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OPTIMAL CONTROL OF ORBITAL TRANSFER VEHICLES*

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Abstract

During the past two decades, considerable research effort has been spent to convincingly prove that the use of aerodynamic forces to assist in the orbital transfer can significantly reduce the fuel consumption as compared to the pure propulsive mode. Since in this aeroassisted mode, preliminary maneuvers in the vacuum effect the resulting performance in the atmospheric phase, and vice versa, the two, space and atmospheric maneuvers, are, to a great extent, coupled. This paper summarizes, via optimal control theory, the fundamental results in the problem of orbital transfer using combined propulsive and aerodynamic forces. For the atmospheric phase, the use of the Chapman's variables reduced the number of the physical characteristics of the vehicle and the atmosphere to a minimum and hence allows a better generalization of the results. The paper concludes with some illustrative examples.

I. Introduction

During the past two decades, considerable research effort has been spent to convincingly prove that the use of aerodynamic forces to assist in the orbital transfer can significantly reduce the fuel consumption as compared to the pure propulsive mode. An excellent review of aeroassisted orbit transfer covering an extensive literature has been presented by Walberg at the AIAA 9th Atmospheric Flight Mechanics Conference.[1] The pioneering research has been geared toward engineering feasibility and design concept based on some basic maneuvers such as in the problems of orbital plane change, return from High Earth Orbit (HEO) to Low Earth Orbit (LEO) and planetary aero-gravity capture. Just as in the sixties, during the period of development of the theory of optimal propulsive orbital transfer, the problems investigated have covered the full range, from high-thrust to low-thrust propulsive systems, from the simple Hohmann transfer to the complex multi-impulse rendezvous problem, it is expected that in the coming years the analysis of aeroassisted orbit transfer will become more and more involved.

Basically, the use of aeroassisted transfer aims at minimizing the fuel consumption which, for a high-thrust propulsive system, is measured by the characteristic velocity, the sum of all velocity changes, produced by applications of the thrust. This is schematically represented in Fig. 1 where S_o and S_f denote the initial and the final state, respectively.

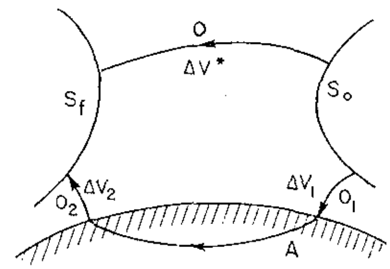

Fig. 1. Orbital maneuvers.

Let A be the space where aerodynamic force is involved. Let O be the best trajectory for a pure propulsive transfer in the vacuum resulting in a total characteristic velocity ΔV^*. Now, in some sense, if the states S_o and S_f are far apart, this characteristic velocity may be excessively high. We may consider an alternative by transferring the vehicle from S_o to A, through trajectory O_1, with a cost ΔV_1, then use aerodynamic maneuver without fuel consumption to move the state closer to S_f for a final transfer, through trajectory O_2, with a cost ΔV_1. This combined aerodynamic propulsive maneuver is optimal if

$$\Delta V_1 + \Delta V_2 < \Delta V^* \tag{1}$$

As a more concrete example, for a transfer from a HEO (state S_o) to a LEO (state S_f), the direction of the trajectory in the space A is such that the energy is decreasing, and furthermore if a plane change is involved (the angular distance between S_o and S_f is large), then this case is definitely a strong candidate for aeroassisted maneuver.

It is proposed, in this paper, to summarize via optimal control theory, the fundamental results in the problem of orbital transfer using combined propulsive and aerodynamic forces.

II. Optimal Control

The general problem is the problem of controlling the Orbital Transfer Vehicle (OTV), through the propulsive force $\vec{T}$, and the aerodynamic force $\vec{A}$, to bring it from the initial state, at time t_o,

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with position vector $\vec{r}_o$, velocity $\vec{V}_o$ and mass m_o, to the final state $\vec{r}_f$, $\vec{V}_f$ and m_f, at the final time t_f, such that a certain performance index, such as the final mass, is maximized (Fig. 2).

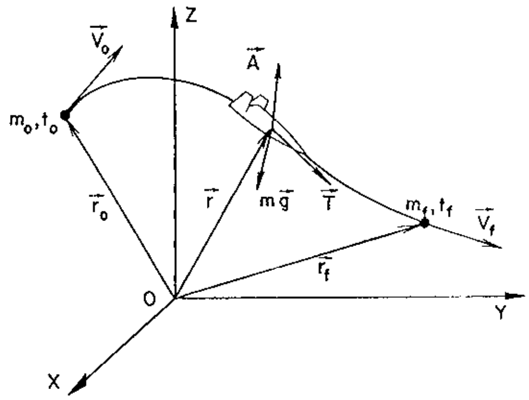

Fig. 2. Transfer trajectory.

The motion of the vehicle, considered as a point mass with varying mass, flying in a general gravitational force field and subject to aerodynamic force and thrusting force, is governed by the equations with standard notation

$$\frac{d\vec{r}}{dt} = \vec{V}$$
$$\frac{d\vec{V}}{dt} = \frac{1}{m}(\vec{T}+\vec{A})+\vec{g}(\vec{r},t) \tag{2}$$
$$\frac{dm}{dt} = -\frac{c}{g_o}T .$$

If we choose the operating mode such that the thrusting phase is always in the vacuum, $\vec{A} = 0$, and the aerodynamic maneuver is performed with power off, $\vec{T} = 0$, then only one vector control, $\vec{T}$ or $\vec{A}$, is active at any given time and we can derive independently the optimal control law for $\vec{T}$ and $\vec{A}$. This is typical for an aeroassisted mission profile. The results can be easily modified to account for a small variation of the mass in the case where a continuous thrust is applied to balance the aerodynamic drag for a turn inside the upper atmosphere.

