

REORIENTATION OF SPACE MULTIBODY SYSTEMS  
MAINTAINING ZERO ANGULAR MOMENTUM

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Abstract

The problem of reorientation of planar multibody systems in space with angular momentum preserving controls is studied. We consider rest-to-rest maneuvers for the absolute orientation of a multibody system which maintains zero angular momentum. We propose a control strategy for a system which is composed of (1)  $N$  planar rigid bodies interconnected by ideal pin joints in the form of an open kinematic chain, (2) joint torque motors which actuate the motions at the joints. The control strategy uses holonomy or geometric phase relationships. A key observation is that the holonomy, the extent to which a loop in the shape space (relative angle space) fails to be lifted to the configuration space (absolute angle space), depends only on the path traversed in the shape space and not on the time history of the joint angular velocities. The control strategy first transfers a given initial condition to the origin of the shape phase space. The control strategy then causes the state to track a loop in the shape space that achieves the desired holonomy. A feedback controller which implements this strategy thus accomplishes the desired objective. The proposed strategy is demonstrated by computer simulations of a three-link example. The theory developed in the paper is applicable to a variety of multibody control problems in space, including space robotics, astronaut maneuvers, satellite antenna deployment, etc., which are briefly described.

1. Introduction

In this paper we discuss control and stabilization problems for a system of  $N$  planar rigid bodies in space which are interconnected by frictionless joints in the form of a topological tree. Angular momentum preserving torques generated by joint motors are considered as controls<sup>8,15,17</sup>. The complete dynamics

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of a planar  $N$ -body system, with  $N \geq 3$ , have several strong accessibility and controllability properties but stabilization of a single equilibrium by smooth feedback is impossible; however, we present an explicit nonsmooth control which stabilizes a single equilibrium configuration. The contribution of this paper is the detailed study of rest to rest maneuvers of a three link configuration in space. The development in this paper is applicable to multibody systems with more than three links since maneuvers of an interconnection of more than three bodies can always be reduced to the case of three bodies by fixing the relative angles at an appropriate number of joints.

This paper is organized as follows. In Section 2, a mathematical model for a planar multibody system is derived. In Section 3, we summarize controllability properties associated with the mathematical model derived in Section 2. A general procedure for constructing a control strategy which transfers an arbitrary initial condition to a single equilibrium solution is described. In Section 4, we use the theoretical ideas introduced previously to construct a necessarily nonsmooth feedback control strategy for rest to rest maneuvers of a three link system in space. Section 5 consists of a summary of the main results and concluding remarks about possible space applications.

2. Problem Formulation

Consider a system of  $N$  planar rigid bodies interconnected by frictionless one degree of freedom joints in the form of an open kinematic chain. The configuration space of the planar  $N$ -body system is  $T^N \times R^2$  where  $T^N$  is the  $N$ -dimensional torus. Without loss of generality, we set the linear momentum of the system to zero and consider only rotation about the center of mass. This is equivalent to moving the frame of reference to the center of mass of the system of rigid bodies and describing the system dynamics with respect to this new frame of reference. Thus the configuration space, for an observer at the center of mass of the  $N$ -bodies, is  $M_q = T^N$ . The configuration space can be coordinatized by the vector of absolute angles of the  $N$  bodies

$$\theta = (\theta_1, \dots, \theta_N)'$$

Note that  $\theta$  is a generalized coordinate vector which completely describes the instantaneous configuration of the  $N$  bodies.

It should be emphasized that the subsequent development is assumed to be carried out for  $N \geq 2$ ; this should be understood even if it is not always explicitly stated.

The Lagrangian, in the absence of potential energy, is defined as a scalar function on the tangent bundle  $TM$ ,  $= T^N \times R^N$ , i.e.  $L: TM_q \rightarrow R$  is the kinetic energy of the system

$$L(\theta, \dot{\theta}) = \frac{1}{2} \dot{\theta}' J(\theta) \dot{\theta}$$

where  $\dot{\theta}$  is the vector of absolute angular velocities of the  $N$  rigid bodies and  $J(\theta)$  is an  $N \times N$  pseudo-inertia matrix which can be described as

$$J_{ij}(\theta) = \begin{cases} \tilde{I}_i & \text{if } i = j \\ \tilde{\lambda}_{ij} & \text{if } i \neq j \end{cases}$$

Here,

$$\tilde{\lambda}_{ij} = \tilde{h}_{ij}^1 \cos(\theta_j - \theta_i) + \tilde{h}_{ij}^2 \sin(\theta_j - \theta_i),$$

where  $\tilde{h}_{ij}^1$  and  $\tilde{h}_{ij}^2$  are constants, and  $\theta_j - \theta_i$  is the relative angle between the  $i$ 'th and  $j$ 'th bodies; also  $\lambda_{ij} = \lambda_{ji}$ . Here  $I_i$ ,  $i = 1, \dots, N$  are constants corresponding to the augmented inertias of the  $N$  bodies. Since we consider an open kinematic chain there are exactly  $(N-1)$  joints. Therefore we define the  $(N-1)$  relative angles (or joint angles) corresponding to the  $(N-1)$  joints:

$$\psi_{i-1} = \theta_i - \theta_{c(i)}, \quad i = 2, \dots, N;$$

where  $c(i)$  denotes the body label for the body contiguous and inboard to body  $i$  as in Sreenath<sup>17</sup>. Following the terminology of Li and Montgomery<sup>8</sup> we define  $M_s = T^{(N-1)}$ , the  $(N-1)$ -dimensional torus, as the shape space which can be coordinatized by the vector of joint angles

$$\psi = (\psi_1, \dots, \psi_{N-1})'$$

Note that the shape space consists of points which describe (angular) positions of the  $N$  bodies relative to each other, whereas the configuration space corresponds to the (angular) positions of the  $N$  bodies with respect to the absolute frame of reference.

