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COMPUTATION OF KUHN-TUCKER TRIPLES IN OPTIMUM DESIGN PROBLEMS IN THE PRESENCE OF PARAMETRIC SINGULARITIES

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Abstract

In the mathematical optimization literature, there has been considerable interest in the role that singularities play when models are subject to parametric deformation. In this paper, our objective is to illustrate the presence of such singularities in applications where the models are smooth in variables and parameters, but the solutions are not always regular. In the first example, an elastic foundation problem is revisited to show that the observed singularities are of the SCC (strict complementarity condition) variety. In the second example, a classical beam-buckling model is augmented with an obstacle in such a way the solution paths have singularities where each of SCC, LICQ (linear independence constraint qualification) and SOSC (second order sufficiency condition) fail individually.

1. Introduction

Consider a typical optimal design problem stated as a standard nonlinear program:

$$\mathbf{P(p)}: \quad \underset{x \in \Re^n}{\text{minimize}} \quad f(x,p)$$
$$\text{subject to} \quad g_i(x,p) \le 0 \quad (i = 1, \dots, l) \qquad (1)$$
$$h_j(x,p) = 0 \quad (j = 1, \dots, m).$$

where x is the vector of design variables, (the scalar) f, g_i, h_j are real valued, and usually smooth (at least C^2), functions defined on $\Re^n \times \Omega$, with $p \in \Omega \subset \Re^r$. As the parameter p is varied in the set Ω, a family of deformed optimal design models is generated which we refer to as $\mathbf{P(p)}$. Typically, the parameter vector p occurs in such models in three ways. Firstly, p may be a naturally occurring parameter, such an imprecise control input, or available resource in a model. Secondly, p may be artificially introduced in the design model, e.g., the parameters used to scalarize a multiobjective problem. Thirdly, p could be a subset of the original variables which are temporarily fixed, as in multilevel decomposition and game theory models.

Such parametric embeddings affect the underlying model in various ways. Consider the following two rather extreme examples:

Example 1

$$\text{minimize } p^2 f(x), \; x \in \Re^n$$

where $f(x)$ is a smooth, strictly convex function. If $N(p)$ denotes the number of minimizing solutions for a given p, then $N(p) = 1$ for $p \neq 0$ and $N(p) = \infty$ for $p = 0$. Note, however, that the number of optimizing function values does not change, as we trivially expect from Sard's theorem.

Example 2

$$\text{minimize } f(x) = (x_1 + x_2)^2 + x_1 \;\text{ s. t. }\; p^2 x_1 \ge 0, \; x \in \Re^2$$

For any $p \neq 0$, $N(p) = 1$, the only solution being $x_1^* = x_2^* = 0 = f^*$. At $p = 0$, the problem is unbounded and $N(0) = 0$.

In general, the most basic question in parametric programming is to establish the various continuity and regularity properties of these point-to-set maps: the feasible solution set $X(p)$, the optimal value function $f^*(p)$, and the optimal solution set $X^*(p)$. The literature on this subject is vast; see e.g., Fiacco et al.[1-3] (and references there in) and Poore et al.[6,7] The theoretical nature of parametric singularities has been examined extensively, which in turn has led to studies on numerical algorithms which can trace parametric solution paths and detect the different singularities along such paths.[4,8] In this paper, our focus is on applications where the model is smooth but where different types of parametric singularities manifest themselves.

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2. The Lagrangian Matrix

As discussed in Fiacco and Liu,[3] a regular or nondegenerate solution to a parametric programming problem is when the linear independence constraint qualification (LICQ), second order sufficiency condition (SOSC) and the strict complementarity condition (SCC) are satisfied. If the standard Lagrangian for Problem $\mathbf{P}(\mathbf{p})$ is

$$L(x,u,v,w,p) = f + \sum_i u_i g_i + \sum_j v_j h_j$$

then the set of $n+l+m$ equations which are part of the KKT (Karush-Kuhn-Tucker) conditions at a candidate solution point are $\nabla_x L = 0$, $u_i g_i = 0$, and $h_j = 0$. Writing these as $F(z,p) = 0$ where $z = (x,u,v)$, the Jacobian matrix $\nabla_z F$, referred to as the Lagrangian matrix, is as follows:

$$\begin{pmatrix} \nabla^2_{xx} L^f & \nabla_x {g_1^f}^T, \dots, \nabla_x {g_m^f}^T & \nabla_x {h_1^f}^T, \dots, \nabla_x {h_p^f}^T \\ u_1 \nabla_x g_1^f & g_1^f \quad\quad 0 & \\ \cdot & \quad\cdot & 0 \\ \cdot & \quad\quad\cdot & \\ u_m \nabla_x g_m^f & 0 \quad\quad g_m^f & \\ \nabla_x h_1^f & & \\ \cdot & 0 & 0 \\ \cdot & & \\ \nabla_x h_p^f & & \end{pmatrix} \quad (2)$$

