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Scaling a Discrete Structural Model to Match Measured Modal Frequencies

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Introduction

AMONG the various possibilities for schemes of its kind, this study deals with the specific case where the computer model of a structure is to be upgraded on the basis of experimental measurements of normal mode frequencies only. It is implicit that the essential forms of the fundamental modes are already adequately identified in the starting computer model. The objective is to establish a rational procedure for predicting the modified structure (stiffness matrix) which correlates most closely with a given set of measured modal frequencies. The form of the computer model itself (conforming elements) is taken to be given and fixed.

An existing procedure for this identification problem, as developed and used in the Viking program, apparently originates with White.[1] Experience with the structural modeling methods used in this program are summarized in a report of Wada et al.[2] Subsequent application of such procedures in the Skylab project is described in a recent report.[3] White's derivation establishes his model modification scheme based on the first-order terms of a Taylor expansion of the altered model about the original one. The scheme allows prediction of a set of scale parameters, in number equal to or less than the number of available test-data points. In the latter case the parameters are determined to minimize the mean-square difference between measured and computed frequencies.

There is no room to improve on White's result within the framework of his analytical derivation. However, by taking a different approach, a more broadly applicable procedure becomes available. Its advantages are that 1) both the number and choice of structural parameters to be varied in the modification are arbitrary, and 2) the possibility is provided that a given revision of the computer model may be approached through a set of lower-order problems with (generally) improved efficiency. The present approach relates to modal energies; its derivation leads to a useful boundedness property for iterate steps toward the upgraded computer model.

Scaling Problem

For the linear eigenvalue problem, modal characteristics ϕ_j and λ_j of the jth mode are associated with the equation

$$([K]-\lambda_j[M])\{\phi_j\}=\{0\} \tag{1}$$

where λ_j represents the square of the modal frequencies. System stiffness $[K]$ and mass $[M]$ are given in terms of element measures k_i and m_i by

$$[K]=\sum_{i=1}^{S}[R_i]^T[k_i][R_i] \tag{2}$$

and

$$[M]=\sum_{i=1}^{S}[T_i]^T[m_i][T_i] \tag{3}$$

for the structural system with S elements. The associated functional, say E_j, is defined:†

$$E_j(K,\lambda_j,\phi_j^*)=\{\phi_j^*\}^T[K]\{\phi_j^*\}-\lambda_j\{\phi_j^*\}^T[M]\{\phi_j^*\} \tag{4}$$

Here ϕ_j^* represents any kinematically admissible field.

The actual eigenvectors ϕ_j are identified with stationary points (local minima) of $E_j(K, \lambda_j, \phi_j^*)$. Also for the actual eigenvalue

$$E_j(K, \lambda_j, \phi_j)=0 \tag{5}$$

Adopting the normalization

$$\{\phi_j^*\}^T[M]\{\phi_j^*\}=I \tag{6}$$

Eq. (4) provides

$$\lambda_j=\{\phi_j\}^T[K]\{\phi_j\} \tag{7}$$

Only a linear scaling is considered in the present development (this corresponds to what is done in prior treatments). That is, the modification of structure, associated with a given application of the procedure, is a linear function of a set of scale parameters δ_i. The structure $[K]$ obtained as the modification of starting structure $[K_0]$ is given by

$$[K]=[K_0]+\sum_{i=1}^{S}\delta_i[R_i]^T[k_i][R_i] \tag{8}$$

(In application the number of scale factors will generally be much less than S; in other words the δ_i would not all be independent.) Symbol $[\Delta]$ is introduced to represent the modification,

$$[\Delta]=\sum_{i=1}^{S}\delta_i[R_i]^T[k_i][R_i] \tag{9}$$

Equations (8) and (9) are used to substitute for $[K_0]$ in the expression for $E_j(K_0,\lambda_j^0,\phi_j^0)$. That is

$$\begin{aligned}E_j(K_0,\lambda_j^0,\phi_j^0)&=\{\phi_j^0\}^T[K_0]\{\phi_j^0\}-\lambda_j^0\\&=\{\phi_j^0\}^T[K]\{\phi_j^0\}-\{\phi_j^0\}^T[\Delta]\{\phi_j^0\}-\lambda_j^0=0\end{aligned} \tag{10}$$

Let the modification represent the difference between starting structure $[K_0]$ and modified structure $[K_e]$ with (test data) frequencies λ_j^e. With the introduction of this notation and the addition and subtraction of λ_j^e, Eq. (10) becomes

$$\{\phi_j^0\}^T[K_e]\{\phi_j^0\}-\lambda_j^e-\{\phi_j^0\}^T[\Delta_e]\{\phi_j^0\}+\lambda_j^e-\lambda_j^0=0$$

or

$$\begin{aligned}\{\phi_j^0\}^T[\Delta_e]\{\phi_j^0\}&=\lambda_j^e-\lambda_j^0+\{\phi_j^0\}^T[K_e]\{\phi_j^0\}-\lambda_j^e\\&=\lambda_j^e-\lambda_j^0+E_j(K_e,\lambda_j^e,\phi_j^0)\end{aligned} \tag{11}$$

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†Only modification of stiffness is to be represented.

Procedure for Computation

Since $\{\phi_j^0\}$ are kinematically admissible for structure $[K_e]$, $E_j(K_e, \lambda_j^e, \phi_j^0) \geq 0$. Also, by Eq. (5), $E_j(K_e, \lambda_j^e, \phi_j^*) \to 0$ as $\phi_j^* \to \phi_j^e$, where ϕ_j^e represent the (actual) eigenmodes for structure $[K_e]$. In view of these qualities, the relation

$$\{\phi_j^{n-1}\}^T[\Delta_n]\{\phi_j^{n-1}\} = \lambda_j^e - \lambda_j^{n-1} \tag{12}$$

is suggested from Eq. (11) as a basis for the iterative computation of modification $[\Delta_e]$. Equation (12) comprises a linear system in the nth iterate of scale factors δ_i.

If the starting solution for iteration meets the additional requirements that for given test information

$$\lambda_j^e - \lambda_j^0 \geq 0 \quad \text{all} \quad j \tag{13}$$

then the iterate values satisfy

$$\lambda_j^0 \leq \lambda_j^1 \leq \ldots \leq \lambda_j^k \leq \ldots \leq \lambda_j^e \tag{14}$$

The proof of this result is given in the Appendix. Condition (13) is easily met, requiring (at most) a single uniform scaling from a proposed set of starting values. The equality in Eq. (14) holds only if all $\delta_i^k = 0$; thus each iteration generally leads to an improved model.

Interpretation

The actual convergence of the computational scheme requires [as does the proof of Eq. (14)] that the trial eigenvector should be sufficiently close to the corresponding modal function $\{\phi_j^e\}$. This requirement is bound to be present in all linear modification schemes that are based on frequencies alone. The implication is simply that the user must be able to identify each test frequency with the appropriate model mode. The details of how this identification may be accomplished are given by White.[1] Difficulty occurs essentially only when two or more modal frequencies are nearly the same. In this case, a crossover of frequencies may occur during the iteration process. Such an occurrence indicates that the premise on which Eq. (14) is based has been violated.

As might be expected, an equation similar to Eq. (12) is common to all linear models for the modification problem. However, the form of application in the present computational procedure is quite distinct. Of course it is useful to have the nondivergence property of Eq. (14). Furthermore, our result frees a prior serious restriction on the permissible form of the modification itself. With the notation NP = number of scaling parameters and NT = number of test data values, White's procedure, for example, requires that $NP \leq NT$. The case $NP = NT$ leads to a determinate linear system. If $NP < NT$, his procedures lead to a least-squares approximation of the test values λ_j^e, i.e., the test values in fact cannot be matched.

In contrast, with the current model of the problem, the number of modification parameters is arbitrary within $NP < S$ = number of structural elements. If one chooses to have $NP \leq NT$, the result will be the same as White's just described. On the other hand in the more likely case of $NP > NT$, each iterate solution is associated with a determinate linear system. The appropriate set of equations for the system are the same as Eq. (12) with the right-hand side defined by:

$$\{\phi_j^{n-1}\}^T[\Delta_n]\{\phi_j^{n-1}\} = \begin{cases} \lambda_j^e - \lambda_j^{n-1} & \text{for} \quad j = 1,\ldots,NT \\ 0 & \text{for} \quad j = NT+1,\ldots,NP \end{cases} \tag{15}$$

Thus, the user has option to choose the modes $(NT+1) \leq j \leq NP$ as well as the correlate modes. That is, the modes for correlation with test data are selected on the basis of relative strain energy participation as indicated in the model solution,‡ and the same criterion is used for determination of the invariance modes $j > NT$.

The iteration based on algorithm (15) converges to match a set of correlate frequencies of arbitrary size. The size of the numerical problem may be controlled by generating the modified structure through a set of cascaded subproblems, with each subproblem based on a limited number of match frequencies. The procedure is called cascaded because each subproblem after the first one would start with the solution obtained in the prior subproblem. Clearly the set of subproblems should be ordered so that the test modal frequencies are taken into account in order of their total energy participation. In each subproblem, the right-hand side (R.S.) of Eq. (12) is given by

$$\text{R.S.} = \begin{cases} \lambda_j^e - \lambda_j^{n-1} & \text{for current match modes} \\ 0 & \text{for } j = (NT+1),\ldots,NP \\ & \text{and for all prior match modes} \end{cases}$$

Note that according to Eq. (14), the modifications achieved through all prior subproblems cannot be degraded.

Summary

An independent formulation has been given for the structure modification problem. The resulting procedure for solving the problem may be interpreted as an extension of White's method. This procedure can accommodate an arbitrary number of scaling parameters, while before the situation $NP > NT$ was classified as "underdetermined," and thus insoluble.

Bounds [Eq. (14)] have been established on iterate values of modal frequencies. Knowledge of these bounds permits flexibility in the application of the solution algorithm. Reduction of the original matching problem to a set of lower-degree problems is an example.

Appendix

The first part of Eq. (14) is proved by contradiction. Suppose

$$\lambda_j^1 < \lambda_j^0$$

From Eq. (7),

$$\{\phi_j^1\}^T[K_1]\{\phi_j^1\} < \{\phi_j^0\}^T[K_0]\{\phi_j^0\} \tag{A1}$$

Also, since ϕ_j^1 is kinematically admissible for structure $[K_0]$,

$$\{\phi_j^0\}^T[K_0]\{\phi_j^0\} \leq \{\phi_j^1\}^T[K_0]\{\phi_j^1\} \tag{A2}$$

Comparing Eqs. (A1) and (A2),

$$\{\phi_j^1\}^T([K_1] - [K_0])\{\phi_j^1\} = \{\phi_j^1\}^T[\Delta_1]\{\phi_j^1\} < 0 \tag{A3}$$

The requirement $\lambda_j^e - \lambda_j^0 \geq 0$ assures through Eq. (12) that $[\Delta_1]$ is a valid stiffness matrix ($[\Delta_1]$ is the structure with modes $\{\phi_j^0\}$ and frequencies $\lambda_j = \lambda_j^e - \lambda^0{}_j$). Therefore

$$\{\phi_j^1\}^T[\Delta_1]\{\phi_j^1\} \geq \{\phi_j^0\}^T[\Delta_1]\{\phi_j^0\} \geq 0$$

This contradicts Eq. (A3), so $\lambda_j^1 \geq \lambda_j^0$. The equality holds only if $\delta_i = 0$ for all i. To prove that $\lambda_j^1 \leq \lambda_j^e$, from Eq. (12)

$$\{\phi_j^0\}^T[\Delta_1]\{\phi_j^0\} = \lambda_j^e - \lambda_j^0 \tag{A4}$$

Substitute for $[\Delta_1]$ from $[K_1] = [K_0] + [\Delta_1]$:

$$\{\phi_j^0\}^T[K_1]\{\phi_j^0\} - \{\phi_j^0\}^T[K_0]\{\phi_j^0\} = \lambda_j^e - \lambda_j^0$$

‡The coefficients of δ_i are measures of unit strain energy of the ith (group of) element(s) in the jth mode.

Eliminate λ_j^0 using Eq. (7); thus,

$$\{\phi_j^0\}^T [K_I] \{\phi_j^0\} = \lambda_j^e \quad \text{(A5)}$$

But, since $\{\phi_j^0\}$ is compatible for structure $[K_I]$,

$$\{\phi_j^0\}^T [K_I] \{\phi_j^0\} \geq \{\phi_j^I\} [K_I] \{\phi_j^I\} = \lambda_j^I \quad \text{(A6)}$$

Between Eqs. (A5) and (A6), $\lambda_j^I \leq \lambda_j^e$.

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Effect of Catalyst Concentration on Burning Rate of Composite Solid Propellants

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Introduction

VARIOUS catalysts/additives have been known to affect the combustion of composite solid propellants and the oxidizer ammonium perchlorate (AP), but the exact mechanism of this phenomenon is by no means clear. Among the various types of catalysts/additives, the transition metal oxides such as Fe_2O_3, MnO_2, etc., form a special class, since they are known to modify AP decomposition, deflagration, and also the combustion of AP based sandwiches and composite propellants. The effect of these oxides has been the subject matter of study by various investigators. The data on this have been reviewed by Jacobs and Whitehead,[1] Hall and Pearson,[2] and also appeared in a recent paper by Boggs et al.[3]

In spite of the availability of much data in open literature on the quantitative effect of the catalysts/additives on the varieties of the propellant systems, no systematic study has been done on the decomposition and burning of the propellants as a function of catalyst concentration. It may be mentioned here that Burnside[4] has studied the burning of the AP/HTPB propellants at higher pressures as a function of Fe_2O_3 concentration. Although Burnsides' studies were on real systems, the effect of the catalyst on AP/HTPB cannot be singled out because the system did contain aluminum. Metal/metal oxide systems are known to form pyrotechnic mixtures.

The study of the catalyst concentration is important because the propellant composition can take only a limited amount of extra solid material apart from the solid oxidizer, which constitutes a major portion of the propellant composition. Second, the catalyst concentration must be as low as possible, so that it does not affect the propellant energy level. The present paper describes the effect of catalyst concentration on the burning rate of solid propellants containing AP and polystyrene (PS) only.

Experimental

The PS/AP propellants were prepared by the method described earlier.[5] Once recrystallized commercial grade AP was used in the propellant formulations. The particle size of AP used was between 53 to 106 μ. Analytical grade Fe_2O_3, Ni_2O_3, Co_2O_3, and MnO_2 of the same particle size as AP were used as catalysts. In the propellant composition, an oxidizer to fuel ratio equal to 3 was maintained constant, while the concentration of the additive alone was varied from 0.5% to 2% of the total propellant weight. AP + catalysts mixtures also were prepared, keeping the proportion of the catalyst the same as in the case of the propellant.

Burning rates of the propellants at ambient pressure and temperature (25°C in air) were determined in the following manner. Cylindrical propellant strands after curing were inhibited by coating with a zinc oxide enamel and then dried. Each strand was marked at two positions, and the distance between the two was measured using vernier calipers. The accuracy in the length measurement was 99%. The time taken for the strand to burn from one mark to the other was measured with a stop watch. The accuracy in the time measurement was 99%. The burning rate was determined from the length time data, and the accuracy achieved in the measurement was 95%.

The thermal decomposition studies on AP with and without catalysts were carried out by differential thermal analysis (DTA). A home-made DTA apparatus was used, and the details are given elsewhere.[6] The temperature was recorded

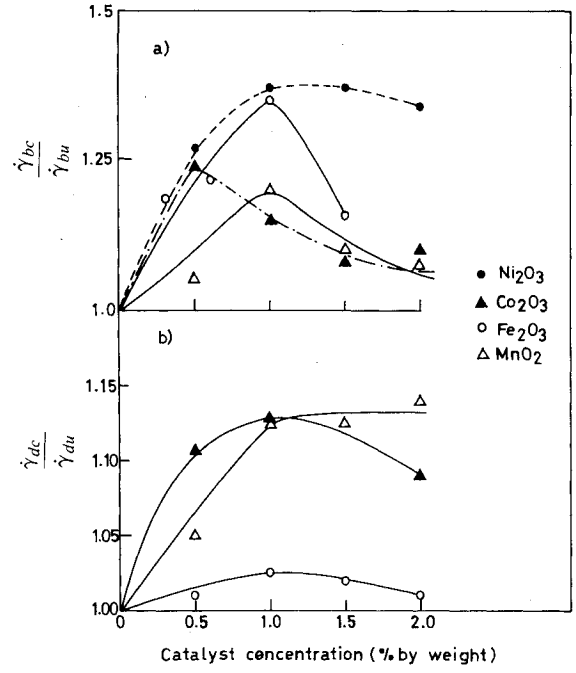

Fig. 1 Dependence of burning rate ratio and thermal decomposition ratio on the catalyst concentration in PS/AP propellant and AP systems, respectively.

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