In the case where the propulsive system is operating in the vacuum, it is advantageous for an analytical study to use the thrust acceleration

$$\vec{\Gamma} = \frac{\vec{T}}{m} \tag{3}$$

as the vector control. The mass, as variable, is then replaced by the characteristic velocity

$$C = \int_0^t |\vec{\Gamma}|\, d\tau \tag{4}$$

and as such, we replace the last equation in system (2) by

$$\frac{dC}{dt} = \Gamma \tag{5}$$

Thrust Control

Using the maximum principle, we introduce the adjoint vectors $\vec{p}_r$, $\vec{p}_V$ and p_c, associated to the state vectors $\vec{r}$, $\vec{V}$, and C, to form the Hamiltonian

$$H = \vec{p}_r \cdot \vec{V} + \vec{p}_V \cdot (\vec{\Gamma}+\vec{g}) + p_c \Gamma \tag{6}$$

The vector thrust acceleration must be selected such that, at each instant, it maximizes the Hamiltonian. Then $\vec{\Gamma}$ must be directed along the vector $\vec{p}_V$, called the primer vector.[2] We then have the maximized Hamiltonian

$$H^* = \vec{p}_r \cdot \vec{V} + \vec{p}_V \cdot \vec{g} + \sup [K\Gamma] \tag{7}$$

where

$$K = p_V + p_c \tag{8}$$

is called the switching function. Since the thrust is bounded

$$0 \le \Gamma \le \Gamma_{max} \tag{9}$$

we have the control law

$$\begin{aligned} &\Gamma = \Gamma_{max} \quad \text{when } K > 0 \text{ (maximum-thrust arc)} \\ &\Gamma = 0 \quad \text{when } K < 0 \text{ (coast arc)} \\ &\Gamma = \text{intermediate when } K \equiv 0 \text{ for a finite time interval.} \end{aligned} \tag{10}$$

Along the optimal trajectory, the state $\vec{x}$, and adjoint $\vec{p}$ must satisfy the canonical system

$$\frac{d\vec{x}}{dt} = \frac{\partial H^*(\vec{p},\vec{x},t)}{\partial \vec{p}}, \quad \frac{d\vec{p}}{dt} = -\frac{\partial H^*(\vec{p},\vec{x},t)}{\partial \vec{x}} \tag{11}$$

The solution to any specified case is obtained by solving a two-point boundary values problem. In particular, p_c = constant = -1 for a minimizing C and $K = p_V - \Gamma$. If the maximum thrust phase is approximated as an impulse, with time interval $\Delta t \to 0$, the magnitude of the primer vector, $p_V = 1$, at the point of the application of the impulse, and between two impulses, on a coast arc, $\Gamma = 0$, and $p_V < 1$. The problem is solved if the vector $\vec{p}_V$ is known on a coast arc.

Aerodynamic Control

If the propulsion system is inactive when $\vec{A} \neq 0$, then m = constant, and from system (2), we consider the Hamiltonian

$$H = \vec{p}_r \cdot \vec{V} + \vec{p}_V \cdot (\vec{a}+\vec{g}) \tag{12}$$

where $\vec{a}$ is the acceleration due to the aerodynamic force. Since the aerodynamic force $\vec{A}$ is decomposed into a drag force $\vec{D}$, opposite to the velocity $\vec{V}$, and a lift force $\vec{L}$ orthogonal to it, to maximize the Hamiltonian, the lift force has to be rotated such that the three vectors $\vec{A}$, $\vec{p}_V$ and $\vec{V}$ are coplanar, that is[3]

$$(\vec{V} \times \vec{p}_V) \cdot \vec{A} = 0 \tag{13}$$

This gives the optimal bank angle and we have the situation in Fig. 3.

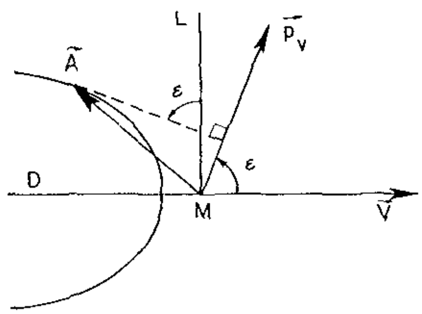

Fig. 3. Optimal lift control .

In the figure, because of the relations

$$L = \frac{1}{2}\rho S V^2 C_L \,, \quad D = \frac{1}{2}\rho S V^2 C_D, \quad C_D = f(C_L) \tag{14}$$

the terminus of the vector $\vec{A}$ moves along a drag polar. If ϵ is the angle between $\vec{V}$ and $\vec{p}_V$, the dot product $\vec{A}\cdot\vec{p}_V$ in the Hamiltonian is maximized when

$$\tan\epsilon = \frac{\partial C_D}{\partial C_L} \tag{15}$$

This formula provides the control law for the lift coefficient. Again, for aerodynamic control, it requires to know the primer vector $\vec{p}_V$ and the solution is obtained by integrating the system (11) subject to boundary constraints.

III. Integrals of the Motion

For the integration of the canonical system (11), the initial state $\vec{x}_0$ is prescribed at t_0 while the *final state* $\vec{x}_f$ is usually partially prescribed. This leads to a certain transversality conditions involving $\vec{x}_f$ and $\vec{p}_f$. The *simplest method is to choose* an initial value $\vec{p}_0$ for the integration and adjust it to ultimately satisfy the final and transversality conditions. However, $\vec{x}_f$ and $\vec{p}_f$ are extremely sensitive with respect to $\vec{p}_0$ and effort has been made to develop direct numerical techniques for the computation of the optimal trajectory. The survey paper by A. Miele contains 147 references on the subject.[4] To alleviate the computational effort, one may search for some exact and explicit relations *between the variables* $\vec{x}$, $\vec{p}$ *and* t. Such relations are called integrals of the motion. For example, m = constant is an integral along a coast arc while $p_V + p_c = 0$ is an integral along an intermediate-thrust arc. For both the thrust control phase and the aerodynamic control phase the most useful integrals are the following.