We can readily observe that if SCC fails, then all the elements in a row of this matrix will become zero. If LICQ fails, then the last $l+m$ columns will be linearly dependent. If LICQ, SOSC and SCC are all satisfied then this matrix can be shown to be non-singular. The implicit function theorem then implies that locally, the KKT triple (x,u,v) is a once continuously differentiable function of the parameter p. This is the essence of the Basic Sensitivity Theorem (BST)[1] which further states that these regularity conditions are satisfied for neighbouring parameter values as well and that such a parametric KKT path may be continued until one of LICQ, SOSC and SCC fails.

3. Illustrative Examples

A Foundation Design Problem

This example is motivated by Shen et al.[10] In that study the authors have used homotopy methods to trace the solution as a resource parameter is varied. The resulting solution paths have singularities when either a variable reaches or leaves a bound or when the structure transitions from a unimodal to bimodal design or vice-versa. As such, this is a non-smooth problem and the results based on BST do not apply directly. However, the solution curves in the cited study[10] appear similar to what we would expect for generic cases when SCC failure occurs. For a simpler version of this foundation design problem, we find that this is indeed the case.

For the column as shown in Figure 1, given a bound on the total available foundation resource, the design goal is to obtain an elastic foundation that maximizes the buckling load. The energy functional for this system is:

$$\begin{aligned} E(u) &= \frac{1}{2}\frac{q}{l^2}\left[(u_2 - 2u_1)^2 + (u_2-u_1)^2 + (u_3-u_2)^2\right] \\ &+ \frac{1}{2}[k_1 u_1^2 + k_2 u_2^2 + k_3 u_3^2] \\ &- Pl[4 - (\cos(\frac{u_1}{l}) + \cos(\frac{u_2-u_1}{l}) \\ &\qquad + \cos(\frac{u_3-u_2}{l}) + \cos(\frac{u_3}{l}))] \end{aligned} \quad (3)$$

Under the small displacements assumption, the stationarity condition of the energy functional $E(u)$ can be written as the eigenvalue problem: $K(x)u = PK_G u$, where $K(x)$ and K_G are the positive-definite 3×3 stiffness and geometric stiffness matrices, respectively. K depends on the design $x = (k_1, k_2, k_3)$, whereas K_G depends on geometry alone. The design problem can now be stated as:

$$\begin{aligned} \max_{x} \ \min_{u} \quad & u^T K(x) u \\ \text{s. t.} \quad & \\ & u^T K_G u - 1 = 0 \\ & \textstyle\sum_i x_i - X = 0 \\ & x^l \le x \le x^u \end{aligned} \quad (4)$$

Our interest here is to observe the model deformation as the resource parameter X is varied. Because of the max-min objective, the model in Eq. (4) is not smooth and the Basic Sensitivity Theorem and related results do not apply directly. Furthermore, we cannot always write the minimum eigenvalue function as a pointwise minimum of smooth functions (see, for example, Overton[5]). In this case, however, by taking a symmetric foundation, i.e., $k_1 = k_3$, writing $k_2 = X - k_1$, and by obtaining the explicit expressions for the three eigenvalues, a one-dimensional smooth problem can be obtained using a "bound formulation" as follows (the upper bound on k_1 is relaxed, $q=1$, $l=0.25$):

$$\begin{aligned} \max_{(k_1,\alpha)} \quad & \alpha \\ \text{s. t.} \quad & \\ g_1: \ & \alpha - (32 + \tfrac{k_1}{2}) \le 0 \\ g_2: \ & \alpha - (\tfrac{64-k_1+X-A}{2}) \le 0 \\ g_3: \ & \alpha - (\tfrac{64-k_1+X+A}{2}) \le 0 \\ g_{4,5}: \ & 0 \le k_1 \le X \end{aligned} \quad (5)$$

where

$$A = \sqrt{2048 + 64k_1 + 5k_1^2 - 64X - 4k_1X + X^2}$$

The solution is indicated in Figure 4. For $X < 32$, the solution is unimodal, g_4 and g_2 are active, with k_1^* is at its lower bound and $k_2^* = X$. At $X = 32$, a transition occurs from unimodal to bimodal solution, and g_1 enters the active set while g_4 leaves the set. This bimodal solution continues until $X = 416$, when a transition occurs to a unimodal solution. At this point, g_1 leaves the active set. These two transitions through SCC singularities are shown in Figure 2.