The relationship between the vectors  $\theta$  and  $\psi$  is given by

$$\psi = P\theta \tag{1}$$

where  $P \in R^{(N-1) \times N}$  is a constant matrix with rows containing a  $(+1)$  and a  $(-1)$  as the only nonzero elements. Construction of  $P$  for a general tree interconnection requires significant additional notation; for the special case of a chain of  $N$  bodies,  $P$  can be expressed as

$$P_{ij} = \begin{cases} -1 & \text{if } j = i \\ +1 & \text{if } j = i + 1 \\ 0 & \text{otherwise} \end{cases}$$

It is easy to show that the matrix  $P$  is full rank.

Equations of motion are defined in terms of the Lagrangian function  $L(\theta, \dot{\theta})$  as

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}}(\theta, \dot{\theta}) - \frac{\partial L}{\partial \theta}(\theta, \dot{\theta}) = P' \tau \tag{2}$$

where  $\tau = (\tau_1, \dots, \tau_{N-1})'$  is the vector of joint torques. We assume each joint is actuated so as to permit free adjustment of the joint angles. We consider angular momentum preserving torques at the body interconnections. Thus the equations of motion are given by

$$J(\theta) \ddot{\theta} + F(\theta, \dot{\theta}) = P' \tau \tag{3}$$

$$\text{where } F(\theta, \dot{\theta}) = \frac{d}{dt} [J(\theta)] \dot{\theta} - \frac{1}{2} \frac{\partial}{\partial \theta} (\dot{\theta}' J(\theta) \dot{\theta}).$$

In order to define an equilibrium configuration, we assume that initial angular momentum is zero. The expression for the conservation of angular momentum is given by

$$1' J(\theta) \dot{\theta} = 0 \tag{4}$$

where  $1 = (1, \dots, 1)'$ . Equation (4) represents a classical nonholonomic constraint for  $N \geq 3$ , since the differential expression (4) is not integrable, i.e. it cannot be written as an exact differential if  $N \geq 3$ . This has important implications in terms of controllability properties of the system as will be shown in the subsequent development.

Any set of generalized velocities for the  $N$  body system must satisfy the angular momentum constraint. Thus, the dynamics of the system can be uniquely described by a set of  $N$  generalized coordinates and  $N-1$  generalized velocities. Hence the phase space of the system is a  $(2N-1)$ -dimensional space which consists of  $N$  independent coordinates and  $N-1$  independent kinematic characteristics for the system. The reduced order equations can be expressed in terms of the variables  $\theta_1, \psi_1, \dots, \psi_{N-1}, \dot{\theta}_1, \dots, \dot{\psi}_{N-1}$ . To obtain the equation corresponding to  $\theta_1$ , we express equation (4) as

$$1'J(\psi)(1e', +S\dot{\psi}) = 0 \quad (5)$$

where  $S \in R^{N \times (N-1)}$  is a suitably defined constant matrix. If the multibody system is a chain interconnection, then the matrix  $S$  is

$$S_{ij} = \begin{cases} 1 & \text{if } i > j \\ 0 & \text{otherwise} \end{cases} .$$

Using (5) we can write the reduced order equations of motion as

$$\dot{\theta}_1 \equiv \frac{1'J(\psi)S}{-1'J(\psi)1} \omega, \quad (6)$$

$$\dot{\psi} = \omega, \quad (7)$$

$$\dot{\omega} = -J_s^{-1}(\psi)F_s(\psi, \omega) + J_s^{-1}(\psi)\tau, \quad (8)$$

where  $J_s^{-1}(\psi) = PJ^{-1}(\psi)P'$  is the inverse of the inertia matrix on the shape space  $M_s$  and the vector function  $F_s(\psi, \omega)$  is given by

$$F_s(\psi, \omega) = \frac{d}{dt}[J_s(\psi)]\omega - \frac{1}{2} \frac{\partial}{\partial \psi}(\omega'J_s(\psi)\omega).$$

It should be noted that equations (6)-(8) completely describe the dynamics of a planar multibody system. Note that equation (6) is a reexpression of conservation of angular momentum which can be regarded as a nonholonomic constraint for  $N \geq 3$ . Note also that the right hand side of this equation is independent of  $\theta_1$ . Mechanical systems having similar properties are referred to as nonholonomic Caplygin Systems in the literature.<sup>1,2,3,13</sup>

Let  $M = T^N \times R^{N-1}$  denote the reduced dimensional state space of the system. Define the following state variables on  $M$

$$x_1 = \theta_1,$$

$$x_2 = (x_{2,1}, \dots, x_{2,N-1})' = (\psi_1, \dots, \psi_{N-1})',$$

$$x_3 = (x_{3,1}, \dots, x_{3,N-1})' = (\dot{\psi}_1, \dots, \dot{\psi}_{N-1})'.$$

