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LONGITUDINAL DYNAMIC STABILITY OF A SHUTTLE VEHICLE

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LONGITUDINAL DYNAMIC STABILITY OF A SHUTTLE VEHICLE†

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Abstract

This paper presents an analytical study of the longitudinal dynamic stability of a non-rolling, lifting vehicle that is gliding at hypersonic speeds. The analysis applies to shuttle vehicles that are designed for operating up to the rim of a planetary atmosphere. A general non-dimensional time transformation is introduced to derive a unified second order linear differential equation for the angle-of-attack, valid for all types of reentry of a general type of vehicle. The stability of motion is discussed for two fundamental regimes of flight that are based on widely different assumptions. For nearly ballistic entry along a straight line trajectory, the equation reduces to a confluent hypergeometric equation, the solution of which can be expressed in terms of the Whittaker's function. Using a theorem in the theory of stability of differential equation, criteria for damped oscillations are derived. The critical case of small static stability derivative C_{m_α}, which may cause instability in pitch, is discussed in detail, and the critical altitude below which the vehicle is unstable is given in explicit form. For gliding entry at small flight path angle, the unified equation reduced to a damped Mathieu's equation with periodic forcing term. Using the method of Krylov-Bogoliubov, an approximate solution is constructed. It is shown that the aerodynamic criteria for stability are the same as for the case of ballistic entry. In addition, for each vehicle configuration, and specified planetary atmosphere, there exists an altitude range where the angle-of-attack frequency is nearly equal to the orbital frequency, causing instability in pitch. This resonance instability is due to the ellipticity of the orbit. Criteria for eccentricity instability are derived.

Notation

a	= constant coefficient, Eq. (31), amplitude, Eq. (68)
A, B, C	= principal moments of inertia
b, c	= coefficients, Eqs. (17), (18). Also Eq. (39)
b, c_1, c_2, c_3	= numerical coefficients, Eq. (60)
C_1, C_2	= constants of integration
C_D	= drag coefficient
C_L	= lift coefficient
C_m	= pitching moment coefficient
$C_{D_\alpha}, C_{L_\alpha}, C_{m_\alpha}, C_{m_{\dot\alpha}}, C_{m_q}$	= stability derivatives
D	= base cone diameter, Fig. 2
$f_i(\xi)$	= coefficients of the unified equation, Eqs. (12), (13)
g	= acceleration due to gravity
ky	= radius of gyration in pitch
k_i, k	= $i = 1, \ldots, 4$ numerical coefficients, Eqs. (25), (28), (31)
L	= characteristic length
m	= mass of vehicle
M	= constant bound, Eq. (42)
n	= angle-of-attack frequency for shallow entry, Section IV
q	= angular velocity in pitch relative to the earth
r	= radial distance from center of earth
s	$= \frac{u_0}{\sqrt{g_0 r_0}}$, speed ratio
S	= reference area
t	= time
u_0	= speed along the reference circular orbit
u	= variable, proportional to the angle-of-attack, Eq. (27)
V	= speed along the trajectory
w	= variable, proportional to the angle-of-attack, Eq. (66)
W	= Whittaker function, Eqs. (35), (36)
x_1, x_2	= variables, Eq. (70)
y	= altitude
y_c	= critical altitude, Eqs. (46) and (49)
Y	= βy, non dimensional altitude
z	= variable, Eq. (27)
α_0	= initial angle-of-attack
$\bar\alpha$	= angle-of-attack
α	= $\bar\alpha - \alpha_0$, variation of the angle-of-attack
β	= constant altitude scale
β_i	= density gradients, Eq. (54)
γ	= flight path angle.
δ, δ_s	= non dimensional mass of the atmosphere, Eqs. (8), (61)
ϵ	= small perturbation, orbit eccentricity
ζ	= damping coefficient, Eq. 60
θ	= angle of pitch, Eq. (6). Phase angle, Eq. (68). Half nose cone angle, Fig. 2
λ	= constant coefficient, Eq. (38). Characteristic root, Eq. (72)
ν	= ratio of moments of inertia, Eq. (8)
ξ	= universal time variable, Eq. (9)
ρ	= air mass density
$\sigma, \bar\sigma$	= inverse non dimensional pitching moment of inertia, Eqs. (8), (61)
τ	= non dimensional time, Eq. (52)
Φ	= central range angle, Fig. 1 Test function, Eq. (40)
ω	= orbit frequency, Eq. (52)
subscript	: Subscript s denotes sea level condition. Subscript 0 denotes condition along the reference flight path.

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I. Introduction

The purpose of this paper is to discuss the longitudinal dynamic stability of a non-rolling, lifting vehicle that is gliding at hypersonic speeds. The analysis applies to shuttle vehicles that are designed for operating up to the rim of a planetary atmosphere.

In earlier studies, it is customary to formulate assumptions for a specific flight regime before deriving the dynamic stability equation. Both Friedrich and Dore[1] and Allen[2] developed their dynamic longitudinal stability equation by considering the zero-thrust flight path trajectory equations which neglect the gravity force, compared to the aerodynamic force. This means that the analysis can only be applied to a portion of ballistic entry along which high deceleration rate is being developed. Laitone then discussed the range of validity of the assumptions[3] by deriving the classical second-order linear ordinary differential equation predicting the oscillations in angle-of-attack by two entirely different approaches. The first method follows the usual procedure of small perturbations in the flight trajectory equations, while the second method utilizes Euler's dynamic equations, for axes that are rigidly fixed in the moving body. It was shown by Allen[2] that, for a large drag, blunt vehicle, having a high rate of deceleration at hypersonic speeds, the equation reduces to a Bessel's equation of order zero. The solution can then be obtained and criteria for stability derived.

Stability for flight path that is nearly parallel to the earth's surface has been studied numerically by Etkin[4] and analytically by Laitone and Chou[5]. Recently, it was shown by Vinh and Dobrzelecki[6] that, if the eccentricity of the flight trajectory is taken into consideration, then the resulting dynamical equation is a damped Mathieu's equation with periodic forcing terms.

