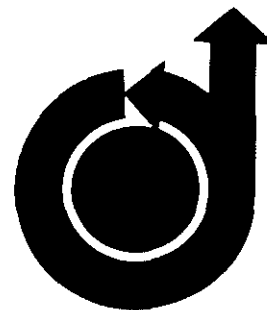# No. 68-169

NEAR WAKE OF THE RAREFIED PLASMA FLOWS AT MESOTHERMAL SPEEDS

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Abstract

This paper discusses the plasma interaction be-
tween a negatively charged body and a tenuous plas-
ma stream. The mean free path of the ions is many
orders larger than the size of the body which is,
in turn, large compared with the Debye length of
the ambient plasma. The speed of the free stream
is much larger than the ion thermal speed and yet
much smaller than the electron thermal speed. The
Vlasov-Poisson system of equations is solved by a
self-consistent method. The results of distribu-
tions of the charged particles and field potential
in the near wake under various ambient plasma and
body environments are presented. Discussion of the
validity of an often-used hydrodynamic approxima-
tion for the ion density and Maxwell-Boltzmann dis-
tribution for the electron density is given. The
significance of the results pertaining to the
ionospheric gasdynamics of an orbiting satellite
is discussed.

I. Introduction

The launching of artificial satellites and
spacecrafts has brought to bear a new challenge to
the aerodynamicists. The motion of a conducting
body, e.g., a satellite, in an extremely tenuous
and ionized gaseous medium such as the upper ionos-
phere, permeated by the geomagnetic field intro-
duces many new aspects to the gasdynamic problem.
It is obvious that the Knudsen number of the flow
($k_n = l/R$) where R denotes a characteristic·length
of the moving body; l the mean free path of the
ambient ions, is many orders larger than unity.
One is thus led to consider the problem in the
light of the gasdynamics of free molecules. The
fact, however, that the free stream particles of
interest are electrically charged adds great com-
plication to the problem in hand. Consider a non-
charged metallic body moving in an ionized gas at
a given temperature. The incident flux of elec-
trons which on the average out-race the ions will
deposit a net amount of negative charge to the
body assuming both species of particles are singly·
charged. The surface charges tend to repel the
electrons and attract the ions. This effect leads
finally to an equilibrium state of the surface po-
tential** (ϕ_s) such that the electron flux balances
the ion flux in the charge input to the body. The
time constant for a satellite to attain such an
equilibrium state is extremely short. The fact
that the body is charged and moves in a medium of
charged particles considerably complicates the
state of the "free molecules," e.g., the motion of
the charged particles is now under the influence
of the surface potential as well as the space
charge potential which, in turn, depends on the
motion of the charged particles themselves. It is
this coupling of the field potential and the part-
icles that gives the new complexion of the ionos-

pheric gasdynamics. The development in a plasma
of significant electric and possibly magnetic
fields that exert a strong and frequently decisive
influence on its motion is a fundamental feature of
ionospheric gasdynamics, distinguishing it from the
dynamics of neutral gases. This feature becomes
manifest to the fullest degree in the motion and
spreading of the free stream plasma into wake cav-
ity right behind a rapidly moving body.

In the above discussion, no specific mention is
made about the effect of an externally applied mag-
netic field which will exert on a moving charge an
additional force, the magnetic component of the
Lorentz force on a charged particle. It is also
noted that the presence of neutral particles in the
free stream is not considered. This stems from the
fact that in the dynamics of the collisionless
flows, the neutral and charged particles can be
treated independently.

This generalization of gasdynamics to the plasma
flows ramifies the gasdynamic variables of interest.
The new quantities include, among others, the dis-
tributions of field potential and charged particle
densities in the disturbed plasma near the body.
The electromagnetic consequences of the flow con-
tribute to much of the apparent anomalous phenomena
pertaining to the electromagnetic scattering from
a satellite moving in the upper ionosphere, often
called the ghost radar cross section phenomenon
and the ionospheric plasma resonances monitored in
the satellite ionograms,[1] etc. In other words, the
crude approximations in the classical electromag-
netic theory which neglects the coupling between
the electromagnetic and the gasdynamic aspects of
the plasma flows needs refinement when the phys-
ical situation demands.

As a modest start on the investigation of the
plasma interaction, the present study limits it-
self to the steady state problem. Consequently
the interesting aspects of instabilities, wave ex-
citations, etc., will be excluded. It is noted
that the disturbed plasma field near a rapidly
moving body can be conveniently divided into two
sub-regional fields: the sheath near the frontal
stagnation point facing the stream and the wake be-
hind the body. They are characterized as the re-
gion of condensation and the region of rarefaction
respectively. It is to the near wake that the
present study is directed.

II. Physical Model

In order to gain physical insight into the in-
teraction between the field potential and the
plasma particles associated with a rapidly moving
body we introduce an idealized physical model.
Consider a body of size R with a constant surface
potential ϕ_s ($\phi_s \leq 0$ = free stream field potential)

which moves in a rarefied plasma of singly-charged particles. It is assumed that the free stream plasma is in a bi-thermal equilibrium which has an electron temperature T_e not necessarily equal to the ion temperature T_i. Note that each species of particles is in thermal equilibrium by itself. This quasi-equilibrium state prevails in the upper ionosphere when it is exposed to the direct solar radiation. The free stream velocity $\underline{V}$ with respect to the coordinates fixed to the body is steady at a mesothermal speed which implies that $C_i \ll V \ll C_e$ where C_i and C_e denote the ion thermal speed $(2kT_i/m_i)^{1/2}$ and the electron thermal speed $(2kT_e/m_e)^{1/2}$, respectively. These conditions comply with typical satellite motions in the upper ionosphere.

The accommodation of the charged particles upon collision with the body must depend on the surface condition as well as the impact energy. For our present purpose where the kinetic energy of the incident particle is not more than a few electron volts, it is reasonable to assume that an electron, which collides with body, becomes absorbed; an ion, neutralized.