Then the state space model associated to equations (6)-(8) is given by

$$\dot{x}_1 = s(x_2)'x_3, \quad (9)$$

$$\dot{x}_2 = x_3, \quad (10)$$

$$\dot{x}_3 = -J_s^{-1}(x_2)F_s(x_2, x_3) + J_s^{-1}(x_2)\tau, \quad (11)$$

where  $s(x_2) \in R^{N-1}$  is given by

$$s(x_2) = -\frac{S'J(x_2)1}{1'J(x_2)1}.$$

Now assume that  $N \geq 3$ . Then equation (9) which corresponds to the conservation law of angular momentum is a nonintegrable scalar equation. The nonintegrability condition can be stated as : the scalar analytic functions

$$H_{ij}(x_2) = \frac{\partial s_i}{\partial x_{2,j}} - \frac{\partial s_j}{\partial x_{2,i}}, \quad x_2 \in M_s, (i, j) \in I^2, \quad (12)$$

where  $I = \{1, \dots, N-1\}$ , cannot all vanish identically. Note also that equations (10)-(11) only, which represent the projection of the motion onto the shape phase space  $TM_s$ , are completely controllable and, consequently, are linearizable using feedback, in the sense that the transformation

$$u = -J_s^{-1}(x_2)F_s(x_2, x_3) + J_s^{-1}(x_2)\tau \quad (13)$$

where  $u \in R^{N-1}$  transforms equations (10)-(11) into (N-1) decoupled double integrators on  $TM_s$ .

Equations (9)-(11), using equation (13), can be expressed in the standard control system form

$$\dot{x} = f(x) + \sum_{i=1}^{N-1} g_i(x)u_i \quad (14)$$

where  $x \in M$ ,  $u \in R^{N-1}$  and

$$f(x) = (s(x_2)'x_3, x_3', 0)'$$

$$g_i(x) = (0, 0', e_i)'$$
 for all  $i \in I$ ,

are vector fields on  $M$ . Here  $e_i$  denotes the  $i$ 'th standard basis vector in  $R^{N-1}$ . Note that an equilibrium solution of equation (14) corresponding to  $u=0$  is given by  $x^e = (x_1^e, x_2^e, 0)'$ , where  $x_1^e \in R$ ,  $x_2^e \in R^{N-1}$ , and hence an equilibrium solution corresponds to a motion of the system for which all the configuration space variables remain constant.

Observe that an equilibrium solution  $x^e$  of equations (9)-(11) corresponding to  $\tau = 0$  has exactly the same form as above. In addition, since we consider  $\tau \in R^{N-1}$ , i.e. no torque constraints, the controllability and stabilizability properties of the system (9)-(11) and the system (14) are identical. This fact allows us to obtain controllability and stabilizability results for the complete dynamics of the planar multibody system by studying the transformed system.

### 3. Controllability and Stabilization

Note that control and stabilization of the relative orientations of the bodies is equivalent to control and stabilization on the shape phase space described by

equations (10),(11),(13); this is a relatively easy problem which can be solved using classical linear control design methods. However, our interest is in control and stabilization of the absolute orientations of the bodies, which requires use of the complete set of equations (9),(10),(11),(13), or equivalently equation (14). As we now demonstrate, this is a difficult control and stabilization problem which is inherently not linear.

The simplest approach to studying controllability of the nonlinear system described by (14) is to consider its linearization. The linearization of (14) about  $x = x^e$  and  $u=0$  can be written as

$$\delta \dot{x} = A \delta x + B \delta u \quad (15)$$

where

$$A = \begin{bmatrix} 0 & 0 & s(x^e)' \\ 0 & 0 & I_{N-1} \\ 0 & 0 & 0 \end{bmatrix},$$

$$B = \begin{bmatrix} 0 \\ 0 \\ I \end{bmatrix}.$$

Obviously, the linearization is not controllable since the Kalman rank condition is not satisfied. Thus the first order test for controllability is inconclusive.

We consider the nonlinear control system (14) and employ certain results of nonlinear control theory. We first recall some relevant definitions and results from the nonlinear control literature<sup>14</sup>.

Let  $\mathcal{R}(p,t)$  denote the reachable set from  $p$  in time exactly  $t$ .

**Accessibility** : The system (14) is accessible if for all  $p \in M$  and any  $T > 0$ ,  $\bigcup_{t \leq T} \mathcal{R}(p,t)$  has a nonvoid interior with respect to  $M$ .

**Strong Accessibility** : The system (14) is strongly accessible if for all  $p \in M$  and any  $T > 0$ ,  $\mathcal{R}(p,T)$  has a nonvoid interior with respect to  $M$ .

**Small-time Local Controllability (STLC)** : The system (14) is STLC from  $p$  if for any  $T > 0$ ,  $p$  is an interior point of  $\bigcup_{t \leq T} \mathcal{R}(p,t)$ .

We now summarize results for a system of  $N$  interconnected bodies of interest in this paper.

**Theorem 1<sup>15</sup>**: The complete dynamics of a planar multibody system, described by equations (9)-(11), is

not accessible for  $N=2$ .

**Theorem 2<sup>15</sup>**: The complete dynamics of a planar multibody system, described by equations (9)-(11), is strongly accessible for  $N \geq 3$ .

**Theorem 3<sup>15</sup>**: The complete dynamics of a planar multibody system, described by equations (9)-(11), is STLC from any equilibrium solution for  $N \geq 3$ .

These controllability properties are suggestive enough for studying the feedback stabilization problem for planar multibody systems with  $N \geq 3$ . We state the problem of feedback stabilization of the system (14) for  $N \geq 3$  as follows.

Consider static state feedback laws

$$u = U(x)$$

which satisfy, for some equilibrium solution  $x^e = (x_1^e, x_2^e, 0)'$ ,  $U(x^e) = 0$ , and which make the equilibrium  $x^e$  of the closed-loop system

$$\dot{x} = f(x) + \sum_{i=1}^{N-1} g_i(x) U_i(x), \quad x \in M$$

asymptotically stable.

