solved simultaneously and thus reduce the processing time for the optimal control and filtering problems of weakly coupled linear systems.

Appendix: Invertibility Proofs

According to Eqs. (21) and (25), it can be seen that

$$\Pi = \begin{bmatrix} I_{n_1} + O(\epsilon^2) & O(\epsilon) & O(\epsilon^2) & O(\epsilon) \\ O(\epsilon) & I_{n_2} & O(\epsilon) & 0 \\ O(\epsilon^2) & O(\epsilon) & I_{n_1} + O(\epsilon^2) & O(\epsilon) \\ O(\epsilon) & 0 & O(\epsilon) & I_{n_2} \end{bmatrix} \tag{A1}$$

$$\Omega = \begin{bmatrix} I_{n_1} & O(\epsilon) & 0 & O(\epsilon) \\ O(\epsilon) & I_{n_2} + O(\epsilon^2) & O(\epsilon) & O(\epsilon^2) \\ 0 & O(\epsilon) & I_{n_1} & O(\epsilon) \\ O(\epsilon) & O(\epsilon^2) & O(\epsilon) & I_{n_2} + O(\epsilon^2) \end{bmatrix} \tag{A2}$$

Therefore,

$$(\Pi_1 + \Pi_2 P) = I_{n_1+n_2} + O(\epsilon) \tag{A3}$$

$$\left(\Omega_1 + \Omega_2 \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix}\right) = I_{n_1+n_2} + O(\epsilon) \tag{A4}$$

There exists $\epsilon_1 > 0$ such that for every $\epsilon \le \epsilon_1$ the required matrices are invertible.

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Small Gain Versus Positive Real Modeling of Real Parameter Uncertainty

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Introduction

THE small gain theorem is one of the principal tools for modeling plant uncertainty. A standard representation[1] of an uncertain plant under feedback is shown in Fig. 1. The plant P and the compensator G are assumed to be known, while the plant uncertainty Δ has been "pulled out" into a fictitious feedback loop. In general, Δ may have a block-diagonal structure composed of scalar and/or matrix blocks. In this Note, we assume Δ is composed of a single matrix block. Figure 2 shows an equivalent representation with $\tilde{P}$ denoting the nominal closed-loop system. Note that r represents a fictitious input that is used as a means of representing the nominal closed loop in feedback with the uncertainty. From the small gain theorem[2] it follows that if $\tilde{P}$ and Δ are stable bounded-gain transfer functions such that $\|\Delta\|_\infty \|\tilde{P}\|_\infty < 1$ for all uncertainties Δ, then the closed-loop system is robustly stable.

Now suppose that the previous uncertainty model is used to represent constant real parameter uncertainty. The inherent conservatism of such a model can be demonstrated in two different ways. From a time-domain point of view, it is shown in Theorem 2.7 of Ref. 3 that the existence of an H_∞ norm bound is equivalent to the existence of a quadratic Lyapunov function that guarantees robust stability with respect to *time-varying* parameter variations. It is well known from the classical analysis of Hill's equation (e.g., the Mathieu equation) that time-varying parameter variations can destabilize a system even when the parameter variations are confined to a region in which constant variations are nondestabilizing.

From a frequency-domain point of view, the uncertainty block Δ satisfying an H_∞ norm bound can represent an arbitrary linear time-invariant transfer function possessing arbitrary frequency-dependent phase characteristics. A constant real parameter variation, however, at least in the scalar case, can be viewed as a transfer function that possesses a constant phase of 0 deg (if positive) or 180 deg (if negative). Thus, H_∞ modeling of real parameter uncertainty permits much larger

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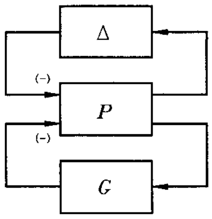

Fig. 1 Standard representation of an uncertain plant under feedback.

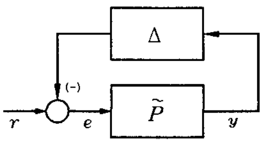

Fig. 2 Nominal closed-loop with feedback uncertainty.

phase variation than actually occurs in constant real parameter variations.

In this Note, we investigate an alternative model for constant real parameter uncertainty proposed in Ref. 4 and show by example that this model can be significantly less conservative than the H_∞ small gain model. The basis for this approach is the well-known fact that a loop consisting of a positive real transfer function and a strictly positive real transfer function is guaranteed to be stable.[2] By applying this principle to Fig. 2 in the case that Δ and $\tilde{P}$ are square, it follows that if $\Delta(j\omega) + \Delta^*(j\omega) \geq 0$, $\omega \in I\!R$, for all uncertainties Δ and if $\tilde{P}$ is a strictly positive real transfer function, then the system is robustly stable. From a frequency-domain point of view this approach to robust stability may be less conservative for constant real parameter uncertainty than a small-gain characterization since (at least in the scalar case) the phase of Δ is now confined to be between -90 and $+90$ deg. This observation thus reinforces the view that real parameter uncertainty can be viewed as a special case of phase information.[5]

The example we consider in this Note involves a lightly damped oscillator with uncertain stiffness. This example was chosen to highlight inherent drawbacks of small-gain principles applied to the analysis of robust controllers for lightly damped flexible structures. If, for this class of problems, uncertainty in the stiffness operator, which may be relatively large, is modeled as a small gain H_∞ block, then the arbitrary phase characteristics of the uncertainty block can contribute to a significant perturbation of the damping operator, which may be relatively small. The following example is intended to illustrate the ramifications of this point. Finally, no claim is made as to the generality of this technique for reducing conservatism in robust analysis for arbitrary systems.