The operational concept of a shuttle vehicle allows more flexibility in flight regimes than the two cases mentioned above. Therefore, there is a need to get a unified dynamic stability equation that is valid for all types of reentry and for a general type of aerospace vehicle. The difficulty in getting such an equation seems to be from the fact that, when it comes to integrating the reduced equation for a specified type of entry trajectory, one has to make a time transformation to replace the real time by an appropriate variable. Thus, for straight line ballistic entry, Allen has used the altitude as independent variable to reduce the equation to a Bessel's type equation, while for shallow glide entry, Vinh and Dobrzelecki elected to use the mean anomaly along the average flight path as a more appropriate variable. It will be shown in this paper that, by using a universal time transformation that replaces the real time by a non-dimensional variable which represents the number of reference lengths travelled along the trajectory of the center of mass, as suggested by Laitone in Reference 3, we get a unified linear differential equation of the second order which describes the variations of the angle-of-attack for all possible reentries. For straight line reentry, the new variable is equivalent to the altitude variable and the equation reduces to a confluent hypergeometric differential equation which, upon simplification for a certain type of pure ballistic missile, becomes the same Bessel's equation of order zero which was considered by Allen. The exact solution for the more general case can be expressed in terms of the confluent hypergeometric function. Using a theorem in stability of differential equation, formulated by Laitone[7], an upper bound for the angle-of-attack oscillations is obtained, and criteria for stability are derived. For shallow entry, the independent variable used is proportional to the mean anomaly along the average flight path and the unified equation becomes a damped Mathieu's equation with periodic forcing terms. The equation is integrated by the method of Krylov-Bogoliubov. It is shown that the aerodynamic criteria for stability are the same as for the case of straight line entry. In addition, there is always present a spiral instability due to aerodynamic drag. Furthermore, for each vehicle configuration and specified planetary atmosphere, there exists an altitude range where the angle-of-attack frequency is nearly equal to the orbital frequency, causing instability in pitch. It is shown that the range of instability gets larger when the orbit eccentricity increases. Criteria for eccentricity instability are then derived.

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II. Unified Dynamic Equation

The motion of a non-rolling, lifting vehicle in a resisting medium and subject to the gravity force of a spherical planet is governed by the system of equations, for an axis system that rotates about the vehicle center of mass (as indicated in Fig. 1) so that the x-axis is always tangent to the instantaneous flight path*

$$\frac{dV}{dt} = -\frac{\rho S C_D V^2}{2m} - g\sin\gamma \tag{1}$$

$$V\frac{d\gamma}{dt} = \frac{\rho S C_L V^2}{2m} - \left(g - \frac{V^2}{r}\right)\cos\gamma \tag{2}$$

$$\frac{dq}{dt} = \frac{\rho S L C_m V^2}{2B} - \frac{3g}{2r}\left(\frac{A-C}{B}\right)\sin 2\theta \tag{3}$$

$$\frac{d\theta}{dt} = q + \frac{V}{r}\cos\gamma \tag{4}$$

$$\frac{dr}{dt} = V\sin\gamma \tag{5}$$

$$\theta = \gamma + \bar{\alpha} \tag{6}$$

The first two equations are, respectively, the drag and lift equations along the tangent and normal to the flight path. The first term on the right-hand side of the pitching moment equation, Eq. 3, expresses the restoring aerodynamic torque, while the second term corresponds to the gravity torque The last three equations are kinematic relations. The mass density, ρ, of the atmosphere, and the acceleration of the gravity, g, are altitude dependent.

The elimination of θ and q results in the following exact equation for the angle-of-attack α.

$$\frac{d^2\alpha}{dt^2} + \frac{S C_L V}{2m}\frac{d\rho}{dt} + \frac{\rho S V}{2m}\frac{dC_L}{dt} - \frac{\rho S L C_m}{2B}V^2$$
$$+ \frac{3g}{2r}\left(\frac{A-C}{B}\right)\sin 2(\gamma+\bar{\alpha}) + \left(\frac{3}{r} - \frac{2g}{V^2}\right) g\sin\gamma\cos\gamma$$
$$- \frac{\rho S C_D}{2m} g\cos\gamma - \frac{\rho^2 S^2}{4m^2} C_D C_L V^2 = 0 \tag{7}$$

Let

$$\delta = \frac{\rho S L}{2m}, \quad \nu = \frac{A-C}{B}, \quad \sigma = \frac{mL^2}{B} \tag{8}$$

and use the time transformation

$$\xi = \frac{1}{L}\int_0^t V(\tau)d\tau, \quad \frac{d\xi}{dt} = \frac{V(t)}{L}, \quad \frac{d}{d\xi}(\,) = (\,)'$$

$$\frac{d}{dt}(\,) = \frac{V}{L}(\,)', \quad \frac{d^2}{dt^2}(\,) = \left(\frac{V}{L}\right)^2(\,)'' + \left(\frac{VV'}{L^2}\right)(\,)' \tag{9}$$

The new independent variable, ξ, can be identified as the number of reference lengths travelled along the trajectory of the center of mass. For small variations of the angle-of-attack, we may assume

$$C_D = C_{D_0} + C_{D_\alpha}\alpha$$
$$C_L = C_{L_0} + C_{L_\alpha}\alpha \tag{10}$$

In hypersonic flight, the aerodynamic derivatives in the linearization of C_D and C_L are approximately independent of flight speed and Mach number. This same assumption also applies to C_m and experience has shown that we must express it as

$$C_m = C_{m_\alpha}\alpha + C_{m_{\dot\alpha}}\left(\frac{L}{V}\right)\dot\alpha + C_{m_q}\left(\frac{L}{V}\right)(\dot\gamma + \dot\alpha) \tag{11}$$

Upon substitution into Eq. (7), and within the validity of the linearized theory, we have the linear differential equation of the second order which governs the angle of attack oscillations

$$\alpha'' + f_1(\xi)\alpha' + f_2(\xi)\alpha = f_3(\xi) \tag{12}$$

where

$$f_1(\xi) = \delta[C_{L_\alpha} - \sigma(C_{m_{\dot\alpha}} + C_{m_q})] + \frac{V'}{V}$$

$$f_2(\xi) = -\delta\left(\sigma C_{m_\alpha} + \frac{gL}{V^2}C_{D_\alpha}\cos\gamma - \frac{V'}{V}C_{L_\alpha}\right) + \delta' C_{L_\alpha}$$
$$- \delta^2(\sigma C_{m_q} C_{L_\alpha} + C_{L_0} C_{D_\alpha})$$
$$+ \frac{3L}{r}\left(\frac{gL}{V^2}\right)\nu\cos 2(\gamma + \alpha_0) \tag{13}$$

$$f_3(\xi) = \delta\left(\frac{gL}{V^2}\right)\left[C_{D_0} - \sigma C_{m_q}\left(1 - \frac{V^2}{gr}\right)\right]\cos\gamma - \delta' C_{L_0}$$
$$-\left(\frac{gL}{V^2}\right)\left[\left(\frac{3L}{2r} - \frac{gL}{V^2}\right)\sin 2\gamma + \frac{3L}{2r}\nu\sin 2(\gamma + \alpha_0)\right]$$
$$+ \delta^2 C_{L_0}(C_{D_0} + \sigma C_{m_q})$$

If the trajectory of the center of mass is known, the elements of the flight path, r, V, and γ, and the atmospheric mass ratio, δ can be evaluated as functions of the independent variable ξ and the unified equation, Eq. 12, uniquely determines the time history of the angle-of-attack oscillations for each prescribed set of initial conditions on α and α'. This assumes the so-called limited problem, that is the angle-of-attack oscillations have negligible effect on the trajectory. A successful analytical integration of Eq. (12) depends on the forms of the functions $f_i(\xi)$, and in general the equation cannot be integrated in closed form. Fortunately, the use of the time transformation defined by Eq. (9) allows the reduction of the unified equation, Eq. (12), to well-known equations in mathematical physics, at least for the two fundamental types of entry trajectory discussed in this paper. Furthermore, the new time variable, ξ, which is monotonically increasing for any flight trajectory, renders the coefficients $f_i(\xi)$ well behaved and most often, with the aid of the vast literature in the theory of stability of ordinary differential equations, stability criteria can be derived without having to integrate the equation. An illustrative example will be given in the next section.