SOSC, SCC and LICQ Singularities

Consider the discrete model of a geometrically nonlinear beam, as shown in Figure 3. This model is well known in singularity theory as a classical example of pitchfork bifurcation (Figure 5) which corresponds to the failure of SOSC. Here, we have slightly modified the system by adding a rigid obstacle as shown in the figure. For this given system, the basic problem is simple - given the size of the particular obstacle and a load P, find the equilibrium position, θ^*, of the beam. Note that we accept as solution to this problem all the θ^* values at which the structure could exist in a stable, physically realizable configuration (i.e., including those to the left of the obstacle which are not normally attainable by incremental loading from the unloaded horizontal configuration). With this broader solution class, we can show that this example illustrates solution singularities when each of LICQ, SOSC and SCC fails.[9]

The model is

$$\begin{aligned} \text{minimize} \quad & f(\theta) = \tfrac{1}{2}k(2\theta)^2 - 2Pl(1-\cos\theta),\ \theta \in \Re \\ \text{s. t.} \quad & \\ & g_1 : (a + b - 2l\cos\theta)(2l\cos\theta - a) \le 0 \quad (6) \end{aligned}$$

The parameters of interest in this model are the load P and the size of the obstacle b. Without the obstacle, the response is as shown in Figure 5. In this case, the stationary condition is

$$df/d\theta := 2\theta - (Pl/k)\cos\theta = 0 \qquad (7)$$

and the SOSC is $(2 - (pl/k)\cos\theta) > 0)$. For $P < (2k/l)$, the only stable solution is $\theta^* = 0$. By increasing p beyond this value, SOSC is no longer satisfied for the stationary solution $\theta = 0$ and the column buckles. The solution bifurcates into two possible stable $\theta^* \neq 0$ values which are obtained by solving Eq. (7). Note that $\theta = 0$ remains a stationary solution at which SOSC is not satisfied.

In the presence of the obstacle, we now consider parametric deformation with respect to both P and b. Consider first the case when P is varied and b is fixed such that $a + b = 2l\cos\beta$ where $0 < \beta < \cos^{-1}(a/2l)$. Due to the presence of the obstacle, there are two stable solutions, one with contact at C where $\theta_1^* = \cos^{-1}(a/2l)$ and another θ_2^* for which contact may or may not happen to the right side of the obstacle. As the load P is increased from zero value, SOSC will first fail when $P = (2k/l)$ and the solution will bifurcate, as before. As P is increased further, the solution θ_2^* will increase monotonically until $\theta_2^* = \beta$ and contact at D will just occur. This value of P causes SCC failure since g_1 is active but the multiplier, which relates to the value of the contact force, is zero. As P is increased further, the stable solution $\theta_2^* = \beta$ remains constant. Note that there are no singularities in this case on the other solution path $\theta_1^*(P)$.

Consider now the case when b is varied and P is fixed at some value $(2k/l) < P < (\pi k/l)$, i.e., at a value at which the structure would have buckled in the absence of the obstacle and let the corresponding buckled configuration be $\theta^* = \gamma$. Let us decrease b from an initial value chosen such that $2l > (a + b) > 2l\cos\gamma$. For this value of b, there will be two solutions, $\theta_1^* = \cos^{-1}(a/2l)$ as before, and $\theta_2^* = \cos^{-1}((a + b)/2l)$. As b is decreased, θ_1^* will be unchanged and θ_2^* will increase until $\theta_2^* = \gamma$ at which time SCC failure occurs and the contact at D just occurs (i.e., without any contact force). As b is decreased further, the solution path $\theta_2^*(b) = \gamma$ will be constant and free from any singularity. However, at $b = 0$, when the obstacle just vanishes, LICQ will fail for the solution path $\theta_1^*(b)$ and this terminates. The full solution set for this example as a function of the two parameters P and $b$is illustrated in Figure 6.

3. Conclusions

Our objective in this paper was to present illustrations of parametric singularities in structural applications. We hope that a better understanding of such singular behavior in design applications will augment the very active research work in theoretical and numerical parametric programming currently underway in the mathematical optimization community. These are by no means isolated examples. Loss of differentiability of structural states in contact problems has been well known, and is readily identified with SCC failure. Simple buckling and the related well-known results in structural stability are examples of SOSC failure. Examples of LICQ failure are not so readily apparent and need to be explored further - this is likely to happen when some design constraints are redundant.

