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In Search Of An Optimal Local Navier-Stokes Preconditioner

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Local preconditioning for the Navier-Stokes equations may be called optimal if it equalizes all propagation and dissipation time-scales, for all combinations of Mach number and Reynolds number. Previously designed preconditioners are ineffective for certain combinations of low Reynolds number and low Mach number; in addition some of these create a growing mode, making the PDE-system unstable. (Users may regain stability through an implicit discretization.)

In this paper we first review the forms and properties of all previously published N-S preconditioners on the basis of the 1-D N-S equations, then derive an optimal preconditioning matrix for these equations. We find again that it creates an unstable mode; a sensitivity analysis shows that optimal preconditioning and stability are mutually exclusive. Two possible remedies are suggested and briefly investigated: (1) to redefine the complex condition number in a way more appropriate for explicit discretizations; (2) to reformulate the N-S equations as a larger first-order system of hyperbolic-relaxation equations and base the preconditioner on this system. The latter approach appears most promising.

Introduction

Local preconditioning of evolutionary PDE's aims at equalizing, or at least reducing the spread among, local time-scales of physical processes [1]. The resulting equations, while no longer time-accurate, are better suited for marching toward a steady solution.

In the past two decades, three sets of PDE's fundamental to CFD have been considered for preconditioning: the Euler equations [1-3], the Navier-Stokes equations [4, 5], and, more recently, the equations of ideal magneto-hydrodynamics [6]. Regarding the Euler equations, our knowledge of local preconditioning is close to complete [7-9], and includes a technique for finding the most efficient preconditioner that can be applied to any hyperbolic system in 2 space dimensions [6, 10]. Application of this design technique to the MHD equations has recently begun [11] and is now in the phase of exploratory 2-D numerical experiments. Because of the complexity of the MHD wave system, no 3-D preconditioner has yet been obtained for these equations. For the Navier-Stokes equations only partially satisfactory preconditioners have been published [4, 12]. All N-S preconditioners lose their efficiency for certain combinations of low Reynolds numbers and low Mach numbers; in addition some of them create a growing mode [4, 5, 12], making the PDE-system unstable. Users may regain computational stability by the grace of an implicit discretization [4], but this negates an important goal of local preconditioning achieved already for the Euler equations: convergence to a steady solution in $O(N)$ operations with a fully explicit method [8, 13].

This paper describes our search for an optimal local N-S preconditioner, that is, one that minimizes the local condition number for all combinations of Mach and Reynolds numbers. All analysis is based on the 1-D N-S equations, for which the fundamental difficulties already are evident. In the process we will review 1-D versions of all published N-S preconditioners and their properties, using a uniform notation and choice of state variables. The results are four-fold:

1. It is possible to put together a local N-S preconditioner that achieves a condition number equal to or close to one in the entire (M,Re)-plane (except for high-Re sonic flow, which is a genuine physical singularity);
2. For low Re there are *three* asymptotic regimes of preconditioning, rather than two, as previously assumed;

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3. This preconditioner again creates a growing mode;
4. Optimal preconditioning and stability of the preconditioned PDE-system appear to be mutually exclusive;

The persistence of the growing mode in the perfect preconditioner is disappointing and disquieting. In any case it means the search is not over. We end the paper discussing two possible remedies:

1. to redefine the complex condition number in a way more appropriate for explicit discretizations;
2. to reformulate the N-S equations as a larger first-order system of hyperbolic-relaxation equations and base the preconditioner on this system [14].

The latter approach appears most promising.

Basics of Local Preconditioning

If we write the Euler or Navier-Stokes equations in the form

$$\frac{\partial \mathbf{U}}{\partial t} = \mathbf{Res}\left(\mathbf{U}\right), \tag{1}$$

where **Res**, the residual, is a spatial differential operator, the preconditioned equations can be written as

$$\frac{\partial \mathbf{U}}{\partial t} = \mathbf{P}\left(\mathbf{U}\right)\mathbf{Res}\left(\mathbf{U}\right), \tag{2}$$

or

$$\mathbf{P}\left(\mathbf{U}\right)^{-1}\frac{\partial \mathbf{U}}{\partial t} = \mathbf{Res}\left(\mathbf{U}\right), \tag{3}$$

where **P**(**U**) is the locally evaluated preconditioning matrix. The goal of including **P** is to equalize the time-scales of the different physical processes described by the system of PDE's, without affecting the steady solutions admitted by the system. In particular, **P** should not reverse the propagation direction of the transient waves, as this would bring along a change in the boundary conditions and therefore change the problem. For this reason **P** is constrained to be positive definite [1, 9]; otherwise the choice is not *a priori* limited.

With all physical time-scales very nearly equalized, the numerical process of marching toward a steady solution becomes significantly more efficient: there are no slow processes dragging behind [1]. In mathematical language: local stiffness has been removed from the equations. It should in principle be possible to convergence to a steady solution in $O(N)$ operations with an explicit preconditioned marching scheme embedded in a multigrid-relaxation strategy [15]. This has indeed been accomplished for discrete Euler solutions [8, 13, 16]; for Navier-Stokes solutions the goal is not yet in sight, as will become evident below.

In designing efficient preconditioners, our foremost tool is a dispersion analysis [4]. When applying such analysis to the Euler equations, it reveals different modes of undamped propagation, some acoustic, some advective. Equalizing the travel-times for these waves, e. g., the time to cross a computational cell, is equivalent to equalizing the propagation speeds, known as the characteristic speeds [1]. This can be very nearly achieved for Mach numbers not close to unity; for $M = 1$ the steady flow equations change type, from mixed elliptic-hyperbolic to purely hyperbolic, and preconditioning can weaken but not avoid this singularity [1].

When including the viscous and conductive terms, the modes revealed by a dispersion analysis of the unpreconditioned PDE's can be anything from damped non-propagating to undamped propagating [3, 4, 12]. The demands on **P** are now raised significantly: the whole pool of propagation and damping time-scales must be equalized [5, 12]. Moreover, in addition to respecting propagation directions, **P** should also respect the direction of dissipation, i. e., not create any growing mode. There is no *ad hoc* criterion to guarantee this; requiring **P** to be positive definite does not suffice.

It turns out that the most powerful N-S preconditioners currently in use create one growing mode [4, 5]. The growth may destabilize flow simulations with an explicit scheme, but can be suppressed by the strong numerical damping present in an implicit scheme for large time-steps. This has been the prevalent strategy.