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*All symbols are defined in notation section.

III. Stability of Nearly Ballistic Entry

For nearly ballistic entry along a flight path with small curvature, at high deceleration rate, the most fundamental assumption is that in Eq. (1) we neglect the contribution of the gravity force. Then

$$\frac{\dot V}{V} \approx -\delta\left(\frac{V}{L}\right)C_D \tag{14}$$

For this type of entry, the aerodynamic force is mainly drag force, and we can write

$$C_D \approx C_{D_0}, \quad C_{L_0} \approx 0 \tag{15}$$

Then, from Eq. (14), with the definition (9)

$$\frac{V'}{V} \approx -\delta C_{D_0} \tag{16}$$

Also, for nearly straight line flight path, $\frac{L}{r} << 1$, and $gL << V^2$. Equation (12) reduces to the classical form

$$\alpha'' + b(\xi)\alpha' + c(\xi)\alpha = 0 \tag{17}$$

where

$$b(\xi) = \delta[C_{L_\alpha} - C_{D_0} - \sigma(C_{m_{\dot\alpha}} + C_{m_q})]$$
$$c(\xi) = -\delta[\sigma C_{m_\alpha} + \delta C_{L_\alpha}(\sigma C_{m_q} + C_{D_0})] + \delta' C_{L_\alpha} \tag{18}$$

The coefficients b and c are solely functions of the atmospheric mass ratio, which can be easily expressed in terms of the independent variable ξ. In this case of straight line ballistic entry, Eq. (17) is equivalent to Allen's equation (6), and the variable ξ is equivalent to the altitude variable. To show this equivalence, let us consider the case of an isothermal atmosphere.

$$\rho = \rho_s e^{-\beta y} \tag{19}$$

where ρ_s is the atmospheric mass density at sea level, β the constant altitude scale, and y the altitude of the vehicle. Let $\gamma_0 < 0$ be the constant flight path angle. By integrating

$$\frac{dy}{dt} = V\sin\gamma_0$$

from the initial time

$$\frac{y - y_0}{L} = \frac{\sin\gamma_0}{L}\int_0^t V(\tau)d\tau = \xi\sin\gamma_0$$

and we have the linear relation between y and ξ

$$y = y_0 + L\xi\sin\gamma_0 \tag{20}$$

Consequently we have for the function $\delta(\xi)$

$$\delta(\xi) = \delta_s e^{-\beta(y_0 + L\xi\sin\gamma_0)} \tag{21}$$

where

$$\delta_s = \frac{\rho_s S L}{2m} \tag{22}$$

is the value of δ evaluated at sea level.

We define the nondimensional altitude Y,

$$Y = \beta(y_0 + L\xi\sin\gamma_0) = \beta y \tag{23}$$

Then, the equation for the angle of attack becomes

$$\frac{d^2\alpha}{dY^2} + 2k_1 e^{-Y}\frac{d\alpha}{dY} + \left(k_2 e^{-Y} + k_3 e^{-2Y}\right)\alpha = 0 \tag{24}$$

where the constants are

$$k_1 = \frac{\delta_s}{2\beta L\sin\gamma_0}\left[C_{L_\alpha} - C_{D_0} - \sigma(C_{m_{\dot\alpha}} + C_{m_q})\right]$$

$$k_2 = -\frac{\delta_s}{\beta^2 L^2\sin^2\gamma_0}\left[\sigma C_{m_\alpha} + \beta L C_{L_\alpha}\sin\gamma_0\right] \tag{25}$$

$$k_3 = -\frac{\delta_s^2 C_{L_\alpha}\left(\sigma C_{m_q} + C_{D_0}\right)}{\beta^2 L^2\sin^2\gamma_0}$$

Equations (24) and (25) are identical to those given by Allen (1957) whose $\theta_0 = -\gamma_0 > 0$. The solution of Eq. (24) can be expressed in terms of known functions in mathematical physics. Allen[2] considered a type of reentry missile such that the term $(k_3 - k_1^2)^{-2Y}$ is small compared to $(k_2 + k_1)e^{-Y}$. Equation (24) is then reduced to a Bessel's equation of order zero, and explicit expression for the angle-of-attack can be obtained as

$$\alpha(y) = e^{k_1 e^{-\beta y}}\left[C_1 J_0\left(2\sqrt{k_1 + k_2}\, e^{-\frac{\beta y}{2}}\right) + C_2 Y_0\left(2\sqrt{k_1 + k_2}\, e^{-\frac{\beta y}{2}}\right)\right] \tag{26}$$

where C_1 and C_2 are constants of integration, and the functions J_0 and Y_0 are the zero order Bessel's functions of the first and second kind, respectively. It is known that, for an equation of the type (24), a change in the constants k_i can produce a profound change in the character of the solutions. Hence, in this paper we shall integrate the full equation, Eq. (24). The analysis is then valid for a more general type of vehicle, the sole condition being that initially it can be trimmed to flight along a nearly straight line trajectory. In Eq. (24) we use the transformation

$$\alpha = e^{k_1(1-k_4)e^{-Y}}u(z) \tag{27}$$
$$z = 2k_1 k_4 e^{-Y}$$

where

$$k_4^2 = \frac{k_1^2 - k_2}{k_1^2} \tag{28}$$

Then, the equation can be put into the familiar form of a confluent hypergeometric equation

$$z\frac{d^2u}{dz^2} + (1-z)\frac{du}{dz} - \frac{1}{2k_4}\left(k_4 - \frac{k_1 + k_2}{k_1}\right)u = 0 \tag{29}$$