If the gravitational field is time invariant, we have the Hamiltonian integral

$$H^* = c_0 \tag{16}$$

where the constant c_0 is zero if the final time is not prescribed.

If the problem has a spherical symmetry, that is if the gravitational acceleration is central and only depends on the radial distance, and if we have similar dependence for the atmospheric density and engine performance, then we have the vector integral

$$\vec{A} = \vec{r}\times\vec{p}_r + \vec{V}\times\vec{p}_V \tag{17}$$

where $\vec{A}$ is a constant vector.

Furthermore, for flight in the vacuum, along a coast arc, T = 0, the motion is Keplerian, the *equations of motion can be integrated, and the* equations for the trajectory is obtained in closed form. By canonical transformation, the adjoint vectors are also obtained. For the time-free case, $H^* = 0$, we have the solution[5]

$$\vec{p}_V = \lambda_1\vec{V} + \vec{\lambda}_2\times\vec{r} + (\vec{r}\times\vec{V})\times\vec{\lambda}_3 + \vec{r}\times(\vec{V}\times\vec{\lambda}_3) \tag{18}$$

where λ_1 is a constant and $\vec{\lambda}_2$, $\vec{\lambda}_3$ are two constant vectors. Since $d\vec{p}_V/dt = -\partial H^*/\partial\vec{V} = -\vec{p}_r$, we deduce

$$\vec{p}_r = -\lambda_1\vec{g} + \vec{V}\times\vec{\lambda}_2 + \vec{V}\times(\vec{\lambda}_3\times\vec{V}) - \vec{g}\times(\vec{r}\times\vec{\lambda}_3) \tag{19}$$

where

$$\vec{g} = -\mu\,\vec{r}/r^3 \tag{20}$$

is the gravitational acceleration for a Newtonian central force field. By forming the cross products *in Eq. (17), we have a relation between the constant* vectors $\vec{A}$ and $\vec{\lambda}_i$

In addition, we have a scalar integral

$$2\,\vec{r}\cdot\vec{p}_r - \vec{V}\cdot\vec{p}_V = B \tag{21}$$

IV. Switching Relations for Impulsive Transfer

The vector integrals for $\vec{p}_V$ and $\vec{p}_r$ and the scalar integral B appear to have no practical value since they were obtained along a coast arc. But due to the outstanding work done by Marec[2] and Marchal,[5] they constitute, together with the Hamiltonian integral $H^* = 0$, and the vector integral $\vec{A}$, *the key to the solution for free-time, optimal im*pulsive transfer.

Consider a transfer orbit O between two impulses I_1 and I_2 . Along this orbit, and with a rotating coordinates system as shown in Fig. 4, the primer vector $\vec{p}_V$ is known, with radial component S, transverse component T and lateral component W. *At the point of the application of an impulse,* $\vec{p}_V$ is a unit vector and its magnitude is maximized for $p_V \le 1$. Hence, $d(p_V^2)/dt = -2\,\vec{p}_V\cdot\vec{p}_r = 0$. It suffices to write the existing relations at the point I_1 and I_2 respectively to have the pertinent relations for the optimal transfer. For example, we have

$$S_1^2 + T_1^2 + W_1^2 = 1 \;, \quad S_2^2 + T_2^2 + W_2^2 = 1 \tag{22}$$

where S_i, T_i, and W_i are the direction cosines of the impulse I_i . By eliminating the constants involved, Marchal has obtained the explicit relations[5]

$$1 + e\cos v_1 = \frac{(1-\cos\Delta)[1 - 2S_2^2 - S_1S_2 + \theta(S_1+S_2)T_2]}{1 + (S_1T_2 - S_2T_1)\sin\Delta - (S_1S_2 + T_1T_2)\cos\Delta - W_1W_2} \tag{23}$$

and

$$1+e\cos v_2 = \frac{(1-\cos\Delta)[1-2S_1^2-S_1S_2-\theta(S_1+S_2)T_1]}{1+(S_1T_2-S_2T_1)\sin\Delta-(S_1S_2+T_1T_2)\cos\Delta-W_1W_2} \qquad (24)$$

where v_1 and v_2 are the true anomalies, along the transfer orbit with eccentricity e, and

$$\Delta = v_2 - v_1 \ , \qquad \theta = \tan(\Delta/2) \qquad (25)$$

with Δ being the transfer angle.

Another useful relation is

$$\theta^3(T_2-T_1)(S_1+S_2)^2+\theta^2(S_1+S_2)[3-2S_1^2-2S_2^2-S_1S_2-3T_1T_2-W_1W_2]+\theta[2T_2-2T_1-T_1S_1^2+3T_1S_2^2-3T_2S_1^2+T_2S_2^2]+(S_1+S_2)[1-2S_1^2-2S_2^2+3S_1S_2-T_1T_2-W_1W_2] = 0 \qquad (26)$$

These relations are the optimal switching relations, and their usefulness will be displayed in the last section.

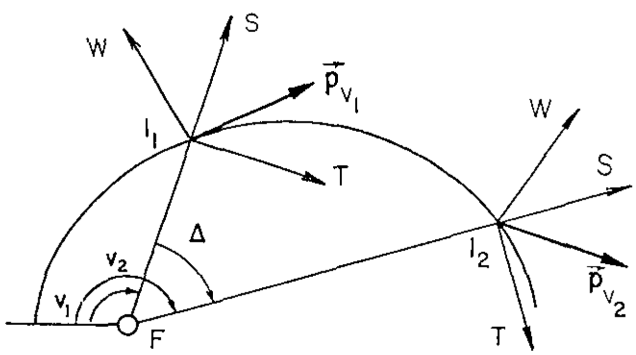

Fig. 4. Optimal switching .

V. Equations for Atmospheric Flight

While in orbital maneuver in the vacuum, especially for the case of time-free, impulsive transfer, the theory is very complete and has explicit solutions for many cases of practical interest,[2] the theory for optimal maneuver in the upper atmosphere using lift and drag modulation is still in the development stage. Although qualitatively the results obtained by various authors on the same type of problem, such as the problem of synergetic plane change, or the problem of aeroassisted return from a geosynchronous orbit (GEO) to a LEO, they are similar and hence, corroborating on each other, it is still difficult from a vast literature surveyed by Walberg[1] to obtain a definite conclusion for each specific problem due to the fact that, for atmospheric flight, the atmospheric density is not precisely known, and furthermore each OTV considered has its own physical characteristics. To have a unified approach, we have suggested using Chapman's variables[3,6]

$$Z = \frac{\rho S C_L^*}{2m}\sqrt{\frac{r}{\beta}} \ , \quad u = \frac{V^2}{gr} \qquad (27)$$

to represent the altitude and the speed variable, and the dimensionless arc length

$$s = \int_0^t \frac{V}{r}\cos\gamma \, dt \qquad (28)$$

to replace the time as independent variable. The drag polar used is the parabolic drag polar

$$C_D = C_{D_o} + K C_L^2 \qquad (29)$$

with the condition at maximum lift-to-drag ratio

$$C_L = C_L^* = \sqrt{\frac{C_{D_o}}{K}} \ , \quad C_D = C_D^* = 2C_{D_o} \ ,$$

$$E^* = \frac{C_L^*}{C_D^*} = \frac{1}{2\sqrt{KC_{D_o}}} \qquad (30)$$

Then, if σ is the bank angle, atmospheric control can be achieved through the modulation of the vertical and the lateral component of the normalized lift

$$\Lambda = \frac{C_L}{C_L^*}\cos\sigma \ , \quad \Omega = \frac{C_L}{C_L^*}\sin\sigma \qquad (31)$$

Then, with a Newtonian gravitational field and a locally exponential atmosphere, that is with the differential variation

$$d\rho = -\beta(r)\,\rho\,dr \qquad (32)$$

and with the spherical coordinates as shown in Fig. 5 for the position and velocity, we have the universal dimensionless equations of motion

$$\begin{aligned}
\frac{dZ}{ds} &= -k^2 Z\tan\gamma \\
\frac{du}{ds} &= -\frac{kZu(1+\Lambda^2+\Omega^2)}{E^*\cos\gamma} - (2-u)\tan\gamma \\
\frac{d\gamma}{ds} &= \frac{kZ\Lambda}{\cos\gamma} + 1 - \frac{1}{u} \\
\frac{d\theta}{ds} &= \frac{\cos\psi}{\cos\phi} \\
\frac{d\phi}{ds} &= \sin\psi \\
\frac{d\psi}{ds} &= \frac{kZ\Omega}{\cos^2\gamma} - \cos\psi\tan\phi
\end{aligned} \qquad (33)$$

In these equations, the only physical characteristic of the vehicle is its maximum lift-to-drag ratio, E^*, and the nature of the atmosphere is specified by the constant value $k^2 = \beta r$, called Chapman's atmospheric parameter. For the Earth's atmosphere, we have the value $k^2 = 900$.

Introducing the adjoint variables p_x, we form the Hamiltonian

$$H = -k^2 Z p_Z \tan\gamma - p_u\left[\frac{kZu(1+\Lambda^2+\Omega^2)}{E^* \cos\gamma} + (2-u)\tan\gamma\right]$$

$$+ p_\gamma\left[\frac{kZ\Lambda}{\cos\gamma} + 1 - \frac{1}{u}\right] + p_\theta \frac{\cos\psi}{\cos\phi} + p_\phi \sin\psi$$

$$+ p_\psi\left[\frac{kZ\Omega}{\cos^2\gamma} - \cos\psi \tan\phi\right] \qquad (34)$$

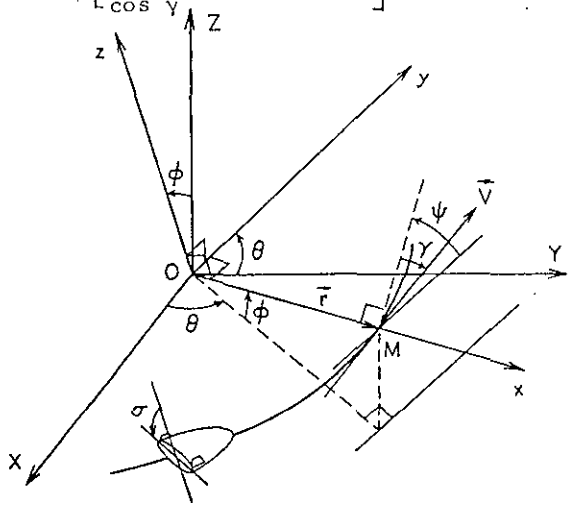

Fig. 5. Spherical coordinates for atmospheric flight.

With respect to the aerodynamic controls, Λ and Ω, the Hamiltonian is maximized either at their boundaries or, in the interior, in a modulated mode, such that

$$\Lambda = \frac{E^* p_\gamma}{2u\, p_u} \quad , \quad \Omega = \frac{E^* p_\psi}{2up_u \cos\gamma} \qquad (35)$$

Hence, as shown in the general vector formulation above, the optimal aerodynamic control is function of the adjoint variables p_u, p_γ, and p_ψ , components of the primer vector $\vec{p}_V$. By writing the adjoint equations, $dp_x / dt = - \partial H/ \partial x$, it is easy to verify the following integrals

$$H = c_0 \,,\quad p_\theta = c_1 \quad,\quad p_\phi = c_2 \sin\theta - c_3 \cos\theta \,,$$

$$p_\psi = c_1 \sin\phi + (c_2 \cos\theta + c_3 \sin\theta) \cos\phi \qquad (36)$$

The last three integrals can be deduced from the vector integral $\vec{A}$, as given in Eq. (17), by canonical transformation.[3]

It remains the integration of the equations for p_Z, p_u and p_γ. Explicitly, we have

$$\frac{dp_Z}{ds} = k^2 p_Z \tan\gamma + p_u \frac{ku(1+\Lambda^2+\Omega^2)}{E^* \cos\gamma} - p_\gamma \frac{k\Lambda}{\cos\gamma} - p_\psi \frac{k\Omega}{\cos^2\gamma}$$

$$\frac{dp_u}{ds} = p_u\left[\frac{kZ(1+\Lambda^2+\Omega^2)}{E^* \cos\gamma} - \tan\gamma\right] - \frac{p_\gamma}{u^2} \qquad (37)$$

$$\frac{dp_\gamma}{ds} = \frac{1}{\cos^2\gamma}\left\{ k^2 Z p_Z + p_u\left[\frac{kZu(1+\Lambda^2+\Omega^2)\sin\gamma}{E^*} + (2-u)\right] - kp_\gamma Z\Lambda \sin\gamma - 2kp_\psi Z\Omega\tan\gamma\right\}$$

Because of the Hamiltonian integral, we can delete one of these adjoint equations, but in practice to generate an optimal trajectory, for a specified initial condition, we can integrate the six state equations (33) and the three adjoint equations (37) while using the Hamiltonian integral as a check for the accuracy of the numerical integration. The three adjoint variables p_θ, p_ϕ and p_ψ are, of course, given by the exact integrals (36).

The optimal equations derived in this section can be used to obtain the solution to any unconstrained reentry problem, and as a special case the problem of plane change in an aeroassisted orbital maneuver.

For example, in the case of optimal aerodynamic turning we have the initial condition at entry

$$s = 0,\; Z_e,\; u_e,\; \gamma_e,\; \theta_e = \phi_e = \psi_e = 0 \qquad (38)$$

The speed u_e and flight path angle γ_e result from the preliminary space maneuver. The value Z_e is evaluated at the top of the sensible atmosphere. It is proposed to use lift and bank modulation to achieve a maximum plane change i , and hence we use the performance index

$$J = -\cos i_f = -\cos\phi_f \cos\psi_f \qquad (39)$$

At the final, exit point, we require that

$$Z_f = Z_e \,,\quad u_f = \text{prescribed},\quad \gamma_f = \text{free} \qquad (40)$$

Since the final time and the final longitude are not prescribed, we have the transversality conditions

$$c_0 = 0 \quad,\quad c_1 = 0 \qquad (41)$$

To start the integration, it requires selecting the values for the constants c_2 and c_3 , in the expressions for p_ϕ and p_ψ and the three initial values for p_Z, p_u and p_γ . But, since in the Hamiltonian and in the adjoint equations, the adjoint variables appear either linearly, or in the form of a ratio, through Λ and Ω as shown in Eq. (35), we can use one of the constants, say c_2, as a normalizing factor. This is achieved by dividing all the equations by c_2 and using the rescaled adjoints $\bar{p}_\phi = p_\phi / c_2, \ldots$, and $\bar{c}_3 = c_3/ c_2$. To simplify the notation, in the following discussion, we shall omit the bar in the variables. Then, we have to estimate four constants, but because of the Hamiltonian integral, only three parameters are truly independent. The problem of sensitivity in selecting these constants can be alleviated by an educated guess based on the knowledge of the control as follows.

From Eq. (36), at the initial time, we have

$$p_\phi(0) = -c_3 \quad,\quad p_\psi(0) = 1 \qquad (42)$$

Next, we notice that

$$\Lambda^2+\Omega^2 = \left(\frac{E^*}{2up_u}\right)^2\left(p_\gamma{}^2 + \frac{p_\psi{}^2}{\cos^2\gamma}\right) = \left(\frac{C_L}{C_L{}^*}\right)^2$$

$$\frac{\Omega}{\Lambda} = \frac{p_\psi}{p_\gamma \cos\gamma} = \tan\sigma \qquad (43)$$

Then, a correct guess of the initial bank angle, σ , and normalized lift coefficient, $C_L/C_L{}^*$, will provide a good estimate of $p_\gamma(0)$, and $p_u(0)$. Furthermore, for short range, $p_\phi \approx$ constant and if the final latitude is free the constant is zero. Hence c_3 is a small constant.

In summary, we have three parameters to be adjusted such that three final and transversality conditions are identically satisfied. The first condition is that when $Z = Z_f = Z_e$, the speed is equal to the prescribed speed u_f. The second condition is that since γ_f is free

$$p_{\gamma_f} = 0 \tag{44}$$

This condition implies that $\sigma_f = 90^\circ$.

For the final transversality condition, based on the performance index (39), we have

$$p_{\phi_f} = \frac{\partial J}{\partial \phi_f} = \sin\phi_f \cos\psi_f = c_2 \sin\theta_f - c_3 \cos\theta_f \tag{45}$$

$$p_{\psi_f} = \frac{\partial J}{\partial \psi_f} = \cos\phi_f \sin\psi_f = (c_2 \cos\theta_f + c_3 \sin\theta_f)\cos\phi_f$$

Taking the ratio of these equations and using c_3 for the ratio c_3/c_2 we have the transversality condition which must be identically satisfied at the final time

$$\tan\psi_f = \frac{1 + c_3 \tan\theta_f}{\tan\theta_f - c_3} \sin\phi_f \tag{46}$$

VI. Examples of Aeroassisted Transfer

As illustrative examples, we shall consider the following two problems.

Planar Rotation of Orbit

It is proposed to rotate, with minimum fuel consumption, the line of apses of an orbit by an angle 2α (Fig. 6).

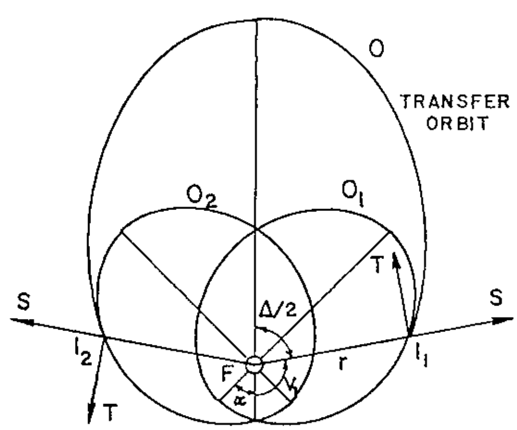

Fig. 6. Planar rotation of orbit.

The optimal solution is either by two impulses or via parabolic orbits.[7] For a two-impulse transfer, the solution is obviously symmetric as shown in the Figure. The given terminal orbits are defined by their common conic parameter P and eccentricity E. The unknown elements are the corresponding parameter p and e for the transfer orbit. If r is the radius at the impulses I_1 or I_2, it is defined by the true anomalies v_1 on the transfer orbit, and $(\alpha + v_1)$ on the initial orbit. We shall use the notation in section IV. In particular, if Δ is the transfer angle between impulses

$$\theta = \tan\frac{\Delta}{2}, \quad \frac{\Delta}{2} + v_1 = \pi, \quad \tan v_1 = -\theta \tag{47}$$

We define the new variable

$$x = \frac{r}{p} = \frac{1}{1 + e\cos v_1} \tag{48}$$

Then

$$e\cos v_1 = \frac{1-x}{x}, \quad e\sin v_1 = \frac{(x-1)\theta}{x} \tag{49}$$

Next, since $r = P/[1 + E\cos(\alpha + v_1)] = px$, we have

$$E\cos(\alpha + v_1) = \frac{y^2}{x} - 1 \tag{50}$$

where

$$y = \sqrt{\frac{P}{p}} \tag{51}$$

is another new variable. We now take benefit of the optimal switching relations in section IV. Let δ be the angle between the T-axis and the impulse I_1. By definition

$$T_1 = \cos\delta, \quad S_1 = \sin\delta \tag{52}$$

Because of the symmetry between I_1 and I_2, we have

$$T_2 = -T_1, \quad S_2 = S_1, \quad W_2 = W_1 = 0, \quad v_2 = -v_1 \tag{53}$$

Using these relations in the optimal condition (26), we have

$$(\theta\tan\delta - 1)[(1 + 2\theta^2)\tan\delta - (1 - \tan^2\delta)\theta] = 0 \tag{54}$$

The first factor is a parasite solution and from the second factor we have

$$\tan 2\delta = \frac{2\theta}{1 + 2\theta^2} \tag{55}$$

which is an interesting relation between the thrust angle and the transfer angle. On the other hand, if we use the symmetric condition in the optimal switching condition (23), we have

$$\frac{1}{x} = \frac{\theta^2(1 - 2\tan^2\delta - 2\theta\tan\delta)}{1 - 2\theta\tan\delta + \theta^2\tan^2\delta} \tag{56}$$

Using the condition (55) for simplification, we obtain

$$\theta\tan\delta = \frac{1}{x+1}, \quad \theta = \frac{(x+2)}{(x-1)}\tan\delta \tag{57}$$

Finally, if $\vec{V}^-$ and $\vec{V}^+$ are the velocity vectors before and after the application of the impulse, we have

$$\Delta\vec{V} + \vec{V}^- = \vec{V}^+ \tag{58}$$

By projecting this vector relation into the S and T axes, we have

$$S_1 \Delta V + \sqrt{\frac{\mu}{P}}\, E\sin(\alpha + v_1) = \sqrt{\frac{\mu}{p}}\, e\sin v_1$$

$$T_1 \Delta V + \frac{\sqrt{\mu P}}{r} = \frac{\sqrt{\mu p}}{r} \tag{59}$$

By eliminating ΔV, we have

$$E \sin(\alpha + v_1) = \frac{y(x+y+1)}{x} \tan \delta \qquad (60)$$

From the second of the Eq. (59), we deduce the total characteristic velocity $2\Delta V$

$$\frac{2\Delta V}{\sqrt{\mu / P}} = \frac{2y(1-y)}{x \cos \delta} \qquad (61)$$

Lawden has derived these same relations using the ordinary theory of maxima and minima.[8] We also *notice that the two equations in (57) is equivalent to* Lawden's equation

$$\tan^2 \delta = \frac{(x-1)}{(x+1)(x+2)} \qquad (62)$$

which shows that the maximum thrust angle is such that $\tan \delta_{max} = \sqrt{3} - \sqrt{2}$, $\delta_{max} = 17.632194°$. The equations are sufficient for solving the problem. For example, by squaring the Eqs. (50) and (60) *and adding, and with the aid of Eq. (62) we have*

$$E^2 = \frac{y^2(x+y+1)^2(x-1)}{x^2(x+1)(x+2)} + \frac{(y^2-x)^2}{x^2} \qquad (63)$$

On the other hand, by eliminating E between these equations, using Eqs. (57) and (62), we have

$$(x-1)[xy^2-(x+1)y-x(x+1)] \tan \alpha = (x+1)[(2x+1)y^2+(x^2-1)y-x(x+2)] \tan \delta \qquad (64)$$

For given E and α, the last two equations can be solved for x and y and the other elements of the transfer orbit can be deduced. The minimum total characteristic velocity, normalized with respect to circular speed at distance P is given by Eq. (61). This minimum cost for two-impulse transfer has to be compared with the cost for transfer via parabolic orbits. This mode, not considered by Lawden, is as follows. An impulse is applied at the perigee of the initial orbit to transfer the vehicle into a parabolic orbit, or in practice into an elongated elliptic orbit. At infinity the rotation of the line of apses is performed, free of fuel consumption since *the speed there is zero. After the rotation, the* vehicle is sent back to the perigee of the final orbit for insertion by another impulse. The total cost for this maneuver is

$$\frac{\Delta V_P}{\sqrt{\mu / P}} = 2\sqrt{2(1+E)} - 2(1+E) \qquad (65)$$

Since the cost is independent of the rotation angle, transfer via parabolic orbits is more economical when α is large.

Another free rotation can be achieved using aeroassisted maneuver, even in the case where the vehicle has no lift capacility. In this case, one decelerative impulse is applied at the apogee to reduce the perigee to the top of the sensible atmosphere. Then with the action of drag near perigee, the orbit will circularize. In the final circular configuration the perigee is open for selection hence, the rotation of the line of apses is achieved free of fuel consumption. A reverse maneuver will put the vehicle into the final orbit. To display explicitly that this aeroassisted maneuver can be optimal, we consider the case of a rotation of 180°, and with a perigee of the initial orbit slightly above the atmosphere. *Then the only significant cost is the cost* of transfer from very low circular orbit to the final elliptic orbit with

$$\frac{\Delta V_A}{\sqrt{\mu / P}} = (1+E) - \sqrt{1+E} \qquad (66)$$

In this special case, the two-impulse transfer becomes the Hohmann transfer, connecting the apogees, with solution $\delta = 0$, $x = 1$, $y = \sqrt{1-E}$. From Eq. (61), we deduce the cost

$$\frac{\Delta V_H}{\sqrt{\mu / P}} = 2\sqrt{1-E} - 2(1-E) \qquad (67)$$

Then, by comparing the last three equations, parabolic mode is better than the Hohmann transfer when $E > 0.535334$, while the Hohmann itself is better than the aeroassisted mode when $E > 0.836842$. Hence the Hohmann transfer is nonoptimal. Between the aeroassisted mode and the parabolic mode, aeroassisted transfer is better when

$$E < 4\sqrt{2}/9 = 0.628539 . \qquad (68)$$

Optimal Transfer from LEO to GEO and Return

The general problem of return from HEO to LEO with plane change is discussed elsewhere.[9] Here, we consider a particular case of minimum fuel transfer from a LEO at distance $r_1 = 6728$ km (i. e., 350 km altitude), with an inclination $i = 28.5°$, to a GEO at $r_2 = 42,241$ km. The radius of the atmosphere is $R = 6498$ km. After completion of its mission, the OTV returns to LEO.

In general, the problem involves the ratios (Fig. 7)

$$n = r_2 / r_1 \quad , \quad c = r_1 / R \qquad (69)$$

besides the plane change i and the maximum lift-to-drag ratio E* of the OTV. The transfer between non coplanar circular orbits is either by two impulses, or three impulses, or via parabolic orbits, depending on the given values n and i .

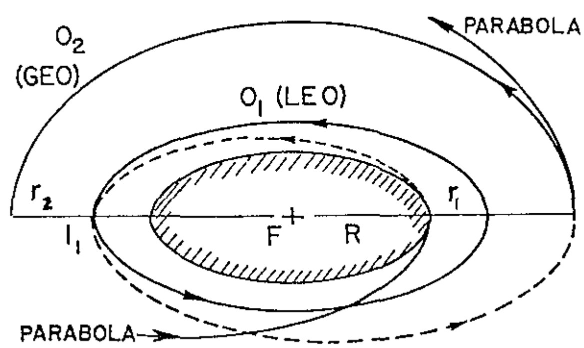

Fig. 7. Transfer between noncoplanar circular orbits .

In the case from LEO to GEO the optimal transfer is by two impulses, via a generalized Hohmann orbit, with plane change made at both impulses.

The switching relations in section IV are given with respect to this transfer orbit. The impulses are perpendicular to the position vectors and hence $S_1 = S_2 = 0$. If δ is the thrust angle with respect to the T axis, then, by definition

$$T_1 = \cos\delta_1,\ W_1 = \sin\delta_1,\ T_2 = \cos\delta_2,\ W_2 = \sin\delta_2 \tag{70}$$

From the velocity diagrams in Fig. 8, we have the relations

$$\frac{\Delta V_1}{V_1^-} = \frac{\sin i_1}{\sin\delta_1},\quad \frac{\Delta V_2}{V_2^+} = -\frac{\sin i_2}{\sin\delta_2} \tag{71}$$

with the constraining relations

$$i_1 + i_2 = i,\quad \frac{\sin(i_1+\delta_1)}{\sin\delta_1} = \sqrt{\frac{2n}{n+1}},$$

$$\frac{\sin(i_2+\delta_2)}{\sin\delta_2} = \sqrt{\frac{2}{n+1}} \tag{72}$$

For given n and i, to solve for the four unknowns δ_1, δ_2, i_1 and i_2, we need one more relation. This is obtained by the optimal switching relations (23) and (24). Since $v_1 = 0$, $v_2 = \pi$, $\Delta = \pi$ we deduce from these relations

$$\begin{aligned} 2\theta(S_1+S_2)T_2 &= (1+e)[1+T_1T_2 - W_1W_2] - 2 \\ -2\theta(S_1+S_2)T_1 &= (1-e)[1+T_1T_2 - W_1W_2] - 2 \end{aligned} \tag{73}$$