This class of N-S preconditioners, due to Merkle and collaborators [4], starts from an efficient Euler preconditioning matrix, then modifies one or more elements in order to introduce *Re*-dependence. The crucial change is in the coefficient of $\partial p/\partial t$ in the equation updating the pressure. For an Euler preconditioner to be effective, this coefficient must equal $1/M^2$ at low Mach numbers [2]; with this choice the preconditioner will also be effective for viscous/conductive flow down to $Re \approx 1$ [5]. For low Reynolds numbers two asymptotic regimes were previously identified [4], requiring different expressions for the coefficient. In this paper we show that there actually are *three* different asymptotic regimes, which explains why earlier preconditioners of this type could not achieve uniformly low condition numbers. In addition they suffer from the growing mode.

A different strategy for N-S preconditioning, due to Godfrey [17-19] and further developed and analyzed by D. Lee, is to combine an effective Euler preconditioner with Jacobi relaxation for just the viscous/conductive terms. This does *not* create a growing mode, and its preconditioning effect is uniformly excellent in three quadrants of the $\left({}^{10}\log M, {}^{10}\log Re\right)$-plane [5, 12]. Only in the region $\left(M<1,\ Re<1\right)$ the condition number remains unbounded.

D. Lee [5] also mixed the above strategies by introducing *Re*-dependence in the entries of the van Leer-Lee-Roe Euler preconditioner while also applying Jacobi relaxation to the viscous/conductive terms. He indicates that this uniformly brings down the condition number to $O(1)$ for all (M, Re)-combinations without creating a growing mode. Unfortunately, we found a region not resolved by Lee, where the condition number increases beyond any bound for sufficiently low M.

Below we shall use the mixed approach to develop the first N-S preconditioner that keeps the condition number bounded for $Re \to 0$, regardless of the value of M (An Optimal N-S Preconditioner). Before doing this we must discuss our chief tool, the dispersion analysis (Dispersion Analysis); to offer a perspective we also will review the properties of the most representative Euler and N-S preconditioners (Review of Euler Preconditioners and Review of N-S Preconditioners). For a review covering *all* previously published local preconditioners we refer to the extensive report [20] prepared by the first author.

Dispersion Analysis

A dispersion analysis tells us how Fourier modes propagate and change amplitude when subjected to the linearized evolutionary PDE. We will start from the 1-D N-S equations, which in conservation form read

$$\mathbf{U}_t + \mathbf{F}_x = \left(\mathbf{C}\mathbf{U}_x\right)_x, \tag{4}$$

where $\mathbf{U}$ is the vector of conserved state quantities, $\mathbf{F}$ is the corresponding flux vector, and $\mathbf{C}$ is the corresponding dissipation-coefficient matrix. While this form is preferred for numerical computations, there are forms better suited for analysis tasks. In particular, there are two preferable forms of the 3-D equations [3, 5, 21] in which all coefficient matrices are symmetric. These are based on different choices of the state variables, known as the Euler-symmetrizing variables and the N-S-symmetrizing variables. They are defined differentially:

$$d\mathbf{U}^{Eu} = \left(\frac{dp}{\rho a}, du, dv, dw, ds\right)^T, \tag{5}$$

$$d\mathbf{U}^{NS} = \left(\frac{a\,d\rho}{\rho\sqrt{\gamma}}, du, dv, dw, \frac{2da}{\sqrt{\gamma(\gamma-1)}}\right)^T; \tag{6}$$

the notation is standard, with s and a representing entropy and sound speed, respectively. In the Euler set, ds is usually replaced by $dp - a^2 d\rho$. Note that the N-S symmetrization utilizes the perfect-gas law as equation of state.

When designing a N-S preconditioner it is advantageous to use the N-S symmetrizing variables, as this diagonalizes the main dissipation-coefficient matrices. The Euler-symmetrizing variables, on the other hand, greatly simplify the inviscid flux Jacobians, although they don't symmetrize the dissipation-coefficient matrices. Since Euler preconditioners are routinely applied to the N-S equations in problems where the cell Reynolds number does not go below 1, we have adopted the Euler-symmetrizing variables as the basis for analyzing *all* preconditioners in this paper; this facilitates comparisons.

Our 1-D dispersion analysis will henceforth be based on the constant-coefficient system

$$\mathbf{U}_t^{Eu} + \mathbf{A}^{Eu}\mathbf{U}_x^{Eu} = \mathbf{C}^{Eu}\mathbf{U}_{xx}^{Eu}, \tag{7}$$

with

$$d\mathbf{U}^{Eu} = \begin{pmatrix} \frac{dp}{\rho a} \\ du \\ dp - a^2 d\rho \end{pmatrix},\ \mathbf{A}^{Eu} = \begin{pmatrix} u & a & 0 \\ a & u & 0 \\ 0 & 0 & u \end{pmatrix},$$
$$\mathbf{C}^{Eu} = \begin{pmatrix} \frac{(\gamma-1)\nu}{Pr} & 0 & \frac{\nu}{\rho a Pr} \\ 0 & \frac{4\nu}{3} & 0 \\ \frac{(\gamma-1)\rho a\nu}{Pr} & 0 & \frac{\nu}{Pr} \end{pmatrix} \tag{8}$$

where ν is the kinematic viscosity coefficient. We now insert the trial solution

$$\mathbf{U}^{Eu} = \mathbf{U}_0^{Eu} e^{i(\omega t - kx)}, \tag{9}$$

where k is the spatial frequency and ω the complex temporal frequency. The resulting dispersion equation is

$$\det\left(\mathbf{A}^{Eu} + ik\mathbf{C}^{Eu} - \frac{\omega}{k}\mathbf{I}\right) = 0, \tag{10}$$

in which the complex wave speed ω/k appears. The real part of this quantity is the real wave speed; a *positive* imaginary part indicates the wave is damped.

Equation (10) is cubic in ω/k and has no elegant solution, unless we introduce a slight simplification.

It is easily verified that one factor could be split off if only the matrix elements $C_{22}^{Eu} = 4\nu/3$ and $C_{33}^{Eu} = \nu/Pr$ were equal. So, following Venkateswaran and Merkle [4], we change 4/3 into 1 and choose $Pr = 1$, assuming that this will not fundamentally change the character of the solution. (This is numerically verified below.)