Spring-Mass Example

Consider the spring-mass system shown in Fig. 3 with constant positive velocity feedback gain G and uncertain stiffness $K + \Delta$, where the uncertainty Δ is a real constant satisfying $|\Delta| \leq \delta$. The closed-loop system

$$M\ddot{x}(t) + G\dot{x}(t) + (K + \Delta)x(t) = 0, \qquad |\Delta| \leq \delta \tag{1}$$

can be represented by the nominal transfer function

$$\tilde{P} = \frac{1}{Ms^2 + Gs + K} \tag{2}$$

with an uncertainty Δ in the feedback configuration as shown in Fig. 2. Note that Eq. (1) can also be interpreted as a spring-mass damper system where G represents viscous damping. However, our main goal is to demonstrate the relationship between the feedback gain G and the parameter uncertainty model. From Fig. 2 it follows that the perturbed transfer function is given by

$$\frac{y}{r} = (1 + \tilde{P}\Delta)^{-1}\tilde{P} = \frac{1}{Ms^2 + Gs + (K + \Delta)} \tag{3}$$

Robust stability is guaranteed by the small gain theorem if

$$\|\Delta\|_\infty < \frac{1}{\|\tilde{P}\|_\infty} \tag{4}$$

where now Δ may denote a stable transfer function. A simple calculation yields

$$\|\tilde{P}\|_\infty = \sup_r \frac{1}{K\sqrt{(1-r^2)^2 + (2\zeta r)^2}} \tag{5}$$

where $r \triangleq \omega/\omega_n$ denotes the frequency ratio, $\omega_n \triangleq \sqrt{K/M}$ denotes the nominal undamped natural frequency, and $\zeta \triangleq G/2\sqrt{MK}$ denotes the nominal closed-loop damping ratio. Noting that the maximum in Eq. (5) occurs at $r = \sqrt{1 - 2\zeta^2}$, it follows that

$$\|\tilde{P}\|_\infty = \frac{1}{2\zeta K\sqrt{1 - \zeta^2}} \tag{6}$$

It now follows from Eq. (4) that stability is guaranteed if

$$\|\Delta\|_\infty < 2\zeta K\sqrt{1 - \zeta^2} \tag{7}$$

Since by assumption Δ satisfies $\|\Delta\|_\infty < \delta$, the bound δ must satisfy

$$\frac{\delta}{K} < 2\zeta\sqrt{1 - \zeta^2} \tag{8}$$

in order to enforce Eq. (7).

Using Eq. (8), we can assess for a given nominal closed-loop damping ratio ζ the maximal allowable uncertainty level δ/K for which H_∞ analysis guarantees robust stability of the uncertain closed-loop system with $\|\Delta\|_\infty \leq \delta$. This result is shown in Fig. 4. Note that under lightly damped conditions, for example, $\zeta = 5\%$, H_∞ analysis allows $\delta = K/10$ stiffness uncertainty for robust stability. In fact, the system is robustly stable for $\delta = K$ (excluding $\Delta = -K$).

Alternatively, using Eq. (8), we can obtain for a given uncertainty level δ/K the minimal closed-loop damping ratio ζ

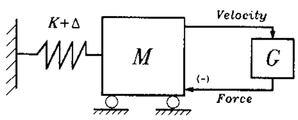

Fig. 3 Spring-mass system with constant velocity feedback.

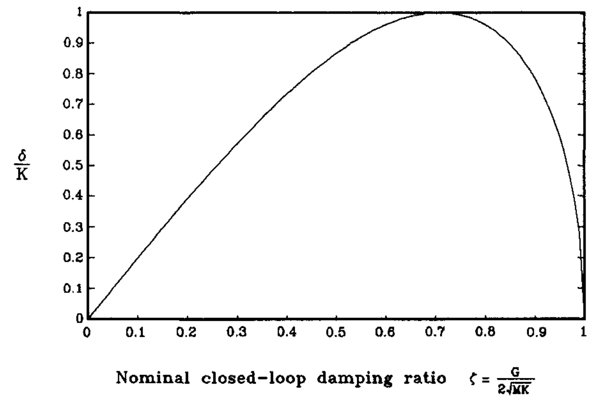

Fig. 4 Stiffness uncertainty vs nominal closed-loop damping.

