NON-LINEAR MODAL ANALYSIS OF STRUCTURAL SYSTEMS USING MULTI-MODE INVARIANT MANIFOLDS

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ABSTRACT

In the past decades, a lot of effort has been put into the study of the behavior of non-linear systems, both qualitatively and numerically. However, due to the fact that superposition does not hold for these systems, no satisfactory method has been developed to determine their response to an arbitrary excitation in a way that would mimic linear modal analysis for linear systems. Therefore, when such a response is needed, a linear modal analysis of the non-linear system is typically performed, and the resulting set of non-linear equations is truncated to retain a small number of linear modes —typically one or two. The resulting reduced-order model may be inaccurate —or even qualitatively incorrect, as in the case of internal resonances—, due to the loss of the non-linear interactions between the modeled and unmodeled linear modes. However, increasing its size to include additional linear modes can yield a computationally expensive model. The recent definition of non-linear normal modes of vibration of non-linear systems as motions occurring on invariant manifolds allows one to incorporate the effects of several linear normal modes into one so-called non-linear normal mode. This is very suitable for a restricted class of motions —namely, those lying on the manifolds characterizing the non-linear normal modes— or if a single-mode model of the system is needed. However, for more general motions or if more modes are to be included in the model, a generalization of this concept is presented herein, which allows for the determination of multi-mode invariant manifolds. These manifolds include the effects of several non-linear normal modes —thus allowing interactions between them—, each of which captures a possibly significant number of linear normal modes, thereby resulting in a substantially smaller reduced-order model, for a given desired accuracy, than that obtained from a linear modal analysis of the non-linear system. In this paper, the multi-mode invariant manifold method is developed and its usefulness is investigated on a case study. The possibility of neglecting some of the interactions between the various non-linear normal modes is also examined, in which case an approximation of general motions could be obtained directly from the single-mode invariant manifolds. Numerical results obtained with the methods presented are described and compared to those obtained with a classical linear modal analysis of the non-linear system, along with a brief discussion of their potential and of on-going work.

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1. INTRODUCTION

The concept of normal modes of motion is well developed for linear oscillatory systems, due to the special features of the linear differential equations governing their dynamics. These features allow for a definition of normal modes in terms of eigenvectors (or eigenfunctions) and the expression of an arbitrary system response as a superposition of modal responses[3]. In particular, given the invariance of the normal modes, truncation procedures have been developed to allow for the reduction of the number of modeled (*i.e.*, simulated) modes, and yet for the elimination, in many cases, of most the contamination of the non-modeled modes.

Many relevant ideas can be generalized to non-linear systems. For example, much work has been done on the existence and stability of normal modes of motion for two-degree of freedom, conservative systems[4, 6, 10]. More recently, new methodologies have been developed[7, 8, 9] to generalize these definitions to a very wide class of systems which includes non-conservative, gyroscopic, and infinite-dimensional systems. Essentially, they define normal modes in terms of motions which occur on low —typically two— dimensional invariant manifolds in the system's phase space. Such a motion must be inherently like that of a lower dimensional system, and this is exactly what is desired for a normal mode motion. A constructive technique for generating such manifolds in terms of asymptotic series, without having to solve the equations of motion, is provided by a simple generalization of the method used in constructing approximate center manifolds in bifurcation theory[2]. Using this approach, it is possible to determine the manifolds which represent the normal modes for weakly non-linear systems. The equations of motion restricted to these manifolds then provide the dynamics of the associated normal modes. The tangent planes to the manifolds at the equilibrium point are the planes on which the usual modal dynamics of the linearized system take place, *i.e.*, they are the familiar eigenspaces. By definition, these non-linear normal mode manifolds are invariant, so that any motion starting exactly in one non-linear normal mode will be comprised only of that one for all time and will not generate any motion in the other non-linear normal modes. On the contrary, a standard linear modal analysis of the system's dynamics on that same manifold —obtained by mere projection of the equations of motion onto the linear modes— would produce a two-way exchange of energy, or *contamination*, between the linear mode tangent to the manifold on which the motion is initiated and the other linear modes, due to the non-linear coupling terms between the obtained projected equations. As was demonstrated previously[1], this may yield inaccurate results if one includes only few linear modes, or expensive solutions if one includes many of them. A summary of this material is provided in Section 2.

Just like the primary use of normal modes of motion of linear systems is the modal analysis associated to them, the concept of non-linear normal modes of vibration suggests the definition of a proper "non-linear modal analysis" in order to be able to obtain the response of a system under general excitation in terms of some non-linear modal coordinates. Moreover, one ought to be able to perform model reductions using the non-linear modal coordinates —as is done for linear systems—, which requires the development of efficient truncation procedures, the ultimate goal being to be able to use fewer non-linear modes than linear ones to perform equally accurate modal analyses of non-linear systems.

Given the definition of the non-linear normal modes in terms of two-dimensional invariant manifolds, it is clear that (1) they will not interact during a pure modal motion, and (2) they are bound to interact during more general motions. Therefore, in order to extend modal analysis ideas to non-linear systems, it is essential to be able to account for the interactions between the various non-linear modes involved in the dynamics of the particular system at hand, which are not readily available with the current formulation. Section 4 of this paper discusses some relevant ideas related to the problems and possibilities allowed by these individually invariant non-linear normal modes. However, it is believed at this point that, even if proper modal interactions could be recovered, the non-modeled non-linear normal modes would certainly be contaminated by this process, which might not allow for reliable low-order models.

Consequently, a new formulation has been developed to ensure the invariance of the set of modeled non-linear modes with respect to the non-modeled ones. This formulation, described in Section 3, essentially generalizes the individually invariant non-linear normal mode manifolds to multi-mode invariant manifolds. A multi-mode manifold is of dimension $2M$ when M non-linear modes are modeled, and includes the influence of all of the M individual non-linear manifolds defined previously. Besides, the interactions between the various modeled non-linear modes are accounted for at the very first stage of the definition process, thus eliminating the need for later work to recover them. The generation of a multi-mode invariant manifold follows very closely that of an individually invariant manifold, and approximations for weakly non-linear systems can be constructed easily using the same method. In the same manner as individually invariant non-linear modes do not interact during pure modal motions, the modes constituting a multi-mode manifold do not interact with the non-modeled ones for motions occurring on that manifold, hence ensuring non-contamination of the non-modeled modes if all relevant modes are embedded in the multi-mode manifold to begin with.

Numerical results have been obtained for the example of a continuous system with a discrete non-linearity. These illustrate the benefits of the formulation compared to classical linear modal analyses of non-linear systems (*i.e.*, projections of equations of motion onto the linear modes). The dynamics recovered by the multi-mode manifold methodology are generally more accurate than those obtained by a linear modal analysis using the same number of linear modes, since the multi-mode manifold reduces to this linear subspace upon linearization. In the worst case (*i.e.*, in the case of linear systems), the results are identical, while they might be much improved when the non-linearities increase. The computational savings thus obtained will of course be case-dependent, but are expected to be significant.

2. INDIVIDUAL NON-LINEAR NORMAL MODES

The equations of motion of the structural systems considered are assumed to be of the form

$$\begin{cases} \dot{x}_i = y_i \\ \dot{y}_i = f_i(x_1, \ldots, x_N, y_1, \ldots, y_N) \end{cases} \qquad i = 1, \ldots, N \tag{1}$$

or

$$\dot{\underset{\sim}{z}} = \underset{\sim}{A}(\underset{\sim}{z}) \tag{2}$$

with $\underset{\sim}{z}^T = [x_1, y_1, \ldots, x_N, y_N]^T$ and $\underset{\sim}{A}(\underset{\sim}{z})^T = [y_1, f_1, \ldots, y_N, f_N]^T$ (where ^{T}denotes a transpose), where any required discretization as been achieved if necessary, for instance using the modes of the linearized system. In the case of a discretized continuous system, $\underset{\sim}{x}^T = [x_1, \ldots, x_N]^T$ and $\underset{\sim}{y}^T = [y_1, \ldots, y_N]^T$ would represent some assumed modal coordinates and velocities, while for discrete systems they would represent generalized coordinates (displacements or rotations) and the corresponding generalized velocities. Furthermore, $\underset{\sim}{f}^T = [f_1, \ldots, f_N]^T$ represents some general forcing on the system.