We then find:

$$\left(\frac{\omega}{k}\right)_{1,3} = u\left[1 + i\frac{\gamma}{2Re}\left(1 \pm \sqrt{1 - \frac{4Re^2}{\gamma^2 M^2}}\right)\right], \qquad (11)$$

$$\left(\frac{\omega}{k}\right)_2 = u\left(1 + \frac{i}{Re}\right), \qquad (12)$$

The Reynolds number in these roots is based on the flow velocity u and the length $1/k$:

$$Re = \frac{u}{k\nu}. \qquad (13)$$

A Reynolds number based on the acoustic velocity a may be called the acoustic Reynolds number:

$$Re_{ac} = \frac{a}{k\nu} = \frac{Re}{M}; \qquad (14)$$

this combination appears in roots 1 and 3.

In the high-Re or Euler limit, Eqs. (11) and (12) yield the usual real characteristic speeds $u - a$, u and $u + a$. For lower Reynolds numbers, as observed by Venkateswaran and Merkle [4], a distinction needs to be made between the acoustics-dominated case, where Re_{ac} is still high, i. e. $M \ll Re$, and the viscosity-dominated case, where even Re_{ac} is low, i. e. $Re \ll M$. When acoustics dominate, roots 1 and 3 still tend to the inviscid characteristic speeds $u \pm a$, so the only damped mode is the one moving with the flow speed. When viscosity dominates Eqn. (11) reduces to

$$\left(\frac{\omega}{k}\right)_{1,3} \approx u\left\{1 + i\frac{\gamma}{2Re}\left[1 \pm \left(1 - \frac{2Re^2}{\gamma^2 M^2}\right)\right]\right\}, \qquad (15)$$

or

$$\left(\frac{\omega}{k}\right)_1 \approx u\left(1 + i\frac{\gamma}{Re}\right), \qquad (16)$$

$$\left(\frac{\omega}{k}\right)_3 \approx u\left(1 + i\frac{Re}{\gamma M^2}\right), \qquad (17)$$

All modes propagate with the flow speed, but there still are two physically distinct cases: $Re \gg M^2$ and $Re \ll M^2$. In the first case all modes are heavily damped, in the second case mode 3 is undamped.

It thus appears that for low Reynolds numbers, $Re < 1$, we need to distinguish three different flow regimes,

$$Re > M, \qquad (18)$$

$$M^2 \le Re \le M, \qquad (19)$$

$$Re < M^2, \qquad (20)$$

with each regime requiring its own distinct preconditioner. This will be confirmed below. What is puzzling is that Venkateswaran and Merkle in their N-S preconditioners [4] switch formulas only according to the value of M^2/Re, although they recognize the parameter M/Re that distinguishes between the acoustics- and viscosity-dominated regimes. Surely, $M^2/Re \gg 1$ is more restrictive than $M/Re \gg 1$; our guess is they did not realize that $M^2/Re \ll 1$ is *less* restrictive than $M/Re \ll 1$, leaving room for combinations of M and Re that should have been given special attention.

For the preconditioned equations

$$\mathbf{P}^{-1}\mathbf{U}_t^{Eu} + \mathbf{A}^{Eu}\mathbf{U}_x^{Eu} = \mathbf{C}^{Eu}\mathbf{U}_{xx}^{Eu} \qquad (21)$$

the dispersion relation becomes

$$\det\left(\mathbf{PA}^{Eu} + ik\mathbf{PC}^{Eu} - \frac{\omega}{k}\mathbf{I}\right) = 0. \qquad (22)$$

The task of finding an effective preconditioner means to solve the inverse problem of finding a **P** such that the roots of the dispersion equation are well-conditioned, i. e.,

$$K = \frac{\max_j \left|\frac{\omega}{k}\right|_j}{\min_j \left|\frac{\omega}{k}\right|_j} = O(1). \qquad (23)$$

Although from a linear-algebra viewpoint this requirement is standard, it needs further discussion from the viewpoint of physics. The above condition makes sense in the Euler limit, when all modes propagate undamped, and in the viscosity-dominated case $M^2 \le Re \le M$, when all modes are heavily damped while hardly propagating, but what about the mixed cases? Should we weight the real and imaginary parts of ω/k equally, as in $|\omega/k|$, or should these get different weights? Should the stability properties of explicit discretizations, which vastly differ between advection and diffusion schemes, be considered? We shall come back to these questions in Fighting the Growing Mode.

In the next two sections we shall visit a selection of Euler and Navier-Stokes preconditioners representing all classes of preconditioners found in the literature. The matrices are presented in the form appropriate for preconditioning the 3-D equations in Euler-symmetrizing flow variables; for some matrices it is assumed that the flow is aligned with the x-axis.

Results are displayed of a 1-D dispersion analysis carried out on a simple difference approximation (upwind inviscid flux, centrally differenced diffusion term); contour and carpet plots of the condition number are also provided.

Review of Euler Preconditioners

The earliest published Euler preconditioners were designed for low-speed flow only, and were inspired by Chorin's [22] method of hyperbolizing the incompressible Euler equations, *viz*. the method of artificial compressibility. Here the time derivative of density in the continuity equation, which under the assumption of incompressibility equals zero, is replaced by $\beta^{-2} p_t$, where β has the dimension of velocity. Fastest convergence to steady solutions is obtained when β is proportional to the flow speed.

This motivated Turkel [2] in 1984 to include the factor β^{-2} in the pressure-evolution equation for low-speed *compressible* flow, with $\beta = M = \sqrt{u^2+v^2+w^2}\big/a$ as the nominal value. When the Euler equations are expressed in terms of the Euler-symmetrizing variables, the entropy equation is completely decoupled and needs no preconditioning; this affords the cleanest form of the Chorin-type preconditioner for low-speed compressible flow:

$$\mathbf{P}_{CH67}^{-1} = \begin{pmatrix} \frac{1}{M^2} & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad M < 1 \qquad (24)$$

or

$$\mathbf{P}_{CH67} = \begin{pmatrix} M^2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad M < 1 \qquad (25)$$

Note that it connects continuously to **P** = **I** for $M \geq 1$. Where the local Mach number gets to close to zero, especially near a stagnation point, the value used in element P_{11} must be limited from below by a threshold value, e. g., $M_{\min} = \varepsilon$, ε small. Euler preconditioners often differ in the way this threshold is evaluated [16, 23].

Another difference among preconditioners of this class arises from applying the above diagonal **P** to a different set of flow variables, such as the set (p,u,v,w,T), which we shall call the "primitive-temperature" variables. When transformed back to the Euler-symmetrizing variables, the preconditioner is no longer diagonal; it has a nonzero element P_{15} representing the coupling between the pressure and temperature equations. This form consistently appears in the work of Merkle and collaborators [24] but is not easily recognized, as the matrices they present include as a factor the Jacobian of the transformation from conservative to primitive-temperature variables (cf. Review of N-S Preconditioners).

Regardless of the choice of the second thermodynamic variable, the above diagonal or "poor people's" preconditioner does an excellent job of reducing the spread among the characteristic speeds for small *M*. An acoustic wave going with/against the flow moves at a speed $\frac{1}{2}\left(\sqrt{5}\pm 1\right)u$ yielding a condition number of 2.62, down from $1/M$ before preconditioning. See Figure 1.

The Chorin preconditioner, in various disguises, is used till this day, in particular by Merkle, Venkateswaran, Weiss and collaborators. For example, the 1999 preconditioner of Weiss, Maruszewski and Smith [25], which has a switch parameter for selecting incompressible or compressible flow, is found, for compressible flow of a perfect gas, to be identical to the above diagonal form, complete with threshold value $M_{\min} = \varepsilon$ after transformation to the Euler-symmetrizing values.

Turkel [2, 26] improved on the diagonal preconditioner by introducing extra nonzero elements:

$$\mathbf{P}_{TL84} = \begin{pmatrix} M^2 & 0 & 0 & 0 & 0 \\ -\frac{u}{a} & 1 & 0 & 0 & 0 \\ -\frac{v}{a} & 0 & 1 & 0 & 0 \\ -\frac{w}{a} & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad M < 1 \qquad (26)$$

the condition number now goes down to 1 for $M \to 0$, indicating perfect preconditioning when approaching incompressible flow. See Figure 2.

The final improvement is due to Van Leer, W.-T. Lee and Roe [1], who derived a preconditioner that achieves the lowest possible condition number for all values of *M*. The matrix is symmetric by design; for the sake of clarity it usually is given in a form valid for flow aligned with the *x*-axis:

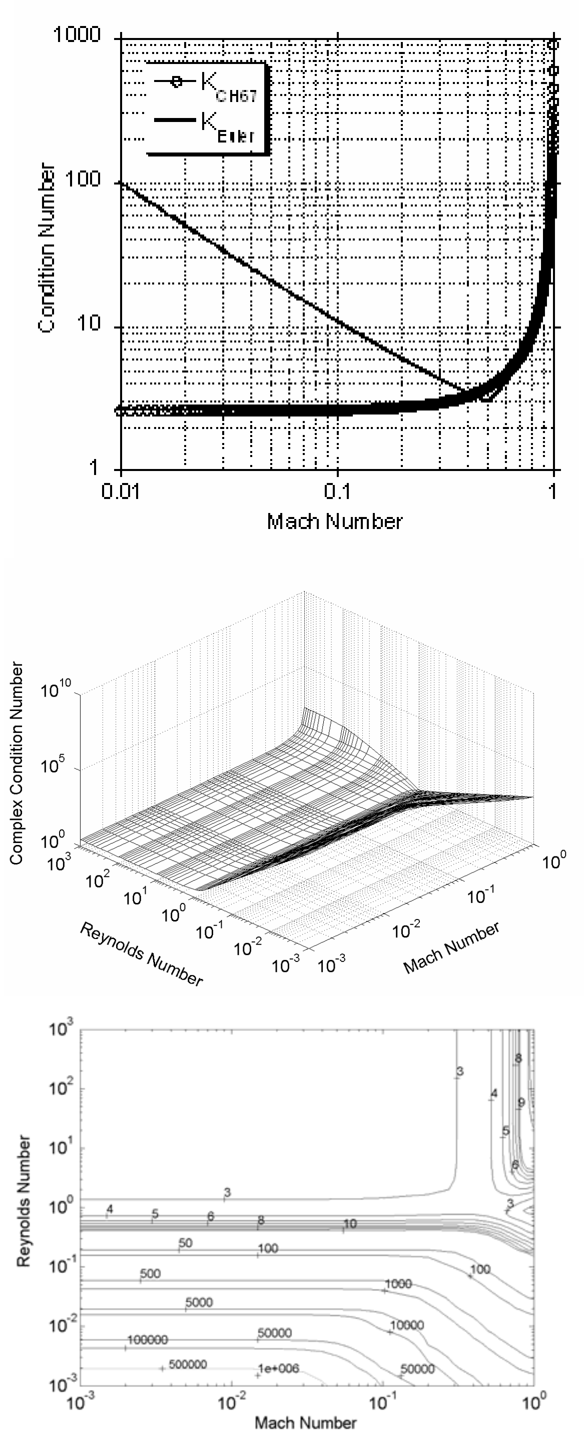

Figure 1: Plots of condition number yielded by 1967 Chorin preconditioner applied to compressible flow. Top: for 1-D Euler equations; bottom: for 1-D N-S equations (carpet and contour plots)

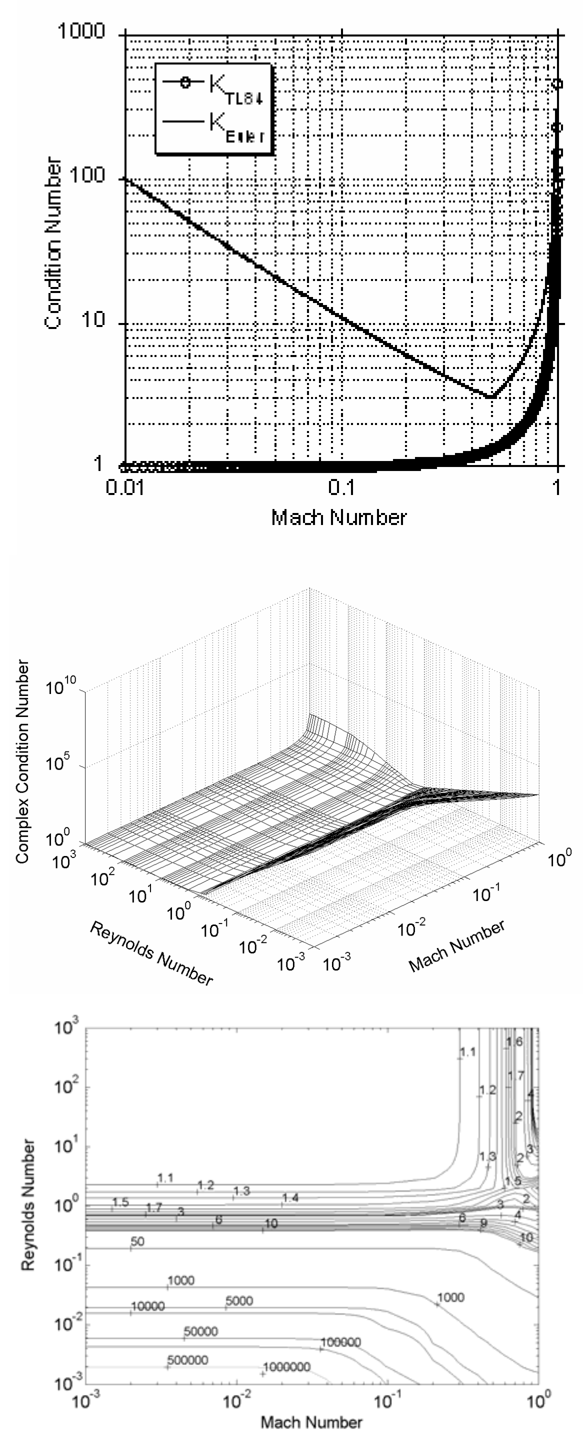

Figure 2: Plots of condition number yielded by Turkel's 1984 preconditioner. Top: for 1-D Euler equations; bottom: for 1-D N-S equations (carpet and contour plots).

$$\mathbf{P}_{VLR91} = \begin{pmatrix} \frac{\tau}{\beta^2}M^2 & -\frac{\tau}{\beta^2}M & 0 & 0 & 0 \\ -\frac{\tau}{\beta^2}M & \frac{\tau}{\beta^2}+\sigma & 0 & 0 & 0 \\ 0 & 0 & \tau & 0 & 0 \\ 0 & 0 & 0 & \tau & 0 \\ 0 & 0 & 0 & 0 & \sigma \end{pmatrix}, \quad (27)$$

with

$$\beta = \sqrt{\left|1-M^2\right|}, \quad (28)$$

$$\tau = \min\left(\beta, \beta/M\right) \text{ in 2D and 3D, 1 in 1D,} \quad (29)$$

$$\sigma = \begin{cases} 1 \quad \text{for PDE 1-D discretization,} \\ \tau\left(1+\dfrac{AR}{\beta}\right) \quad \text{for 2-D discretization,} \\ ? \text{ for 3-D discretization.} \end{cases} \quad (30)$$

The dependence of σ on a cell's aspect ratio AR is due to D. Lee [5, 27] and shows that a preconditioner based on the continuum PDE's needs modification for best performance with the discretized PDE's.

The condition number achieved by this preconditioner depends on the dimensionality of the equations:

$$K_{VLR91} = \begin{cases} 1 \text{ for all } M \text{ (1D),} \\ \frac{1}{\beta} \text{ for } M<1, \text{ 1 for } M>1 \text{ (2D),} \\ \frac{1}{\tau} \text{ for all } M \text{ (3D).} \end{cases} \quad (31)$$

Although the condition number still blows up near $M=1$, the growth is reduced from $\sim\left|1-M\right|^{-1}$ to $\sim\left|1-M^2\right|^{-1/2}$. See Figure 3.

By including β and τ in the entries of the Turkel matrix (26), the latter can be made to achieve the same condition number as $\mathbf{P}_{VLR91}$ for all M [3, 15].

Review of N-S Preconditioners

Much of the literature on Navier-Stokes preconditioning is due to C. L. Merkle and collaborators, whose papers date back as far as 1985 [28]. Typical of their approach is to write the preconditioned equations in the form

$$\mathbf{\Gamma}\frac{\partial \mathbf{U}^T}{\partial t} = \mathbf{Res}\left(\mathbf{U}\right), \quad (32)$$

mixing the residuals of the conservative equations (3) with time derivatives of the primitive-temperature variables $\mathbf{U}^T$. Comparing Eqn. (32) with Eqn. (3) we see that

$$\mathbf{\Gamma} = \mathbf{P}_c^{-1}\frac{d\mathbf{U}}{d\mathbf{U}^T}, \quad (33)$$

where $\mathbf{P}_c$ is the preconditioner for the conservative equations; hence,

$$\mathbf{P}_c = \frac{d\mathbf{U}}{d\mathbf{U}^T}\mathbf{\Gamma}^{-1}. \quad (34)$$

For example, in a 1991 paper by Choi and Merkle [29], we find

$$\mathbf{\Gamma}_{CM91} = \begin{pmatrix} \frac{1}{\beta M^2} & 0 & 0 & 0 & 0 \\ \frac{u}{\beta M^2} & 1 & 0 & 0 & 0 \\ \frac{v}{\beta M^2} & 0 & 1 & 0 & 0 \\ \frac{w}{\beta M^2} & 0 & 0 & 1 & 0 \\ \frac{\rho E+p}{\rho\beta M^2}-1 & \rho u & \rho v & \rho w & \frac{\gamma\rho R}{\gamma-1} \end{pmatrix}, \; M<1 \quad (35)$$

where E denotes specific total energy and R is the gas constant; the parameter β equals a^2. After extracting the preconditioner and transforming it to the Euler-symmetrizing variables, the result is

$$\mathbf{P}_{CM91} = \begin{pmatrix} M^2 & 0 & 0 & 0 & -\frac{M^2}{\rho a} \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \; M<1 \quad (36)$$

This is basically the Chorin Euler preconditioner and has no special merit when applied to the N-S equations. See Figure 4.

In other papers, earlier [30] as well as later [31], this group of authors includes in β some dependence on the cell-Reynolds number. This dependence usually appears to be motivated by numerical experimentation, except in the paper by Venkateswaran and Merkle [4], where the modification of the entry M^2 is based on the 1-D dispersion analysis presented in Dispersion Analysis. This is by far the most important contribution by these authors, since it offers a tool for designing N-S preconditioners beyond the method of trial and error.

As explained in Dispersion Analysis, Venkateswaran and Merkle distinguish for small Re only two asymptotic regimes. Correspondingly, they recommend the following dependence of P_{11} on the Reynolds number:

$$P_{11} = \begin{cases} O\left(\frac{M^2}{Re^2}\right), \frac{M^2}{Re} \ll 1 \\ O\left(\frac{1}{Re}\right), \frac{M^2}{Re} \gg 1 \end{cases}, \quad (37)$$

These asymptotic cases are then incorporated in a continuous switch for computational practice:

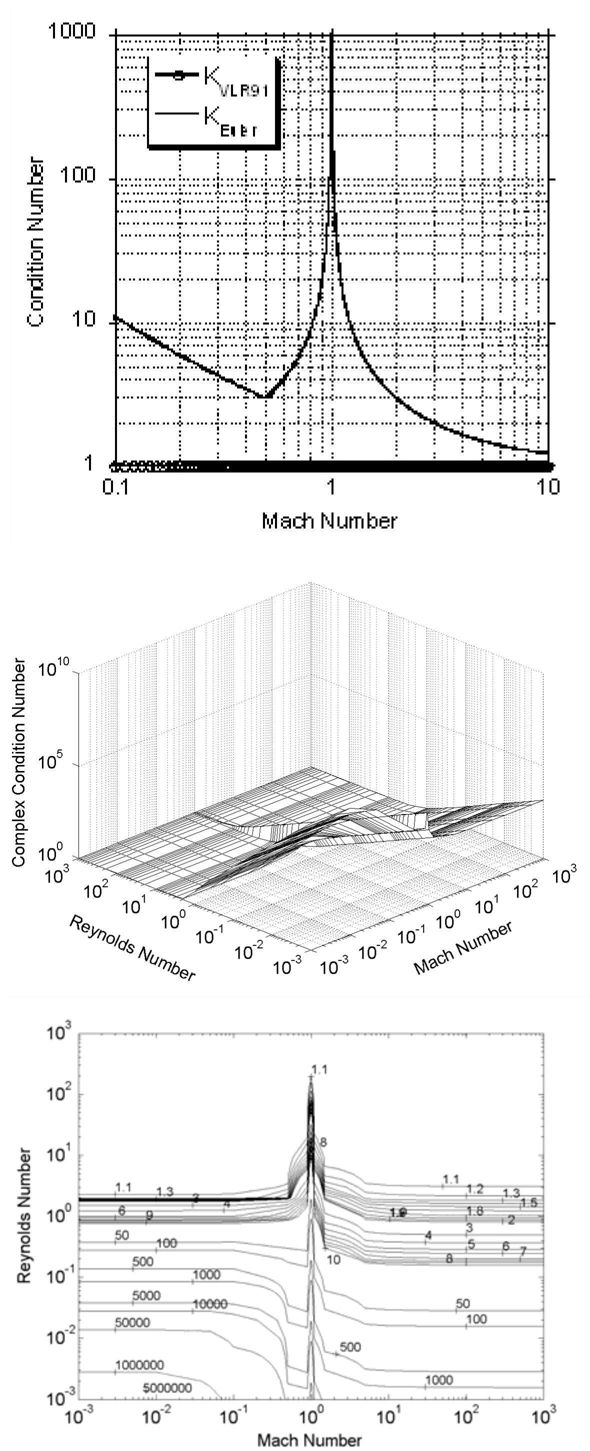

Figure 3: Plots of condition number yielded by 1991 van Leer-Lee-Roe preconditioner. Top: for 1-D Euler equations; bottom for 1-D N-S equations (carpet and contour plots).

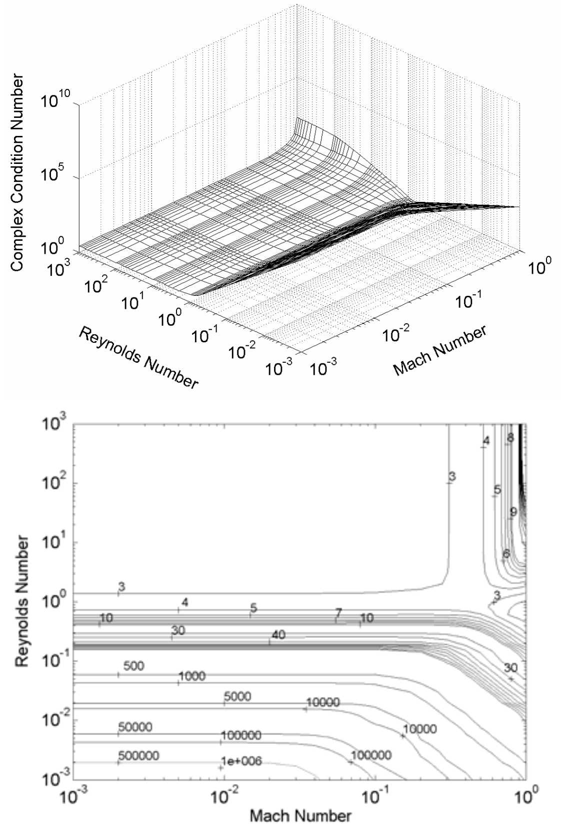

Figure 4: Plots of condition number yielded by 1991 Choi and Merkle preconditioner. Carpet and contour plots for 1-D N-S equations.

$$P_{11} = \max\left\{ M^2, \frac{M^2\left(\frac{1}{Re}-1\right)}{Re\left[1+M^2\left(\frac{1}{Re}-1\right)\right]} \right\}, Re < 1 \qquad (38)$$

for $Re \geq 1$ Eqn. (38) connects to the Euler choice $P_{11} = M^2$.

Outside the incorrectly treated domain $M \geq Re \geq M^2$, this choice of P_{11} in the Chorin preconditioner brings the condition number down to $O(1)$ for low *Re*. Unfortunately, a dispersion analysis of the 1-D preconditioned equations shows that for $M^2/Re \gg 1$ the complex wave speed $(\omega/k)_3$ has a negative real part, i. e. the originally undamped wave becomes a growing wave.

A different low-*Re* technique, contributed by Godfrey [17-19], is motivated by the discretized N-S equations. The idea is to combine an efficient Euler preconditioner, such as $\mathbf{P}_{VLR91}$, with point-implicit or Jacobi relaxation. This type of relaxation may be regarded as an exact implicit solver for data that only contain the "checkerboard mode" (odd-even decoupling): it damps this mode in one step.

Jacobi relaxation has the effect of moving the matrix **C** to the left-hand side of the discretized Eqn. (21). This observation suggests the following way of enriching an Euler preconditioner with *Re*-dependence:

$$\left(\mathbf{P}_{Eu}^{-1} + \frac{2\mathbf{C}}{\Delta x}\right)\mathbf{U}_t + \mathbf{A}\mathbf{U}_x = \mathbf{C}\mathbf{U}_{xx}, \qquad (39)$$

Evidently, the N-S preconditioner is defined by

$$\mathbf{P}_{NS}^{-1} = \mathbf{P}_{Eu}^{-1} + \frac{2\mathbf{C}}{\Delta x}, \qquad (40)$$

Note that **C** contains the factor ν, which we may take out; then the following ratio appears:

$$\frac{\nu}{\Delta x} = \frac{u}{Re_{\Delta x}}, \qquad (41)$$

In a dispersion analysis, Δx may be replaced by $1/k$, and the Reynolds number becomes $\mathrm{Re} = u/k\nu$ as before.