(or gain G) required by H_∞ analysis for robust stability of the uncertain closed-loop system with $\|\Delta\|_\infty \le \delta$. This follows from Eq. (8) and is given by

$$\zeta = \frac{G}{2\sqrt{MK}} = \frac{1}{\sqrt{2}}\left[1 - \sqrt{1 - (\delta/K)^2}\right]^{\frac{1}{2}} \tag{9}$$

It can be seen from Fig. 5 that as the uncertainty level δ/K increases, H_∞ analysis requires a substantial increase in the nominal closed-loop damping ratio (and thus gain G) to guarantee robust stability.

To illustrate the ramifications of this result, we can determine the minimal damping ratio ζ of the nominal closed-loop system *actually* required to guarantee a worst-case closed-loop damping ratio δ_0 with Δ assumed to be a constant real perturbation satisfying $|\Delta| \le \delta$. To do this, rewrite Eq. (1) as

$$\ddot{x}(t) + 2\zeta_a\omega_{na}\dot{x}(t) + \omega_{na}^2 x(t) = 0 \tag{10}$$

where the actual closed-loop damping ratio ζ_a and natural frequency ω_{na} satisfy

$$2\zeta_a\omega_{na} \triangleq \frac{G}{M}, \qquad \omega_{na}^2 \triangleq \frac{K+\Delta}{M} \tag{11}$$

which imply that the actual closed-loop system damping ratio ζ_a is given by

$$\zeta_a = \frac{G}{2\sqrt{M(K+\Delta)}} \tag{12}$$

Requiring $\zeta_a \ge \zeta_0$ for all $|\Delta| \le \delta$, it follows from Eq. (12) with $\Delta = \delta$ (worst-case damping) that

$$\zeta = \frac{G}{2\sqrt{MK}} \ge \zeta_0\sqrt{1 + (\delta/K)} \tag{13}$$

The lower bound in Eq. (13) is plotted in Fig. 5 for two typical cases of ζ_0 corresponding to 5 and 10%. Comparing these curves with the small gain result clearly shows the conservatism of H_∞ theory for constant real parameter uncertainty.

Positive Real Parameter Uncertainty Model

In this section we consider an alternative uncertainty model for the uncertainty Δ. For this model we shift the nominal value of K so that K is positive but arbitrarily close to zero and let the real constant perturbation of K be denoted by Δ where now $0 \le \Delta \le \infty$. Again Fig. 2 can be used to represent this situation. Now, however, we replace Δ by Δ/s and $\tilde{P}$ by $s\tilde{P}$ in Fig. 2. Next note that the effective uncertainty block Δ/s is positive real since $\Delta/j\omega + \Delta/(-j\omega) = 0$, $\omega \in I\!R$, and, furthermore, the effective plant $s\tilde{P}$ is strictly positive real since

$$j\omega\tilde{P}(j\omega) + [j\omega\tilde{P}(j\omega)]^*$$
$$= \frac{4\zeta\omega_n\omega}{(\omega_n^2 - \omega^2) + (2\zeta\omega_n\omega)^2} > 0, \qquad \omega \in I\!R$$

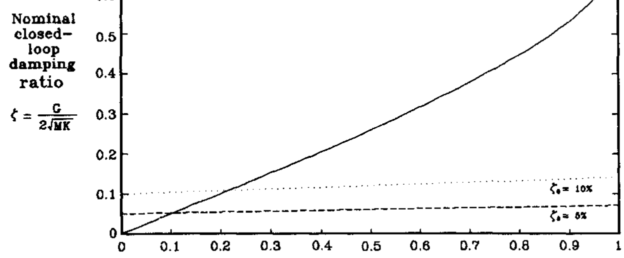

Fig. 5 Nominal closed-loop damping vs stiffness uncertainty.

Hence by the positivity theorem[2] the system is robustly stable for all $\Delta \in [0, \infty)$. Consequently, the closed-loop system is guaranteed to be unconditionally stable for all constant positive velocity feedback gains G. Thus the positive real uncertainty model is, for this example, nonconservative with respect to constant real parameter uncertainty.

Conclusion

We have shown by means of a lightly damped oscillator example with uncertain stiffness that small gain modeling of constant real parameter uncertainty can be extremely conservative. An alternative uncertainty modeling approach involving positive real transfer functions and the positivity theorem was shown to be significantly less conservative.

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Model Order Effects on the Transmission Zeros of Flexible Space Structures

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Introduction

THE dynamics of flexible space structures (FSS) are generally quite poorly known before launch. In particular, most preflight dynamic analysis and controller design for these vehicles is based on approximate finite-dimensional models obtained by finite element methods. However, flexible structures are distributed parameter systems, and so are essentially infinite dimensional. Two implications of this are that the choice of dimension for an approximate finite element model of an FSS is somewhat arbitrary, and that only the lower-frequency approximate modes will tend to be accurate estimates for the corresponding true values.

Recently, it has been shown[1] that the transmission zeros[2] of any finite-dimensional model for a flexible structure with

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