Eliminating the factor $\theta(S_1+S_2)$, which obviously has a finite limit, and noticing that $e = (n-1)/(n+1)$, we have

$$n\sin\delta_1 + \sin\delta_2 = 0 \tag{74}$$

The solution is obtained by solving the Eqs. (72) and (74). By combining these equations it can be shown that

$$\sin^2(\delta_1+\delta_2+i) = \frac{2n\{1-[2\sqrt{n}/(n+1)]\cos(\delta_1+\delta_2+i)\}}{[n^2+1-2n\cos(\delta_1+\delta_2)]}\sin^2(\delta_1+\delta_2) \tag{75}$$

For given n and i, we solve for $(\delta_1+\delta_2)$ and deduce the other elements of the transfer orbit. For the case considered, $n = 6.278389$, $i = 28.5^o$, we have $i_1 = 2.212^o$, $i_2 = 26.288^o$, $\delta_1 = 7.002^o$, $\delta_2 = -49.943^o$, with the resulting characteristic velocity, normalized with respect to circular speed at distance R

$$\frac{\Delta V_H}{\sqrt{\mu/R}} = 0.538068 \tag{76}$$

To return from GEO to LEO, for a pure propulsive mode, the reverse operation applies, and we have the same characteristic velocity.

Now, if aeroassist is considered, then for the return trajectory the following parabolic-aero-assisted mode is the absolute optimal. In this mode, an accelerative impulse is applied at GEO, to send the vehicle into a parabola. At infinity, the plane change, for any amount, is performed

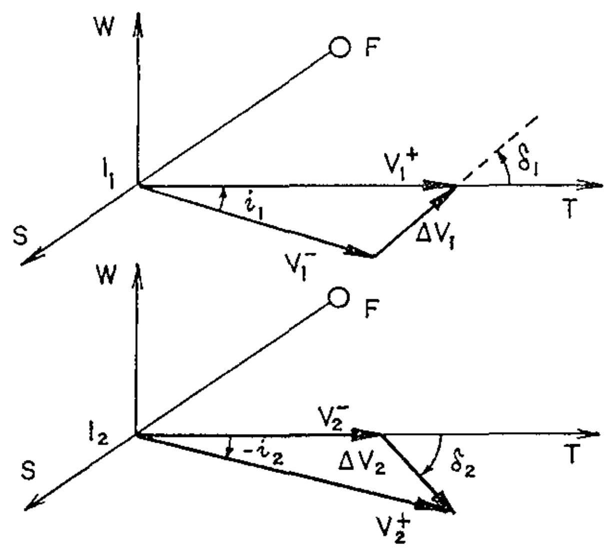

Fig. 8. Velocity diagrams as viewed toward center of attraction.

without fuel consumption. The vehicle then returns along a parabolic orbit grazing the atmosphere at distance R. Following an orbit contraction due to drag force acting at perigee, the apogee will decrease progressively to the LEO level where an accelerative impulse is applied for circularization. The total characteristic velocity for this mode is

$$\frac{\Delta V_{PA}}{\sqrt{\mu/R}} = \frac{1}{\sqrt{c}}\left[\sqrt{\frac{2}{n}} - \sqrt{\frac{1}{n}} + 1 - \sqrt{\frac{2}{c+1}}\right] \tag{77}$$

where n and c are the ratios defined in Eq. (69). With the given radii, this normalized characteristic velocity has the value 0.171043 which is only 32% of the cost for pure propulsive maneuver. Again, we notice that this parabolic-aeroassisted mode is independent of the amount of plane change and only requires drag capacity for the OTV. If we want an immediate return from GEO, then with a decelerative impulse applied at GEO, the vehicle returns with a plane change i_1 to enter the atmosphere at distance R with a small entry angle γ_e and a relatively high entry speed $u_e = V_e^2/gR$. The equations and method displayed in section V allows the computation of the atmospheric plane change, with exit speed sufficient for climbing to LEO altitude for circularization. In this case, we also have plane change without fuel consumption in the amount of $i_A = 22.5^o$. The remaining angle $i_1 = 6^o$ must be performed at GEO. The computation has been done with an OTV having a maximum lift-to-drag ratio $E^* = 1.5$. The total characteristic velocity, also normalized with respect to circular speed at distance R, is now 0.2040, and hence at about 38% of the cost for pure propulsive maneuver.

References

1. Walberg, G. D., "A Review of Aeroassisted Orbit Transfer," AIAA paper 82-1378, AIAA 9th Atmospheric Flight Mechanics Conference, San Diego, California, August 1982.

2. Marec, J. P., Optimal Space Trajectories, Elsevier Scientific Publishing Co., Amsterdam and New York, 1979.

3. Vinh, N. X., Optimal Trajectories in Atmospheric Flight, Elsevier Scientific Publishing Co., Amsterdam and New York, 1981.

4. Miele, A., "Survey paper--Recent Advances in Gradient Algorithms for Optimal Control Problems," JOTA, Vol. 17, 1975.

5. Marchal, C., "Optimization of Space Trajectories," IAF paper 76-010, 27th Congress of the IAF, Anaheim, California, October 1976.

6. Chern, J.S., and Vinh, N.X., "Optimum Reentry Trajectories of a Lifting Vehicle," NASA CR-3236, 1980.

7. Johannesen, J.R., Ph.D. thesis, The University of Michigan, in preparation, 1983.

8. Lawden, D.F., "Impulsive Transfer Between Elliptical Orbits," in Optimization Techniques, G. Leitmann Ed., Academic Press, New York and London, 1962.

9. Vinh, N.X., and Hanson, J.M., "Optimal Aeroassisted Return from High Earth Orbit with Plane Change," IAF paper 83-330, 34th Congress of the IAF, Budapest, Hungary, October 1983.