We now ask the specific question : " Does there exist a  $C^1$ -feedback law which makes an equilibrium point  $x^e$  asymptotically stable? ". To give an answer to this question, we first examine the linearization (15). It is obvious that the linearization (15) has an uncontrollable eigenvalue at the origin; therefore the first order test for smooth stabilization is inconclusive, except for the fact that exponential stability cannot be achieved. Hence we require more sophisticated tests.

We now state a general result for planar multibody systems.

**Theorem 4<sup>15</sup>**: The complete dynamics of a planar multibody system, described by equations (9)-(11), can not be asymptotically stabilized by a  $C^1$  feedback law to any equilibrium solution.

This negative result implies that we must restrict study to the class of nonsmooth feedback controllers that may stabilize (9)-(11). Clearly, traditional methods (linearization, center manifold and zero dynamics approaches) are of no use. However, a nonsmooth feedback controller which asymptotically stabilizes an equilibrium can be constructed via the guidance of

the controllability properties possessed by the system. We state this fact as a formal result.

**Theorem 5:** The complete dynamics of a planar multibody system, described by equations (9)-(11), can be controlled from any initial state to any equilibrium solution for  $N \geq 3$ .

**Proof:** There is no loss of generality in assuming that the equilibrium solution is the origin, since any equilibrium solution  $x^e$  can be transformed to the origin via a change of coordinates. Therefore we shall prove this result by constructing a feedback control strategy that transfers any initial condition to the origin of the state space M.

Consider the system (14). Let  $N \geq 3$  and let  $x^0 = (x_1^0, x_2^0, x_3^0)'$  denote an initial condition.

We describe two steps involved in construction of a controller.

**Step 1 :** Bring the system to the origin of TM, , i.e. find a control which transfers the initial state  $(x_1^0, x_2^0, x_3^0)'$  to  $(x_1^1, 0', 0)'$  in a finite time  $t_1$ , for some  $x_1^1$ .

**Step 2 :** Traverse a closed path (a simply-connected closed curve) in the shape space  $M_s = T^{N-1}$  to produce the desired holonomy value  $-x_1^1$  in the configuration space  $M_q = T^N$ , thus guaranteeing that  $(x_1^1, 0', 0)'$  is transferred to  $(0, 0', 0)'$ .

For Step 1 we choose feedback functions given as

$$U_i^0(x) = -k_i \text{sign}(x_{2,i} + x_{3,i}|x_{3,i}|/2k_i) \quad (16)$$

where  $k_i, i \in I$ , are constants, which guarantee that at time  $t_1$  the joint angles and joint angular velocities are transferred to their desired values, i.e. to the origin of the shape phase space by the feedback control  $u = (U_1^0(x), \dots, U_{N-1}^0(x))$ .

For Step 2 the desired holonomy condition is given by

$$-x_1^1 = \oint_{\Gamma} s(x_2)' dx_2$$

where  $\Gamma$  denotes the loop traversed in  $M_s$ . The value of the line integral in the above expression is referred to as the holonomy or geometric phase of the closed path  $\Gamma$ . For notational simplicity in presenting the main idea, we assume that the desired holonomy can be obtained by a single closed path. In certain cases, traversal of multiple closed paths may be required to

produce the desired holonomy; for such cases  $\Gamma$  can be viewed as a concatenation of a series of closed paths.

Now let  $(i_0, j_0) \in I^2$  denote a pair of joints. Assume that for  $t > t_1$  only this pair of joints are actuated while all the other joints are kept fixed. This is equivalent to locking all the joints except the ones labelled  $i_0$  and  $j_0$  and treating the N bodies as three interconnected bodies, for  $t \geq t_1$ . In this case the desired holonomy condition can be written as

$$-x_1^1 = \oint_{\Gamma} \tilde{s}_{i_0}(x_{2,i_0}, x_{2,j_0}) dx_{2,i_0} + \tilde{s}_{j_0}(x_{2,i_0}, x_{2,j_0}) dx_{2,j_0}$$

where the scalar functions  $\tilde{s}_{i_0}(x_{2,i_0}, x_{2,j_0})$  and  $\tilde{s}_{j_0}(x_{2,i_0}, x_{2,j_0})$  are obtained by evaluating  $s_{i_0}(x_2)$  and  $s_{j_0}(x_2)$  at  $x_{2,i} = 0, \forall i \in I$  where  $i \neq i_0, j_0$ .

Consider the square path  $\Gamma^*$  on the  $x_{2,i_0}x_{2,j_0}$ -plane formed by the line segments  $\Gamma_1^*$  from  $(0, 0)$  to  $(z^*, 0)$ ,  $\Gamma_2^*$  from  $(z^*, 0)$  to  $(z^*, z^*)$ ,  $\Gamma_3^*$  from  $(z^*, z^*)$  to  $(0, z^*)$  and  $\Gamma_4^*$  from  $(0, z^*)$  to  $(0, 0)$ . Here  $z^*$  denotes the parameter describing the square path which yields the desired holonomy and can be precalculated as a function of  $x_1^1$  to achieve the desired holonomy. From now on we assume that the function  $z^*(x_1^1)$  is known.