In plasma dynamics, the Debye length $d = (kT/4\pi e^2 n_o)^{1/2}$ is an important characteristic length which denotes the distance over which a static electric charge is screened by polarization of the ambient plasma with density n_o. We shall further assume that the present model satisfies the characteristic conditions: $l \gg R \gg d$ which comply with the typical satellite motions in the upper ionosphere. As an illustration we may cite a satellite[2] moving with a speed $V \approx 8$ km/sec at an altitude of 1000 km where $T_i \approx T_e \sim 3000°K$ ($C_i \sim 2$ km/sec, $C_e \sim 350$ km/sec), $d = 1$ cm, $l = 8$ km and ion Larmor radius $r_i = 10$ m. If the ion Larmor radius corresponding to the local geomagnetic field is small compared with R, it is expected that the magnetic field effect on the plasma disturbance in the near wake would be negligible. The condition $l \gg R$ decouples the gasdynamic effects of the neutral and the charged particles and makes it possible to treat the charged particles as collision-free.

In the following analysis we shall use the normalized quantities; e.g., the linear displacement $\underline{r}$ is in units of R; field potential ϕ, in kT_i/e (e = electron charge); velocity $(2kT_i/m_i)^{1/2}$; number density n_i in n_o, the free stream electron density.

III. Electron Distribution

The parameter $(V/C_e)^2$ can be considered as an index for the comparison of the rates of generation and annihilation of the electron gasdynamic disturbances as the body moves. Note that the thermal speed indicates the capacity of randomization which tends to restore the particle motions to their equilibrium state for a closed system. The given conditions* $l \gg R$ and $C_e \gg V$ make it possible to obtain simple approximations for the electron distribution in the following cases.

A. The Maxwell-Boltzmann Distribution

In the statistical mechanics of a thermal equi-

librium state, the distribution of the particles is governed by the Maxwell-Boltzmann law. Under the condition $V \ll C_e$, the electron arrangement can be approximated by the Maxwell-Boltzmann distribution provided the detailed balancings of the particle motions can be maintained. It is noted that an electron of sufficiently high energy can reach the body against the repelling surface potential and is lost to the system of the equilibrium free stream without replenishment. Consequently, the higher the value of the repelling surface potential, the smaller the loss of the high energy particles to the equilibrium system which is governed by the Maxwell-Boltzmann distribution

$$n_e = \exp(\phi T_i/T_e) \tag{1}$$

The electron loss to the body has been estimated.[3]

B. Field-Free Approximation

When the field potential is uniformly small, it is valid to approximate the electron distribution as if they were neutral particles. The neutral particle distribution in the wake can be obtained from integration of the Maxwellian distribution over the velocity space considering the interception of the particles by the body which, for a sphere,

$$N_{sph} = \frac{1}{2}[1 + \sqrt{1-(\rho^2+z^2)^{-1}}] \tag{2}$$

and for a long cylinder,

$$N_{cyl} = \frac{1}{2}[1 + \sqrt{1-(y^2+z^2)^{-1}}] \tag{3}$$

when the free stream speed is negligibly small compared with the thermal speed of the neutral particles. Adopting the Boltzmann factor, we can approximate the electron distribution in the wake of a sphere,

$$(n_e)_{sph} = N_{sph} \exp(\phi T_i/T_e); \tag{4}$$

of a long cylinder,

$$(n_e)_{cyl} = N_{cyl} \exp(\phi T_i/T_e) \tag{5}$$

provided that $V \ll C_e$ and $|\phi| \ll 1$.

Although the above approximation appears reasonable for bodies with zero surface potential, it turns out from the self-consistent analysis that the field potential $|\phi(\underline{r})|$ can have high values in the wake even with $\phi_s = 0$ as will be shown later.

IV. Equations Governing the Ion Distribution

In view of the condition $V \gg C_i$, the ion distribution in the wake is expected to deviate significantly from its equilibrium state. Under the disturbance of the moving body, the ions will stream in space with variable accelerations which

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*The mean free path of the electrons is larger than that of the ions.

depend upon the local field potential. An adequate coordinate system to represent the ion distribution must be, at least, a six-dimensional phase space $(\underline{r},\underline{c})$. With a rarefied plasma where the binary collisions are negligible, the ion distribution $f(\underline{r},\underline{c})$ which represents the phase density in the phase space is governed by the collisionfree Boltzmann equation together with the Poisson equation for the field potential $\phi(\underline{r})$ as follows:[4]

$$\underline{c}\cdot(\partial f/\partial \underline{r}) - \frac{1}{2}(\partial\phi/\partial\underline{r})\cdot(\partial f/\partial\underline{c}) = 0 \quad (6)$$

$$(d/R)^2 \Delta\phi = n_e - n_i \quad (7)$$

where $n_i = \int f d\underline{c}$. This system of equations, often called the Vlasov-Poisson system, when solved simultaneously with appropriate boundary conditions gives the self-consistent solution of the particle-field distributions for a given electron distribution $n_e(\phi)$.

The inherent mathematical difficulty of treating such a system of coupled, nonlinear equations has induced the use of various ad hoc approximations to decouple the system. One of the often-used scheme is to approximate the ion distribution as if they were neutral particles hence ignoring the field effect on their motion. Equation (7) is then used to solve for the field potential $\phi(\underline{r})$. The invalidity of such pseudo-neutral approximation to the near wake has been discussed and illustrated.[2] Note that with the sheath, which is sandwiched between a mesothermal stream and a blunt body facing the stream, the field component most influential to the ion motion aligns with the free stream. Inasmuch as the field effect on the ion motion which is assumed the same order as the ion thermal speed is small compared with the free stream ion speed and is justifiably neglected. On the other hand, with the wake where the cavity-filling action is primarily attributed to the thermal velocity normal to the free stream velocity there is no justification for ignoring the field effect in comparison with the thermal velocity effect.

An approach which gives self-consistent solutions for the ion density $n_i(\underline{r})$ and the field potential $\phi(\underline{r})$ has been introduced[2,5] and will be used here.