For a non-linear, autonomous, oscillatory system such as that defined above, a normal mode of motion is a motion which takes place on a two-dimensional invariant manifold in the system's phase-space. This manifold passes through a stable equilibrium point $(\underline{x},\underline{y}) = (0,0)$ of the system and it is tangent to an eigenspace of the system linearized about that equilibrium[7]. Therefore, an invariant manifold and the dynamics on it can be described by a pair of independent coordinates, which can be chosen to be a single displacement-velocity pair (note that in some degenerate cases, some pairs may not be suitable for such a description, in which case the procedure has to be modified[7, 8]; the procedure has also been applied to the case of internally resonant systems where the dimension of the invariant manifolds has to be augmented[4]). For the *kth* non-linear normal mode, it is a natural choice to define $u_k = x_k$ and $v_k = y_k$, so that all displacements and velocities can be related to (u_k, v_k) only —thus enforcing the two-dimensionality and the invariance of the motion— as

$$\begin{cases} x_i = X_i(u_k, v_k) \\ y_i = Y_i(u_k, v_k) \end{cases} \qquad i = 1,\ldots,N \qquad i \neq k \tag{3}$$

Substitution into the equations of motion yields a set of constraint equations which describe the geometry of the non-linear invariant manifold, as

$$\begin{cases} \dfrac{\partial X_i}{\partial u} \times v_k + \dfrac{\partial X_i}{\partial v} \times f_k = Y_i \\ \dfrac{\partial Y_i}{\partial u} \times v_k + \dfrac{\partial Y_i}{\partial v} \times f_k = f_i \end{cases} \qquad i = 1,\ldots,N \qquad i \neq k \tag{4}$$

where use has been made of the *kth* pair of equations of motion, *i.e.*, $\dot{u}_k = v_k$ and $\dot{v}_k = f_k$. Notice no assumption has yet been made on u_k and v_k, and therefore Eq. (4) describes the *k*th non-linear normal mode in a non-local sense. Thus, if one can find the exact solution of Eq. (4), this solution will describe the exact shape of the manifold. However, solving Eq. (4) is in general not possible.

For weakly non-linear systems, an approximate local solution can be computed by assuming a Taylor series expansion of X_i and Y_i with respect to u_k and v_k up to the desired order as

$$\begin{cases} X_i = a^k_{1,i}u_k + a^k_{2,i}v_k + a^k_{3,i}u_k^2 + a^k_{4,i}u_kv_k + a^k_{5,i}v_k^2 \\ \qquad + a^k_{6,i}u_k^3 + a^k_{7,i}u_k^2v_k + a^k_{8,i}u_kv_k^2 + a^k_{9,i}v_k^3 + \ldots \\ Y_i = b^k_{1,i}u_k + b^k_{2,i}v_k + b^k_{3,i}u_k^2 + b^k_{4,i}u_kv_k + b^k_{5,i}v_k^2 \\ \qquad + b^k_{6,i}u_k^3 + b^k_{7,i}u_k^2v_k + b^k_{8,i}u_kv_k^2 + b^k_{9,i}v_k^3 + \ldots \end{cases} \tag{5}$$

Substituting Eq. (5) into Eq. (4) and equating coefficients of like powers in u_k and v_k yields a set of linear equations which can be solved, one order at a time, for the $a^k_{j,i}$'s and $b^k_{j,i}$'s. These represent the non-linear corrections (at various orders) in the *kth* non-linear normal mode due to the i^{th} linear mode.

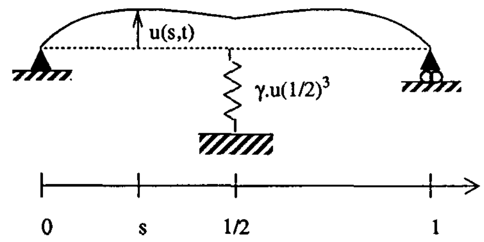

Fig. 1: Simply-supported Euler-Bernoulli (linear) beam connected to a purely cubic spring.

The dynamics on one particular non-linear normal mode are found by back substitution of the X_i's and Y_i's into Eq. (1) for $i = k$, determination of the dynamics of $u_k(t)$ (for example by numerical integration), and then by recombination of the motion using Eq. (5). This process requires solving only one non-linear ordinary differential equation (ODE) to determine the motion on each non-linear normal mode, as compared to N coupled ODE's involved in a direct integration of Eq. (1). Besides, it can be shown that, for systems where the lowest non-linearity is of order Q, the order of approximation of the dynamics is $N'+Q-1$ where N' is the order of approximation of the manifold. In the case of general (*e.g.*, quadratic) non-linearities, this means that the approximation of the dynamics is one order higher than that of the manifold itself, and two orders higher in the case of odd non-linearities.

Example : A Simply Supported Euler-Bernouilli Beam Constrained by a Non-Linear Spring

The above methodology is applied to a homogeneous, simply supported Euler-Bernoulli beam with a non-linear cubic spring attached at its middle —see Fig. 1. While the beam itself is assumed to deform in the linear range, the spring is chosen as purely cubic so that the linearized system's normal modes are those of the simply supported beam alone (*i.e.*, pure sine waves). With this choice the influence of the various linear modes on the non-linear ones can be visualized easily. Notice that since the spring is located at a node of the antisymmetric (even) modes, it does not affect them, so that the antisymmetric modes of the non-linear system are the same as those of the linearized system. Therefore, only the symmetric (odd) modes are influenced by the non-linear spring and, furthermore, they feature only contributions of the symmetric linear modes.

If the beam is of length $l = 1$, the equation of transverse motion of the system can be shown to be, in non-dimensional form:

$$\ddot{u} + \alpha u_{,ssss} + \beta u^3 \delta\left(s - \frac{1}{2}\right) = 0, \qquad s \in \;]0,1[\tag{6}$$

where $\alpha = EI/m$, $\beta = \gamma/m$, E is the Young's modulus, I is its second moment of area, m is its mass per unit length, γ is the non-linear stiffness of the spring, s represents the abscissa along the beam, $u(s,t)$ is the transverse deflection, $\bullet_{,s}$ denotes a derivative with respect to s, an overdot represents a derivative with respect to time and δ is the Dirac function. The associated boundary conditions are $u(0) = u(1) = 0$ and $u_{,ss}(0) = u_{,ss}(1) = 0$.

The beam deflection, $u(s,t)$, is first discretized using the natural modes of the linearized system, $\phi_j(s) = \sin(j\pi s)$, as[9]

$$u(s,t) \equiv \sum_{j=1}^{N} \eta_j(t)\,\phi_j(s) \tag{7}$$

where N is the number of terms in the expansion, *i.e.*, the number of terms that would be retained for a linear modal analysis of the non-linear system. Projection of equation of motion onto the i^{th} linear mode yields

$$\ddot{\eta}_i + \alpha(i\pi)^4\eta_i + 2\beta\left[\sum_{j=1}^{N}\eta_j \sin\left(j\frac{\pi}{2}\right)\right]^3 \sin\left(i\frac{\pi}{2}\right) = 0 \tag{8}$$

for $i = 1, \ldots, N$, which can be written in first-order form as

$$\begin{cases} \dot{x}_i = y_i \\ \dot{y}_i = f_i(x_1, \ldots, x_N, y_1, \ldots, y_N) \end{cases} \qquad i = 1, \ldots, N \tag{9}$$

where $x_i = \eta_i$, $y_i = \dot{\eta}_i$, and

$$f_i = -\alpha(i\pi)^4 x_i - 2\beta\left[\sum_{j=1}^{N} x_j \sin\left(j\frac{\pi}{2}\right)\right]^3 \sin\left(i\frac{\pi}{2}\right)$$

The set of differential equations, Eq.(9) is what is simulated for a typical linear modal analysis. Alternatively, the procedure described earlier in this section can be applied to Eq. (9) to determine the third- or higher order approximation of the non-linear normal modes of the system. The kth non-linear mode is given by, to fifth-order[1] :

$$\eta_k = u_k, \qquad \dot{\eta}_k = v_k$$

and for $i = 1,\ldots,N$, $i \neq k$:

$$\begin{cases} \eta_i = \dot{\eta}_i = 0 & (i \text{ even}) \\ \eta_i = a_6^i u_k^3 + a_8^i u_k v_k^2 & \\ \qquad + a_{15}^i u_k^5 + a_{17}^i u_k^3 v_k^2 + a_{19}^i u_k v_k^4 + \ldots & (i \text{ odd}) \\ \dot{\eta}_i = b_{7,i}^k u_k^2 v_k + b_{9,i}^k v_k^3 & \\ \qquad + b_{16,i}^k u_k^4 v_k + b_{18,i}^k u_k^2 v_k^3 + b_{20,i}^k v_k^5 + \ldots & (i \text{ odd}) \end{cases} \tag{10}$$

where if k is even $a_{6,i}^k = a_{8,i}^k = a_{15,i}^k = a_{17,i}^k = a_{19,i}^k = 0$, and if k is odd, for $i = 1,\ldots,N$, $i \neq k$, i odd

$$\begin{cases} a_{6,i}^k = 2\beta(-1)^{\frac{k+i}{2}} \dfrac{[i^4 - 7k^4]}{\alpha\pi^4[i^4 - k^4][i^4 - 9k^4]} \\ a_{8,i}^k = -12\beta(-1)^{\frac{k+i}{2}} \dfrac{1}{\alpha^2\pi^8[i^4 - k^4][i^4 - 9k^4]} \end{cases} \tag{11}$$

