* *truncation error from physically irrelevant terms.*

Linear Acoustics

The simplest model problem having any relevance to this is two-dimensional linear acoustics

$$\begin{aligned} p_t + \rho a^2(u_x + v_y) &= 0 \\ u_t + (1/\rho)p_x &= 0 \\ v_t + (1/\rho)p_y &= 0 \end{aligned} \qquad (2)$$

Defining $\omega - v_x - u_y$ it is an easy deduction, if ρ, a are constants, that

$$\omega_t = 0. \qquad (3)$$

On the other hand, if ρ is a given function of x and y then we have production of vorticity via the barotropic mechanism. The principle above states that in the constant density case we should seek a scheme that generates no vorticity, and in the variable density case the only production should be recognizably barotropic. In Morton and Roe,[12] a family of Lax-Wendroff schemes were reviewed for their effect on the discrete vorticity[3] $\omega' = \mu_y\delta_x v - \mu_x\delta_y u$. Almost all members of the family generated vorticity, on a square grid with mesh spacing h, at a rate proportional to h^3 (and related first-order schemes generated vorticity at a rate proportional to h^2). In a student project directed by Professor Alain Lerat, these predictions were verified, and initial data corresponding to a Lamb vortex was quite rapidly dissipated. One single member of the family was predicted to conserve the vorticity, and indeed did so in practice. We will now describe and motivate that particular Lax-Wendroff scheme.

A model problel and solution

Consider solving the problem (2) on a square grid with spacing h by a finite-volume method. Variables p, u, v are stored in cell centers, and in this context may be regarded as conserved variables. The fluxes of the velocities are U, V and of the pressures P, Q as shown in Figure 5. Auxiliary quantities are stored at the vertices and denoted p', u', v'. Update of the velocities takes place through

$$\begin{aligned} \delta_t u &= (k/h)\delta_x P \qquad (4) \\ \delta_t v &= (k/h)\delta_y Q \qquad (5) \\ & \qquad (6) \end{aligned}$$

where k is the time step, and so the vorticity is updated through

$$\begin{aligned} \delta_t\omega &= \mu_y\delta_x\delta_t v - \mu_x\delta_y u \\ &= -(k/h)[\mu_y\delta_x\delta_y Q - \mu_x\delta_y\delta_x P] \\ &= -(k/h)\delta_x\delta_y[\mu_y Q - \mu_x P] \end{aligned} \qquad (7)$$

[3]Clearly we need to choose *some* definition of discrete vorticity, just as we need to define *some* quadrature formula before we can discuss conservation. The exact choice should not be crucial. If vorticity behaves perfectly according to some reasonable definition, then it cannot behave too badly under any any reasonable definition.

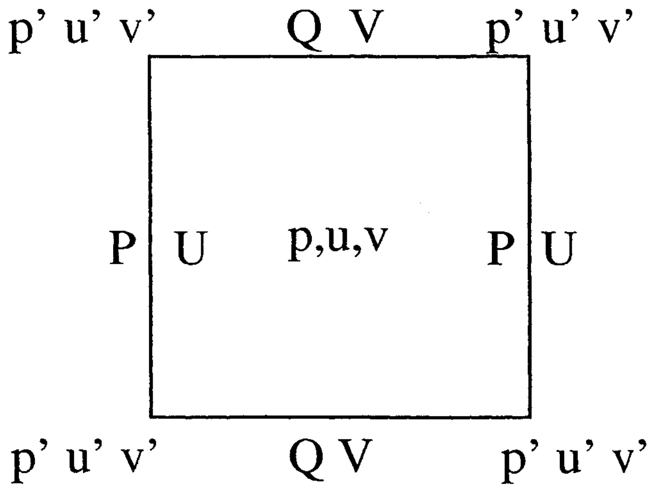

Fig. 5 Storage for computing the acoustic equations.

To ensure that this vanishes, we can take $P = \mu_y p', Q = \mu_x p'$, and there is no other solution without enlarging the stencil. The striking conclusion of this very simple analysis is that the fluxes P, Q must be computed in two-dimensional mannner. No flux formula, no matter of what order, that is obtained from one-dimensional interpolation, can have this property.

To obtain second-order accuracy in time the intermediate quantities must be evaluated midway through the time step, and there is only one compact way to do this.

$$p' = \mu_x\mu_y p - (k/2h)[\mu_y\delta_x u + \mu_x\delta_y v] \qquad (8)$$

Note, however, that p' could be obtained from a random number generator, and there would still be bo vorticity generated! This provides a mechanism for inserting non-linear limiters. In fact the two-dimensional algorithm can be constructed so that, for one-dimensional data, it collapses to a favored high-resolution scheme. Therefore, for thye model proble, there is no conflict between ensuring conservation, monotonocity, and vorticity preservation.

The update of pressure does not in itself affect vorticity, but can be achieved in a neatly symmetric manner by choosing $U = \mu_y u', V = \mu_x v'$ with

$$\begin{aligned} u' &= \mu_x\mu_y u - (k/2h)[\mu_y\delta_x p] \qquad (9) \\ v' &= \mu_x\mu_y v - (k/2h)[\mu_x\delta_y p] \qquad (10) \end{aligned}$$

This completes the description of a second-order scheme of Lax-Wendroff type[4] It can be identified with the Rotated Richtmyer scheme,[13] which has been neglected because it suffers from an odd-even spurios mode. It is also identical, on regular grids, to the Ni scheme.[14] Among the many Lax-Wendroff schemes that are possible these two are optimal with respect to

[4]The basic strategy involved is aplicable also to semidiscrete schemes.

stability and isotropy. The odd-even decoupling can be dealt with by the limiter.

One explanation for the freedom from vorticity is that the analytical identity $\partial_x(\partial_y) = \partial_y(\partial_x)$ is irrored at the discrete level by the identity $\mu_y\delta_x(\mu_x\delta_y) = \mu_x\delta_y(\mu_y\delta_x)$. In Morton and Roe[12] it is shown that the same scheme, applied in a natural way to the case of variable density, does produce vorticity, and that a discrete version of Kelvins Circulation Theorem,

$$\partial_t\Gamma_C - \int_{\delta C}\frac{dp}{\rho} \qquad (11)$$

can be found. In other words, the circulation around any contour C depends correctly on the events on the boundary of C and not on any events within C. This is moreover true on unstructured grids, if the control volumes of concern are chosen correctly. It is not an easy task to extend this analysis to the full nonlinear Euler equations in three dimensions. Nor is it self-evident that a code designed in this way will be free of all problems concerning vorticity, any more than conservation form by itself solves all problems in shock-capturing. Nevertheless, a start has been made on adressing an issue that impacts many current corcerns in computational aerodynamics.

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