References

[1] Fiacco A.V., 1976, "Sensitivity Analysis for Nonlinear Programming Using Penalty Methods", *Math. Programming*, 10(3), pp. 287-311.

[2] Fiacco A. V., 1983, *Introduction to Sensitivity and Stability Analysis in Nonlinear Programming,* Academic Press, NY.

[3] Fiacco A. V., Liu J., 1993, "Degeneracy in NLP and the development of results motivated by its presence," *Annals of Operations Res., 46,* pp. 61-80.

[4] Lundberg, B. N., and A. B. Poore, "Numerical Continuation and Singularity Detection Methods for Parametric Nonlinear Programming," *SIAM J. Optimization, Vol. 3, No. 1,* pp. 134-154.

[5] Overton M. L., 1992, "large-Scale Optimization of Eigenvalues," *SIAM J. Optimization, Vol. 2, No. 1,* pp. 88-120.

[6] Poore A. B., and C. A. Tiahrt, 1987, "Bifurcation Problems in Nonlinear Parametric Programming," *Mathematical Programming,* Vol. 39, pp. 189-205.

[7] Poore A. B., and C. A. Tiahrt, 1990, "A Bifurcation Analysis of the Nonlinear Programming Problem," *Mathematical Programming, Vol. 47,* pp. 117-141.

[8] Rao, J. R.J., and Panos Y. Papalambros, 1989, " A Nonlinear Programming Continuation Strategy for One Parameter Design Optimization Problems," in *Advances in Design Automation (B. Ravani, Ed.),* ASME Publ. No. DE-Vol.19-2, pp. 77-89.

[9] Rao, J. R. J., Bradlaw R. B., 1994 "Parametric Singularities in Optimum Design Problems in the Presence of Obstacles," *Jl. of Structural Optimization* (in preparation).

[10] Shin Y. S., Haftka R. T., Watson L. T., and R. H. Plaut, 1988, "Tracing Structural Optima as a Function of Available Resources by a Homotopy Method," *Comp. Methods in Applied Mech. and Engg., 70,* pp.151-164.

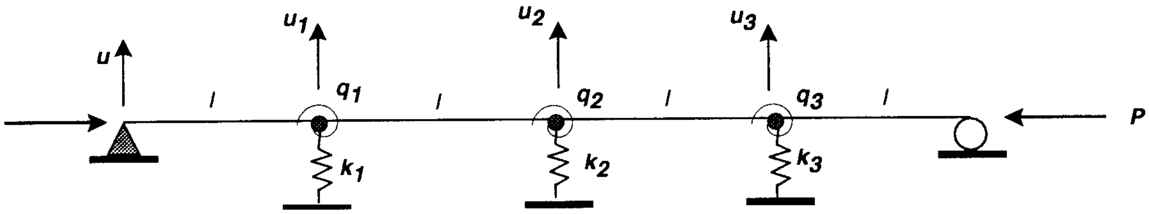

Figure 1: A discrete foundation design problem.

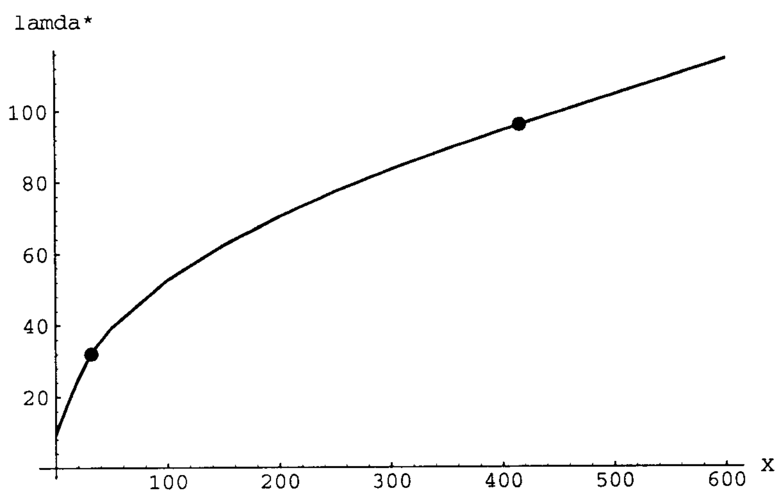

Figure 2: The optimal value function in the foundation problem.