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The appropriate solution of Eq. (29) that will lead to a *physically valid angle-of-attack variation for* Eq. (24) is given by the confluent hypergeometric function

$$u(z) = {}_1F_1(a,1,z) = 1 + az + \frac{a(a+1)z^2}{(2!)^2} + \frac{a(a+1)(a+2)z^3}{(3!)^2} + \cdots \qquad (30)$$

where

$$a = \tfrac{1}{2} - k, \quad k = \frac{k_1 + k_2}{2k_1k_4} = \frac{k_1 + k_2}{2\sqrt{k_1^2 - k_3}} \qquad (31)$$

as given by Slater (Ref. 8, p. 2). For large values of the constant "a", Slater (1960, p. 68) shows that the asymptotic representation of Eq. (30) is given by

$${}_1F_1(a,1,z) \rightarrow e^{\frac{z}{2}} J_0\left(2\sqrt{kz}\right)\left[1 + 0\left(\frac{1}{\sqrt{k}}\right)\right] \qquad (32)$$

Upon introducing the above into Eq. (27) we obtain an asymptotic solution of Eq. (24) as identical to Eq. (26) which was obtained by Allen (1957) under *more restrictive assumptions*. As shown by Eqs. (31) and (32) the asymptotic representation, given by Eq. (26), is valid for large values of k which occur whenever either k_1 or k_4 are much smaller than k_2. For the solid nose cone considered by Allen (1957), these values are given in Fig. 2, and it is seen that $k > 10^4$ for $21° < \theta < 75°$, primarily because $k_2 > 10^5$ while $-k_1 < 12$ and $-k_4 < 3$. Consequently, Allen's approximate solution as given by Eq. (26) is very satisfactory for this type of a body at hypersonic speeds. However, for other types of bodies for which k, as defined by Eqs. (25) and (31) is not large, we must return to the exact solution of Eq. (24) which is now given by Eqs. (27) and (30) as

$$\alpha(y) = \alpha_0 e^{k_1(1-k_4)e^{-\beta y}} {}_1F_1\left[\left(\tfrac{1}{2} - k\right), 1, 2k_1k_4e^{-\beta y}\right] \qquad (33)$$

This exact solution provides an oscillatory variation for the angle-of-attack only if

$$\left|2k - \sqrt{1+4k^2}\right| < \left|2k_1k_4e^{-\beta y}\right| < \left|2k + \sqrt{1+4k^2}\right| \qquad (34)$$

as shown by Slater (1960, p. 118). However this relation, and the question of stability of the oscillation, are better analyzed by putting the confluent hypergeometric equation, Eq. (29), into the form of the Whittaker's equation. Whittaker's transformation is

$$u = e^{\frac{z}{2}} z^{-\frac{1}{2}} W(z) \qquad (35)$$

The final equation is the Whittaker's equation

$$\frac{d^2W}{dz^2} + \left(-\frac{1}{4} + \frac{k}{z} + \frac{1}{4z^2}\right)W = 0 \qquad (36)$$

Again, in this form, we can see the importance of the constant coefficient k. By a proper selection *of the sign of* k_4, *defined by Eq. (28)*, we can always make both the constant k, and the argument z, positive in the differential equation. It now appears that the damping constant k_1 and the parameter k are the two important parameters for *the stability of the entry vehicle. Allen's assump*tion fails when the moment stability derivative $C_{m_{\dot{\alpha}}}$ is approaching zero. In this case the constant k can be small. Explicitly, we have, by taking $C_{m_{\dot{\alpha}}} \approx 0$

$$k^2 = \frac{(k_1+k_2)^2}{4(k_1^2 - k_3)} \approx \frac{\left[C_{D_0} + C_{L_\alpha} + \sigma C_{m_q} + \dfrac{2\sigma C_{m_{\dot{\alpha}}}}{\beta L \sin\gamma_0}\right]^2}{4\left[C_{D_0} + C_{L_\alpha} + \sigma C_{m_q}\right]^2}$$

or

$$k^2 = \tfrac{1}{4}(1+\lambda)^2 \qquad (37)$$

with the constant λ defined as

$$\lambda = \frac{2\sigma C_{m_{\dot{\alpha}}}}{\left(C_{D_0} + C_{L_\alpha} + \sigma C_{m_q}\right)\beta L \sin\gamma_0} \qquad (38)$$

In our sign convention, $\gamma_0 < 0$, and for the validity of the high deceleration assumption $\gamma_0 \sim -45°$. For a reference length of the order of 5 feet, and taking $\sigma = 1$, $\beta^{-1} = 22{,}000$ ft, the coefficient $2\sigma/\beta L \sin\gamma_0$ is of the order of -1.25×10^4. For all practical cases, the static stability derivative C_{m_α} is negative, the constant k is large, and Bessel solution for α is valid. Since it is unlikely that $C_{m_{\dot{\alpha}}}$ be allowed to have positive value, the minimum value of $C_{m_{\dot{\alpha}}}$ for a certain position of the center of mass would be zero and the limiting value of k is 1/2. For small value of k, Whittaker solution can be unstable at a certain altitude and it is interesting to investigate the case of instability of flight due to the position of the center of mass, that is the case where Allen's assumption is no longer valid. This can be done by using Whittaker's equation, Eq. (36), and a theorem in the boundedness of solutions of *ordinary linear differential equations due to Lai*tone[7]. Following Laitone, we consider the linear differential equation

$$W'' + b(z)W' + c(z)W = 0 \qquad (39)$$

Then it is shown that if

$$\Phi(z) = c(z) - \tfrac{1}{4}b^2(z) - \tfrac{1}{2}b'(z) \geq \Phi_{min} > 0 \qquad (40)$$

and as long as $\Phi(z)$ varies monotonically, that is it is either never decreasing or never increasing, even though varying, the magnitude of W is bounded by

$$|W(z)| \leq M \exp\left(-\tfrac{1}{2}\int b(z)dz\right) \qquad (41)$$

where

$$M = \sqrt{\frac{\Phi(z_0)W^2(z_0) + \left[W'(z_0) + \frac{1}{2}W(z_0)b(z_0)\right]^2}{\Phi_{min}}} \qquad (42)$$

Now, applying criteria (40) and (41) to Eq. (36), we have

$$\Phi(z) = -\frac{1}{4} + \frac{k}{z} + \frac{1}{4z^2} > 0 \qquad (43)$$

Since both k and z are positive, the monotonicity of the variations of $\Phi(z)$ is assured. For positiveness of $\Phi(z)$, we must have

$$z^2 - 4kz - 1 < 0 \qquad (44)$$

or

$$z < 2k + \sqrt{4k^2 + 1} \approx 4k \qquad (45)$$

That is

$$y_c > \frac{1}{\beta}\log\frac{k_1 + k_2}{4k^2} \qquad (46)$$

This inequality gives the limit for the altitude, under which the boundedness of the Whittaker's solution is not assured. As has been mentioned, the criterion applies to the case where $C_{m_{\dot{\alpha}}}$ is negligibly small. In general, $k_1 + k_2 < 4k^2$, and condition (46) holds at all altitudes, and Laitone's criteria lead to the bound

$$|\alpha(y)| \leq Me^{\left(\frac{1}{2}\beta y + k_1 e^{-\beta y}\right)} \qquad (47)$$