$$\begin{cases} a_{15}^i = \Big[-\alpha^2\pi^8(i^4 - 17k^4)(i^4 - 13k^4)\mu_1^i \\ \qquad + 2\alpha^3\pi^{12}k^8(i^4 - 13k^4)\mu_2^i - 72\alpha\pi^4k^8\mu_1^i\Big]/\Delta \\ a_{17}^i = \Big[-\alpha^2\pi^8(i^4 - 5k^4)(i^4 - 13k^4)\mu_2^i \\ \qquad + 20\alpha\pi^4(i^4 - 13k^4)\mu_1^i\Big]/\Delta \\ a_{19}^i = \dfrac{-120\mu_1^i + 6\alpha\pi^4(i^4 - 5k^4)\mu_2^i}{\Delta} \end{cases} \tag{12}$$

and, for all k and $i = 1,\ldots,N$, $i \neq k$, i odd:

$$\begin{cases} b_{7,i}^k = -2\alpha\pi^4k^4a_{8,i}^k + 3a_{6,i}^k \\ b_{9,i}^k = a_{8,i}^k \end{cases} \tag{13}$$

$$\begin{cases} b_{16,i}^k = 5a_{15,i}^k - 2\alpha(k\pi)^4a_{17,i}^k - 4\beta a_{8,i}^k \\ b_{18,i}^k = 3a_{17,i}^k - 4\alpha(k\pi)^4a_{19,i}^k \\ b_{20,i}^k = a_{19,i}^k \end{cases} \tag{14}$$

where, for the fifth-order terms,

$$\begin{cases} \Delta = \alpha^3\pi^{12}[(i^4 - 5k^4)(i^4 - 17k^4)(i^4 - 13k^4)\mu_2^i \\ \qquad + 32k^8(i^4 + 5k^4)] \\ \mu_1^i = 2\beta\left[3\left(\sum_{j\neq k} a_6^i \sin\left(j\frac{\pi}{2}\right)\right)\sin\left(i\frac{\pi}{2}\right) + 2a_8^i - b_7^i\right] \\ \mu_2^i = 2\beta\left[3\left(\sum_{j\neq k} a_8^i \sin\left(j\frac{\pi}{2}\right)\right)\sin\left(i\frac{\pi}{2}\right) - 22a_8^i\right] \end{cases} \tag{15}$$

Consequently, the deflection of the beam in the kth non-linear mode, $u^k(s,t)$, can be expressed in terms of the kth non-linear modal coordinate, $u_k(t)$, and the associated modal velocity, $v_k(t)$, as

$$u^k(s,t) = u_k \sin(k\pi s) + \sum_{\substack{i\,odd \\ i\neq k}} [a_6^i u_k^3 + a_8^i u_k v_k^2 + a_{15}^i u_k^5 + a_{17}^i u_k^3 v_k^2 + a_{19}^i u_k v_k^4] \sin(i\pi s) + \ldots \tag{16}$$

while the dynamics of the non-linear modal coordinate itself is governed by

$$\ddot{u}_k + \alpha\,(k\pi)^4 u_k + 2\beta u_k^3 \Bigg[\sin\left(k\frac{\pi}{2}\right) + \left(3\sum_{\substack{j\,odd \\ j\neq k}} [a_6^j u_k^2 + a_8^j v_k^2 + a_{15}^j u_k^4 + a_{17}^j u_k^2 v_k^2 + a_{19}^j v_k^4] \sin\left(j\frac{\pi}{2}\right)\right) + 3\sin\left(k\frac{\pi}{2}\right) \left(\sum_{\substack{j\,odd \\ j\neq k}} [a_6^j u_k^2 + a_8^j v_k^2] \sin\left(j\frac{\pi}{2}\right)\right)^2 \Bigg] \sin\left(k\frac{\pi}{2}\right) + \ldots = 0 \qquad (17)$$

for $k = 1,\ldots,N$. Here, $u^k(s,t)$ refers to the deflection of the point of abscissa s at time t when the system undergoes a motion in the *kth* non-linear normal mode. It should not to be confused with $u_k(t)$, which is the non-linear modal coordinate and is not meant to represent the motion of any particular point. Note that the dynamics of the N non-linear modal oscillators are individually decoupled from one another, which accounts for the invariance of the non-linear normal modes.

As noted above, one can obtain the dynamics of the *kth* non-linear mode up to an accuracy of fifth-order with only a third-order accurate invariant manifold, as is apparent from Eq. (17) (retaining only the cubic coefficients $a_{6,j}^k$'s and $a_{8,j}^k$'s yields the complete fifth-order dynamics), and up to seventh-order with a fifth-order accurate manifold.

Figures 2, 3 and 4 display results obtained using the procedure presented herein, along with results obtained with classical linear modal analyses of the non-linear system performed with various number of modeled linear modes. In these figures, the "exact" solution was determined using a linear modal analysis with 25 linear modes. In this particular case it appears at least three to five linear modes are necessary to achieve an accuracy comparable to that obtained with the seventh- or fifth-order dynamics as obtained above. Bearing in mind that the latter results are obtained by simulation of one differential equation only (Eq. (17)), it is evident that the non-linear normal mode approach is a better candidate than linear modal analysis of the non-linear system for the generation of reduced-order models consisting of only one mode. In the case of single-mode linear modal analysis the influence of the other linear modes would be missing whereas it is embedded in the non-linear normal mode (see Fig. 4 which represents simulations all utilizing only one ODE).

3. MULTI-MODE INVARIANT MANIFOLDS

The potential of non-linear normal modes is evident from the previous section. However, it is important to note that, by definition, they are only *individually* invariant. Therefore, they do not interact when the system undergoes a motion in any one of the modal manifolds, but nothing prevents them from interacting during an arbitrary motion. This immediately reminds one of the problems encountered in the linear modal analysis of the non-linear system —where contamination between the various linear normal modes almost inevitably occurs—, which were at the origin of the definition of the non-linear normal modes as tools to try to eliminate the phenomenon of contamination. An attempt at utilizing these individual non-linear normal modes to obtain directly the dynamics of the system undergoing an arbitrary motion will be presented in Section 4. In this approach the interaction between the non-linear modes is essentially ignored, thereby allowing for the direct use of the single-mode non-linear manifold results. The remainder of this section, however, concentrates on completely removing this contamination (to a given order), which, at this point, requires additional work.

In order to properly ensure the non-contamination of the non-linear modes which are not included in the reduced-order model, a new formulation is necessary, which generalizes the individually invariant non-linear normal modes and reduces to them in special cases. The underlying idea is to generate high-dimensional invariant manifolds, referred to as *multi-mode manifolds*, essentially in the same manner as the individually invariant manifolds were produced in the Section 2. These multi-mode manifolds, when comprising the influence of M non-linear modes, are of dimension $2M$ in the phase-space for the oscillatory systems typically of interest in structural dynamics. Evidently, these multi-mode manifolds are still not completely invariant —in the sense that two different multi-mode manifolds would interact during a general motion, as the non-linear normal modes did— but, for motions on a given multi-mode manifold, invariance is ensured between itself and the rest of the (non-modeled) non-linear modes —essentially in the same manner as the non-linear normal modes were not interacting during purely modal motions.

Consequently, for a system for which M non-linear modes are to be modeled and for which the remaining ones are to be merely ignored, the multi-mode manifold should comprise all M modes, so that (1) the interactions between those M modes can be accounted for, and (2) the interactions with the non-modeled modes can be completely removed. If a mode is non-modeled despite an internal resonance with a modeled one, the mathematical process of generating the multi-mode manifold will become singular, thereby detecting the anomaly.

The procedure to determine multi-mode invariant manifolds follows closely the one presented in Section 2. If S_m denotes the subset of indices corresponding to the modeled modes, and $\underline{u}_m$ and $\underline{v}_m$ represent the vectors of the corresponding non-linear modal coordinates and velocities, then the various linear modal coordinates are expressed as functions of the modeled modes as

$$\begin{cases} x_k = u_k \\ y_k = v_k \end{cases} \quad \text{for} \quad k \in S_m \qquad (18)$$

$$\begin{cases} x_j = X_j(\underline{u}_m, \underline{v}_m) \\ y_j = Y_j(\underline{u}_m, \underline{v}_m) \end{cases} \quad \text{for} \quad j \notin S_m \qquad (19)$$