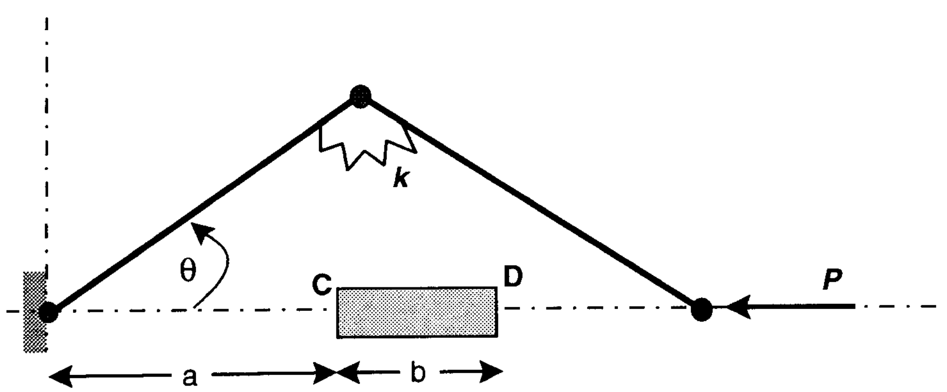

Figure 3: Buckling of a discrete, nonlinear beam.

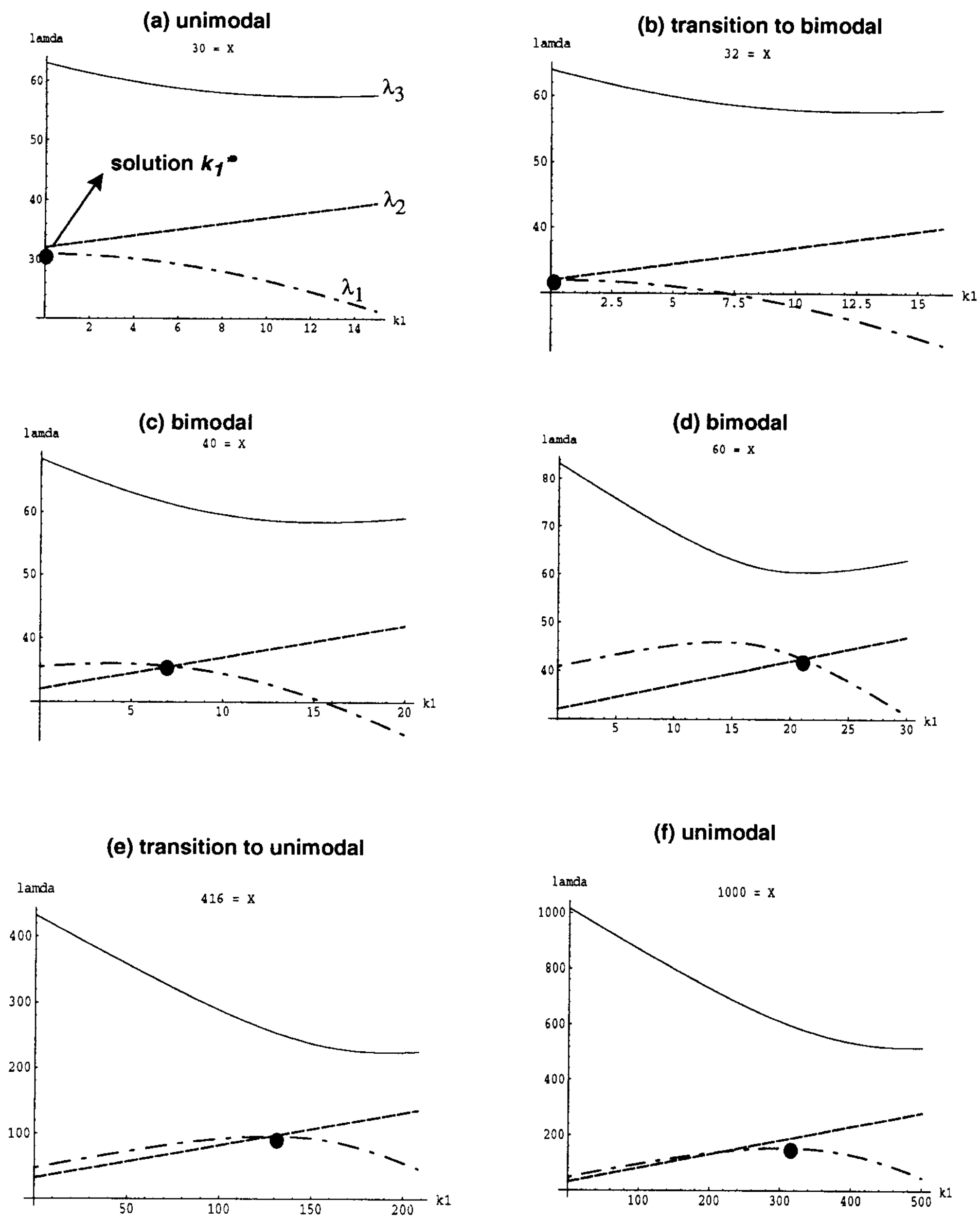

Figure 4: Unimodal-bimodal transitions through SCC singularities.