The square path  $\Gamma^*$  can be regarded as a subset of  $M_s$ . In this case  $\Gamma^*$  is described by the four corner points

$$x_2^1 = 0, x_2^2 = z^* e_{i_0}, x_2^3 = z^*(e_{i_0} + e_{j_0}), x_2^4 = z^* e_{j_0};$$

where  $e_{i_0}$  and  $e_{j_0}$  are the  $i_0$ 'th and  $j_0$ 'th standard basis vectors in  $R^{N-1}$ . Then the four straight line segments,  $\Gamma_1^*, \Gamma_2^*, \Gamma_3^*$  and  $\Gamma_4^*$  connect  $x_2^1$  to  $x_2^2, x_2^2$  to  $x_2^3, x_2^3$  to  $x_2^4$  and  $x_2^4$  to  $x_2^1$ , respectively.

Note that since equations describing the motion in the shape space are completely controllable, a (non-smooth) feedback controller that causes the shape variables to track  $\Gamma^*$  can be constructed. We now present such a discontinuous feedback controller. Define on appropriate subsets of M, the feedback functions  $U^k(x) = (U_1^k(x), \dots, U_{N-1}^k(x))'$  for  $k = 1, 2, 3, 4$ , by

$$U_i^1(x) = -k_{i_0} \delta_{i i_0} \text{sign}(x_{2,i_0} - z^* + x_{3,i_0}|x_{3,i_0}|/2k_{i_0}), \quad (17)$$

$$U_i^2(x) = -k_{j_0} \delta_{i j_0} \text{sign}(x_{2,j_0} - z^* + x_{3,j_0}|x_{3,j_0}|/2k_{j_0}), \quad (18)$$

$$U_i^3(x) = -k_{i_0} \delta_{i i_0} \text{sign}(x_{2,i_0} + x_{3,i_0}|x_{3,i_0}|/2k_{i_0}), \quad (19)$$

$$U_i^4(x) = -k_{j_0} \delta_{i j_0} \text{sign}(x_{2,j_0} + x_{3,j_0}|x_{3,j_0}|/2k_{j_0}), \quad (20)$$

where

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

Here  $k_i$ ,  $i \in \mathbf{I}$  are arbitrary positive constants, and  $z^*$  is chosen such that the desired holonomy condition is satisfied.

Under the ideal model assumptions, the feedback control  $u = U^0(x)$  steers all of  $M$  to the origin of the shape phase space in finite time, and the feedback control formed by  $u = U^k(x)$ ,  $k = 1, 2, 3, 4$ , produce the desired holonomy, thereby transferring the system to the origin. In particular,  $u = U^0(x)$  transfers the initial condition to some equilibrium state  $(x_1^1, 0', 0')$ .  $u = U^1(x)$  transfers  $(x_1^1, x_2^1, 0')$  to  $(x_1^2, x_2^2, 0')$ ;  $u = U^2(x)$  transfers  $(x_1^2, x_2^2, 0')$  to  $(x_1^3, x_2^3, 0')$ ;  $u = U^3(x)$  transfers  $(x_1^3, x_2^3, 0')$  to  $(x_1^4, x_2^4, 0')$  and finally  $u = U^4(x)$  transfers  $(x_1^4, x_2^4, 0')$  to  $(0, 0', 0')$ , i.e. to the origin. Here  $x_1^k$ ,  $k = 1, 2, 3, 4$  denote the absolute angular positions (of body 1) corresponding to the four corner points of the desired closed path  $\Gamma^*$ . The feedback controller  $u$  can be expressed as a piecewise analytic function as follows;

$$u = \begin{cases} 0 & ; x = 0 \\ U^0(x) & ; x \in M - \{x | (x_2', x_3') \in T\Gamma^*\} \\ U^1(x) & ; (x_2', x_3') \in T\Gamma_1^* \\ U^2(x) & ; (x_2', x_3') \in T\Gamma_2^* \\ U^3(x) & ; (x_2', x_3') \in T\Gamma_3^* \\ U^4(x) & ; (x_2', x_3') \in T\Gamma_4^* \end{cases}$$

where we have used the notation  $TU$ , to denote the tangent bundle to a subset  $U_s$  of the shape space  $M_s$ . Note that the control torque  $\tau$  corresponding to the above transformed control  $u$  can be computed using equation (13). It should be clear that the constructed control torque transfers any initial condition of the system (9)-(11) to the origin. This completes the proof of Theorem 4.

#### 4. Example of Maneuvering a 3-Body System

In this section, the theory developed in the previous section is used to study a specific class of automated space maneuvers for interconnected multibodies using only torque inputs at the joint connections. As discussed previously, general planar maneuvers cannot be achieved using two or fewer interconnected bodies. An interconnection of three bodies provides complete maneuvering capability; consequently that is the case considered here. Maneuvers of an interconnection of more than three bodies can always be reduced to the case of three bodies by fixing the relative angles at an appropriate number of joints.

For simplicity, we consider rest to rest planar maneuvers of an interconnection of three bodies using

torque inputs at the joint connections. Our interest is in several types of space based applications where gravity is ignored. For illustration purposes we consider a three link model which consists of three bars of equal length, mass and body moment of inertia. The system parameters corresponding to the model are taken as follows:

$$\bar{l}_1 = \bar{l}_3 = 4, \quad \bar{l}_2 = 8 ;$$

$$\bar{h}_{12}^1 = \bar{h}_{23}^1 = 3, \quad \bar{h}_{31}^1 = 1 ;$$

$$\bar{h}_{12}^2 = \bar{h}_{23}^2 = \bar{h}_{31}^2 = 0 .$$

Using the notation introduced previously with  $N = 3$ , the following are the reduced order equations of motion.