Since $e^{\frac{1}{2}\beta y}$ is decreasing, when $k_1 < 0$, the angle-of-attack oscillation is certainly stable. Then, we can say that, when $C_{m_{\dot{\alpha}}}$ is not nearly zero, as long as

$$\left[C_{L_\alpha} - C_{D_0} - \sigma\left(C_{m_{\dot{\alpha}}} + C_{m_q}\right)\right] > 0 \qquad (48)$$

the angle-of-attack is bounded, and a damped stable oscillation will occur during rapid descent along a straight line through an isothermal earth atmosphere.

When k_1 is positive, $\alpha(y)$ can still be bounded by a decreasing function if the exponential in (47) is decreasing. The function passes through a minimum when

$$\tfrac{1}{2} - k_1 e^{-\beta y} = 0$$

This gives the critical altitude for stability

$$y_c = \frac{1}{\beta}\log 2k_1 \qquad (49)$$

The oscillation is stable above this altitude, and unstable below. The critical altitude obtained by Allen[2] is

$$y_c = \frac{1}{\beta}\log 4k_1 \qquad (50)$$

Thus, our criteria gives a larger margin, even for the general case, for the critical altitude than the one obtained with Allen's simplified analysis.

In summary, when $C_{m_{\dot{\alpha}}}$ is not nearly zero, inequality (48) gives the stability criteria. When condition (48) is violated, the critical altitude, above which the reentry is stable is given by (49). When $C_{m_{\dot{\alpha}}}$ is nearly zero, condition (49) has to be replaced by the more restricted condition (46). The same type of analysis applies to the case where $C_{L_0} \neq 0$. In this case a small forcing term is present due to the perturbed lifting force.

Also, it should be mentioned that, as has been pointed out by Allen, for high drag shapes, the velocity quickly approaches the so-called terminal velocity for which the drag and the weight are equal, the reentry angle is getting steeper at lower altitude, and below a certain altitude the assumptions of high deceleration rate, straight line reentry are no longer valid. The transonic-to-low-hypersonic flight is in itself a different flight regime and deserves a separate study.

IV Stability of Shallow Gliding Entry

For gliding entry with a flight path that is nearly parallel to the earth's surface, it has been shown by Laitone and Chou[5], and Vinh and Dobrzelecki[4] that the trajectory is a descending spiral which, for a few revolutions required for a stability analysis, can be approximated by a nearly circular orbit with equation

$$r = \frac{r_0(1-\epsilon^2)}{1 + \epsilon\cos\tau} \qquad (51)$$

where r_0 is the radius of the reference circular orbit, ϵ a small quantity which denotes here the eccentricity of the orbit and τ the true anomaly which defines the position of the vehicle along the orbit. *In general, τ is a transcendental function of the* mean anomaly which, in turn, is proportional to the time. For nearly circular orbit, τ is equal to the mean anomaly and we have[6]

$$\tau = \frac{g_0}{u_0}\omega t, \quad \omega^2 = (1 - s^2)(2 - \beta_1 s^2) + s^4, \quad s^2 = \frac{u_0^2}{g_0 r_0} \qquad (52)$$

where subscript zero denotes the condition along the reference circular orbit. The quantity ω is the nondimensional orbit frequency, and the ratio s of the velocity along the reference circular orbit to the circular velocity without drag, at the distance r_0, can be evaluated from

$$1 - s^2 = \frac{\rho_0 S C_{L_0} r_0}{2m} s^2 \qquad (53)$$

The coefficient β_1 denotes the first atmospheric mass density gradient. In general

$$\frac{\rho(r)}{\rho_0} = 1 + \beta_1\left(\frac{r - r_0}{r_0}\right) + \beta_2\left(\frac{r - r_0}{r_0}\right)^2 + \cdots$$

$$\beta_1 = \left(\frac{d\rho}{dr}\right)_0 \frac{r_0}{\rho_0}, \quad \beta_2 = \frac{1}{2}\left(\frac{d^2\rho}{dr^2}\right)_0 \frac{r_0^2}{\rho_0}, \ldots \qquad (54)$$

It is important, for an order of magnitude analysis, to mention that the coefficients β_i are large. β_1 is of the order of -10^3 as shown in Fig. 3. The coefficients are calculated from an inverse polynomial representation of the earth's atmosphere as given in the U. S. Standard Atmosphere Supplements, 1966. The values of β_1 are in excellent agreement with the values calculated from tabulated data in the altitude range 100-600 thousand feet.

To the order ϵ, along the gliding trajectory, we have

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$$r = r_0(1-\epsilon\cos\tau) \quad , \quad V = u_0(1+\epsilon\cos\tau)$$
$$\sin\gamma = \epsilon\frac{\omega}{s^2}\sin\tau \quad , \quad \cos\gamma = \frac{\omega}{s^2}(1-2\epsilon\cos\tau) \tag{55}$$
$$g = g_0(1+2\epsilon\cos\tau) \quad , \quad \delta = \delta_0(1-\epsilon\beta_1\cos\tau)$$

If the reference length L in the definition (9) of our independent variable ξ is taken as the radius r_0 of the reference circular orbit, we have

$$\xi = \frac{1}{r_0}\int_0^t V(t)dt = \frac{s^2}{\omega}(\tau + \epsilon\sin\tau + \cdots) \tag{56}$$

After each revolution, the contribution of the periodic terms in the preceding relation averages to zero, and to the accuracy of this analysis we can take

$$\xi = \frac{s^2}{\omega}\tau \tag{57}$$

Thus, for shallow entry, our variable is proportional to the mean anomaly along the orbit. Using the elements of the orbit as given by Eq. (55), and the new variable τ, as related to ξ by Eq. (57), we can now rewrite the unified equation, Eq. (12), to apply to the case of shallow entry. It is important to mention the following facts:

The elements of the orbit, as given in Eq. (55), are good approximation above 300 thousand feet. They will be used mainly to evaluate the forcing terms, namely the function $f_3(\tau)$ in the unified equation, Eq. (12). For the damping and frequency, namely the functions $f_1(\tau)$ and $f_2(\tau)$, it is important to show the effect of the drag by using the drag equation, Eq. 1, with the small gravity component neglected. Then we have

$$\frac{V'}{V} = \frac{1}{V}\frac{dV}{d\tau}\frac{d\tau}{d\xi} \approx -\delta_0 C_{D_0} \tag{58}$$

The gliding flight regime requires an aerodynamic configuration such that $C_{L_0} \neq 0$. We can select the reference body axes such that this value of C_{L_0} is also the value of C_L along the reference circular flight path with $\alpha_0 = 0$. The eccentricity of the orbit ϵ and the mass ratio of the atmosphere δ_0 are small, so that we neglect their second and higher order, but since the density gradient β_1 is large, it is necessary to retain some higher order terms having β_1 as coefficient. With these considerations we have the equation governing the angle-of-attack for gliding entry

$$\frac{d^2\alpha}{d\tau^2} + 2\zeta\frac{d\alpha}{d\tau} + \left(\frac{n^2}{\omega^2} - \epsilon b\cos\tau\right)\alpha$$
$$= c_1 + \epsilon c_2\sin\tau + \epsilon c_3\cos\tau \tag{59}$$

where

$$\zeta = \frac{\delta_0 s^2}{2\omega}[C_{L_\alpha} - C_{D_0} - \bar{\sigma}(C_{m_{\dot\alpha}} + C_{m_q})]$$
$$n^2 = 3\nu\frac{\omega^2}{s^2} - \delta_0(\bar{\sigma}s^4 C_{m_\alpha} + \omega C_{D_\alpha})$$
$$b = \frac{9\nu}{s^2} - \frac{\delta_0}{\omega^2}\beta_1(\bar{\sigma}s^4 C_{m_\alpha} + \omega C_{D_\alpha}) \tag{60}$$
$$c_1 = \frac{\delta_0}{\omega}C_{D_0}$$
$$c_2 = \frac{2}{s^4} - \frac{3(1+\nu)}{s^2} - \frac{s^2}{\omega}\delta_0\beta_1 C_{L_0}$$
$$c_3 = -\frac{\delta_0\beta_1}{\omega}[C_{D_0} - (1-s^2)\bar{\sigma}C_{m_q}]$$

The rescaling of the inertia constant $\bar{\sigma}$ is necessary to have the moment stability derivative C_{m_α} expressed numerically, in the usual way, with body length as reference length. Hence, in this section, we have the definitions

$$\delta_0 = \frac{\rho_0 S r_0}{2m}, \quad \bar{\sigma} = \frac{m r_0 L}{B}, \quad C_{m_q} = \frac{L}{r_0}\frac{\partial C_m}{\partial\left(\frac{L}{u_0}q\right)} \tag{61}$$

The governing equation is a linear, second order differential equation with periodic coefficients, and periodic forcing terms. Its form is to be expected by the almost periodic nature of the trajectory. For an understanding of the different terms, let us consider the case of circular orbit, $\epsilon = 0$. Of course, this can happen only when the aerodynamic forces are vanishingly small. Then, we have

$$\frac{d^2\alpha}{d\tau^2} + 2\zeta\frac{d\alpha}{d\tau} + \frac{n^2}{\omega^2}\alpha = c_1 \tag{62}$$

Neglecting the second order of the damping ζ, the general solution for the angle of attack is

$$\alpha(\tau) = e^{-\zeta\tau}[C_1\cos\frac{n}{\omega}\tau + C_2\sin\frac{n}{\omega}\tau] + \frac{\omega}{n^2}\delta_0 C_{D_0} \tag{63}$$

The condition for stability is

$$[C_{L_\alpha} - C_{D_0} - \bar{\sigma}(C_{m_{\dot\alpha}} + C_{m_q})] > 0 \tag{64}$$

which is the same as condition (48) for ballistic entry. The angle-of-attack has a damped oscillation with frequency n/ω and its value tends asymptotically to $(\omega/n^2)\delta_0 C_{D_0}$. The constant forcing term c_1 can be seen as the drag force which induces the spiral decay of the orbit Another instability with dynamic nature will arise when we consider the ellipticity of the orbit. For ϵ non vanishing, we first integrate the homogeneous equation in Eq. (59)

$$\frac{d^2\alpha}{d\tau^2} + 2\zeta\frac{d\alpha}{d\tau} + \left(\frac{n^2}{\omega^2} - \epsilon b\cos\tau\right)\alpha = 0 \tag{65}$$

Using the Liouville transformation

$$\alpha = e^{-\zeta\tau}w(\tau) \tag{66}$$

the equation is transformed into a Mathieu's equation

$$\frac{d^2w}{d\tau^2} + \left(\frac{n^2}{\omega^2} - \epsilon b\cos\tau\right)w = 0 \tag{67}$$

where a small constant of order δ_0^2 has been omitted in the coefficient of w. The Mathieu's equation possesses periodic solutions only when the constants satisfy a certain relation. In general, the solution is not periodic and if the damping condition (64) is satisfied, the solution for α in (65) is a damped Mathieu's solution with negligible damping at high altitude. In Eq. (67) for w, it can be seen that, when $\epsilon = 0$, the solution is a pure harmonic function. For small ϵ, the solution is oscillatory although not periodic in general. For large ϵ the solution may become unstable. It is possible to determine the zone of instability by the method of Krylov-Bogoliubov(9). Following Bogoliubov, for the second approximation, we have

$$w(\tau) = a\cos(\tau+\theta) + \frac{\epsilon ab\omega}{2(\omega+2n)}\cos(2\tau+\theta)$$
$$+ \frac{\epsilon ab\omega}{2(\omega-2n)}\cos\theta \tag{68}$$

where the amplitude a, and the phase angle θ, considered as functions of τ, must be determined by the equations of second approximation

$$\frac{da}{d\tau} = \frac{\epsilon^2 ab^2\omega}{8(2n-\omega)}\sin 2\theta$$
$$\frac{d\theta}{d\tau} = \frac{n}{\omega} - 1 - \frac{\epsilon^2 b^2\omega^2}{4(4n^2-\omega^2)} + \frac{\epsilon^2 b^2\omega}{8(2n-\omega)}\cos 2\theta \tag{69}$$

To solve the system of equations (69) we introduce new variables x_1 and x_2 according to the relations

$$x_1 = a\cos\theta$$
$$x_2 = a\sin\theta \tag{70}$$

Then, it can be shown that the system reduces to *the linear system with constant coefficients*

$$\frac{dx_1}{d\tau} = \left[\frac{\epsilon^2 b^2\omega(2n+3\omega)}{8(4n^2-\omega^2)} - \left(\frac{n}{\omega}-1\right)\right]x_2$$
$$\frac{dx_2}{d\tau} = \left[\frac{\epsilon^2 b^2\omega(2n-\omega)}{8(4n^2-\omega^2)} + \left(\frac{n}{\omega}-1\right)\right]x_1 \tag{71}$$

The characteristic equation of the system is

$$\lambda^2 = \frac{\epsilon^4 b^4\omega^2(2n-\omega)(2n+3\omega)}{64(4n^2-\omega^2)^2} - \left(\frac{n}{\omega}-1\right)^2 + \frac{\epsilon^2 b^2\omega(n-\omega)}{2(4n^2-\omega^2)} \tag{72}$$

Hence, the general solution of the system is

$$x_1 = C_1 e^{\lambda\tau} + C_2 e^{-\lambda\tau}$$
$$x_2 = \frac{8\omega(4n^2-\omega^2)}{8(\omega-n)(4n^2-\omega^2) + \epsilon^2 b^2\omega^2(2n+3\omega)}[C_1\lambda e^{\lambda\tau} - C_2\lambda e^{-\lambda\tau}] \tag{73}$$

where C_1 and C_2 are constants of integration. The amplitude a and the phase angle θ in the solution (68) for w are then given by

$$a = \sqrt{x_1^2 + x_2^2}$$
$$\theta = \tan^{-1}\frac{x_2}{x_1} \tag{74}$$

It is clear from (73) that, if the roots of the characteristic equation (72) are imaginary, the amplitude a will be a bounded function of the time. The condition for dynamic instability, that is the condition for λ to be real is

$$1 + \frac{\epsilon^2 b^2\omega^2}{2(4n^2-\omega^2)} - \frac{\epsilon^2 b^2\omega}{4(2n-\omega)} < \frac{n^2}{\omega^2} < 1 + \frac{\epsilon^2 b^2\omega^2}{2(4n^2-\omega^2)} + \frac{\epsilon^2 b^2\omega}{4(2n-\omega)} \tag{75}$$

This happens for near resonance, when the proper angle-of-attack frequency of oscillation, n, is near the frequency ω of the forcing periodic term. Therefore, since $n \approx \omega$ near resonance, condition (75) can be simplified to

$$1 - \frac{\epsilon^2 b^2}{12} < \frac{n^2}{\omega^2} < 1 + \frac{5\epsilon^2 b^2}{12} \tag{76}$$

Fig. 4 plots the zone of instability as function of the eccentricity. It is clear that, for circular orbit, resonance is not observed, and the zone of instability gets larger when the orbit eccentricity increases. It should be mentioned that this analysis applies to nearly circular orbit. When ϵ is large, the trajectory extends over a large range of the altitude, and higher order gradients of the mass density of the atmosphere should be included. The altitude where $n = \omega$, called the resonance altitude, is obtained by solving

$$\omega^2 = 3\nu\frac{\omega^2}{s^2} - \delta_0(\bar{\sigma}s^4 C_{m_\alpha} + \omega C_{D_\alpha}) \tag{77}$$

Resonance occurs at high altitude, and we can approximately take $\omega \approx 1$, $s \approx 1$. This leads to the relation, by neglecting the contribution of C_{D_α} and using subscript s to denote condition at sea level

$$g_s\rho_0 r_0^2 = \frac{2k_y^2(3k_0-1)(W/S)_s}{LC_{m_\alpha}}, \quad k_y^2 = \frac{B}{m} \tag{78}$$

The left-hand side of the formula above is solely dependent on the planetary atmosphere. This very simple function, as varying with the altitude, for the earth's atmosphere, is plotted in Fig. 5. The plot can be used to compute graphically the resonance altitude for any type of vehicle, as characterized by the right-hand side of Eq. (78). The *criterion (76) shows that, near this resonance altitude* the angle-of-attack oscillation is dynamically unstable. For any given vehicle, and orbit eccentricity, the exact altitude range for instability can be computed numerically by using equality signs in (76).

The complete solution for the angle-of-attack is the sum of the general solution of the damped Mathieu's equation, Eq. (65), and a particular solution of the non-homogeneous equation, Eq. (59). To construct the particular solution we can neglect the small damping and consider the equation

$$\frac{d^2\alpha}{d\tau^2} + \left(\frac{n^2}{\omega^2} - \epsilon b\cos\tau\right)\alpha = c_1 + \epsilon c_2\sin\tau + \epsilon c_3\cos\tau \tag{79}$$

Following Poincaré(10), we seek the following series solutions for the different forcing functions

$$\alpha_1 = c_{10} + \sum_{p=1}\epsilon^p c_{1p}\cos p\tau$$
$$\alpha_2 = \sum_{p=1}\epsilon^p c_{2p}\sin p\tau \tag{80}$$
$$\alpha_3 = c_{30} + \sum_{p=1}\epsilon^p c_{3p}\cos p\tau$$

To the second order in ϵ, we have the particular solutions

$$\alpha_1 = \frac{\omega^2 c_1}{n^2} + \frac{\epsilon\omega^2 b c_1}{n^2 - \omega^2}\cos\tau + \frac{\epsilon^2\omega^4 b^2 c_1}{2(n^2-\omega^2)(n^2-4\omega^2)}\cos 2\tau$$

$$\alpha_2 = \frac{\epsilon\omega^2 c_2}{n^2 - \omega^2}\sin\tau + \frac{\epsilon^2\omega^4 b c_2}{2(n^2-\omega^2)(n^2-4\omega^2)}\sin 2\tau \qquad (81)$$

$$\alpha_3 = \frac{\epsilon^2\omega^4 b c_3}{2n^2(n^2-\omega^2)} + \frac{\epsilon\omega^2 c_3}{n^2 - \omega^2}\cos\tau + \frac{\epsilon^2\omega^4 b c_3}{2(n^2-\omega^2)(n^2-4\omega^2)}\cos 2\tau$$

The particular solution due to eccentricity oscillation is then

$$\alpha_e = \alpha_1 + \alpha_2 + \alpha_3 \qquad (82)$$

Again, by the form of the solutions (81) we can see that resonance occurs when we have $n \approx \omega$. The amplitude of the force oscillations becomes increasingly large, and linearized theory no longer holds. Fortunately for practical aerodynamic configuration, resonance occurs at high altitude, and during entry, the vehicle is quickly spiraling through this critical altitude in less time than for the resonance to build up.

V. Conclusion

In this paper, we have presented an analytical study of the longitudinal dynamic stability of a hypervelocity vehicle during its descent through a planetary atmosphere. A general non-dimensional variable has been introduced to replace the real time. This new variable, defined as the number of reference lengths travelled in time allows the derivation of a unified equation of motion for the angle-of-attack, valid for all types of reentry of a general type of reentry vehicle. Two flight regimes of *fundamental importance have been discussed.*

In the first case, the steep reentry along a nearly straight line flight path is analyzed. It is shown that the general solution can be obtained in terms of the confluent hypergeometric function. Using a theorem in the theory of stability of differential equations, simple criteria for boundedness of the oscillations have been obtained. In general, for normal, nearly ballistic vehicle configuration, the motion of the angle-of-attack is stable. When the position of the center of mass of the vehicle is such that the static stability derivative C_{m_α} is small, the vehicle motion becomes unstable below a certain altitude. Explicit expression for this critical altitude is derived.