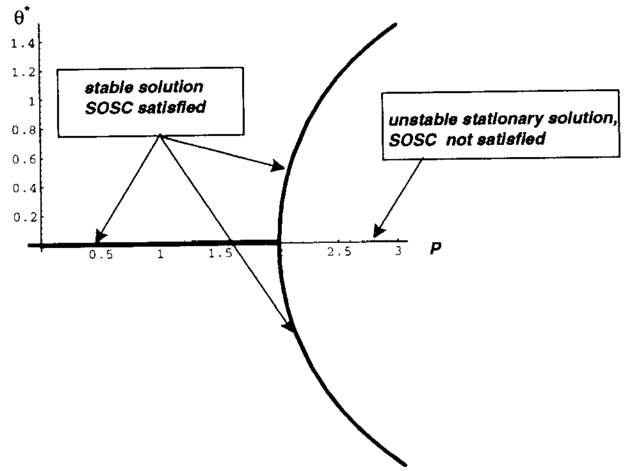

Figure 5: Pitchfork bifurcation in the nonlinear beam

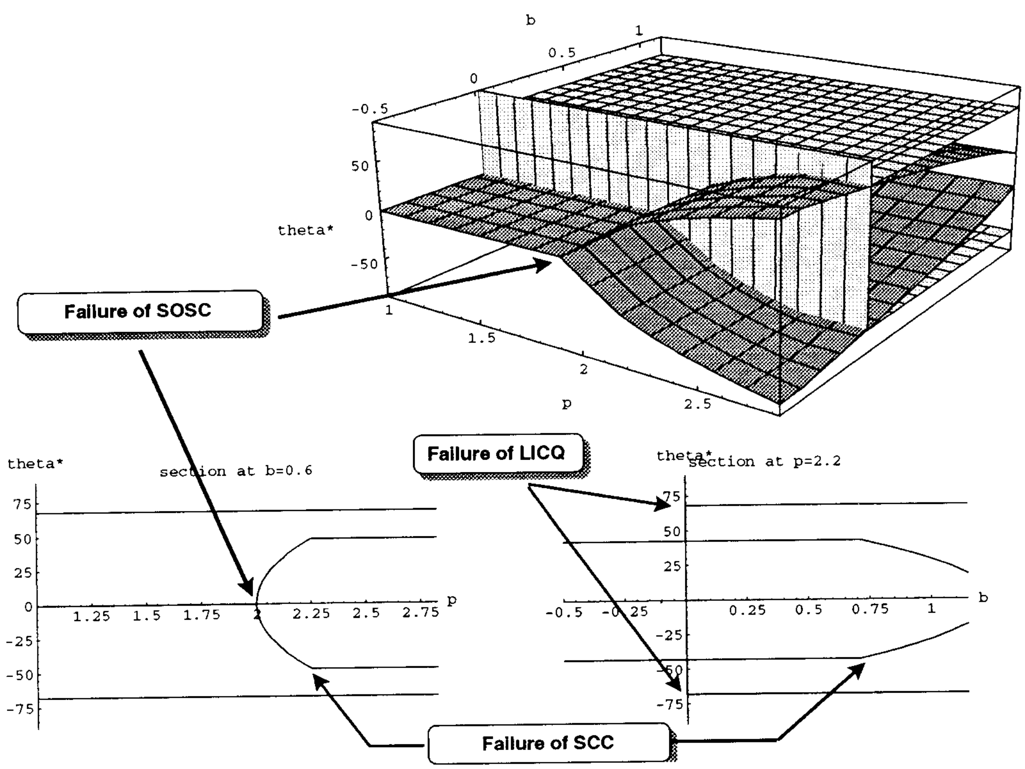

Figure 6: LICQ, SOSC and SCC singularities in the beam buckling example.