V. Ion Distributions

In the self-consistent approach[2,5] a formal solution to Equation (6) for the ion distribution f is obtained. It is then integrated over the velocity space with appropriate limits to determine the ion density functional $n_i(\phi)$. Equation (7), after substitution of $n_e(\phi)$ and $n_i(\phi)$, becomes a nonlinear, integro-differential equation for the field potential $\phi(\underline{r})$. The solution for $\phi(\underline{r})$ must be constructed which satisfies the boundary conditions at the body and the free stream.

To construct the formal solution for f from the collisionfree Boltzmann equation or Vlasov equation (6), we suppose to have an ion system s which is in steady state. As the ions in the system

move, the corresponding phase points will describe phase paths in the six-dimensional phase space $(\underline{r},\underline{c})$. Equation (6) represents the fact that the phase density is constant along the phase paths.* Within a specific $f = f(\underline{r},\underline{c})$ solving Equation (6) we can consider the field of directions defined by the tangent vectors $(\underline{c}, -\frac{1}{2}\partial\phi/\partial\underline{r})$. This field of directions is composed of the tangents of a one-parameter family of phase paths in that surface, called characteristics, which are determined by the system of ordinary differential equations[4]

$$-d\underline{r}/\underline{c} = d\underline{c}/\frac{1}{2}(\partial\phi/\partial\underline{r}) \quad (8)$$

Any phase path solving the system of Equation (8) is called the a characteristic of the first order quasi-linear partial differential equation (6). It is indeed obvious that Equation (8) is equivalent to the equations of motion of an ion. The integrals of Equation (8) will give us the integrals of motion of an ion. We shall proceed to evaluate these integrals for the following potential fields of specific symmetry:

1. Axisymmetric field which is appropriate for the wake behind a sphere or a circular disk placed normal to the free stream. In this case, Equation (8) in vectorial form is expanded in terms of the component phase coordinates c_ρ, c_θ, c_z; ρ, z (see the insert in Fig. 1). Two of the integrals of motion which are easily obtained are: (a) the conservation of energy,

$$c_\rho^2 + c_\theta^2 + c_z^2 + \phi = E, \quad (9)$$

where E denotes the energy constant of an ion; (b) the conservation of angular momentum with respect to the axis of symmetry (z),

$$\rho c_\theta = I_z \quad (10)$$

where I_z denotes the angular momentum of an ion. A third integral is needed to determine the ion orbit with given initial conditions for the axisymmetric problem in hand. Equation (8) does not, however, yield a third integral with a universal constant of motion as in (9) or (10). To search for a quasi-integral or locally constant of motion, we simplify the integration by carrying it out for small phase intervals during which the local values of c_ρ and c_z are assumed quasi-constant, thus obtaining

$$\rho - c_\rho z/c_z + (2c_z c_\rho)^{-1}\int\int(\partial\phi/\partial\rho)d\rho dz$$
$$- (2c_\rho c_z)^{-1}I_z^2\int\rho^{-2}dz = I_3 \quad (11)$$

Note that I_3 is only locally constant.

2. Two-dimensional symmetric field (see insert in Fig. 2) which is appropriate for the wake behind a long cylinder with axis along x-direction and the free stream along z-direction; or behind a long strip placed in the x,y-plane. Here the

*In statistical mechanics, it is known as Liouville's theorem.

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equation (8) is expanded in term of the component phase coordinates c_y, c_z; y, z (see Fig. 2). Again the relation of energy conservation is easily obtained

$$c_y^2 + c_z^2 + \phi = E . \qquad (12)$$

Inasmuch as the conservation of angular momentum does not lead to a nontrivial relation, we must develop a second integral to determine the ion orbit with given initial conditions for the problem in question. Again, Equation (8) does not yield another integral with a universal constant of motion. A search for a quasi-integral of motion with similar procedure as in the axisymmetric problem, i.e., by assuming c_y and c_z quasi-constants, we obtain

$$y - c_y z/c_z + (2c_y c_z)^{-1} \iint \frac{\partial \phi}{\partial y} \, dz dy = I_2 \qquad (13)$$

where I_2 is locally constant.

It is noted that with the assumption $V \gg C_1$, we expect that $(c_\theta^2, c_z^2, c_y^2) \ll c_z^2$ on the average.

The general solution of the partial differential equation (6) is an arbitrary function of the integrals of the system of ordinary differential equation (8).[4] We can thus construct the formal solution $f(\underline{r},\underline{c})$ as a function of the constants of motion (9), (10), and (11) for the wake with an axisymmetric field potential namely $f(E,I_z, I_3)$ behind a sphere; (12) and (13), with a two-dimensional symmetric field namely $f(E,I_2)$, behind a long cylinder. Such an arbitray function $f(E,I_z, I_3)$ or $f(E,I_2)$ would be of no interest to us unless it satisfies the boundary conditions prescribed according to the physical conditions of the problem in question. We shall determine the specific functional forms of the ion distribution f for the following physical problems:

A. Wake Behind a Sphere

Two physical requirements of the wake which must be satisfied are the undisturbed plasma state at infinity and the ion neutralizing sphere surface. Relative to a coordinate system fixed to the sphere, the undisturbed plasma state at infinity implies that the ion distribution function at infinity is a Maxwellian displaced by the free stream velocity $\underline{V}$, namely

$$f_s = \pi^{-3/2} \exp[-w^2-(c_z-V)^2-\phi] \qquad (14)$$

The ion distribution at the plane $z = 0$ outside of the sphere of radius R, which is large compared with the thickness of the local sheath, is also represented by (14). On the body surface,

$$f(\underline{r},\underline{c})|_{r=1,c_n > 0} = 0 \qquad (15)$$

where c_n denotes the ion velocity normal to the wall.

A solution to Equation (6) which satisfies the boundary conditions (14) and (15) is the following

ing

$$f = \frac{1}{2}(1+\text{sign } F)f_s \qquad (16)$$

where[6] sign $F = 1$ for $F > 0$ and sign $F = -1$ for $F < 0$. From the condition (15) and the assumption $|\phi| \ll V^2$, it can be shown that

$$F = |\rho - c_\rho z/c_z| - 1 = I_3^2 + (z^2/\rho^2)(I_z^2/E) - 1 \qquad (17)$$

Eq(16) degenerates to the displaced Maxwellian at the free stream where $\phi = 0$. Note with the condition $|\phi| \ll V^2$ where $|\phi|$ is of the order as the ion thermal energy, the f given in (16) is a function of the three integrals of motion (9), (10), and (11).

B. Wake of a Long Cylinder Placed Transversed to a Plasma Stream

Following a procedure and approximations similar to those for the wake of a sphere, we obtain the ion distribution in the wake behind a long cylinder,

$$f = \frac{1}{2}[1+\text{sign } G]f_c \qquad (18)$$

where

$$G = |y-c_y z/c_z|-1 = I_2^2 - 1 \qquad (19)$$

and

$$f_c = \pi^{-1}\exp[-c_y^2-(c_z-V)^2-\phi] \qquad (20)$$

VI. Self-Consistent Solutions

A. Equations of Field Potentials

The equation governing the field potential in a wake is given by the Poisson equation (7) in which the ion density $n_i(r) = \int f(\underline{r},\underline{c})d\underline{c}$.

1. Wake Behind a Sphere

The number density of ions in the wake behind a sphere is obtained by integrating the distribution function (16) over velocity space, where the velocity range is delineated by the condition $F = 0$ or

$$I_3^2 + (z^2/\rho^2)(I_z^2/E) - 1 = 0 \qquad (21)$$

Equation (21), after retaining only linear terms in ϕ in the integrals $\iint(\partial\phi/\partial\rho)d\rho dz$ and $\int \rho^{-2}dz$ which appear in I_3, is solved for c_z to obtain the limits of integration for c_z. In performing the integration for $n_i(\underline{r})$, it is convenient to consider the cases $\rho < 1$ and $\rho > 1$ separately.[5] For $\rho < 1$

$$n_1(\phi) = \frac{e^{-\phi}}{\pi} \int_0^\infty w e^{-w^2} dw \int_0^{2\pi} erfc\left\{ V + (1-\rho^2)^{-1} \right.$$

$$\times \left[\rho\left(zw \cos \alpha - (2w \cos \alpha)^{-1} \iint \frac{\partial \phi}{\partial \rho} d\rho dz \right. \right.$$

$$\left. + \frac{\rho^2 w \sin^2\alpha}{2 \cos \alpha} \int \frac{dz}{\rho^2}\right) - (z^2 w^2(1-\rho^2\sin^2\alpha)$$

$$\left. \left. - 2\phi(\rho^2-1)^2)^{1/2} \right] \right\} d\alpha \tag{22}$$

and for $\rho > 1$,

$$n_1(\phi) = \frac{e^{-\phi}}{\pi} \int_0^\pi w e^{-w^2} dw \left\{ 2(\pi - \sin^{-1}\rho^{-1}) \right.$$

$$+ \frac{1}{2} \int_{-\sin^{-1}\rho^{-1}}^{\sin^{-1}\rho^{-1}} erfc\left\{ V - (\rho^2-1)^{-1}\left[\rho\left(zw \cos \alpha \right. \right. \right.$$

$$- (2w \cos \alpha)^{-1} \iint \frac{\partial \phi}{\partial \rho} d\rho dz + \frac{\rho^2 w \sin^2\alpha}{2 \cos \alpha} \int \frac{dz}{\rho^2}\right)$$

$$\left. \left. - (z^2 w^2(1-\rho^2\sin^2\alpha) - 2\phi(\rho^2-1)^2)^{1/2}\right] \right\} d\alpha$$

$$+ \frac{1}{2} \int_{-\sin^{-1}\rho^{-1}}^{\sin^{-1}\rho^{-1}} erfc\left\{ (\rho^2-1)^{-1}\left[\rho\left(zw \cos \alpha \right. \right. \right.$$

$$- (2w \cos \alpha)^{-1} \iint \frac{\partial \phi}{\partial \rho} d\rho dz + \frac{\rho^2 w \sin^2\alpha}{2 \cos \alpha} \int \frac{dz}{\rho^2}\right)$$

$$\left. \left. - (z^2 w^2(1-\rho^2\sin^2\alpha) - 2\phi(\rho^2-1)^2)^{1/2}\right] - V \right\} d\alpha \right\} \tag{23}$$

2. Wake of a Long Cylinder

The number density of ions in the wake behind a cylinder which is tranversed to the free stream (V) is determined with a procedure and approximations similar to those for the wake of a sphere. For $y < 1$,

$$n_1(\phi) = e^{-\phi}\left\{ 1 - \frac{1}{2\sqrt{\pi}} \int_0^\infty dc_z e^{-(c_z-V)^2}\left\{erf\left[\frac{1}{z}\right. \right. \right.$$

$$\times \left(1+y+(2c_z^2)^{-1} \iint \frac{\partial \phi}{\partial y} dydz\right) c_z\right] + erf\left[\frac{1}{z}\right.$$

$$\left. \left. \left. \times \left(1-y-(2c_z^2)^{-1} \iint \frac{\partial \phi}{\partial y} dydz\right) c_z\right]\right\}\right\} \tag{24}$$

and for $y > 1$,

$$n_1(\phi) = e^{-\phi}\left\{ 1 - \frac{1}{2\sqrt{\pi}}\left\{\int_0^{-\infty} dc_z e^{-(c_z-v)^2}\left\{erf\left[\frac{1}{z}\left(1+y\right. \right. \right. \right. \right.$$

$$+ (2c_z^2)^{-1}\iint \frac{\partial \phi}{\partial y} dydz\right) c_z\right] + erf\left[\frac{1}{z}\left(1-y-(2c_z^2)^{-1}\right.$$

$$\left. \times \iint \frac{\partial \phi}{\partial y} dydz\right) c_z\right]\right\} + \int_0^\infty dc_z e^{-(c_z-v)^2}\left\{erf\left[\frac{1}{z}\left(1\right. \right. \right.$$

$$+ y+(2c_z^2)^{-1} \iint \frac{\partial \phi}{\partial y} dydz\right) c_z\right] + erf\left[\frac{1}{z}\left(1-y-(2c_z^2)^{-1}\right.$$

$$\left. \left. \left. \left. \times \iint \frac{\partial \phi}{\partial y} dydz\right) c_z\right]\right\}\right\}\right\} \tag{25}$$

The substitution of the ion density functional in the Poisson equation will result in a partial integro-differential equation in the field ϕ which must be iterated numerically to obtain a solution.

B. Iteration for Self-Consistent Solution

The self-consistent solution to the field potential and ion density is obtained by a specially devised numerical process of successive iteration (see Appendix). Briefly, the boundary value problem of the field potential ϕ for the semi-infinite wake region is mapped into a semi-circular region by coordinate inversion. A special numerical analysis which combines the techniques of over-relaxation and alternating-direction-implicit iteration is used to iterate the field potential equation successively until a self-consistent state is reached.

VII. Results and Discussions

The computed results of ion density, electron density and field potential distributions from the self-consistent solutions reveal some insights of the field-particle interactions in the near wake. We shall itemize them and discuss their significances.

A. The Potential Valleys in the Wake

When a large conducting body ($R \gg d$) with a negative surface potential moves at a mesothermal speed in a tenuous plasma ($R \ll l$), it is expected that both ions and electrons from the free stream, will rush into the wake-cavity created by the rapidly moving body. The electrons, on the average, move faster; consequently the electron and ion components of the plasma always tend to separate. This gives rise to an uncompensated electric charge which produces an internal electric field (the space charge potential), the latter retards the faster particles, namely the electrons and accelerates the slow ones, i.e., the ions thus hindering the charge separation. As a result, the inhomogeneous front moves and spreads with essentially a common mean speed, immediate between those of the ions and the electrons; the internal electric field exerts in this case a strong influence on the particle motion as a whole in the near wake. The potential valleys in the wake of a sphere (see Fig. 1) and of a long cylinder (see Fig. 2) are associated with the above-mentioned fronts of electron-rich mixture. The potential minimum forms a conically shaped surface based on the spherical moving body in question; a double wedge, on a two-dimensional cylindrical moving body. It is seen from the figures that the depth of the potential valley decreases as the body size (R/d) decreases.

The presence of such potential behind a satellite orbiting in the upper ionosphere suggests the possible existence of trapped ions in the valley.[1] The specific modes of oscillations of the trapped ions and electrons, plus the electron cyclotron modes, if the geomagnetic field is taken into account, appear to have shed light on the anomalous oscillations observed in the Ariel I and Alouette I and II satellites.

B. Comparison of the Approximations for Electron Distributions

It is recalled from the discussions in Section III that either the approximation (1) or (2) for the electron distribution is valid under restricting conditions. The results of self-consistent solutions with electron distributions prescribed by the two different approximations are compared (see Fig. 3) and show nontrivial discrepencies. It is worth noting that in view of an intense potential minimum, when $R \gg d$, even with vanishing surface potential, it appears more justifiable to use the Maxwell-Boltzmann approximation (1).

C. Gurevich's Approximation for Ion Density

In view of the tedious effort in obtaining a self-consistent solution of a plasma wake, it would be desirable to have a simpler approximation with reasonable accuracy. An idea of approximating the field effect on ion motion, first proposed by Gurevich,[2] is worthy of further pursuit. Comparing the free expansions of the charged and the neutral particles into the wake-cavity, the difference stems primarily from the lack of field action on the neutral particles. To estimate the influence of the field potential on the ions, one notes that the field potential which is partly caused by the electrons tends to increase the ion pressure. A total pressure of $k(n_e T_e + n_i T_i)$ to replace the ion pressure $k n_i T_i$ can be used with advantage to estimate the ion density in the far wake using free molecular flow theory of a neutral gas. It is of interest to compare the results from such an approximation and self-consistent solutions of a near wake (see Fig. 3). Note that the approximation becomes better for a bi-thermal ionosphere with $T_e > T_i$. "Hydrodynamic approximation" is also used.

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Appendix. Self-Consistent Solution by Iteration

The two basic steps include coordinate inversion in the unit circle and the iterative solution of Poisson's equation.

A. Coordinate Inversion

We transform Poisson's equation (7) from the physical plane into the image plane by inversion defined by Equations (A.1), (A.2), or (A.3), and (A.4), thereby converting the semi-infinite wake domain into a semi-circle image domain. For the sphere wake, coordinates transform as (primed quantities denote physical coordinates)

$$\rho = \frac{\rho'}{\rho'^2 + z'^2} \text{ and } z = \frac{z'}{z'^2 + \rho'^2} \qquad (A.1)$$

and potential transforms as

$$\psi(\rho,z) = (\rho^2 + z^2)^{1/2} \phi(\rho',z') \qquad (A.2)$$

where ψ is the image potential and ϕ is the electric field potential with coordinates related by Equation (A.1). For the cylinder wake, coordinates transform as

$$y = \frac{y'}{y'^2 + z'^2} \text{ and } z = \frac{z'}{z'^2 + y'^2} \qquad (A.3)$$

and potential transforms as

$$\psi(y,z) = \phi(y',z') \qquad (A.4)$$

where ψ is the image potential and ϕ is the electric field potential with coordinates related by Equation (A.3).

B. Iterative Solution of Poisson's Equation[7]

The nonlinear potential equations solved iteratively are, for the sphere

$$(d/R)^2 (\rho^2 + z^2)^{5/2} \Delta \psi = e^{(\rho^2 + z^2)^{1/2} \psi} - n_i(\psi) \qquad (A.5)$$

where $n_i(\psi)$ comes from Equations (22) and (23); and, for the cylinder

$$(d/R)^2 (y^2 + z^2)^2 \Delta \psi = e^{\psi} - n_i(\psi) \qquad (A.6)$$

where $n_i(\psi)$ comes from Equations (24) and (25).

The calculation was carried out numerically in finite differences as a discretized Dirichlet boundary value problem for the domain bounded by $\rho^2 + z^2 = 1$, $0 \le z \le 1$, and $0 \le \rho \le 1$, with boundary conditions $\psi = \psi_s = $ constant on $\rho^2 + z^2 = 1$, $\psi = 0$ along $0 \le \rho < 1$, and $\partial \psi / \partial \rho = 0$ along $0 < \rho < 1$, for the sphere; and the domain bounded by $y^2 + z^2 = 1$, $0 \le z \le 1$, and $0 \le y \le 1$, with boundary conditions $\psi = \psi_s = $ constant on $y^2 + z^2 = 1$, $\psi = 0$

along $0 \leq y < 1$, and $\partial \psi / \partial y = 0$ along $0 < z < 1$, for the cylinder. In each case, sphere or cylinder, to begin the iteration, an initial input $\psi^{(o)}$ from solving the Dirichlet problem for Laplace's equation $\Delta \psi = 0$, was computed.[3] Using these $\psi^{(o)}$ values, initial electron and ion densities were calculated. After this, iteration was carried out on Poisson's equation by a combined successive-overrelaxation and alternating-direction iterative technique. The solution ϕ for the electric potential was computed from the converged image potential ψ by use of the inversion relations (A.1), (A.2), or (A.3), and (A.4).

We shall now briefly describe the mathematics of the solution method itself. Let us record the five-point finite difference equations for uniform mesh h. For sphere wake, we have

$$(d/R)^2 (z^2+\rho^2)_o^{5/2} \{ (2\psi_o - \frac{2\rho+h}{2\rho} \psi_2 - \frac{2\rho-h}{2\rho} \psi_4) + (2\psi_o - \psi_1$$

$$-\psi_3)\} \; = \; h^2 [n_1(\psi_o) - e^{(z^2+\rho^2)_o^{1/2}} \psi_o] \qquad (A.7)$$

For cylinder wake, we have

$$(d/R)^2 (z^2+y^2)_o^2 \{ (2\psi_o - \psi_2 - \psi_4) + (2\psi_o - \psi_1 - \psi_3) \} = h^2 [n_1(\psi_o)$$

$$- e^{\psi_o}] \qquad (A.8)$$

In Equations (A.7) and (A.8) subscripts denote neighboring points.

In discretizing the Dirichlet boundary value problem for Poisson's equation, we replace the partial differential equation by a finite difference equation and we replace the region of interest by a set of mesh points in the region. In other words, we solve a finite set of simultaneous difference equations instead of the partial differential equation.

The totality of difference equations describing the discrete problem can be written as a vector equation

$$A \psi \; = \; \gamma \qquad (A.9)$$

In Equation (A.9) A is a square matrix whose entries are coefficients in the finite differences equation, the column vector ψ denotes the unknown vector whose components consist of values of ψ on the set of mesh points, and γ denotes the column vector whose components are boundary values of ψ and net charge densities computed from a previous iterate.

An effective iterative method of solving the wake potential problem in the above form is the following. We split the nonsingular matrix A into a sum of nonsingular matrices

$$A \; = \; D - E - F \qquad (A.10)$$

This can always be done.[7] In Equation (A.10), D denotes a diagonal matrix, E is a lower triangular matrix, and F is an upper triangular matrix. Let

$L \equiv D^{-1}E$ and $U \equiv D^{-1}F$. Then it can be shown that successive overrelaxation iteration is defined by

$$\psi^{(m+1)} \; = \; (I-\omega L)^{-1} \{ (1-\omega)I + \omega U \} \psi^{(m)} + \omega (I-\omega L)^{-1} D^{-1} \gamma \qquad (A.11)$$

In Equation (A.11), I denotes the identity matrix, ω is an overrelaxation factor, and $(I-\omega L)^{-1}$ and D^{-1} are nonsingular inverse matrices. Associated with a given difference operator, the nonsingular matrix A has also the natural splitting

$$A \; = \; H + V \qquad (A.12)$$

where, for the sphere from Equation (A.7)

$$H\psi_o \; = \; (d/R)^2 (z^2+\rho^2)_o^{5/2} \{ -\psi_1 + 2\psi_o - \psi_3 \} \qquad (A.13)$$

and

$$V\psi_o \; = \; (d/R)^2 (z^2+\rho^2)_o^{5/2} \{ - \frac{2\rho+h}{2\rho} \psi_2 + 2\psi_o - \frac{2\rho-h}{2\rho} \psi_4 \} \qquad (A.14)$$

and, for the cylinder from Equation (A.8)

$$H\psi_o \; = \; (d/R)^2 (z^2+y^2)_o^2 \{ -\psi_1 + 2\psi_o - \psi_3 \} \qquad (A.15)$$

and

$$V\psi_o \; = \; (d/R)^2 (z^2+y^2)_o^2 \{ -\psi_2 + 2\psi_o - \psi_4 \} \qquad (A.16)$$

Then it can be shown that an alternating-direction iteration is defined by

$$\psi^{(m+1)} \; = \; P_\alpha \psi^{(m)} + g_\alpha(\gamma), \quad m \geq 0 \qquad (A.17)$$

where matrix P_α and vector $g_\alpha(\gamma)$ have the definitions

$$P_\alpha \; \equiv \; (V+\alpha I)^{-1} (\alpha I - H)(H+\alpha I)^{-1} (\alpha I - V) \qquad (A.18)$$

and

$$g_\alpha(\gamma) \; \equiv \; (V+\alpha I)^{-1} \{ (\alpha I - H)(H+\alpha I)^{-1} + I \} \gamma \qquad (A.19)$$

The nonsingular matrix P_α is called the Peaceman-Rachford matrix, where α denotes an acceleration parameter.

The coupling of successive overrelaxation to alternating direction was done through introducing the residuals from overrelaxation into the vector $g_\alpha(\gamma)$ of alternating direction. This coupled procedure has been proposed as a method of attack on the Dirichlet boundary value problem in nonrectangular regions for mildly nonlinear elliptic difference equations. In each calculation in this

paper, the region was the interior of a unit quarter circle and the equation was Poisson's equation governing the wake electric field potential in the image space. The uniform mesh constant was h = .04.

FIGURE 1. WAKE OF A SPHERE

$$\left[\text{TAN}^{-1}\left(\frac{\rho}{z}\right) = \overset{\circ}{1}, \; T_i/T_e = 1, \; V/C_i = 8, \; \phi_s = -1 (\text{NORM})\right]$$

$$r = (z^2 + \rho^2)^{\frac{1}{2}}$$

1A   FIELD POTENTIALS

1B ION DENSITIES

1C ELECTRON DENSITIES
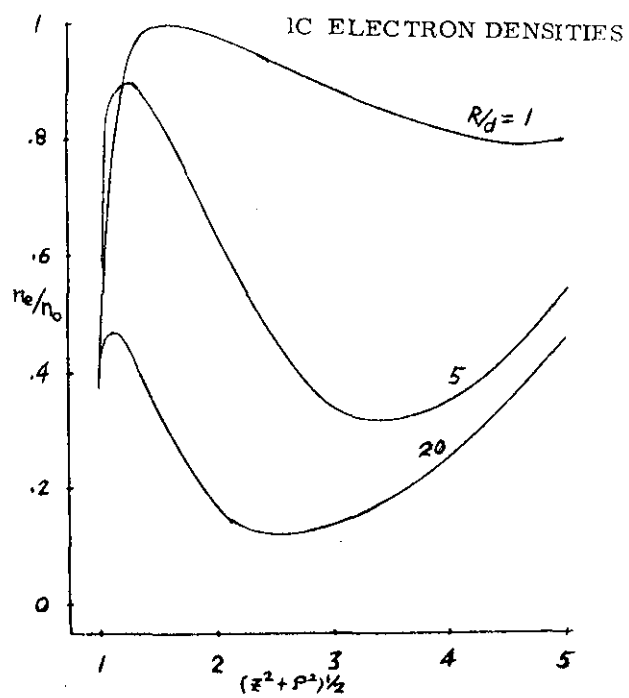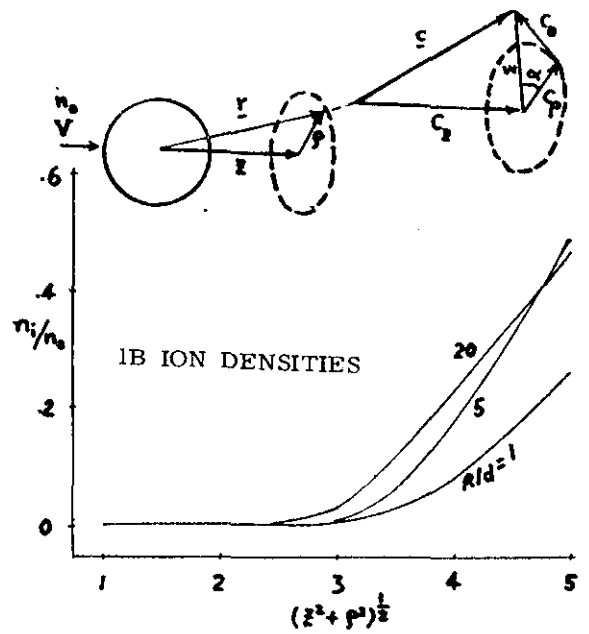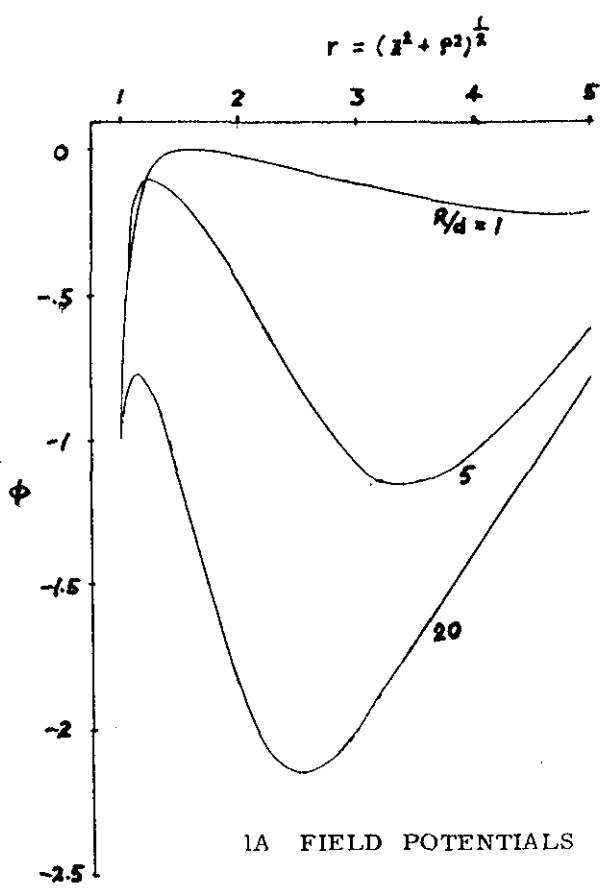
FIGURE 2. WAKE OF A CYLINDER

$$\left[\text{TAN}^{-1}\left(\frac{y}{z}\right) = \overset{\circ}{0}, \; T_i/T_e = 1, \; V/C_i = 8, \; \phi_s = -1 (\text{NORM})\right]$$

