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A TWO-PULSE SCHEME FOR THE TIME-OPTIMAL ATTITUDE CONTROL OF A SPINNING MISSILE

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Abstract

The problem of minimum-time attitude control of a spinning missile is addressed. The missile is modeled as a rigid body which is symmetric about one axis. The missile is assumed to have a large roll rate about this axis of symmetry. Control is achieved by a single reaction jet which, when fired, provides a constant moment about a transverse axis. Disturbance torques are assumed to be zero. The equations of motion are written under these assumptions. The missile is assumed to have some arbitrary initial transverse angular velocity and it is desired to take it to some final attitude in minimum time while reducing the transverse angular velocity to zero. This problem is formulated as an optimal control problem. Instead of taking the conventional approach of solving a two point boundary value problem, we consider an alternative approach. This approach deals with the specific case where only two thruster firings are sufficient to change the attitude of the missile in minimum time. By iterating on the switch times and integrating the state equations, we can compute the thruster firing times for a given set of boundary conditions. Some examples are included to illustrate the application of the concepts presented. We conclude by proposing a mechanization of this control scheme and pointing out some further research directions.

1. Introduction

Over the past three decades many papers and reports have treated various aspects of homing schemes and trajectory control associated with these schemes. Most of these papers consider surface-to-air or air-to-air missiles which use aerodynamic forces for trajectory control. With the advent of SDI, much attention has been focused on the interception of satellites or ICBM's outside the sensible atmosphere. Hence, aerodynamic forces cannot be generated for vehicle control. Instead, the thrust of a rocket engine is used to provide the necessary maneuver forces, with vehicle attitude control employed to point the thrust in the desired direction. Conventional thrust vector control systems tend to add both weight and complexity, and as a result counter the objective of minimizing the weight of the guided warhead. The simplest control involves a single thruster at right angles to the spin axis of the missile. In this scheme, the missile is given a large roll rate and the thruster is turned on for a fraction of each revolution in roll and at the right time during each roll cycle so that the desired attitude changes are achieved. Meanwhile the main thruster, by producing a thrust component perpendicular to the flight path, provides the necessary trajectory changes.

The problem of attitude control of spinning rigid bodies has not received much attention recently, although some research has been reported on this topic in the 1960's. The reorientation problem of a spinning rigid body is conceptually different than the simple rest-to-rest maneuver of a non-spinning rigid body. Because of the spin of the body, application of any moment about the transverse axes generates a precessional motion. If the initial transverse angular velocity is not zero, the problem becomes even more difficult because the problem loses its symmetry.

Athans and Falb[2] consider the problem of time-optimal velocity control of a rotating body with a single axis of symmetry. They show that for a single fixed control jet, the system has the properties of a harmonic oscillator. Thus, a switching curve can be derived to implement the control scheme. The cases of a gimballed control jet and two control jets are also considered. No mention is made of the complete attitude reorientation problem, however. Howe[9] proposes an attitude control scheme for sounding rockets. The main feature of this scheme is that it uses a single control jet. The control jet is fired for a fixed duration whenever certain conditions on direction cosines or transverse angular velocity are satisfied.

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This results in the alternate reduction of attitude error and transverse angular velocity, finally ending in a limit cycle. Some other references[16,4,1,6,7,8,12,15,13] discuss the problem of reorienting a rotating rigid body which has no initial transverse angular velocity. Windenknecht'' proposes a simple system for sun orientation of spinning satellites. In this scheme, the desired attitude is achieved by a succession of 180° precessional motions, each resulting in a small attitude change (small-angle approximations assumed valid), until the spin axis arrives at an attitude corresponding to the dead zone of the sun sensors. Cole *et al.*[4] prescribe the desired attitude change and solve for the necessary torques but give no details on mechanization. Other papers which propose active attitude control systems for spin stabilized vehicles have been published by Adams[1], Freed', and Grasshoff[7], but none of these explicitly discusses the reorientation problem. Grubin[8] uses the concept of finite rotations to mechanize a two-impulse scheme for reorienting the spin axis of a vehicle. If the torques are ideally impulsive, then the scheme is theoretically perfect. But in the case of finite-duration torquing, considerable errors can result. Wheeler[15] extends Grubin's work to include asymmetric spinning satellites, but the underlying philosophy is the same. Porcelli and Connolly[13] use a graphical approach to obtain control laws for the reorientation of a spinning body. Their results are only valid for small angles and small angular velocities. For this linearized case they prove that a two-impulse control scheme is fuel-optimal. Two sub-optimal control laws are then derived for the case of limited thrust based on the two-impulse solution.

None of the above papers consider time-optimal reorientation of a spinning space body. The control laws derived are based on small angle and/or impulsive torque approximations. For large angle maneuvers with limited thrust, sizeable errors can result because of these approximations. In the present work we examine a practical scheme for the attitude control of a spinning missile. The control scheme proposed is not limited to small angles and small angular velocities and the initial transverse angular velocity can be arbitrary, *i.e.*, it is not assumed to be zero. We have assumed no disturbances such as aerodynamic forces, gravity, solar radiation pressures, or structural damping. Because of the short flight times, these disturbances have negligible effect on the dynamics of the missile. These assumptions yield a simple mathematical model described by five state equations, *viz.*, two dynamical equations involving the transverse angular velocities and three kinematical equations giving the rates of change of Euler angles. The theory of optimal control is used to find a minimum-time control law.

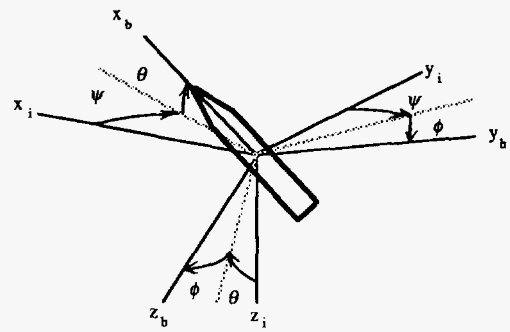


Figure 1: Axes systems.