Taking the time-derivatives for $j \notin S_m$ yields :

$$\begin{cases} \dot{X}_j = \sum_{k \in S_m} \frac{\partial X_j}{\partial u_k} v_k + \frac{\partial X_j}{\partial v_k} f_k \\ \dot{Y}_j = \sum_{k \in S_m} \frac{\partial Y_j}{\partial u_k} v_k + \frac{\partial Y_j}{\partial v_k} f_k \end{cases}$$

which can be substituted into the j^{th} pair of equations of motion to produce equations resembling Eq. (4). In most cases (namely, for weakly non-linear systems), approximations will be sought in a series expansion form. One has, to third order :

$$\begin{aligned} X_j(\underline{u}_m, \underline{v}_m) = & \sum_{k \in S_m} a_{1,j}^k \cdot u_k + a_{2,j}^k \cdot v_k \\ & + \sum_{k \in S_m} \sum_{l \in S_m} a_{3,j}^{k,l} \cdot u_k u_l + a_{4,j}^{k,l} \cdot u_k v_l + a_{5,j}^{k,l} \cdot v_k v_l \\ & + \sum_{k \in S_m} \sum_{l \in S_m} \sum_{q \in S_m} a_{6,j}^{k,l,q} \cdot u_k u_l u_q + a_{7,j}^{k,l,q} \cdot u_k u_l v_q \\ & \qquad + a_{8,j}^{k,l,q} \cdot u_k v_l v_q + a_{9,j}^{k,l,q} \cdot v_k v_l v_q + \dots \end{aligned} \quad (20)$$

$$\begin{aligned} Y_j(\underline{u}_m, \underline{v}_m) = & \sum_{k \in S_m} b_{1,j}^k \cdot u_k + b_{2,j}^k \cdot v_k \\ & + \sum_{k \in S_m} \sum_{l \in S_m} b_{3,j}^{k,l} \cdot u_k u_l + b_{4,j}^{k,l} \cdot u_k v_l + b_{5,j}^{k,l} \cdot v_k v_l \\ & + \sum_{k \in S_m} \sum_{l \in S_m} \sum_{q \in S_m} b_{6,j}^{k,l,q} \cdot u_k u_l u_q + b_{7,j}^{k,l,q} \cdot u_k u_l v_q \\ & \qquad + b_{8,j}^{k,l,q} \cdot u_k v_l v_q + b_{9,j}^{k,l,q} \cdot v_k v_l v_q + \dots \end{aligned} \quad (21)$$

Note that this decomposition is not unique, and that the number of coefficients of order p when M non-linear modes are modeled is, for each X_j and Y_j :

$$C_p^{2M-1+p} = \frac{(2M-1+p)!}{(2M-1)!p!}$$

which increases very rapidly with both p and M. Substituting Eqs. (20) and (21) into the j^{th} pair of equations of motion and equating like powers in $\underline{u}_m$ and $\underline{v}_m$, one obtains the first- and higher-order coefficients sequentially, one order at a time. If one uses the linear normal modes to discretize the continuous system (or, equivalently casts the linearized discrete system in terms of the linear modal coordinates), the first-order coefficients can be shown to vanish for all $j \notin S_m$. For systems with purely cubic non-linearities (which is the case of the example studied below), all second-order coefficients are zero, while the equations for the third-order coefficients can be put in matrix form as

$$\left[A_j\right] \underline{a}_j^{(3)} = \underline{b}_j^{(3)} \quad (22)$$

$$\left[A_j\right] \underline{b}_j^{(3)} = \underline{f}_j^{(3)} \quad (23)$$

where $\underline{a}_j^{(3)}$ and $\underline{b}_j^{(3)}$ represent the third-order coefficients and $\underline{f}_j^{(3)}$ is problem dependent and is linear in both $\underline{a}_j^{(3)}$ and $\underline{b}_j^{(3)}$ ($\left[A_j\right]$, $\underline{a}_j^{(3)}$, and $\underline{b}_j^{(3)}$ are given in the Appendix for the example of two-mode model). Combining Eqs. (22) and (23) then yields

$$\left[A_j\right]^2 \underline{a}_j^{(3)} = \hat{\underline{f}}_j^{(3)} \quad (24)$$

where $\hat{\underline{f}}_j^{(3)}$ is in general linear in $\underline{a}_j^{(3)}$ (the hat on $\hat{\underline{f}}_j^{(3)}$ denotes the fact that Eq. (22) has been used wherever necessary). Equation (24) can be solved for $\underline{a}_j^{(3)}$ (using Maple™ for example), and $\underline{b}_j^{(3)}$ is then obtained using Eq. (22). Higher-order approximations of the multi-mode manifold can be computed sequentially in the same manner.

Once the multi-mode manifold of interest has been approximated to the desired order, the dynamics of the system on it are obtained by solving the reduced set of equations of motion corresponding to the modeled modes, *i.e.*,

$$\begin{cases} \dot{u}_k = v_k \\ \dot{v}_k = f_k(\underline{u}_m, \underline{v}_m) \end{cases} \quad \text{for } k \in S_m \quad (25)$$

where Eqs. (20) and (21) have been utilized where necessary, and then by recombining the linear modal amplitudes using Eqs. (18) and (19). At this point, the manner in which the contamination with the non-modeled modes has been removed becomes evident. On the one hand, it is clear from Eq. (25) that the dynamics on the multi-mode manifold itself depend only on the non-linear modal coordinates corresponding to non-linear modes that constitute it. On the other hand, the non-modeled non-linear modes can be viewed in two generic ways : either as a whole (*i.e.*, as another multi-mode manifold, constituted of all the non-modeled modes), in which case their dynamics (dictated by equations resembling Eq. (25)) are independent of those of the modeled modes; or as individual non-linear normal modes (as defined in Section 2), which can merely be considered as special cases of the multi-mode invariant manifold concept, in which case the previous remark still applies. Consequently, if the initial conditions are given in terms of the modeled non-linear modes only (while the non-modeled ones are initially zero —which is the case when one merely ignores them), the non-modeled modes will remain quiescent for all time even if their dynamics are simulated, and therefore their contributions will not be missing.

Example : A Simply Supported Euler-Bernouilli Beam Constrained by a Non-Linear Spring

In the particular case of the system depicted in Fig. 1, a two-mode invariant manifold is computed with the aid of the symbolic manipulation package Maple™. In this case, the first-order terms vanish except for those corresponding directly to the linear modes (since the linear modes are used to discretize the system), and all second-order terms are zero (no quadratic non-linearities). If $S_m = \{k, l\}$, the vector $\hat{\underline{f}}_j^{(3)}$ in Eq. (24) reduces to

$$\hat{f}_{j}^{(3)} = -\alpha (j\pi)^4 a_j^{(3)} - 2\beta \sin\left(j\frac{\pi}{2}\right) \begin{bmatrix} \sin\left(k\frac{\pi}{2}\right)^3 \\ 3\sin\left(k\frac{\pi}{2}\right)^2 \sin\left(l\frac{\pi}{2}\right) \\ 3\sin\left(k\frac{\pi}{2}\right)\sin\left(l\frac{\pi}{2}\right)^2 \\ \sin\left(l\frac{\pi}{2}\right)^3 \\ 0 \\ \dots \\ 0 \end{bmatrix} \quad (26)$$

It is then found that $a_{7,j}^{p,q,r} = a_{9,j}^{p,q,r} = 0$ for all $\{p, q, r\} \in S_m$, and

$$a_{6,j}^{k,k,k} = 2\frac{(-j^4 + 7k^4)\beta \sin\left(k\frac{\pi}{2}\right)^3 \sin\left(j\frac{\pi}{2}\right)}{\alpha\pi^4 (j^8 - 10j^4k^4 + 9k^8)} \quad (27)$$

$$\begin{aligned} a_{6,j}^{k,k,l} = {} & 6\beta (j^8 - 6j^4l^4 - 2j^4k^4 - 2k^4l^4 + l^8 + 8k^8) \sin\left(k\frac{\pi}{2}\right)^2 \\ & \times \sin\left(l\frac{\pi}{2}\right)\sin\left(j\frac{\pi}{2}\right) / [(\alpha\pi^4 (-j^4 + l^4) \\ & \times (-8k^4l^4 - 8j^4k^4 + 16k^8 + j^8 - 2j^4l^4 + l^8)] \end{aligned} \quad (28)$$

$$a_{8,j}^{k,k,k} = 12\frac{\beta \sin\left(k\frac{\pi}{2}\right)^3 \sin\left(j\frac{\pi}{2}\right)}{\alpha^2\pi^8 (j^8 - 10j^4k^4 + 9k^8)} \quad (29)$$

$$\begin{aligned} a_{8,j}^{k,l,l} = {} & -12\beta (3k^4 + j^4 - 4l^4) \sin\left(l\frac{\pi}{2}\right)^2 \sin\left(j\frac{\pi}{2}\right) / [\alpha^2\pi^8 \\ & \times (-j^4 + k^4) \times (-8k^4l^4 - 8j^4l^4 + 16l^8 + j^8 - 2j^4k^4 + k^8)] \end{aligned} \quad (30)$$

$$a_{8,j}^{k,k,l} = 24\frac{\beta \sin\left(k\frac{\pi}{2}\right)^2 \sin\left(l\frac{\pi}{2}\right)\sin\left(j\frac{\pi}{2}\right)}{\alpha^2\pi^8 (-8k^4l^4 - 8j^4k^4 + 16k^8 + j^8 - 2j^4l^4 + l^8)} \quad (31)$$

with similar relations when k and l are switched, from which one obtains, $b_{6,j}^{p,q,r} = b_{8,j}^{p,q,r} = 0$ for all $\{p, q, r\} \in S_m$, and

$$b_{7,j}^{k,k,k} = 6\frac{(-j^4 + 3k^4)\beta \sin\left(k\frac{\pi}{2}\right)^3 \sin\left(j\frac{\pi}{2}\right)}{\alpha\pi^4 (j^8 - 10j^4k^4 + 9k^8)} \quad (32)$$

$$\begin{aligned} b_{7,j}^{k,k,l} = {} & 6\beta \sin\left(k\frac{\pi}{2}\right)^2 \sin\left(l\frac{\pi}{2}\right)\sin\left(j\frac{\pi}{2}\right) \\ & \times (j^8 - 2j^4k^4 - 2j^4l^4 - 6k^4l^4 + l^8 + 8k^8) / [\alpha\pi^4 (-j^4 + l^4) \\ & \times (-8k^4l^4 - 8j^4k^4 + 16k^8 + j^8 - 2j^4l^4 + l^8)] \end{aligned} \quad (33)$$

$$b_{7,j}^{k,l,k} = 12\frac{(-l^4 - j^4 + 4k^4)\beta \sin\left(k\frac{\pi}{2}\right)^2 \sin\left(l\frac{\pi}{2}\right)\sin\left(j\frac{\pi}{2}\right)}{\alpha\pi^4 (-8k^4l^4 - 8j^4k^4 + 16k^8 + j^8 - 2j^4l^4 + l^8)} \quad (34)$$

$$b_{9,j}^{k,k,k} = 12\frac{\beta \sin\left(k\frac{\pi}{2}\right)^3 \sin\left(j\frac{\pi}{2}\right)}{\alpha^2\pi^8 (j^8 - 10j^4k^4 + 9k^8)} \quad (35)$$

$$\begin{aligned} b_{9,j}^{k,k,l} = {} & \beta \sin\left(k\frac{\pi}{2}\right)^2 \sin\left(l\frac{\pi}{2}\right)\sin\left(j\frac{\pi}{2}\right) (-l^4 - 3j^4 + 4k^4) \\ & / [\alpha^2\pi^8 (-j^4 + l^4) (-8k^4l^4 - 8j^4k^4 + 16k^8 + j^8 - 2j^4l^4 + l^8)] \end{aligned} \quad (36)$$

with, again, similar relations when k and l are switched.

It can be noted by inspection of Eqs. (27), (29) and (11) that

$a_{6,j}^{k,k,k} = a_{6,j}^{k}$ as obtained for the *kth* non-linear normal mode (37)

and

$a_{8,j}^{k,k,k} = a_{8,j}^{k}$ as obtained for the *kth* non-linear normal mode (38)

This is expected since the multi-mode manifold reduces to the *kth* non-linear normal mode when $S_m=\{k\}$. Consequently, an alternative to directly solving Eq. (24) is first to solve for the individual non-linear normal modes (as in Section 2), and then to use all single-mode coefficients as known coefficients, thereby somewhat reducing the size of the system in Eq. (24) (see Appendix). Although linear sets of equations such as Eq. (24) can be dealt with very efficiently with symbolic manipulation packages such as Maple™ or Mathematica™, it should always be kept in mind that the number of coefficients involved at each step increases very rapidly with both the order of approximation and the number of modeled modes, so that the use of relations such as Eqs. (37) and (38) should be made wherever applicable. For instance, while there are 20 cubic coefficients involved in a two-mode model, there are 56 of them for a three-mode model. In the latter case, directly solving for those coefficients using Eq. (24) would result in a 56x56 linear system of equations, while making use of the above remark would result in solving successively a 4x4, a 12x12, and an 8x8 system of equations corresponding to the cubic orders of the single-, two- and three- mode models, respectively. Along the same line, it should be noted that, regardless of the number of modeled modes in the multi-mode manifold at hand, each cubic coefficient will always involve no more than three modes at a time, and therefore all cubic coefficients are known for any number of modeled modes as soon as the three-mode model has been solved to cubic order. For example, if one was to construct a five-mode model, there would be 220 cubic coefficients, resulting in a 220x220 system of equations to be solved using the brute force approach, whereas in fact no work at all should be required once the three-mode model has been solved for to cubic order analytically!

Results of simulations performed using either the above multi-mode manifold procedure or a linear modal analysis of the non-linear system are shown on Figs. 5-10 for two different sets of initial conditions on a three-mode manifold. In these examples, the three-mode model is composed of the first three modes and can therefore be obtained directly from the two-mode model involving only the first and third modes —since all the coefficients corresponding to the added even mode vanish. As expected from the theory, a given number of non-linear modes embedded in the multi-mode manifold yields better results than the same number of linear modes used in a linear modal analysis procedure, all the more so as the influence of the non-linearity increases.

Note that for this example system, as was the case for single-mode manifolds, the dynamics are obtained at order $N'+2$ when the order of approximation of the multi-mode manifold is N'. In general the dynamics are of order $N'+Q-1$ when the lowest non-linearity is of order Q. In the present case, the dynamics are obtained at fifth-order by the coupled equations

$$\begin{cases} \dot{u}_k = v_k \\ \dot{v}_k = -\alpha(k\pi)^4 u_k - 2\beta \sin\left(k\frac{\pi}{2}\right)\left[\sum_{l\in S_m} u_l \sin\left(l\frac{\pi}{2}\right)\right]^3 \\ \quad -6\beta\sin\left(k\frac{\pi}{2}\right)\left[\sum_{l\in S_m} u_l \sin\left(l\frac{\pi}{2}\right)\right]^2\left[\sum_{j\in S_m} X_j^{(3)}\sin\left(j\frac{\pi}{2}\right)\right] + \dots \end{cases} \quad (39)$$

for $k \in S_m$, where $X_j^{(3)}$ represents the cubic part of X_j. Note that, in contrast with the case of single-mode manifolds, the dynamics of the various modeled non-linear modes are coupled, so that essential interactions between them are allowed. However, the dynamics are uncoupled from that of the non-modeled modes.

4. NON-LINEAR MODAL ANALYSIS REVISITED

The multi-mode procedure presented in Section 3 allows for complete removal of the contamination of the non-modeled non-linear modes (to a given order), and yet for proper interaction between the modeled ones. However, an alternative method was previously introduced[7], in which an attempt at superposition was proposed to recombine the linear modal components directly from the single-mode non-linear components. This can be formalized as

$$\underline{z} = \underline{M}(\underline{w}) = \left(\left[M_0\right] + \left[M_1(\underline{w})\right] + \left[M_2(\underline{w})\right] + \dots\right)\underline{w} \quad (40)$$

where $\underline{w}^T = (u_1, v_1, \dots, u_N, v_N)^T$, $\left[M_0\right]$ is the identity matrix (if one uses the linear normal modes in the discretization process), $\left[M_1(\underline{w})\right]$ is linear in $\underline{w}$ (and is identically zero when no quadratic non-linearities are present), and $\left[M_2(\underline{w})\right]$ is quadratic in $\underline{w}$ and is assembled from the $2N$x2 matrices representing the cubic part of each individual non-linear normal mode (see reference [7] for more details). This method has not been fully investigated yet, and some work is currently under way, but some general ideas will be outlined here.

From Eqs. (40) and (2), the equations of motion become

$$\dot{\underline{w}} = \left[\frac{\partial \underline{M}}{\partial \underline{w}}\right]^{-1} \cdot \underline{A}(\underline{M}(\underline{w})) \quad (41)$$

Note that Eq. (41) requires one to model as many non-linear modes as linear ones in order to obtain a square matrix inversion. The efficiency of this process might be improved by use of the generalized inverse when fewer non-linear modes than linear ones are modeled. In such a case,

$$\underline{z} = \underline{N}(\underline{q}) \quad (42)$$

where $\underline{q}$ is the restriction of $\underline{w}$ to the modeled non-linear modes, and $\left[\partial \underline{N}/\partial \underline{q}\right]$ is now rectangular. The equations of motion therefore become

$$\dot{\underline{q}} = \left[\frac{\partial \underline{N}}{\partial \underline{q}}\right]^{-} \cdot \underline{A}(\underline{N}(\underline{q})) \quad (43)$$

where

$$\left[\frac{\partial \underline{N}}{\partial \underline{q}}\right]^{-} = \left[\left[\frac{\partial \underline{N}}{\partial \underline{q}}\right]^T\left[\frac{\partial \underline{N}}{\partial \underline{q}}\right]\right]^{-1}\left[\frac{\partial \underline{N}}{\partial \underline{q}}\right]^T$$

The above approach essentially consists of a direct extension of the ideas used in modal truncation of linear systems. However, it is based on non-linear modes which are *individually* invariant, but whose behavior in this context is as of now largely unknown. The reduced-order models thus obtained possess the desirable property of accounting for some non-linear coupling between the various modeled non-linear modes, but the influence of the modal contamination of the non-modeled modes is yet to be determined. These issues are currently under investigation. In particular, viewing the non-linear modal coordinates as curvilinear coordinates along some particular directions (*i.e.*, along the non-linear manifolds), it may be possible to determine a "non-linear projection" so that the linear modal coordinates of the system can be decomposed on this set of curvilinear coordinates. Utilizing this non-linear projection to replace Eq. (40), the procedure could, possibly without too much computations, provide a most accurate description of the interactions between the individual non-linear modes, although the issue of the contamination of the non-modeled modes would probably still not be addressed.

It should also be noted that this procedure will not necessarily always be less demanding than the one presented in Section 3, since it is very likely to involve the numerical factorization of a non-square matrix at each time-step during the simulation, whereas the former procedure, once the multi-mode manifold is determined, consists of explicit simulations followed by simple (but possibly long) recombinations. These issues are also currently under investigation.

5. CONCLUSION

The developments in Sections 2 and 3 suggest that the concept of invariant manifolds has potentially important implications for non-linear structural dynamics problems. This is the first time that the problem of defining a non-linear modal analysis for non-linear systems is tackled effectively, in the sense that proper interaction between the various modeled modes is allowed and accounted for, while contamination with the non-modeled modes is ensured to be eliminated (*i.e.*, even if they were simulated, they would remain quiescent for all time in the absence of resonances). This property, which is essential for proper simulation of the dynamics of a system once its most important modes have been selected, is an extension to non-linear systems of what exists for the modal analysis of linear systems.

Besides, when the original equations of motion are given in terms of the linear modal coordinates, each non-linear mode or multi-mode manifold is certain to comprise at least the contributions of the linear modes to which it reduces upon linearization, which guarantees, for a fixed number of modeled modes, to obtain results at least as good as those from a linear modal analysis of the non-linear system with the same number of modes. In general, however, the results obtained by the proposed method will be better than those obtained with the same number of linear modes, since part of the influence of some higher linear modes is included in the non-linear normal mode or multi-mode model considered.

Regarding the generation of those multi-mode manifolds, it should be emphasized that the use of symbolic manipulation packages can greatly reduce the amount of work required to obtain a multi-mode model at a given order. Specifically, if the determination of all the lower-dimensional multi-mode manifolds is preliminary carried out analytically, many of the coefficients involved for the desired number of modeled modes are known by inspection.

Finally, an alternative method based on the individual non-linear normal modes was presented, which, by neglecting some of the interactions between the various non-linear modes, allows for an approximate non-linear modal analysis of the system. Since the contamination with the non-modeled is not removed in this case, the accuracy of the reduced-order model is not clear as of now. A generalization of this approach by use of curvilinear coordinates and a "non-linear projection" may yield some interesting qualitative results concerning the nature of the interactions between the individual non-linear modes. This work is still in progress.

The methods proposed herein to generate reduced-order models have potentially important implications for many areas, including structural dynamics and control, where accurate low-order models are of interest.

6. REFERENCES

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7. APPENDIX : MATRICES AND VECTORS INVOLVED IN THE DETERMINATION OF THE CUBIC PART OF A TWO-MODE INVARIANT MANIFOLD

The following shows the matrices involved in solving the full 20x20 system of equations (Eq. (24)) for the cubic coefficients of a two-mode invariant manifold in the Euler-Bernoulli beam case-study. Alternately, a reduced system can be used if one takes advantage of the results obtained for the single-mode calculations. The corresponding matrices are also provided here.

The equations for the cubic coefficients are put in matrix form as shown in Eqs. (22) and (23), where the components of $\underset{\sim}{a}_j^{(3)}$, and $\underset{\sim}{b}_j^{(3)}$ are ordered in the same manner, as follows :

$$(\underset{\sim}{a}_j^{(3)})^T = [a_{6,j}^{k,k,k}, a_{6,j}^{k,k,l}, a_{6,j}^{k,l,l}, a_{6,j}^{l,l,l}, a_{7,j}^{k,k,k}, a_{7,j}^{k,k,l}, a_{7,j}^{l,l,k}, a_{7,j}^{l,l,l}, a_{7,j}^{k,l,k}, a_{7,j}^{k,l,l},$$
$$a_{8,j}^{k,k,k}, a_{8,j}^{k,l,l}, a_{8,j}^{l,k,k}, a_{8,j}^{l,l,l}, a_{8,j}^{k,k,l}, a_{8,j}^{l,k,l}, a_{9,j}^{k,k,k}, a_{9,j}^{k,k,l}, a_{9,j}^{k,l,l}, a_{9,j}^{l,l,l}]^T$$

Following this ordering, the matrix $\left[A_j\right]$ is given by

$$\left[A_j\right] = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}$$

where $A_1 = \left[A_1^{left}\ A_1^{right}\right]$, $A_2 = \left[A_2^{left}\ A_2^{right}\right]$, $A_4 = \left[A_4^{left}\ A_4^{right}\right]$, as

$$A_1^{left} = \begin{bmatrix} 0 & 0 & 0 & 0 & -\alpha(k\pi)^4 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\alpha(l\pi)^4 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \end{bmatrix}$$

$$A_2^{right} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -\alpha(k\pi)^4 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\alpha(l\pi)^4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -\alpha(l\pi)^4 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\alpha(k\pi)^4 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A_1^{right} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\alpha(k\pi)^4 & 0 \\ -\alpha(k\pi)^4 & 0 & 0 & -\alpha(l\pi)^4 \\ 0 & -\alpha(l\pi)^4 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A_2^{left} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -2\alpha(k\pi)^4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2\alpha(l\pi)^4 \\ 0 & 0 & -2\alpha(k\pi)^4 & 0 \\ 0 & -2\alpha(l\pi)^4 & 0 & 0 \end{bmatrix}$$

$$A_4^{left} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

$$A_4^{right} = \begin{bmatrix} -3\alpha(k\pi)^4 & 0 & 0 & 0 \\ 0 & 0 & -\alpha(k\pi)^4 & 0 \\ 0 & -\alpha(l\pi)^4 & 0 & 0 \\ 0 & 0 & 0 & -3\alpha(l\pi)^4 \\ 0 & -2\alpha(k\pi)^4 & 0 & 0 \\ 0 & 0 & -2\alpha(l\pi)^4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

In the case where one takes advantage of the knowledge of the single-mode manifolds, $\left[A_j\right]$ would be replaced by a 12x12 matrix as :

$$\left[\tilde{A}_j\right] = \begin{bmatrix} \tilde{A}_1 & \tilde{A}_2 \\ \tilde{A}_3 & \tilde{A}_4 \end{bmatrix}$$

where

$$\tilde{A}_1 = \begin{bmatrix} 0 & 0 & -\alpha(l\pi)^4 & 0 & -\alpha(k\pi)^4 & 0 \\ 0 & 0 & 0 & -\alpha(k\pi)^4 & 0 & -\alpha(l\pi)^4 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\tilde{A}_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\alpha(k\pi)^4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2\alpha(k\pi)^4 & -\alpha(l\pi)^4 & 0 & 0 & 0 \\ -2\alpha(l\pi)^4 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\tilde{A}_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\tilde{A}_4 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -\alpha(k\pi)^4 \\ 0 & 0 & 0 & 0 & -\alpha(l\pi)^4 & 0 \\ 0 & 0 & 0 & 0 & -2\alpha(k\pi)^4 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2\alpha(l\pi)^4 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

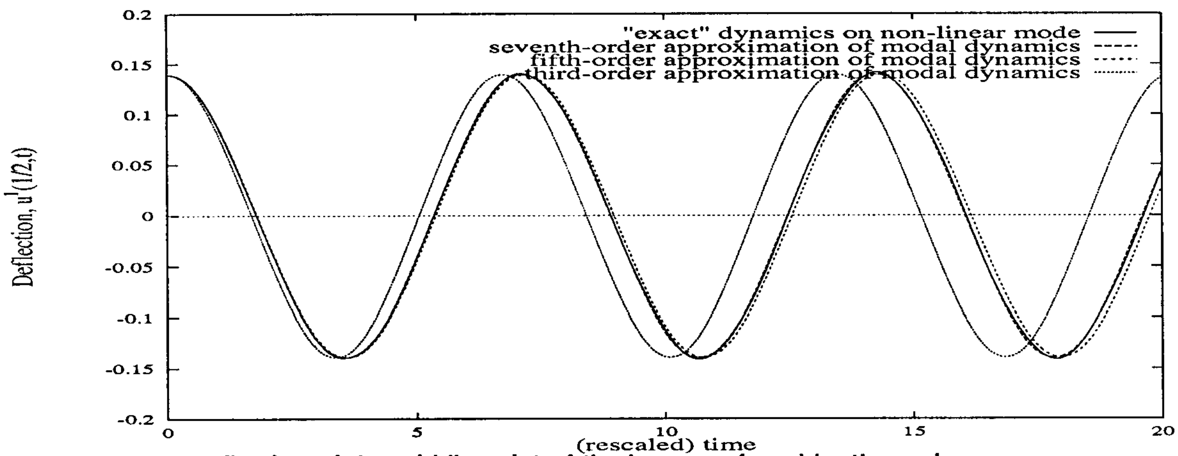

Fig. 2: Deflection of the middle-point of the beam as found by the various non-linear normal mode dynamics approximations (initiated on the fifth-order accurate first non-linear normal mode manifold). $\alpha = 1, \beta = 10^4$, $u_1(t{=}0) = 0.15, v_1(t{=}0) = 0$.

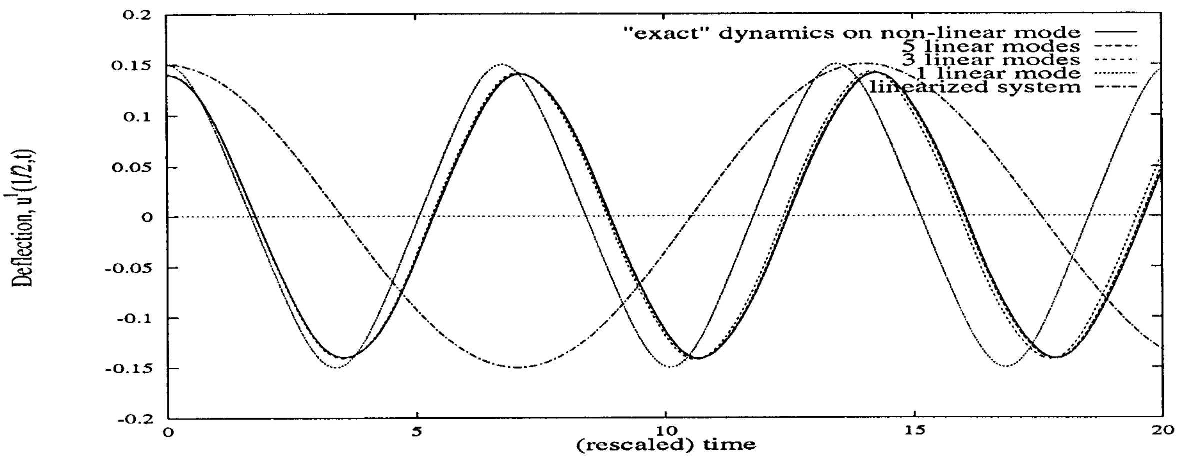

Fig. 3: Deflection of the middle-point of the beam as obtained by various linear modal analysis simulations initiated on the fifth-order approximation of the first non-linear normal mode manifold. $\alpha = 1, \beta = 10^4$, $u_1(t{=}0) = 0.15, v_1(t{=}0) = 0$.

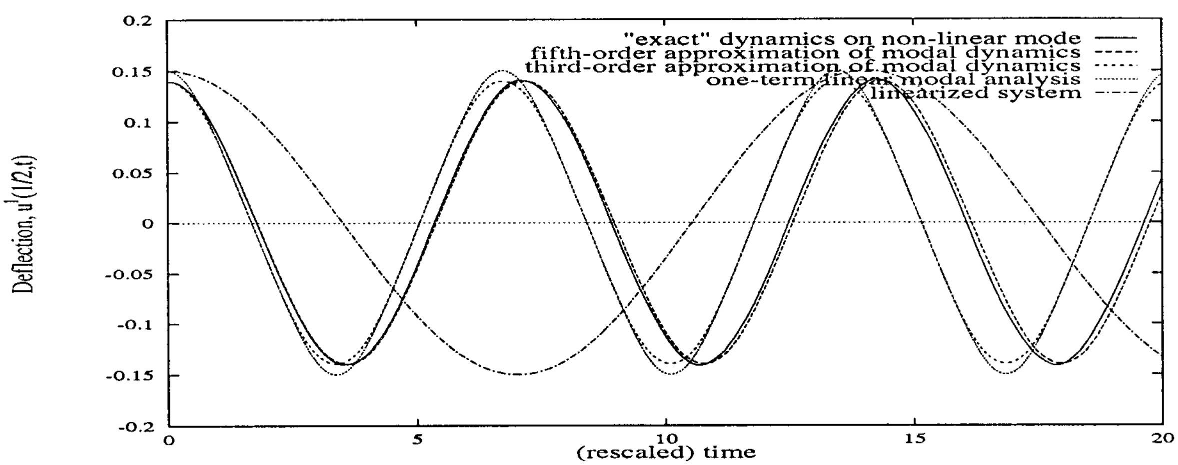

Fig. 4: Deflection of the middle-point of the beam as obtained by the various simulations initiated on the third-order approximation of the non-linear normal mode manifold. All curves correspond to the simulation of only one ODE $\alpha = 1, \beta = 10^4$, $u_1(t{=}0) = 0.15, v_1(t{=}0) = 0$.

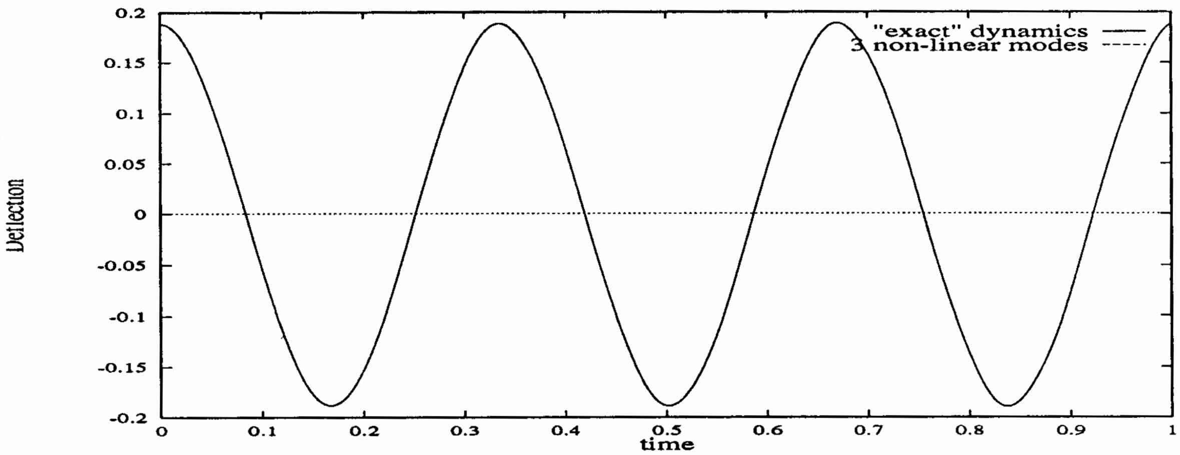

Fig. 5: Deflection of the middle-point of the beam as obtained by a third-order accurate three-mode invariant manifold. $\alpha = 1, \beta = 5000$, $u_1(t{=}0) = 0.2$, $u_2(t{=}0) = 0.1$, $u_3(t{=}0) = 0.01$, $v_1(t{=}0) = v_2(t{=}0) = v_3(t{=}0) = 0$.

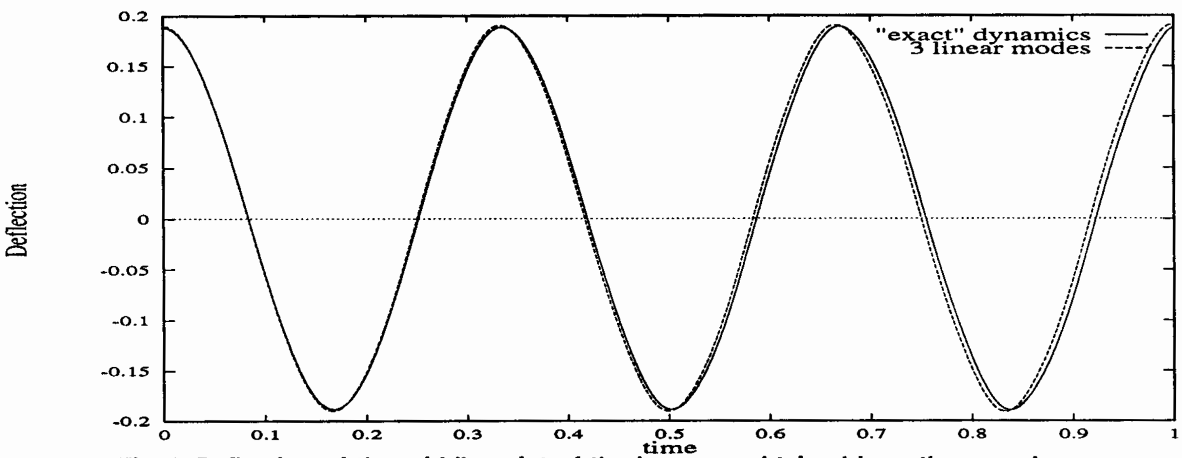

Fig. 6: Deflection of the middle-point of the beam as obtained by a three-mode linear modal analysis of the non-linear system. $\alpha = 1, \beta = 5000$, $u_1(t{=}0) = 0.2$, $u_2(t{=}0) = 0.1$, $u_3(t{=}0) = 0.01$, $v_1(t{=}0) = v_2(t{=}0) = v_3(t{=}0) = 0$.