$$(z^2 + y^2)^{\frac{1}{2}}$$

2A FIELD POTENTIALS

n_o
V

n_i/n_o

.6

.4

.2

0

20
5
$R/d=1$

1 2 3 4 5

$(z^2 + y^2)^{\frac{1}{2}}$
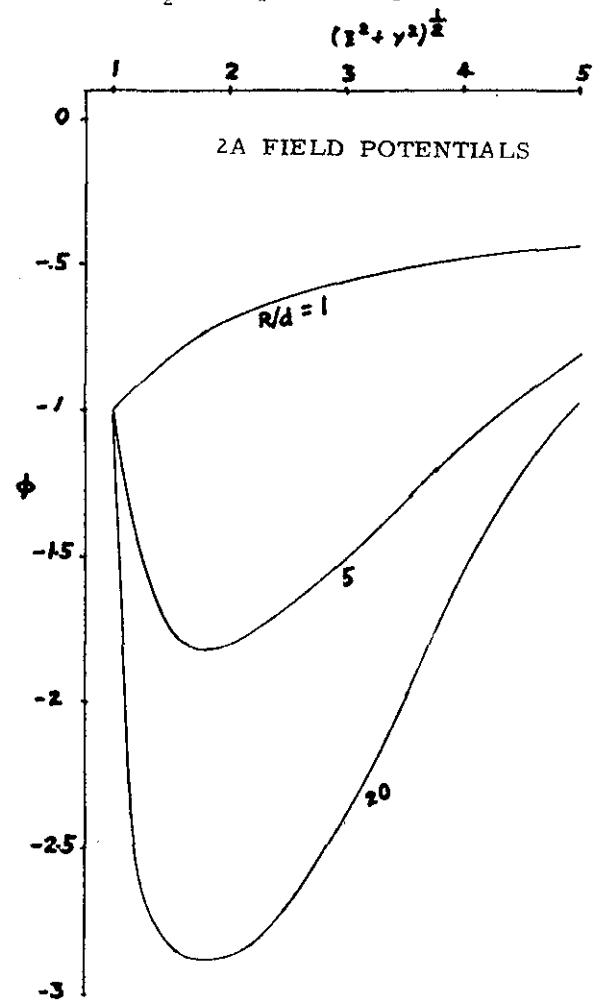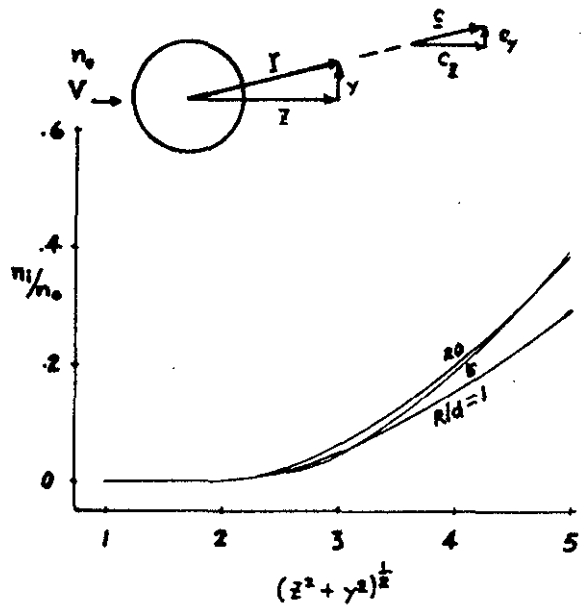
2B. ION DENSITIES

FIGURE3 COMPARISON OF ELEC. DISTRIB.

CYLINDER, $R/d=20$, $T_i/T_e=1$, $V/C_i=8$, $\phi_s=0$

AT $z=1$

n_e ———
n_i – – –

1.2

1

.8

.6

.4

.2

0

n_e/n_o, n_i/n_o

M-B

M-B

F-F

F-F

F-F = FIELD-FREE

M-B

F-F

M-B

0 .5 1 1.5 2

Y

1

.8

.6

.4

.2

0

n_e/n_o

$R/d=1$

5

20

1 2 3 4 5

$(z^2 + y^2)^{\frac{1}{2}}$

2C ELECTRON DENSITIES
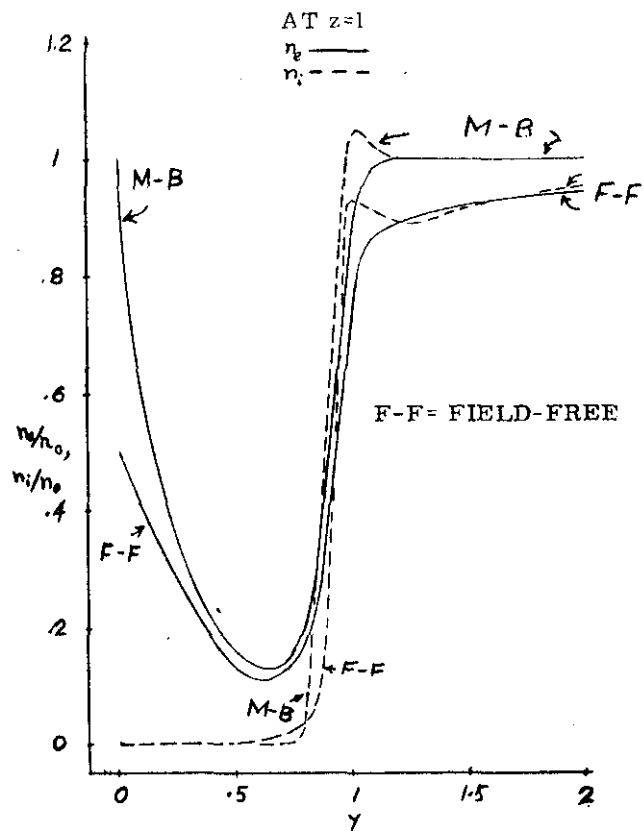
FIGURE4 COMPARISON OF APP.

CYLINDER, $R/d=20$, $V/C_i=8$, $\phi_s=-1$

$TAN^{-1}(\frac{y}{z})=0°$

1

.8

.6

.4

.2

0

n_i/n_o

S-C= SELF-CONSISTENT
P-N= PSEUDO-NEUTRAL

S-C ($T_e=5T_i$)

S-C ($T_e=T_i$)

Hyd. APP. ($T_e=5T_i$)

P-N ($T=T_i=T_e$)

1 2 3 4 5

$(z^2 + y^2)^{1/2}$

9