In the second case, the descent is achieved along a spiral flight path with small angle of inclination. The general equation is reduced to a damped Mathieu's equation with periodic forcing term. Using the method of Krylov-Bogoliubov, approximate solution is constructed, and criteria for stability derived. It is shown that, first the vehicle should be designed such that a certain aerodynamic criterion be satisfied. It is the same as the stability criterion for ballistic entry. There is always present a small spiral instability due to the effect of drag. Also, there exists an altitude range in which the motion is unstable if the vehicle is uncontrolled for a certain length of time. This altitude range is a function of both the characteristics of the vehicle and the atmosphere. This resonance phenomenon is due to the commensurability between the frequency of oscillation of the vehicle and the orbital frequency. Simple expression for the altitude in which the two frequencies are equal is derived.

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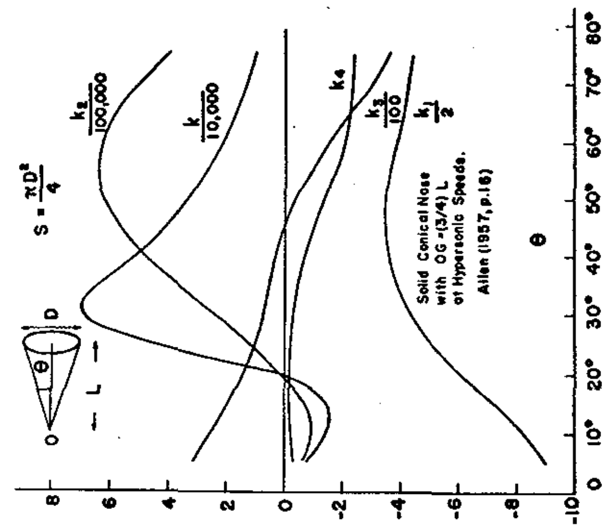

Fig. 2. Numerical Values of the Constant Coefficients For a Solid Nose Cone

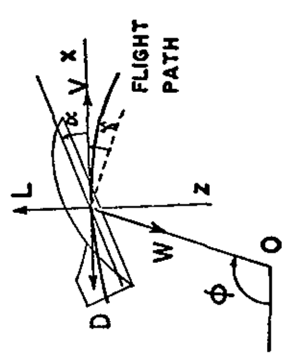

Fig. 1. Axis System and Nomenclature

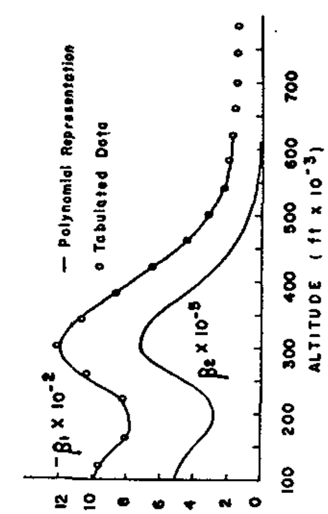

Fig. 3. Non-Dimensional Density Gradients

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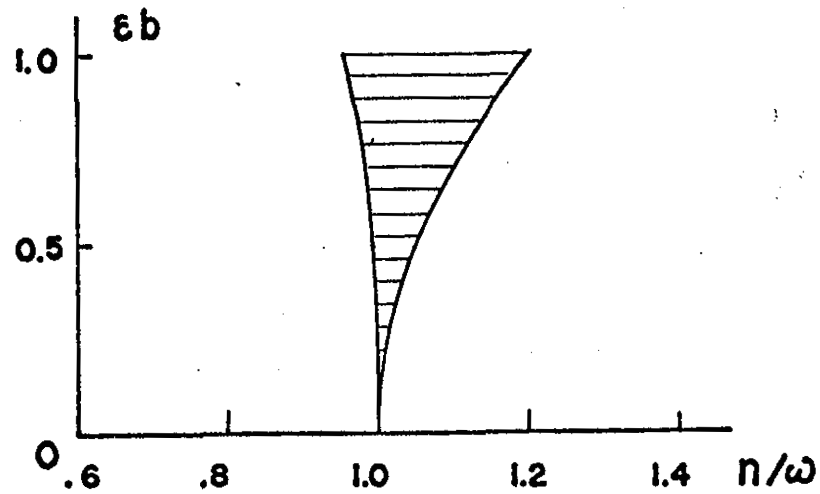

Fig. 4. Zone of Dynamic Instability as Function of the Eccentricity

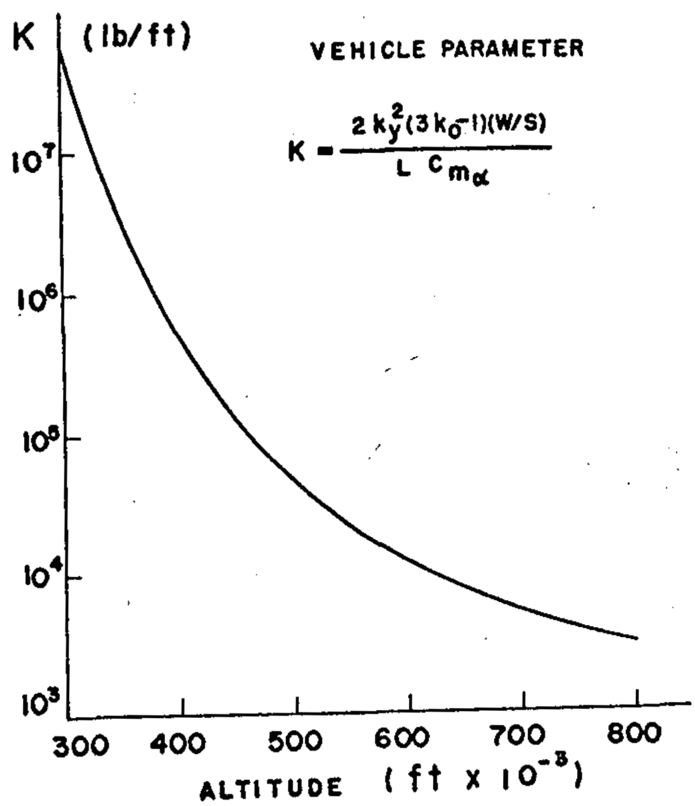

Fig. 5. Resonance Altitude