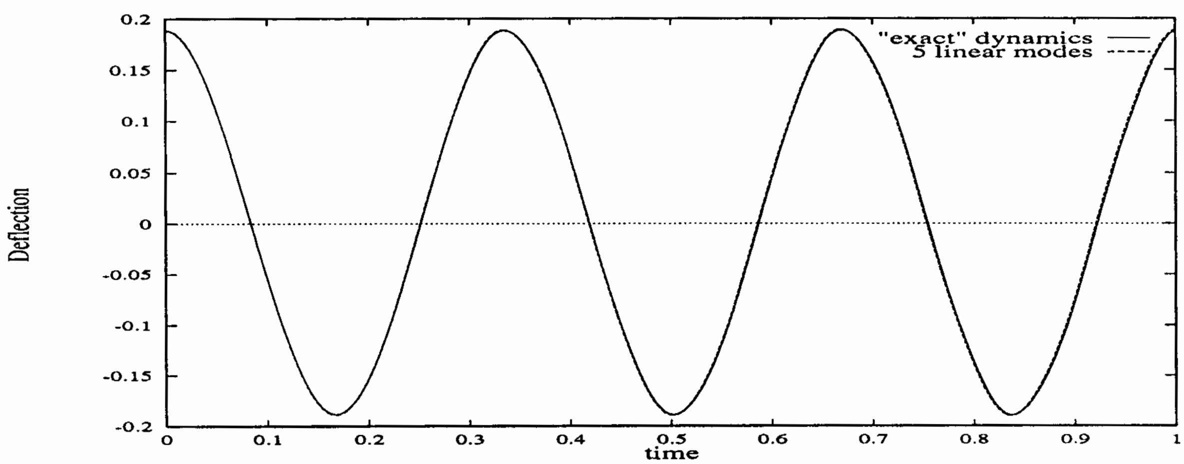

Fig. 7:Deflection of the middle-point of the beam as obtained by a five-mode linear modal analysis of the non-linear system. $\alpha = 1, \beta = 5000$, $u_1(t{=}0) = 0.2$, $u_2(t{=}0) = 0.1$, $u_3(t{=}0) = 0.01$, $v_1(t{=}0) = v_2(t{=}0) = v_3(t{=}0) = 0$.

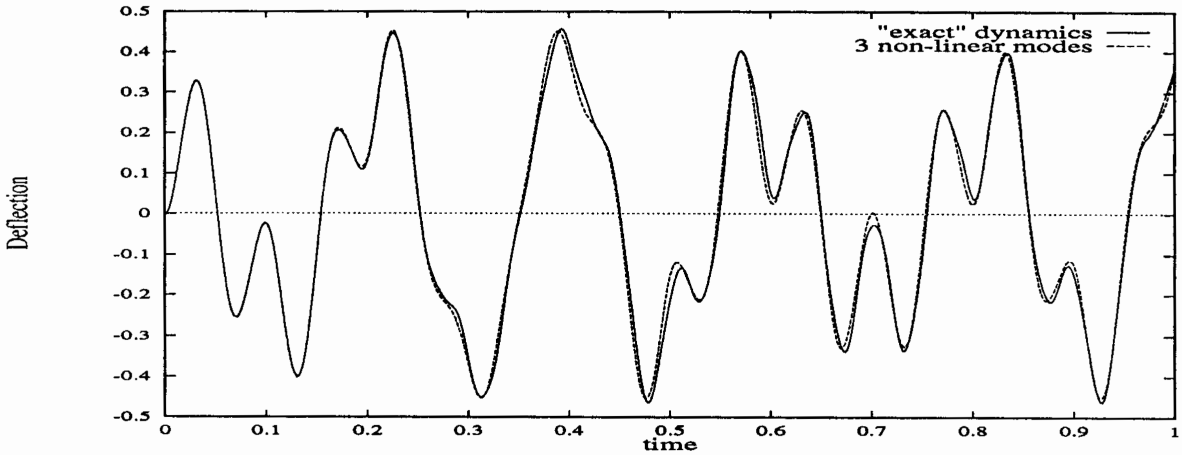

Fig. 8: Deflection of the middle-point of the beam as obtained by a third-order accurate three-mode invariant manifold. $\alpha = 1, \beta = 5000$, $u_1(t=0) = u_3(t=0) = 0.2$, $u_2(t=0) = 0.1$, $v_1(t=0) = v_2(t=0) = v_3(t=0) = 0$.

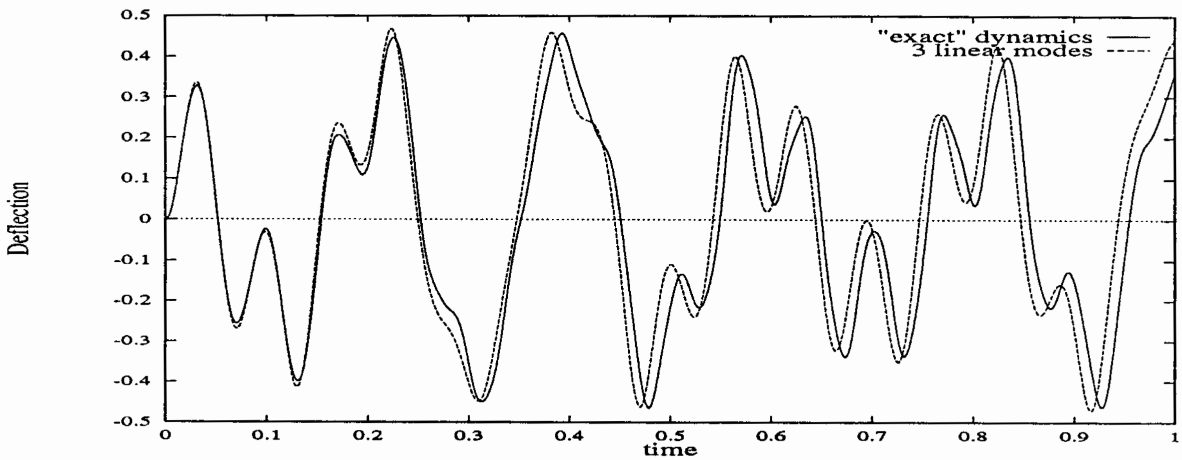

Fig. 9: Deflection of the middle-point of the beam as obtained by a three-mode linear modal analysis of the non-linear system. $\alpha = 1, \beta = 5000$, $u_1(t=0) = u_3(t=0) = 0.2$, $u_2(t=0) = 0.1$, $v_1(t=0) = v_2(t=0) = v_3(t=0) = 0$.

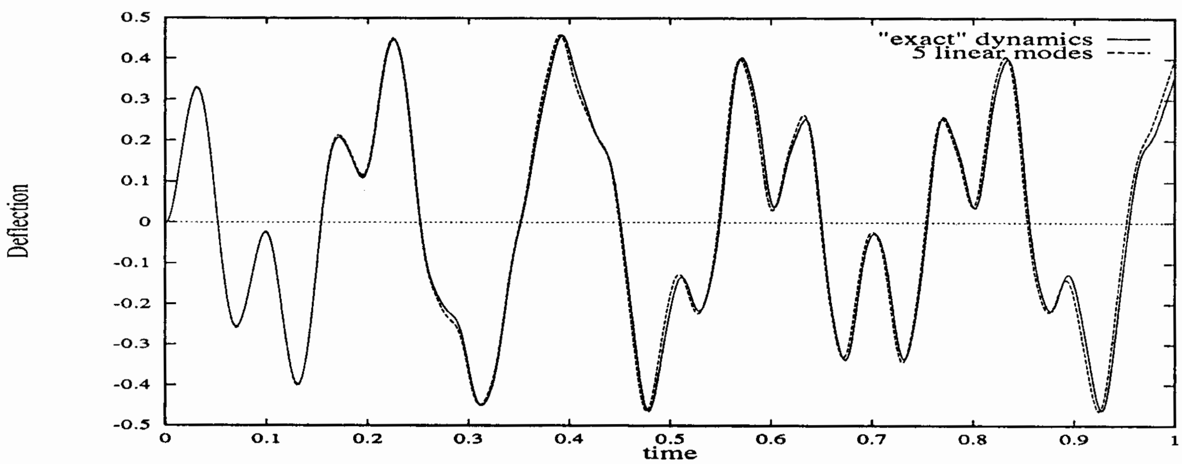

Fig. 10: Deflection of the middle-point of the beam as obtained by a five-mode linear modal analysis of the non-linear system. $\alpha = 1, \beta = 5000$, $u_1(t=0) = u_3(t=0) = 0.2$, $u_2(t=0) = 0.1$, $v_1(t=0) = v_2(t=0) = v_3(t=0) = 0$.