While Godfrey [17-19] tested this approach in explicit and numerical calculations, most analysis is due to D. Lee [5, 12]. He found that the matrix (40), with $\mathbf{P}_{VLR91}$ as the Euler preconditioner, does an excellent job in the domain $M \geq 1$, $Re < 1$, the fourth quadrant of the log-log plane; in the third quadrant, though, the matrix is not effective. See Figure 5.

There is one great plus to this approach: it does not create any growing modes. D. Lee [5] shows how the mode (M = 0.1, Re = 0.001), unstable with the use of $\mathbf{P}_{VM95}$, becomes stabilized, though not sufficiently damped to yield a low condition number.

D. Lee then tried to get even more improvement by incorporating *Re*-dependence in the van Leer-Lee-Roe preconditioner in the manner of Eqn. (37), while also adding the viscous Jacobian as in Eqn. (40). This somewhat reduces the condition number in the third quadrant, but not down to $O(1)$; meanwhile, unnoticed by D. Lee, a growing mode has been introduced in the acoustics-dominated regime $Re > M$.

It is at this point that our search takes off.

An Optimal N-S Preconditioner

We were certain that a 1-D N-S preconditioner yielding uniformly low condition number was within reach, but were not certain that it would leave the equations stable. We adopted D. Lee's latest technique of superimposing $\mathbf{P}_{VLR91}^{-1}$, modified by *Re*-dependence, and the viscous Jacobian; once the results for $Re < 1$ began to take shape, the approach became blurred, eventually leading to a single *Re*-dependence matrix.

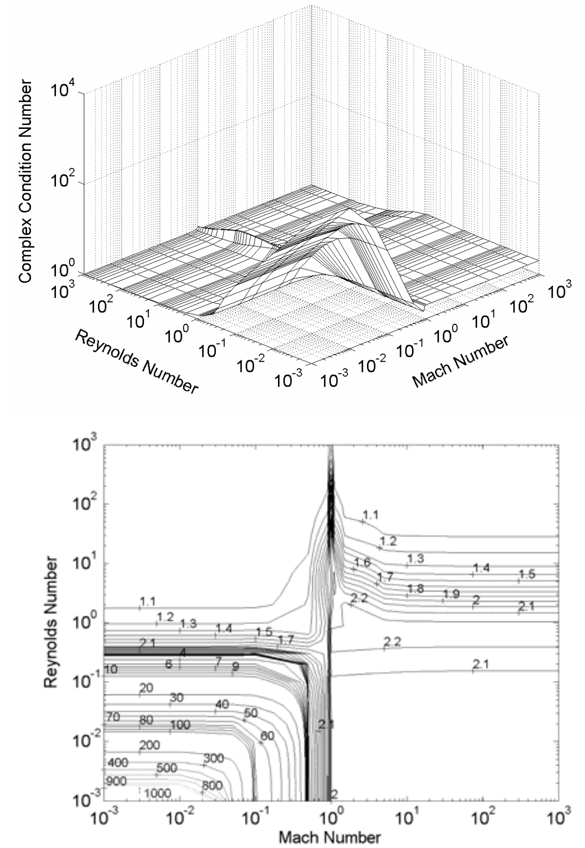

Figure 5: Plots of condition number yielded by Godfrey/Lee preconditioner. Carpet and contour plots for 1-D N-S equations.

We started out by modifying the (1,1) element of the Euler preconditioner so as to get the lowest possible condition number for all combinations of M and Re in the elusive third quadrant. This meant computing the variation of the condition number K with M and Re, and then compensating for it by a suitable functional dependence on M and Re in the element. Not only was it possible to determine the correct power of M and Re to be used, but we could also determine the correct coefficients. The constants appearing in the elements of **A** and **C** are γ, Pr and 4/3; by determining the sensitivity of K to each of these quantities their place in the function became evident.

After exhausting the possibilities provided by varying the (1,1) element, other elements were considered. Some made a clear difference, others hardly influenced the condition number. It was at this stage that the third asymptotic region was *numerically* identified; its derivation from the dispersion analysis came after the fact.

The results are summarized below. Note that down to $Re = 1$ the superposition approach of Godfrey/D. Lee is entirely satisfactory, while in the 4th quadrant $(Re < 1,\ M \geq 1)$ it can be slightly improved. In the most problematic third quadrant, $(Re < 1,\ M < 1)$, including the Euler preconditioner is no longer useful; a single matrix with three branches is presented.

First and second quadrant: $(0 < M < \infty,\ Re \geq 1)$

$$\mathbf{P}_{DV02}^{-1} = \mathbf{P}_{DL96}^{-1} = \mathbf{P}_{VLR91}^{-1} + \frac{2\mathbf{C}}{\Delta x}, \tag{42}$$

Fourth quadrant: $(M \geq 1,\ Re < 1)$

$$\mathbf{P}_{DV02}^{-1} = \mathbf{P}_{VLR91}^{-1} + \frac{2\mathbf{C}'}{\Delta x} \tag{43}$$

with

$$\mathbf{C}' = \nu \begin{bmatrix} \frac{3Re}{8} + \frac{\gamma-1}{Pr} & 0 & \frac{\frac{1}{Pr} - \frac{3Re}{8}}{\rho a} \\ 0 & \frac{4}{3} & 0 \\ (\gamma-1)\left(\frac{1}{Pr} - \frac{3Re}{8}\right)\rho a & 0 & \frac{(\gamma-1)3Re}{8} + \frac{1}{Pr} \end{bmatrix}, \tag{44}$$

Third quadrant: $(M < 1,\ Re < 1)$

$$\mathbf{P}_{DV02}^{-1} = \frac{2\mathbf{C}''}{\Delta x}, \tag{45}$$

where the formula for matrix $\mathbf{C}''$ has three branches:

$$\mathbf{C}_I'' = \nu \begin{bmatrix} \frac{\gamma(2\gamma-1)}{2(\gamma-1)Pr} & 0 & \frac{\gamma}{4(\gamma-1)\rho a Pr} \\ 0 & \frac{4}{3} & 0 \\ \frac{\gamma \rho a}{4Pr} & 0 & 0 \end{bmatrix}, \frac{M}{Re} < 1 \tag{46}$$

$$\mathbf{C}_{II}'' = \nu \begin{bmatrix} \frac{\left[PrRe + \gamma(\gamma-1)\right]}{\gamma Pr} & 0 & \frac{(\gamma - PrRe)}{\gamma \rho a Pr} \\ 0 & \frac{4}{3} & 0 \\ \frac{(\gamma - PrRe)(\gamma-1)\rho a}{\gamma Pr} & 0 & \frac{\left[PrRe(\gamma-1)+\gamma\right]}{\gamma Pr} \end{bmatrix}, \frac{M^2}{Re} \geq 1 \tag{47}$$

$$\mathbf{C}_{III}'' = \nu \begin{bmatrix} \frac{\left[3PrRe^2 + 4M^2\gamma^2(\gamma-1)\right]}{4M^2\gamma^2 Pr} & 0 & \frac{\left(4M^2\gamma^2 - 3PrRe^2\right)}{4M^2\gamma^2 \rho a Pr} \\ 0 & \frac{4}{3} & 0 \\ \frac{\left(4M^2\gamma^2 - 3PrRe^2\right)(\gamma-1)\rho a}{4M^2\gamma^2 Pr} & 0 & \frac{\left[3PrRe^2(\gamma-1) + 4M^2\gamma^2\right]}{4M^2\gamma^2 Pr} \end{bmatrix}, \frac{M}{Re} \geq 1 \text{ and } \frac{M^2}{Re} < 1 \tag{48}$$

Carpet and contour plots of the condition number are given in Figure 6. It is seen that the condition number is essentially equal to 1 everywhere, except at the seams of the different sub domains, where it is elevated (though bounded). In the future these ridges will be removed or smoothed by continuous blending of the formula branches.

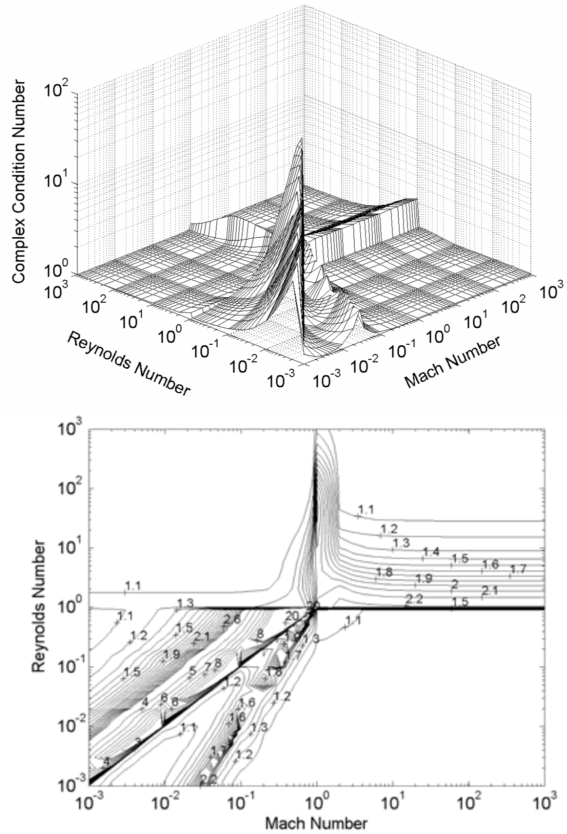

Figure 6: Plots of the condition number yielded by Depcik and van Leer preconditioner. Carpet and contour plots for 1-D N-S equations.

Fighting the Growing Mode

Having found that the perfect 1-D N-S preconditioner creates a growing mode in the acoustics-dominated regime (see Figure 7), we investigated the sensitivity of the growth factor, i. e. the negative imaginary part of ω/k, to the value of the elements of the matrix in that regime. Just as for the condition number, we found that some made a clear difference, others hardly influenced the growth rate. Ultimately we had to conclude that getting rid of the negative imaginary part means giving up the $O(1)$ value of the condition number.

We have tried to come up with remedies; so far we see two possibilities.

1. To redefine the complex condition number in a way more appropriate for explicit discretizations. From the viewpoint of stability of explicit advection and diffusion schemes, it would be more appropriate to use the following weighted L_1 norm (rather than the Euclidian norm) for the complex wave number:

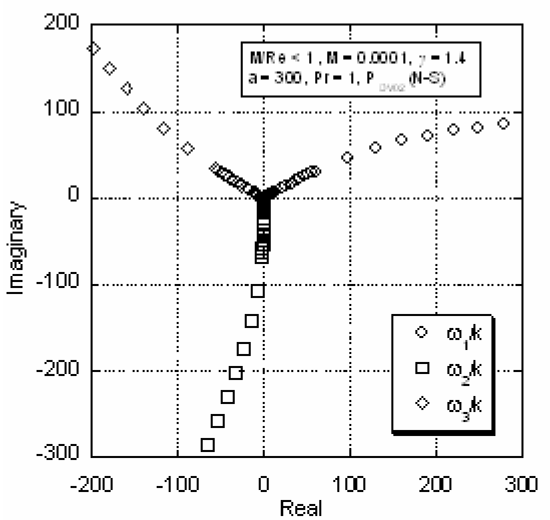

Figure 7: The three roots of the dispersion equation for the 1-D N-S equations preconditioned by the new preconditioner, plotted in the complex plane for a range of *Re*-values. The mode with the negative imaginary part is the growing mode.

$$\left\|\frac{\omega}{k}\right\| = \left|\Re\left(\frac{\omega}{k}\right)\right| + 2\left|\Im\left(\frac{\omega}{k}\right)\right|, \qquad (49)$$

which in turn would change the value of the condition number and therefore the matrix-optimization results. This remains to be investigated, but we doubt the effect would be strong enough to remove the growing mode.

2. To reformulate the N-S equations as a larger first-order system of hyperbolic-relaxation equations [14, 32, 33] and design the preconditioner for this system. This is a much more drastic measure; it is part of a broader philosophy that recognizes there are a host of advantages, both numerical and computer-science-based, to using the lowest possible order of PDE's when modeling physics [14], i. e. the first order. A preliminary investigation suggests that with the first-order system we have ample control over damping rates after preconditioning, so it may be possible to prevent growing modes without compromising the condition number.

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