$$\dot{x}_1 = s_1(x_{2,1}, x_{2,2})x_{3,1} + s_2(x_{2,1}, x_{2,2})x_{3,2} , \quad (21)$$

$$\dot{x}_{2,1} = x_{3,1} , \quad (22)$$

$$\dot{x}_{2,2} = x_{3,2} , \quad (23)$$

$$\dot{x}_{3,1} = u_1 , \quad (24)$$

$$\dot{x}_{3,2} = u_2 . \quad (25)$$

where

$$s_1(x_2) = -\frac{12 + 3 \cos x_{2,1} + 6 \cos x_{2,2} + \cos(x_{2,1} + x_{2,2})}{16 + 6 \cos x_{2,1} + 6 \cos x_{2,2} + 2 \cos(x_{2,1} + x_{2,2})}$$

$$s_2(x_2) = -\frac{4 + 3 \cos x_{2,2} + \cos(x_{2,1} + x_{2,2})}{16 + 6 \cos x_{2,1} + 6 \cos x_{2,2} + 2 \cos(x_{2,1} + x_{2,2})}$$

We note that, according to the previous results, the origin is strongly accessible, small time locally controllable, and any initial state can be controlled to the origin in an arbitrary time interval. As in the general development given previously, we construct a feedback controller which (1) transfers the initial state to the origin of the  $x_{2,1} - x_{2,2}$  shape space and (2) controls the motion along an appropriately defined closed path in the shape space to achieve the desired phase shift for  $x_1$ . We consider square paths defined by the parameter  $z$ . The holonomy function is then defined by

$$\alpha(z) = \int_0^z (s_1(x, 0) + s_2(z, x) - s_1(x, z) - s_2(0, x)) dx .$$

We vary  $z$  from  $-2\pi$  to  $2\pi$  by increments of 0.01 and calculate the corresponding holonomy using composite Simpson integration method. Intermediate points are calculated via Lagrange interpolation. The holonomy function  $\alpha(z)$  is computed and is shown graphically in Figure 1.

The specific feedback control functions that we use in the simulations are given by

$$\begin{aligned}
 U^0(x) &= \begin{pmatrix} -k_1 \operatorname{sign}(x_{2,1} + x_{3,1}|x_{3,1}|/2k_1) \\ -k_2 \operatorname{sign}(x_{2,2} + x_{3,2}|x_{3,2}|/2k_2) \end{pmatrix}, \\
 U^1(x) &= \begin{pmatrix} -k_1 \operatorname{sign}(x_{2,1} - z^* + x_{3,1}|x_{3,1}|/2k_1) \\ 0 \end{pmatrix}, \\
 U^2(x) &= \begin{pmatrix} 0 \\ -k_2 \operatorname{sign}(x_{2,2} - z^* + x_{3,2}|x_{3,2}|/2k_2) \end{pmatrix}, \\
 U^3(x) &= \begin{pmatrix} -k_1 \operatorname{sign}(x_{2,1} + x_{3,1}|x_{3,1}|/2k_1) \\ 0 \end{pmatrix}, \\
 U^4(x) &= \begin{pmatrix} 0 \\ -k_2 \operatorname{sign}(x_{2,2} + x_{3,2}|x_{3,2}|/2k_2) \end{pmatrix},
 \end{aligned}$$

where the functions are defined on appropriate subsets of the state space  $\mathbf{as}$  in the general development. The control parameters  $k_1$  and  $k_2$  are set to unity. The parameter  $z^*$  is chosen such that the desired holonomy condition

$$x_1^1 \dagger \alpha(z^*) = 0$$

is satisfied. Here  $x_1^1$  corresponds to the absolute angular position of body 1 when the origin of the shape space is reached. Expressions for the required joint torques can be obtained by using the general expression (13).

We present a representative simulation example for a rest-to-rest maneuver. The maneuver is defined by an initial rest configuration given by the initial condition  $(\pi/4, 0, 0, 0, 0)$  and a final rest configuration at the origin. In geometric terms, the initial configuration is that all links are oriented at  $45^\circ$  in a straight line  $\mathbf{as}$  shown in Figure 3; the final configuration is that all links are oriented at  $0^\circ$  in a straight line  $\mathbf{as}$  shown in Figure 6.

The time responses for  $\theta_1, \psi_1, \psi_2$  are shown in Figure 2. In Figures 3-6, the configuration of the three links is shown for a sequence of uniformly spaced time instants; the figures are scaled so that the center of mass of the three links is always at the origin. Figure 3 represents the motion along the first segment of the shape space square, Figure 4 represents the motion along the second segment of the shape space square, etc.

This example maneuver illustrates the complexity of the required motions. The value of the theory that has been introduced should be clear.

In this paper we have examined controllability and stabilizability properties of planar multibody systems in space with angular momentum preserving control torques. We have assumed zero initial momentum to study control and stabilization problems to an equilibrium configuration. For more than two bodies, we have shown that the system is strongly accessible, small-time locally controllable and that smooth stabilization to an equilibrium is impossible. We have described a means of constructing a discontinuous feedback controller which stabilizes an arbitrary equilibrium configuration. The results have been applied to a specific space maneuver of a three body interconnection.

We have developed a theory for reorientation maneuvers of multibody systems in space. These results have potential applications in a number of areas; a few of these possibilities are now mentioned.