2. Equations of Motion

Figure 1 shows the orientation of the moving body axes x_b, y_b, z_b relative to the inertial reference axes x_i, y_i, z_i, and also the Euler angles ψ, θ, ϕ relating the two axis systems. The body axes origin is at the missile c.g. with the x_b-axis assumed to be the axis of symmetry; the y_b- and z_b-axes lie in a plane perpendicular to the longitudinal axis, x_b. The missile is modeled as a rigid cylindrical body. We also assume that the control jet is located in the x_b-z_b plane and pointed in the direction of the z_b-axis. When fired, the control jet generates a constant positive moment about the y_b-axis.

Since no moment is applied about the x_b-axis, and since $I_y = I_z$ (the moments of inertia about the y_b- and z_b-axes are equal for a missile that is axially symmetric about its x_b-axis), it turns out that w,, the missile angular velocity component along the x_b-axis, is a constant equal to the initial spin velocity of the missile. We then obtain a set of five state equations: two dynamical equations involving the transverse angular velocities and three kinematical equations giving the rates of change of Euler angles. Thus

$$\dot{\omega}_y = \left(1 - \frac{I_x}{I_y}\right)\omega_x\omega_z + \frac{M_y}{I_y} \tag{1}$$

$$\dot{\omega}_z = -\left(1 - \frac{I_x}{I_y}\right)\omega_x\omega_y \tag{2}$$

$$\dot{\psi} = (\omega_y \sin\phi + \omega_z \cos\phi)\sec\theta \tag{3}$$

$$\theta = \omega_y \cos\phi - \omega_z \sin\phi \tag{4}$$

$$\phi = \omega_x + (\omega_y \sin\phi + \omega_z \cos\phi)\tan\theta \tag{5}$$

where

$\omega_y, \omega_z =$ trasverse angular velocity components along the y- and z-axes, respectively,

$\psi, \theta, \phi =$ Euler angles corresponding to yaw, pitch and roll, respectively,

I_x, $I_y =$ the moments of inertia about the longitudinal and transverse axes, respectively,

$M_y =$ the thruster torque about the y-axis.

For convenience we choose to write Eqs. **(1)-(5)** in terms of dimensionless variables and parameters in accordance with the following definitions:

$$\Omega_y = \frac{\omega_y}{\omega_x},\ \Omega_z = \frac{\omega_z}{\omega_x}$$
$$A = 1 - \frac{I_x}{I_y},\ \Lambda = \frac{M_y}{I_y \omega_x^2}$$
$$\text{dimensionless time } T = \omega_x t$$

Now, if we redefine the − operator as differentiation with respect to the dimensionless time T, the equations become

$$\Omega_y = \text{AR}, + \Lambda \tag{6}$$

$$\Omega_z = -A\Omega\text{Y} \tag{7}$$

$$\psi = (\Omega_y \sin\phi + \Omega_z \cos\phi) \sec \mathbf{0} \tag{8}$$

$$\text{t?=}\ \Omega_y \cos\phi - \Omega_z \sin\phi \tag{9}$$

$$\phi = 1 + (\Omega_y \sin\phi + \Omega_z \cos\phi) \tan\theta \tag{10}$$

We assume that at the initial time, the missile body axis system coincides with the inertial axis system. The initial transverse angular velocity of the missile, however, is non-zero. We thus obtain the following initial conditions:

$$\Omega_z(T_0) = \Omega_{z0},\ \Omega_y(T_0) = \Omega_{y0}$$
$$\psi(T_0) = 0,\ \theta(T_0) = 0,\ \phi(T_0) = 0$$

The desired final conditions on the state variables are given by:

$$\Omega_z(T_f) = 0,\ \Omega_y(T_f) = 0$$
$$\psi(T_f) = \psi_d,\ \theta(T_f) = \theta_d,\ \phi(T_f) = \text{free}$$

The numerical values for the two parameters, A and Λ which will be used later in examples, are

$$A = 0.9,\ \Lambda = 0.02$$

This value of A corresponds to a length to diameter ratio of **3.775** for a cylindrical body of uniform density. **A** missile weighing 10 lbs. and having a uniform mass density of aluminum would have the following dimensions:

length = **12.30** in., diameter = **3.26** in.

If the moment arm is half the length and the spin velocity is 50 rad/sec., $\Lambda = 0.02$ corresponds to a thrust of **2.79** lbs.

3. Solution for the Linearized System

The equations of motion given by Eqs. (6)-(10) are nonlinear and no analytic solution can be found for the general case of arbitrary angles and angular velocities. Considerable simplification can be achieved, however, by assuming small angles and small transverse angular velocity compared to the axial spin velocity. For this linearized case, the equations of motion can be analytically integrated. These assumptions also yield some analytic results for the time history of the time-optimal control.

3.1. System Equations and Optimal Control Description

In order to get a simplified set of equations which can be easily integrated and which yield analytic solutions for the time optimal control, we assume that R,, Ω_z and θ remain small during the attitude change maneuver. **Eqs.** (8)-(10) can then be written as

$$\psi = \Omega_y \sin\phi + \Omega_z \cos\phi \tag{11}$$

$$\theta = \Omega_y \cos\phi - \Omega_z \sin\phi \tag{12}$$

$$\dot{\phi} = 1 \tag{13}$$

Note that no assumption has been made on the magnitude of $ or ϕ. **Eq. (13)** can be integrated with the initial condition $\phi(T_0) = 0$ to give

$$\phi = T - T_0 \tag{14}$$

We assume $T_0 = 0$ without loss of generality. Substituting this into Eqs. (11) and (12) we obtain the linearized equations of motion for the four state variables Ω_y, Ω_z, ψ, and θ as

$$\Omega_y = A\Omega_z + \Lambda \tag{15}$$

$$\Omega_z = -A\Omega_y \tag{16}$$

$$\psi = \Omega_y \sin \text{T} + \Omega_z \cos T \tag{17}$$

$$\theta = \Omega_y \cos T - \Omega_z \sin T \tag{18}$$

These four equations can be written in the standard state-space form by defining the state vector $\mathbf{x}$ as

$$\mathbf{x} = \begin{bmatrix} \Omega_y & \Omega_z & \psi & \mathbf{0} \end{bmatrix}^T$$

and the control u as

$$u = \Lambda$$

The standard form for the equations is

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}u \tag{19}$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & A & 0 & 0 \\ -A & 0 & 0 & 0 \\ \sin T & \cos T & 0 & \mathbf{0} \\ \cos T & -\sin T & 0 & 0 \end{bmatrix} \tag{20}$$

$$\mathbf{b} = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}^T \tag{21}$$

The initial state $\mathbf{x}_0$ is given by

$$\mathbf{x}_0 = \left[\begin{array}{cccc} x_{1,0} & x_{2,0} & 0 & 0 \end{array}\right]^T$$

We want to find a control which will take this initial state to the desired state $\mathbf{x}_d$

$$\mathbf{x}_d = \left[\begin{array}{cccc} 0 & 0 & x_{3,d} & x_{4,d} \end{array}\right]^T$$

while minimizing the time.

The question of existence of an optimal control for this class of problems is discussed by Cesari[3]. It is proven that a bang-bang optimal solution exists for the system described by $x = \mathbf{A}(t)\mathbf{x}(t)+\mathbf{B}(t)u(t)$ for a class of performance indices and constraints. He also shows that there may well be optimal solutions which are not bang-bang. However, if the opimal solution is unique, it must be bang-bang (under certain conditions on the states and constraints). At this time no general theorems arc available on the uniqueness of optimal solutions for the one-sided controls, *i.e.*, $0 \leq u \leq u_{max}$. Therefore, we can only give necessary conditions for u^* to be an optimal control.

Proceeding with the derivation of the necessary conditions on the time-optimal control, we write the performance index

$$J = \int_{T_0}^{T_f} 1 dt \tag{22}$$

with the given constraints

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}u \tag{23}$$

$$0 \leq u \leq u_{max} \tag{24}$$

The Hamiltonian can be written as

$$H = \mathbf{p}^T\dot{\mathbf{x}} - 1 \tag{25}$$

where $\mathbf{p}$ is the costate vector. The necessary conditions for u^* to be an optimal control are

$$\dot{\mathbf{x}}^* = \frac{\partial H}{\partial \mathbf{p}} = \mathbf{A}\mathbf{x}^* + \mathbf{b}u^* \tag{26}$$

$$\dot{\mathbf{p}}^* = -\frac{\partial H}{\partial \mathbf{x}} = -\mathbf{A}^T\mathbf{p}^* \tag{27}$$

$$u^* = \begin{cases} u_{max} & \text{if } p_1^* > 0 \\ 0 & \text{if } p_1^* < 0 \end{cases} \tag{28}$$

and

$$\mathbf{x}(T_0) = \mathbf{x}_0 \tag{29}$$

$$\mathbf{x}(T_f) = \mathbf{x}_d \tag{30}$$

$$\mathbf{p}(T_0) = \text{free} \tag{31}$$

$$\mathbf{p}(T_f) = \text{free} \tag{32}$$

$$H(T_f) = 0 \tag{33}$$

Eqs. (26) and **(27)** are the differential equations for the state and costate vector, Eq. (28) is derived from the optimality condition *i.e.* maximizing the Hamiltonian H. Eqs. (29) and (30) are the given boundary conditions and Eqs. (31) and (32) are derived from the transversality conditions. Hence, in the problem we have 8 differential equations (Eqs. (26)-(27)) with 8 boundary conditions (Eqs. (29)-(30)) constituting a TPBVP (two point boundary value problem). Eq. (33) is used to determine T_f.

Eq. (27) can be solved analytically to obtain $\mathbf{p}$ as an analytical function of the dimensionless time, T and the initial condition, $\mathbf{p}(0)$. The expression for the switching function p_1 is

$$\begin{aligned} p_1 &= \rho_1 \cos AT + \rho_2 \sin AT - \frac{\rho_3}{1-A}(\cos AT \\ &\quad - \cos T) + \frac{\rho_4}{1-A}(\sin AT - \sin T) \end{aligned} \tag{34}$$

where the initial conditions $\rho_1 = p_1(0)$, $\rho_2 = p_2(0)$, $\rho_3 = p_3(0)$, and $\rho_4 = p_4(0)$ are constants to be determined.

The expression for the control u^* can now be written as

$$u^* = \begin{cases} u_{max} & \text{if } S > 0 \\ 0 & \text{if } S < 0 \end{cases} \tag{35}$$

where for convenience we have defined $S = p_1$. Eq. (35) states that the optimal control is a bang-bang type. **A** singular solution is not possible because if $S \equiv 0$ over some interval, that implies that $\rho_1 = \rho_2 = \rho_3 = \rho_4 = 0$. This means that $H \equiv -1$ for all T. This contradicts the transversality condition $H(T_f) = 0$.

The switching function S will be periodic if A is a rational number. However, it does not go through zero at regular intervals, *i.e.*, the intervals between successive times when $S = 0$ are not uniform. Thus, depending on the boundary conditions, the constants ρ_1, ρ_2, ρ_3 and ρ_4 will be different and each turn-on and turn-off interval may be of a different duration. Figure 2 shows the switching curve for $A = 0.9$, for some values of the costate initial conditions ρ_1, ρ_2, ρ_3 and ρ_4.

3.2. Two-pulse Solution

Starting at $T_0 = 0$, typical time history curves for the control u and the transverse angular velocity components x_1 and x_2, are shown in Figure 3 for the case where the final target state is reached with two thruster firings. The switch times, T_1, T_2, T_3 and T_f, are the four unknowns. The thruster is fired from T_1 to T_2 and then from T_3 to T_f.

The equations for $\dot{x}_1$ and $\dot{x}_2$, which are exact, can be integrated starting from any arbitrary initial conditions at T_i to obtain

$$x_1(T) = x_{1,i} \cos A(T - T_i)$$

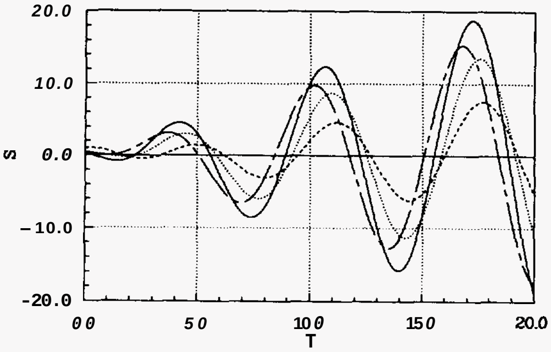


Figure 2: Switching function S for four different sets of costate initial conditions.

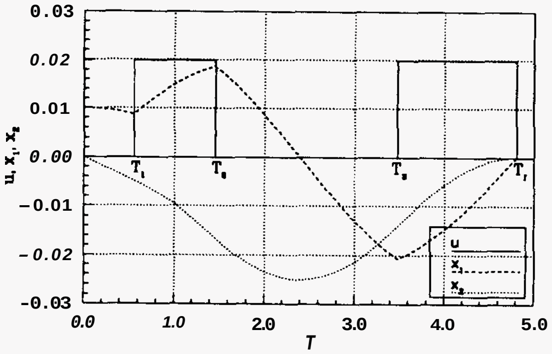


Figure 3: u, x_1 and x_2 vs. T

$$+\left(x_{2,i}+\frac{u}{A}\right)\sin A(T-T_i) \quad (36)$$

$$x_2(T) = \left(x_{2,i}+\frac{u}{A}\right)\cos A(T-T_i) - x_{1,i}\sin A(T-T_i) - \frac{u}{A} \quad (37)$$

The subscript i refers to arbitrary initial conditions. Substituting Eqs. **(36)** and **(37)** into the equations for $\dot{x}_3$ and $\dot{x}_4$ in **Eq.** (20), we obtain

$$\begin{aligned}\dot{x}_3 &= x_{1,i}\sin((1-A)T+AT_i) \\ &+x_{2,i}\cos((1-A)T+AT_i) \\ &+\frac{u}{A}(\cos((1-A)T+AT_i)-\cos T) \quad (38)\end{aligned}$$

$$\begin{aligned}\dot{x}_4 &= x_{1,i}\cos((1-A)T+AT_i) \\ &-x_{2,i}\sin((1-A)T+AT_i) \\ &+\frac{u}{A}(-\sin((1-A)T+AT_i)+\sin T) \quad (39)\end{aligned}$$

These can be easily integrated with the given initial conditions on x_3 and x_4. Thus we obtain

$$\begin{aligned}x_3(T) &= x_{3,i}+\frac{2}{1-A}\sin a(1-A) \\ &[x_{1,i}\sin(b-aA)+x_{2,i}\cos(b-aA)] \\ &+\frac{2u}{A}\left[\frac{1}{1-A}\cos(b-aA)\sin a(1-A)\right. \\ &\left.-\cos b\sin a\right] \quad (40)\end{aligned}$$

$$\begin{aligned}x_4(T) &= x_{4,i}+\frac{2}{1-A}\sin a(1-A) \\ &[x_{1,i}\cos(b-aA)-x_{2,i}\sin(b-aA)] \\ &+\frac{2u}{A}\left[\frac{-1}{1-A}\sin(b-aA)\sin a(1-A)\right. \\ &\left.+\sin b\sin a\right] \quad (41)\end{aligned}$$

where

$$a=\frac{T-T_i}{2} \qquad b=\frac{T+T_i}{2}$$

Since we now have expressions for x_1, x_2, x_3, and x_4 as functions of the running time T and the given initial conditions, we can obtain expressions for these variables at $T=T_f$. The procedure is to use Eqs. **(36)-(37)** and (40)-(41) with $T_i=T_0$ and $u=0$ to obtain the state variables at $T=T_1$. Then we let $T_i=T_1$ and use the associated initial conditions with $u=u_{max}$ to obtain the new conditions at $T=T_2$, and *so* on. Finally, we end up with $x_1(T_f)$, $x_2(T_f)$, $x_3(T_f)$, and $x_4(T_f)$ as functions of the boundary conditions and T_1, T_2, T_3 and T_f. Since the boundary conditions at $T=T_0$ and $T=T_f$ are known, this procedure yields four equations in four unknowns, T_1, T_2, T_3 and T_f. The problem can be simplified, however, by noting that T_3 and T_f can be chosen such that $x_1(T_f)=x_2(T_f)=0$. In this way T_3 and T_f can be expressed as functions of T_1 and T_2 and the problem is reduced to finding T_1 and T_2 such that $x_3(T_f)=x_{3,d}$ and $x_4(T_f)=x_{4,d}$. Since T_1 and T_2 cannot be expressed as explicit functions of the initial conditions, the problem must be solved iteratively.

This procedure of writing the nonlinear equations $\mathbf{x}(T_f)=\mathbf{f}(\mathbf{x}_0,T_1,T_2,T_3,T_f)$ can be thought of **as** a numerical integration method which uses the state transition matrix of the system to integrate the state from $\mathbf{x}(T_0)$ to $\mathbf{x}(T_f)$ in four time intervals of variable duration, where over each interval the control u remains constant. The state transition method to simulate linear systems is discussed by Howe[10].

In Eqs. **(36)** and (37) we let $T_i=T_2, T=T_3$ and $u=0$ and obtain

$$x_{1,3}=x_{1,2}\cos A(T_3-T_2)+x_{2,2}\sin A(T_3-T_2) \quad (42)$$

$$x_{2,3}=-x_{1,2}\sin A(T_3-T_2)+x_{2,2}\cos A(T_3-T_2) \quad (43)$$

Now letting $T_i=T_3, T=T_f$ and $u=u_{max}$, we obtain

$$\begin{aligned}x_1(T_f) &= x_{1,2}\cos A(T_f-T_2)+x_{2,2}\sin A(T_f-T_2) \\ &+\frac{u_{max}}{A}\sin A(T_f-T_3) \quad (44)\end{aligned}$$

$$x_2(T_f) = x_{2,2}\cos A(T_f - T_2) - x_{1,2}\sin A(T_f - T_2) + \frac{u_{max}}{A}(\cos A(T_f - T_3) - 1) \quad (45)$$

Setting $x_1(T_f) = x_2(T_f) = 0$ and using simple trigonometric relationships, we finally obtain

$$T_3 = T_2 + \frac{1}{A}\left(\tan^{-1}(-) - \sin^{-1}\left(\frac{A\sqrt{x_{1,2}^2 + x_{2,2}^2}}{2u_{max}}\right)\right) \quad (46)$$

$$T_f = T_2 + \frac{1}{A}\left(\tan^{-1}\left(\frac{-x_{2,2}}{-x_{1,2}}\right) + \sin^{-1}\left(\frac{A\sqrt{x_{1,2}^2 + x_{2,2}^2}}{2u_{max}}\right)\right) \quad (47)$$

Clearly, adding $\frac{2n\pi}{A}$ to T_3 and T_f would still make $x_1(T_f) = x_2(T_f) = 0$. However, from Eq. **(35)** and Figure **2,** we observe that the intervals between switches are never more than one period, *i.e.*, $\frac{2\pi}{A}$. Hence, we limit $T_1 \in (0, \frac{2\pi}{A})$ and $T_3 - T_2 \in (0, \frac{2\pi}{A})$. In the next section a condition is derived which provides **a** check for the time-optimality of a two-pulse solution.

3.3. Optimality of the Two-pulse Solution

Figure **3** shows the time history of the control for the two-pulse solution. Again, T_1 and T_3 represent the first and second turn-on times, and T_2 and T_f represent the first and second turn-off times, respectively. From Eq. (35) (the necessary condition on control), we know that

$$u^* = \begin{cases} u_{max} & \text{if } S > 0 \\ 0 & \text{if } S < 0 \end{cases}$$

Since the boundary conditions are satisfied by the two-pulse solution, we only have to check for the above necessary condition. Thus, in order for the two-pulse solution to be time-optimal, it should satisfy the following condition on S.

$$S \begin{cases} < 0 & \text{if } T \in (T_0, T_1) \text{ or } T \in (T_2, T_3) \\ > 0 & \text{if } T \in (T_1, T_2) \text{ or } T \in (T_3, T_f) \end{cases} \quad (48)$$

Eq. (48) gives the necessary condition for the time-optimality of a given two-pulse solution. We now present a procedure to determine if the switching times obtained satisfy the necessary condition on the optimal control. We compute $S(T)$ such that $S(T_1) = S(T_2) = S(T_3) = 0$. Then $S(T)$ is plotted to see whether $S(T)$ goes through zero at only $T =$

Boundary	$x_{1,0}$	0.0666666701	0.0282842708
	$x_{2,0}$	−0.0222222235	0.0282842708
	$x_{3,d}$	**−0.0041141893**	**0.3913032919**
	$x_{4,d}$	**0.4101158694**	**0.1566834726**
Turn-on and Turn-off Times	T_1	**1.72042500**	**5.46257075**
	T_2	**5.27150921**	**6.28533040**
	T_3	**8.71603244**	**9.65781759**
	T_f	**10.47287320**	**12.13034180**

T_1, T_2 or T_3 or whether there are other $T \in [T_0, T_f]$ such that $S(T) = 0$.

From Eq. **(48)** we **know** that $S(T_1) = S(T_2) = S(T_3) = 0$. Thus, we can write

$$\begin{bmatrix} \cos AT_1 & \sin AT_1 & \frac{\cos T_1 - \cos AT_1}{1-A} & \frac{\sin AT_1 - \sin T_1}{1-A} \\ \cos AT_2 & \sin AT_2 & \frac{\cos T_2 - \cos AT_2}{1-A} & \frac{\sin AT_2 - \sin T_2}{1-A} \\ \cos AT_3 & \sin AT_3 & \frac{\cos T_3 - \cos AT_3}{1-A} & \frac{\sin AT_3 - \sin T_3}{1-A} \end{bmatrix} \begin{bmatrix} \rho_1 \\ \rho_2 \\ \rho_3 \\ \rho_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad (49)$$

One alternative way of writing this equation is

$$\begin{bmatrix} \cos AT_1 & \sin AT_1 & \frac{1}{1-A} \\ \cos AT_2 & \sin AT_2 & \frac{\cos T_2 - \cos AT_2}{} \\ \cos AT_3 & \sin AT_3 & \frac{\cos T_3 - \cos AT_3}{1-A} \end{bmatrix} \begin{bmatrix} \frac{\rho_1}{\rho_4} \\ \frac{\rho_2}{\rho_4} \\ \frac{\rho_3}{\rho_4} \end{bmatrix} = \begin{bmatrix} \frac{\sin T_1 - \sin AT_1}{1-A} \\ \frac{\sin T_2 - \sin AT_2}{1-A} \\ \frac{\sin T_3 - \sin AT_3}{1-A} \end{bmatrix} \quad (50)$$

This equation can be solved for $\frac{\rho_1}{\rho_4}$, $\frac{\rho_2}{\rho_4}$ and $\frac{\rho_3}{\rho_4}$ if the **3** x **3** matrix on tlie left hand side is non-singular. Once ρ_1, ρ_2, ρ_3 and ρ_4 are known, S can be plotted as a function of the dimensionless time T and the two-pulse solution can be checked to see if it satisfies the necessary conditions on the optimal control. If $S = 0$ for some $T \in [T_0, T_f], T \# T_1, T_2, T_3$ then the necessary condition is violated and we can reject the switch times as non-optimal. However, if $S = 0$ only for $T = T_1, T_2, T_3$ and $T \in [T_0, T_f]$ then the switch times obtained remain candidates for being time-optimal.

Two examples are presented here. The initial and the desired final conditions and the corresponding switch times for the two examples are given in Table 1. The first example is shown as the solid curve in Figure **4.** It can be seen from this plot that $S = 0$ only at the switch times T_1, T_2 and T_3 given in Table 1 for Example 1. Therefore, the solution is time-optimal. The dashed line in Figure 4 shows the second example, where the two-pulse solution obtained results in $S(T) = 0$ when $T \in [T_0, T_1)$. Therefore, this solution is not time-optimal.

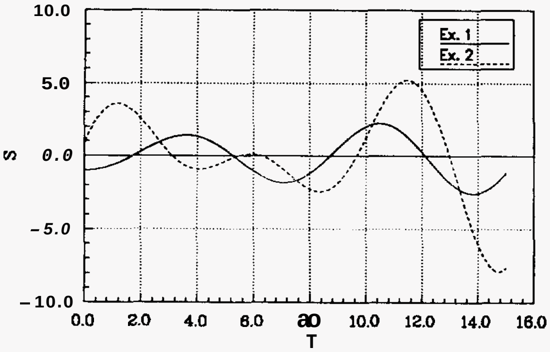


Figure 4: Switching function S vs. T.

4. Solution for the Nonlinear System

In the previous section we linearized the system equations and obtained some simple expressions for the time-optimal control. A two-pulse solution, based on analytic integration of state equations, was derived. These analytic expressions are only valid for transverse angular velocities much smaller than the axial spin velocity, and for small Euler angles. When the transverse angular velocity is not of negligible magnitude compared to the axial spin velocity or the angles get relatively large, the analytic solutions of Section 3 yield poor results. We consider the complete nonlinear equations of motion in this section and derive the necessary conditions for u^* to be an optimal control.

4.1. System Equations and Optimal Control Description

We wish to find the time history of the control $\mathbf{u}$ which takes our initial state to the desired state in minimum time. In order to write a state variable description of the system, we define the state $\mathbf{x}$ of the system as

$$\mathbf{x} = [\ \Omega_y \quad \Omega_z \quad \psi \quad \theta \quad \phi\ ]^T$$

and the control $\mathbf{u}$ as

$$u = \lambda_y$$

Eqs. (6)-(10) can now be written in the standard form.

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{g}u \tag{51}$$

where

$$\mathbf{f}(\mathbf{x}) = \begin{bmatrix} Ax_2 \\ -Ax_1 \\ (x_1 \sin x_5 + x_2 \cos x_5) \sec x_4 \\ x_1 \cos x_5 - x_2 \sin x_5 \\ 1 + (x_1 \sin x_5 + x_2 \cos x_5) \tan x_4 \end{bmatrix} \tag{52}$$

$$\mathbf{g} = [\ 1 \quad 0 \quad 0 \quad 0 \quad 0\ ]^T \tag{53}$$

The initial state $\mathbf{x}_0$ is given by

$$\mathbf{x}_0 = [\ x_{1,0} \quad x_{2,0} \quad 0 \quad 0 \quad 0\ ]^T$$

We want to find a control which will take this initial state to the desired state $\mathbf{x}_d$

$$\mathbf{x}_d = [\ 0 \quad 0 \quad x_{3,d} \quad x_{4,d} \quad \text{free}\ ]^T$$

while minimizing the total maneuver time.

Filippov[5] gives a theorem and proves the existence of an optimal control for a Mayer problem. This theorem covers the more specific case of time-optimal control of Eq. (4.1) under the given constraints and the boundary conditions. We note that the existence of an optimal control for the linearized system discussed in Chapter 3 can also be proven using the more general Filippov's theorem.

In order to derive an expression for the time-optimal control, we write the performance index as

$$J = \int_{T_0}^{T_f} 1 dt \tag{54}$$

with the given constraints

$$\mathbf{x} = \mathbf{f}(\mathbf{x}) + \mathbf{g}u \tag{55}$$

$$0 \le u \le u_{max} \tag{56}$$

We can also write the Hamiltonian

$$H = \mathbf{p}^T \dot{\mathbf{x}} - 1 \tag{57}$$

where $\mathbf{p}$ is the costate vector. The necessary conditions for u^* to be an optimal control are

$$\dot{\mathbf{x}}^* = \frac{\partial H}{\partial \mathbf{p}} = \mathbf{f}(\mathbf{x}^*) + \mathbf{g}u^* \tag{58}$$

$$\dot{\mathbf{p}}^* = -\frac{\partial H}{\partial \mathbf{x}} = \mathbf{H}(\mathbf{x}^*)\mathbf{p}^* \tag{59}$$

$$u^* = \begin{cases} u_{max} & \text{if } p_1^* > 0 \\ 0 & \text{if } p_1^* < 0 \end{cases} \tag{60}$$

and

$$\mathbf{x}(T_0) = \mathbf{x}_0 \tag{61}$$

$$x_1(T_f) = 0, x_2(T_f) = 0, x_3(T_f) = x_{3,d}, x_4(T_f) = x_{4,d} \tag{62}$$

$$p_5(T_f) = 0 \tag{63}$$

$$H(T_f) = 0 \tag{64}$$

Eqs. (58) and (59) are the differential equations for the state and costate vector. Eq. (60) is derived from the optimality condition, *i.e.*, maximizing the Hamiltonian H. Eqs. (61) and (62) are the given boundary conditions and Eqs. (63) and (64) are derived from the transversality conditions. Furthermore, we note from the theory of necessary conditions that

$$\frac{\partial H(\mathbf{x}^*,\mathbf{p}^*,T)}{\partial T} = \frac{dH(\mathbf{x}^*,\mathbf{p}^*,T)}{dT} = 0$$

This, in addition to Eq. (64), shows that

$$H(\mathbf{x}^*,\mathbf{p}^*,T) = 0 \text{ for all } T \in [T_0, T_f]$$

Hence, the time-optimal control problem is described by 10 scalar differential equations given by Eqs. (58) and (59). The boundary conditions on the state and costate variables are given by Eqs. (61)-(63). This constitutes a TPBVP (two point boundary value problem). Eq. **(64)** is used to determine T_f. We compare the results obtained in this section with the results that were obtained for the linearized equations of motion in Section **3.1.** The control obtained in both cases is bang-bang and the switching is determined by the costate variable corresponding to $x_1(=\Omega_y)$. However, the expression for the derivative of the costate vector in the nonlinear case cannot be integrated analytically. For this reason, no simple test for the optimality of the two-pulse solution for the nonlinear case (to be discussed in Section **4.2)** analogous to Section **3.2** can be devised. Also, unlike the linearized system where the Hamiltonian was a function of time, $\frac{\partial H}{\partial T} = 0$ and therefore $H \equiv 0$.

4.2. Two-pulse Solution

In Section 4.1 the necessary conditions for u^* to be an optimal control are derived. We find that the solution to the time-optimal problem involves integrating 10 differential equations with split boundary conditions and an unknown T_f. Instead of trying to solve this complex TPBVP (two point boundary value problem), we propose a method which requires integration of only the state equations with the unknown switch times.

In this section we follow the same procedure used previously in Section **3.2** to obtain the optimal switch times T_1, T_2, T_3, and T_f, except here we integrate the state equations numerically instead of analytically. This removes the required assumption of small angles and small transverse angular rates and still leads to the calculation of the two-pulse switch times T_1, T_2, T_3, and T_f. However, no optimality test, analogous to the one in Section **3.3,** can easily be devised because of our inability to integrate the costate equations analytically. Nevertheless, if the solution to the nonlinear problem is close to the solution of the time-optimal linearized problem, it is likely that the solution will be time-optimal. To verify this hypothesis, we generated several optimal trajectories by varying the initial costate variables. By comparing these trajectories with the trajectories generated by the two-pulse solution, it is verified that this hypothesis, *i.e.*, the two-pulse solution to the control of the nonlinear problem is time-optimal if it is close to the time-optimal solution of the linearized system, is indeed true.

4.2.1. Algorithm to Compute the Two-pulse Solution

The procedure used here is basically the same **as** in Section **3.2.** However, instead of working with linearized equations by assuming small angles and small transverse angular velocities, we will employ here the complete nonlinear equations of motion and integrate them numerically. We can solve this problem by assuming initial trial values for T_1, T_2, T_3 and T_f, integrating Eq. (51) numerically from T_0 to T_f, and then updating the four time parameters T_1 through T_f based on the difference of the desired final conditions and the computed final conditions, *viz.*, $x_1(T_f) - x_{1,d}$, $x_2(T_f) - x_{3,d}$, $x_3(T_f) - x_{3,d}$ and $x_4(T_f) - x_{4,d}$. However, **as** in Section **3.2,** we can separate the problem into two parts. The parameters T_1 and T_2 affect only the final Euler angles $x_3(T_f)$ and $x_4(T_f)$, whereas T_3 and T_f are chosen such that the final transverse angular velocity components $x_1(T_f)$ and $x_2(T_f)$ are zero. The parameters T_3 and T_f, as given by Eqs. (46) and (47), are simple analytic functions of T_1 and T_2. It should be noted that these equations involve no approximations. These have been obtained by integrating the transverse angular velocity equations, which are unaffected by small angle and small transverse angular velocity assumption. The algorithm to find T_1, T_2, T_3 and T_f is the following:

1. Assume T_1 and T_2
2. Integrate Eq. (51) from T_0 to T_1 with $u = 0$ and from T_1 to T_2 with $u = u_{max}$ (In order to avoid discontinuities in the middle of an integration step, integration is carried out in patched intervals with an integer number of steps in each interval)
3. Calculate T_3 and T_f from Eqs. (46) and **(47)**
4. Integrate Eq. (51) from T_2 to T_3 with $u = 0$ and from T_3 to T_f with $u = u_{max}$
5. If $|x_3(T_f) - x_{3,d}| < \epsilon$ and $|x_4(T_f) - x_{4,d}| < \epsilon$ then stop. Else
6. Update T_1 and T_2 (For simplicity, the Newton-Raphson update scheme is used)
7. Goto **2**

		Example 3	Example 4
Boundary Conditions	$x_{1,0}$	0.0666666701	-0.0209311595
	$x_{2,0}$	-0.0222222235	-0.0651306177
	$x_{3,d}$	-0.0041141893	-0.2981263563
	$x_{4,d}$	0.4101158694	-0.0595274245
Linearized System	T_1	1.72042500	0.34050073
	T_2	5.27150921	3.44888255
	T_3	8.71603244	6.75078968
	T_f	10.47287320	8.72622400
Nonlinear System	T_1	1.74532925	0.29424471
	T_2	5.23598775	3.54733834
	T_3	8.72664626	6.69524732
	T_f	10.47197550	8.72664626

Table 2: Comparison of linearized and nonlinear systems.

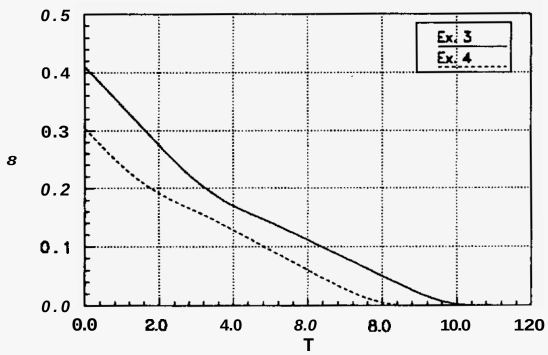


Figure 6: Total angle α vs. T.

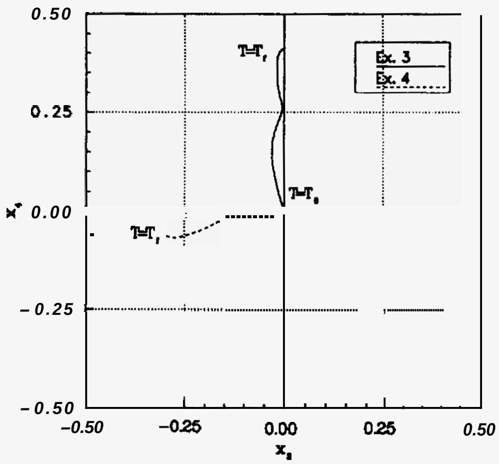


Figure 5: The path of x_b-axis in the 23-24 plane.

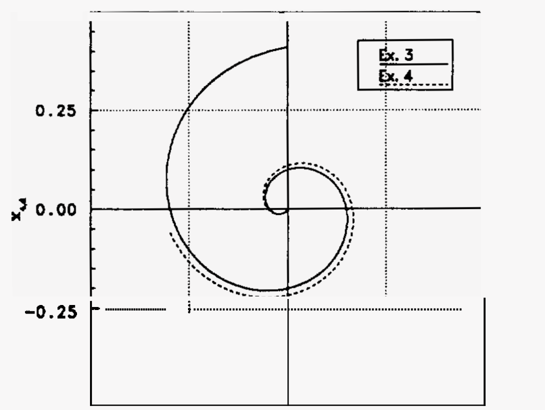


Figure 7: Path of the target in the $x_{3,d}$-$x_{4,d}$ plane.

5. Examples

We consider two examples here. The given initial conditions and the desired final conditions for the two examples are listed in Table 2. Also shown are the thruster turn-on and turn-off times obtained from the solution of both the linearized and the nonlinear problem for the two examples. It can be seen that the results for the nonlinear system are close to the results for the linearized system. Since we know that the results for the linearized system minimize the maneuver time, we conclude that the results for the nonlinear system also minimize T_f.

Figure 5 shows the path of the tip of a unit vector along the missile x_b-axis in the x_3-x_4 space, where x_3 and x_4 are the yaw and pitch angles measured with respect to the missile body axes at the start of the maneuver. The position of the target with respect to the moving missile body axis system can also be shown. As the missile x_b-axis moves toward the target direction, the angles $x_{3,d}$ and $x_{4,d}$ change with time. We define $a = \cos^{-1}(\cos x_{3,d} \cos x_{4,d})$. In other words, α is the total angular distance of the target direction with respect to the missile x_b-axis. Figure 6 shows the angle α as a function of the dimensionless time T. We see that the attitude change maneuver is completed in about 1.5 roll revolutions. In Figure 7 the position of the target direction relative to the moving missile body axis system, given by the yaw angle $x_{3,d}$ and the pitch angle $x_{4,d}$, is plotted as the maneuver proceeds. An observer fixed in the missile body will see the target move in this fashion. The attitude change maneuver is completed when $x_{3,d} = x_{4,d} = 0$.

The total transverse angular velocity $\Omega = \sqrt{x_1^2 + x_2^2}$ is plotted as a function of the dimensionless time T in Figure 8, where we recall $x_1 = \Omega_y$ and $x_2 = \Omega_z$. As expected, Ω becomes zero at the same time $a = 0$. Figure 9 shows the time history of x_1 and x_2 in the x_1-x_2 plane. When the x_1, x_2 trajectory radius, given by Ω, is constant in Figure 9, the missile coasts. Conversely, when the radius Ω changes, it means that the thruster is on.

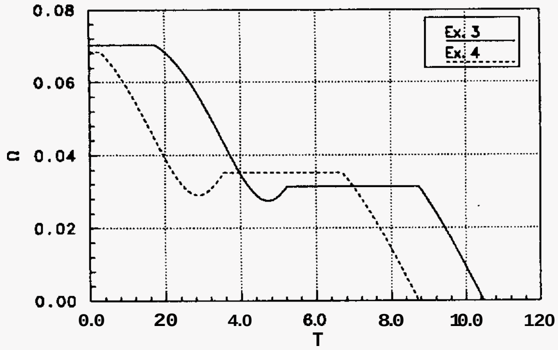
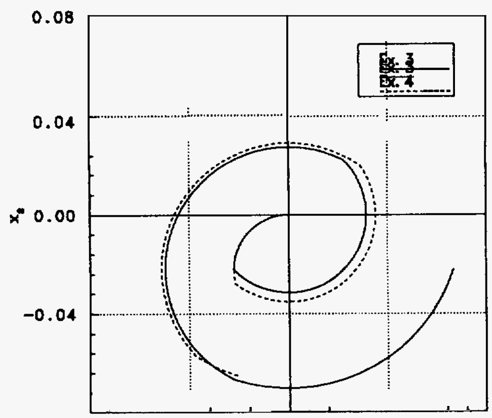


Figure 8: Total transverse angular velocity Ω vs. T.



6. Mechanization of the Control Scheme

The algorithm to find the thruster switch times which minimize the total maneuver time requires iterations. These iterations can be costly in terms of the time required for the solution to converge and also in terms of the complexity of the iterative procedure. Hence, this procedure cannot be used in real-time situations. The switch times can be stored on an on-board computer as functions of the boundary conditions. Table look-up and interpolation can then be used to compute the switch times and implement the attitude change maneuver.

In the exact solution, we compute T_1, T_2, T_3 and T_f as functions of the initial angular velocities and the desired Euler angles. A control law, however, can be devised based on T_1 and T_2 only. After the first thruster firing has been completed, we can measure the state variables at T_2. The switch times T_1 and T_2 can now be recomputed based on this measured state. These new T_1 and T_2 correspond to T_3 and T_f, respectively, for the previous T_1 and T_2. Thus for the new T_3 and T_f, $T_f - T_3 = 0$. In the presence of interpolation, numerical, or measurement errors this will not be quite true. Nevertheless, in reality this scheme would probably be superior because it can correct for system and measurement errors by introducing a feedback based on the latest state information.

7. Future Research

If the boundary conditions happen to lie outside the subset of the state space within which a two-pulse solution is time-optimal, the scheme given in Section 6 cannot be used. We can, however, use the dissertation[11] as well as in other future research papers.

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