Space robots have been envisioned to carry out construction, maintenance and repair tasks in an external space environment. A free flying robot is essentially a multibody system satisfying the assumptions of this paper. In order to carry out the desired tasks they must be capable of performing a variety of reorientation maneuvers. There has been previous research on maneuvering of free flying space robots<sup>4,10,12,16</sup>. In these papers, maneuvers are studied which achieve desired orientation of some of the bodies, e.g. the robot end effector, while the orientation of some of the remaining bodies cannot be specified, at least using the methodologies employed. Our approach is more general, in that maneuvers can be defined which achieve a desired reorientation for all of the space robot links and bodies. Such additional flexibility in performing reorientation maneuvers should have great practical significance for completion of robotic tasks in space.

Another interesting application is the performance by astronauts of reorientation maneuvers in space. Although it is well known that astronauts in space can perform a variety of complicated reorientation maneuvers, without the use of thrusters, the theoretical basis for such maneuvers is incomplete. Again we note that an astronaut in space can be considered  $\mathbf{as}$  a multibody system which satisfy all of the assumptions of this paper. Consequently, our theory is applicable to study of the maneuvering capability of astronauts in space. Previous research in this area<sup>7</sup> has emphasized dynamics issues. Other closely related research

has focused on describing the reorientation maneuvers of a falling cat<sup>6</sup> and the reorientation maneuvers of divers<sup>5</sup>.

Finally, we mention another area of potential application of the results of this paper, namely the development of deployment maneuvers for multibody antennas connected to a spacecraft. If deployment maneuvers for an antenna, or other deployable structure, are performed using only torque motors at the joints of the antenna segments, then the spacecraft-antenna system is a multibody system which satisfies the assumptions of this paper. Consequently, our results can be used to develop efficient antenna deployment maneuvers. The importance of such deployment maneuvers is that they can be designed so that they do not change the final orientation of the spacecraft or the total angular momentum of the spacecraft-antenna system, thereby reducing the requirements of the spacecraft momentum management system. To our knowledge, such active control approaches to antenna deployment have not yet been studied. It is expected that such an approach would have many advantages over the use of existing passive antenna deployment mechanisms<sup>11</sup>.

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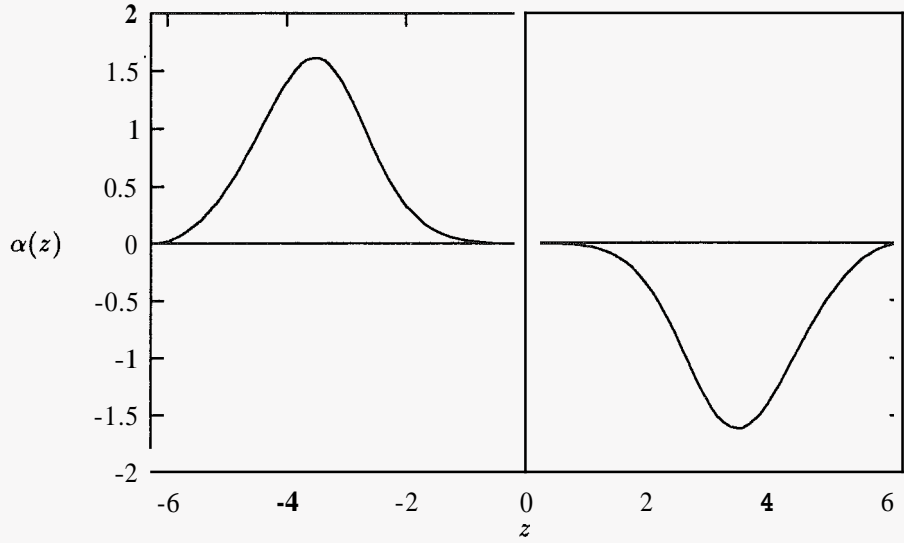


Figure 1:

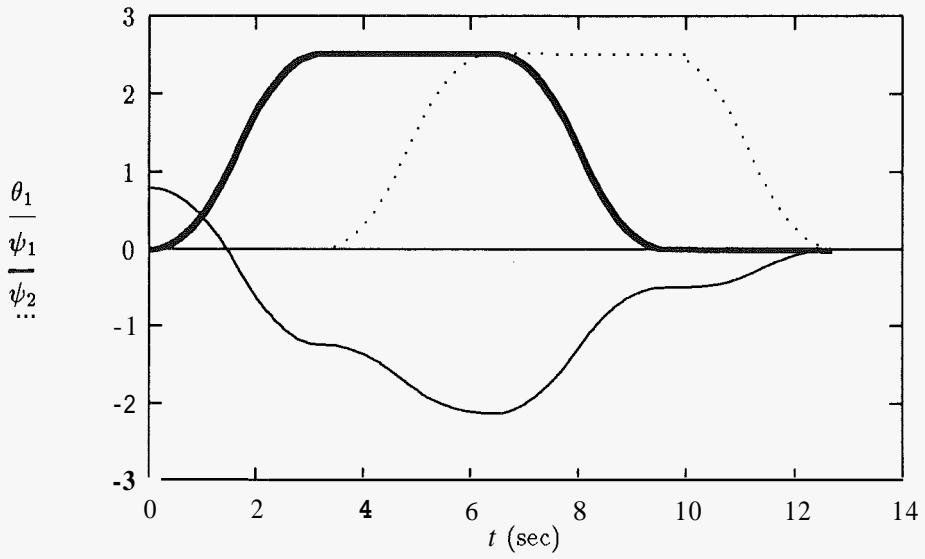


Figure 2:

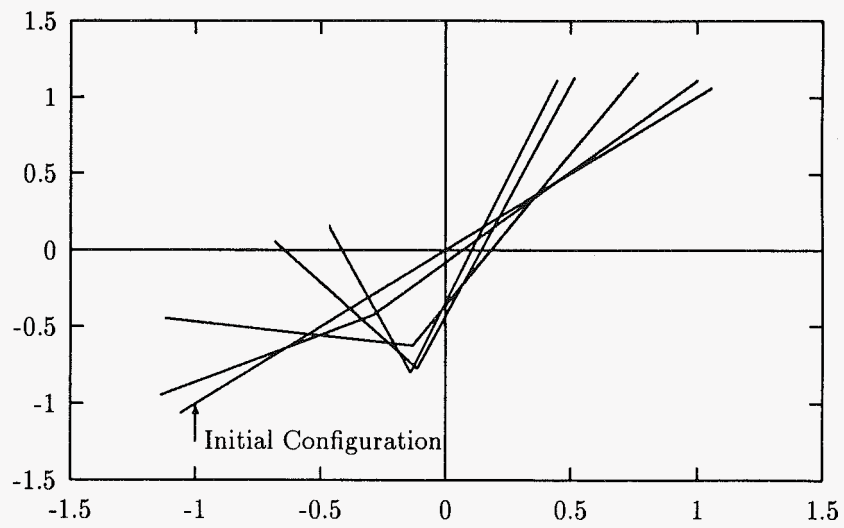


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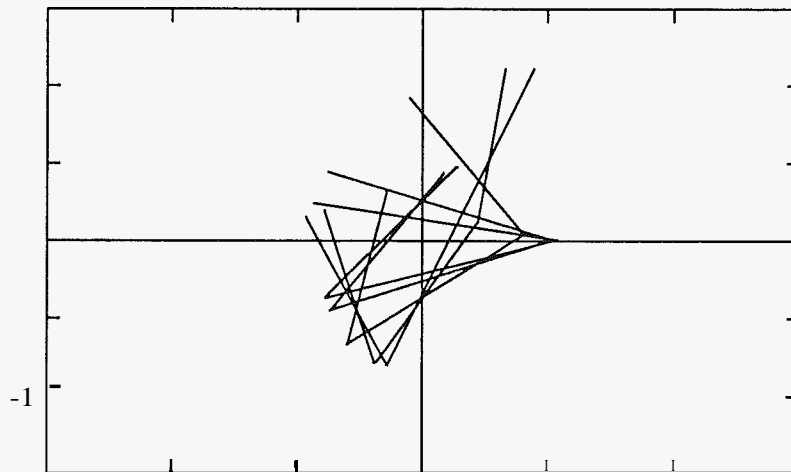


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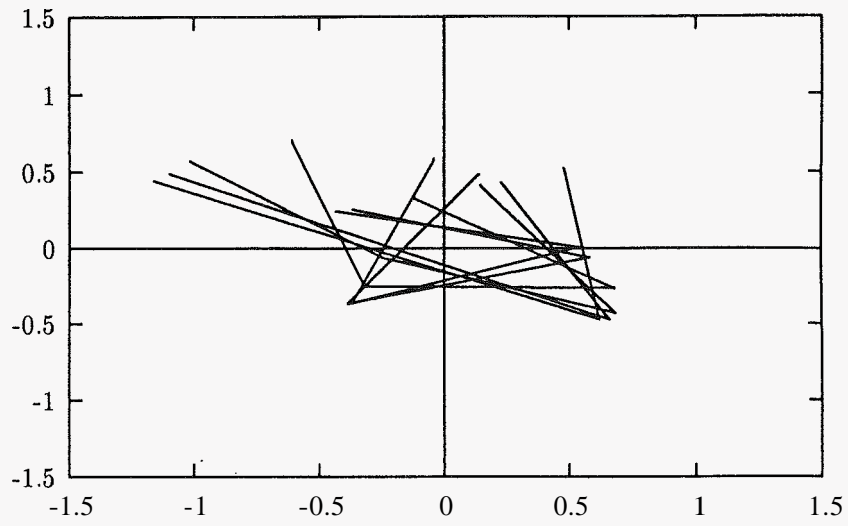


Figure 5:

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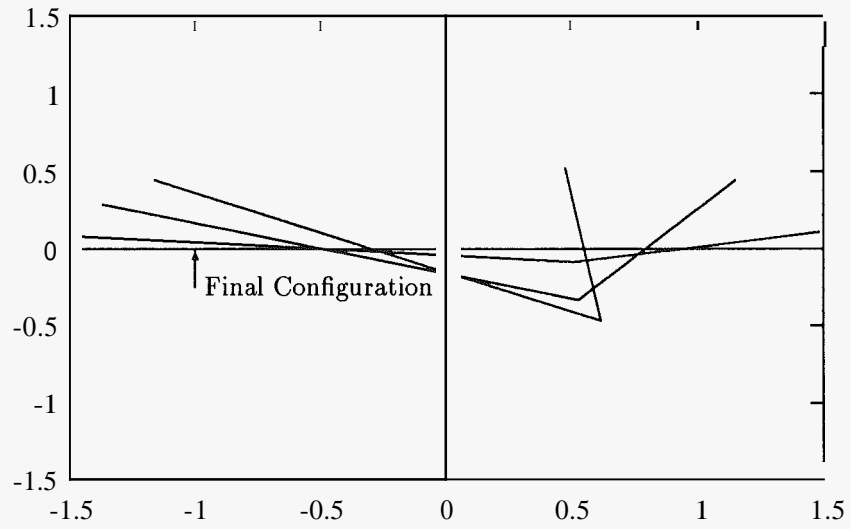


Figure 6: