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OPTIMAL PLACEMENT OF STATISTICAL MANEUVERS IN AN UNSTABLE ORBITAL ENVIRONMENT *

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ABSTRACT

The optimal placement of statistical control maneuvers are analyzed for maintaining position near an unstable equilibrium point. We develop this idea for the libration points in the Hill 3-body problem, but the analysis can be generalized to other unstable systems, and is applied to the restricted 3-body problem as well. This paper first reviews the basics of statistical fuel usage in the context of orbit determination errors and their mapping in time. Using linear theory we derive several explicit targeting formulae for driving a spacecraft back to a fixed point. The mean and standard deviation of these schemes are analyzed for our special case, and explicit solutions for them are found. Using these results we can derive the fuel-optimal spacing of the maneuvers in time in order to control a spacecraft to the vicinity of an unstable libration point.

Statistical maneuver design

To plan for the navigation of a spacecraft, it is necessary to develop a statistical model for the amount of fuel that will be needed to keep the spacecraft on course. This problem has been considered in the standard 2-Body Problem by Battin [3], and has been applied to analysis of motion in unstable libration point orbits by Farquhar [4] and Gomez et.al [5]. In this paper we reconsider the question of optimal timing of control maneuvers given statistical errors in orbit determination [1]. In its most general form, the problem can be stated as follows. Given an error in position and velocity relative to the nominal trajectory at time t_o of the form $\delta\mathbf{r}_o$, $\delta\mathbf{v}_o$, what is the mean and variance in the cost of the maneuvers to reduce the system back to $\delta\mathbf{r} = \delta\mathbf{v} = 0$ at some future time. Generally, the errors in position and velocity arise from the previous maneuver and can be thought of as errors in knowledge of the spacecraft state. Practically, maneuver execution errors must also be incorporated, but we ignore these in this paper.

For a general trajectory, a minimum of two maneuvers are required to get back on track. One maneuver to target back to the trajectory at some future time, and a second maneuver to reduce the relative velocity to zero at that crossing. Of course, at the time when the trajectory crossing occurs, errors from the epoch of the last maneuver manifest themselves in a new set of dispersions, which must themselves be corrected. By considering the new dispersions to be uncorrelated with the initial dispersions (a conservative assumption in general), we can isolate these effects from each other and perform an analysis on the two maneuvers alone. For the design of these maneuvers, we have two free parameters, the time at which we perform the first correction maneuver, t_1, and the time at which the trajectory crossing (and the second maneuver) will occur, t_2. For a more general approach, we also derive a discrete linear quadratic controller and apply it to our problem, where now the control sequence consists of a maneuver of the same form after a characteristic time interval.

The cost of a general correction maneuver can always be expressed as a formula of the form:

$$\Delta V_i = |\Psi_i \mathbf{x}_o| \tag{1}$$

where Ψ_i is a time varying matrix in general and $\mathbf{x}_o$ represents the state deviation measured at some initial epoch t_o. To compute the statistical cost of these maneuvers requires us to compute the mean and variance:

$$\overline{\Delta V} = \int_\infty \Delta V f(\mathbf{x}_o) d\mathbf{x}_o \tag{2}$$

$$\sigma^2_{\Delta V} = \int_\infty \left(\Delta V - \overline{\Delta V}\right)^2 f(\mathbf{x}_o) d\mathbf{x}_o \tag{3}$$

$$= \overline{(\Delta V)^2} - \overline{\Delta V}^2 \tag{4}$$

Assuming that the measurement noise has zero mean and a gaussian distribution, the probability density function of the initial conditions can be written as:

$$f(\boldsymbol{x}_0) = \frac{1}{(2\pi)^{N/2}\sqrt{|\Lambda_0|}} e^{-\frac{1}{2}\boldsymbol{x}_0^T \Lambda_0 \boldsymbol{x}_0} \tag{5}$$

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where N is the total dimension of the system phase space and Λ_0 is the initial information matrix.

If we implement a series of M such maneuvers, each with the same assumed statistical and dynamical representation, the total mean maneuver cost is $M\overline{\Delta V}$ and the total variance is $M\sigma_{\Delta V}^2$. Thus, if we wish to estimate the statistical cost of performing this sequence of maneuvers to within an n-sigma probability value (1-D Gaussian), we find:

$$\Delta V_{stat} = M\left[\overline{\Delta V} + \frac{n}{\sqrt{M}}\sigma_{\Delta V}\right] \quad (6)$$

Thus, as the number of maneuvers becomes large we see that the total (predicted) statistical cost can be approximated as:

$$\Delta V_{stat} \sim M\overline{\Delta V} \quad (7)$$

Assume we wish to control a trajectory over an extended period of time T_∞, and that we perform a maneuver (or repeat a maneuver sequence) after every time T, resulting in a total of $M = T_\infty/T$ maneuvers. Then the total statistical cost of this sequence of maneuvers is:

$$\Delta V_{stat} \sim T_\infty \frac{\overline{\Delta V}}{T} \quad (8)$$

and for an arbitrary length of time T_∞ we see that to optimize fuel usage we need to choose the time interval between maneuvers T to minimize $\overline{\Delta V}/T$. In the following discussion we will use a set definition for T equal to the time between the determination of the state error $\mathbf{x}_o$, t_o, and the first maneuver of our sequence, t_1.

Hill Model and Motion about the Libration Points

Scaling of the equations

Let us introduce the 3-dimensional equations for the Hill 3-Body problem [6]:

$$\ddot{x} - 2\omega\dot{y} = -\frac{\mu}{r^3}x + 3\omega^2 x \quad (9)$$

$$\ddot{y} + 2\omega\dot{x} = -\frac{\mu}{r^3}y \quad (10)$$

$$\ddot{z} = -\frac{\mu}{r^3}z \quad (11)$$

where ω is the rotation rate of the secondary around the primary, $\mu = GM$, M is the mass of the secondary and x,y,z denote the position of the spacecraft in the vicinity of the secondary.

Transforming these equations into nondimensional form gives us the length scale: $l = \left(\frac{\mu}{\omega^2}\right)^{1/3}$ and the time scale $\frac{1}{\omega}$ (for the Restricted 3-Body Problem, the length scale is instead the distance between the two primaries). Let Λ_0 be the information matrix corresponding to these nondimensional equations, then:

$$\Lambda_0 = \begin{bmatrix} 1/{\sigma_r}^2 & 0 \\ 0 & 1/{\sigma_v}^2 \end{bmatrix} \quad (12)$$

with $\sigma_r = \frac{\sigma_{rd}}{l}$ and $\sigma_v = \frac{\sigma_{vd}}{l\omega}$, σ_{rd} and σ_{vd} correspond to the variance of the error in position and velocity, respectively, due to orbit determination errors. In the following, we will take $\sigma_{rd}/\sigma_{vd} = 10^6$. All the following computations have been made using a nondimensional information matrix $\Lambda = \sigma_r^2\Lambda_0$:

$$\Lambda_0 = \begin{bmatrix} 1 & 0 \\ 0 & \omega_E^2(\sigma_{rd}/\sigma_{vd})^2 \end{bmatrix} \quad (13)$$

The results of the nondimensional statistical analysis will be the mean maneuver cost rate with units of length over time squared, represented as $\frac{\Delta v}{\Delta \tau}$, where Δv is the mean maneuver cost and $\Delta\tau$ is the time interval between maneuvers. Even though the normalized Hill problem has no parameter, there is a parameter in the information matrix, $\omega^2\left(\frac{\sigma_{rd}}{\sigma_{vd}}\right)^2$, which varies according to the ratio of position to velocity accuracy and with the rotation rate of the secondary about the primary. An additional parameter is $\sigma_r = \sigma_{rd}/l$. Unless otherwise specified, the plots show the value $\frac{\Delta v}{\Delta \tau}$. To transform these to dimensional costs in $\mathrm{km/s}^2$, we use:

$$\frac{\Delta v_d}{\Delta t} = \sigma_{rd}\omega^2\frac{\Delta v}{\Delta \tau} \quad (14)$$

To get the maneuver rate in (km/s)/secondary period:

$$\frac{\Delta v_d}{\Delta t} = 2\pi\sigma_{rd}\omega\frac{\Delta v}{\Delta \tau} \quad (15)$$

Libration Points of the Hill Problem

The general nondimensional equations of motion for the Hill 3-Body Problem are:

$$\ddot{x} - 2\dot{y} = -\frac{x}{r^3} + 3x \quad (16)$$

$$\ddot{y} + 2\dot{x} = -\frac{y}{r^3} \quad (17)$$

$$\ddot{z} = -\frac{z}{r^3} \quad (18)$$

This system has two equilibrium points: $x = \pm 3^{-1/3}$, $y = z = 0$. As the linear motion along the z axis is decoupled from linearized motion in the x-y plane, the following analysis will only consider motion in the x-y plane. Let us linearize the system around the equilibrium points $x = \pm 3^{-1/3}$, using $\delta\boldsymbol{x} = \left[\begin{array}{cccc} \delta x & \delta y & \delta\dot{x} & \delta\dot{y} \end{array}\right]^T$:

$$\delta\dot{\boldsymbol{x}} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 9 & 0 & 0 & 2 \\ 0 & -3 & -2 & 0 \end{bmatrix} \delta\boldsymbol{x} \tag{19}$$

Let us call the above matrix U. Integrating this equation gives $\delta\boldsymbol{x} = \Phi(t, t_0, \boldsymbol{x}_0)\delta\boldsymbol{x}_0$ with:

$$\Phi(t, t_0, \boldsymbol{x}_0) = e^{U(t-t_0)} \tag{20}$$

The eigenvalues of U are $\pm\lambda_1 = \pm\sqrt{1+2\sqrt{7}}$ and $\pm j\lambda_2 = \pm j\sqrt{2\sqrt{7}-1}$. The eigenvectors corresponding to these eigenvalues define the direction of the stable and unstable manifolds in phase space.
Writing $e^{Ut} = \alpha_1 U^3 + \alpha_2 U^2 + \alpha_3 U + \alpha_4 I$, for each eigenvalue λ we find:

$$e^{\lambda t} = \alpha_1\lambda^3 + \alpha_2\lambda^2 + \alpha_3\lambda + \alpha_4 \tag{21}$$

Solving this system leads to:

$$\alpha_1 = \frac{\sinh(\lambda_1 t)/\lambda_1 - \sinh(\lambda_2 t)/\lambda_2}{\lambda_1^2 + \lambda_2^2} \tag{22}$$

$$\alpha_2 = \frac{\cosh(\lambda_1 t) - \cosh(\lambda_2 t)}{\lambda_1^2 + \lambda_2^2} \tag{23}$$

$$\alpha_3 = \frac{\lambda_2^2 \sinh(\lambda_1 t)/\lambda_1 + \lambda_1^2 \sinh(\lambda_2 t)/\lambda_2}{\lambda_1^2 + \lambda_2^2} \tag{24}$$

$$\alpha_4 = \frac{\lambda_2^2 \cosh(\lambda_1 t) + \lambda_1^2 \cosh(\lambda_2 t)}{\lambda_1^2 + \lambda_2^2} \tag{25}$$

Computing U^2 and U^3 and replacing the α_i by their values gives $\Phi(t, t_0, \boldsymbol{x}_0) = [\Phi_1 \quad \Phi_2]$:

$$\Phi_1 = \frac{1}{2\sqrt{7}} \begin{bmatrix} \sqrt{7}a + 4b & -3c \\ -9c & \sqrt{7}a - 2b \\ 9(2c + \sqrt{7}d) & -3b \\ -9b & 3(4c - \sqrt{7}d) \end{bmatrix} \tag{26}$$

$$\Phi_2 = \frac{1}{2\sqrt{7}} \begin{bmatrix} 2c + \sqrt{7}d & b \\ -b & -(4c - \sqrt{7}d) \\ \sqrt{7}a + 2b & c + 2\sqrt{7}d \\ -(c + 2\sqrt{7}d) & \sqrt{7}a - 4b \end{bmatrix} \tag{27}$$

where:
$a = \cosh(\lambda_1 t) + \cos(\lambda_2 t)$
$b = \cosh(\lambda_1 t) - \cos(\lambda_2 t)$
$c = \dfrac{\sinh(\lambda_1 t)}{\lambda_1} - \dfrac{\sin(\lambda_2 t)}{\lambda_2}$
$d = \dfrac{\sinh(\lambda_1 t)}{\lambda_1} + \dfrac{\sin(\lambda_2 t)}{\lambda_2}$

As this gives us the general expression of $\Phi(t, t_0, \boldsymbol{x}_0)$, the computation of the matrix exponential is not needed.

Control Strategies

2-maneuver control sequence

In this section, the statistical cost of two maneuvers performed at different times is computed. A maneuver is made at $t_1 = n\Delta\tau$ based on a solution at time t_0 to null the error in position at time t_2, and a maneuver is made at $t_2 = n\Delta\tau + (t_2 - t_1)$ to null the error in velocity.
Let us give the general formula to compute the first correction maneuver ΔV_1 to null the spacecraft position error at t_2, and the second correction maneuver ΔV_2 to null the relative speed of the spacecraft at t_2. Let:

$$\Phi = \begin{bmatrix} \phi_{rr} & \phi_{rv} \\ \phi_{vr} & \phi_{vv} \end{bmatrix} \tag{28}$$

$$\phi_1(t_2, t_1) = \phi_{rv}^{-1}(t_2, t_1)\phi_{rr}(t_2, t_1) \tag{29}$$

$$\begin{aligned} \phi_2(t_2, t_1) = \; & \phi_{vv}(t_2, t_1)\phi_{rv}^{-1}(t_2, t_1)\phi_{rr}(t_2, t_1) \\ & -\phi_{vr}(t_2, t_1) \end{aligned} \tag{30}$$

Then:

$$\begin{aligned} \Delta V_1 = \; & -[\phi_1(t_2, t_1)\phi_{rr}(t_1, t_0) + \phi_{vr}(t_1, t_0)]\delta\boldsymbol{r}_0 \\ & -[\phi_1(t_2, t_1)\phi_{rv}(t_1, t_0) + \phi_{vv}(t_1, t_0)]\delta\boldsymbol{v}_0 \end{aligned} \tag{31}$$

$$\Delta V_2 = \phi_2(t_2, t_1)[\phi_{rr}(t_1, t_0)\delta\boldsymbol{r}_0 + \phi_{rv}(t_1, t_0)\delta\boldsymbol{v}_0] \tag{32}$$

Using the notation of the Appendix, the computation of $\overline{|\Delta V|}$ is made using:
$K = \Lambda$
For $\overline{|\Delta V_1|}$: $\Psi = \Psi_1$ ($\Delta V_1 = \Psi_1 \delta x_0$).
For $\overline{|\Delta V_2|}$: $\Psi = \Psi_2$ ($\Delta V_2 = \Psi_2 \delta x_0$).
The general formula used is:

$$\overline{\Delta V} = \sqrt{\frac{2}{\pi}}\sqrt{\lambda + \mu} \quad \text{EllipticE}\left(\sqrt{\frac{2\mu}{\lambda + \mu}}\right) \tag{33}$$

with:

$$\lambda = (a_1^T K^{-1} a_1 + \overline{a_2}^T K^{-1} \overline{a_2})/2 \quad (34)$$

$$\mu = \sqrt{(a_1^T K^{-1} a_1 - \overline{a_2}^T K^{-1} \overline{a_2})^2/4 + (a_1^T K^{-1} \overline{a_2})^2} \quad (35)$$

$$\Psi = \begin{bmatrix} a_1 \\ \overline{a_2} \end{bmatrix} \quad (36)$$

where EllipticE is the complete elliptic integral of the second kind.
Figure [1] shows the variation of the minimum of the costs when the interval $t_2 - t_1$ is varying. For each value of $t_2 - t_1$, the total cost $(\overline{|\Delta V_1|} + \overline{|\Delta V_2|})/(t_1 - t_0)$ is computed for a range of values of $t_1 - t_0$ to find the minimum cost. This plot shows that the influence of this period is only important when $t_2 - t_1$ is such that the matrix ϕ_{rv} is singular. At $t_2 - t_1$ constant, the minimum occurs for $\Delta\tau = 0.4$, quite close to the characteristic time of the unstable mode, $1/\lambda_1$. Then, the minimum of the cost varying $t_2 - t_1$ is the same in each interval, about 46.2. The cost has been computed taking the nondimensional norm of the velocity needed to correct the position divided by the nondimensional period between the manœuvers $\Delta\tau$.

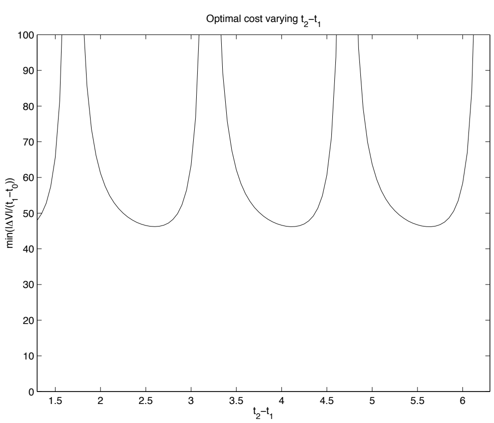

Figure 1: Separated maneuvers: optimal cost

Overlaid 2-maneuver control sequences

The maneuvers to null the position at $t_2 + \Delta\tau$ and to null the velocity at t_2 can be performed simultaneously. As only the magnitude of the velocity is important for computing the optimal cost, we hope to find a better minimum for $|\Delta V_1 + \Delta V_2|$ than for $|\Delta V_1| + |\Delta V_2|$. More generally, the case $t_2 - t_1 = k(t_1 - t_0)$ where k is an integer will also be investigated.

Correction in the computation of the mean and the variance

At time $t_2 = n\tau$ two maneuvers are performed simultaneously: one maneuver to null the position at a time τ later and one to null the velocity due to the previous correction. The new correction is $\Delta V = \Delta V_1 + \Delta V_2$ where $\Delta V_1 = \Psi_1 \delta x_1$ is the correction to null the state error δx_1 determined τ before and $\Delta V_2 = \Psi_2 \delta x_0$ is the correction to null the velocity due to the error δx_0 determined 2τ before. Therefore, the new correction ΔV depends on two independant gaussian variables δx_1 and δx_0.
Using the notation of the appendix, the computation is made using (33):
$\Psi = [\Psi_1 \Psi_2]$ 8×2 matrix
$K = \begin{bmatrix} \Lambda & 0 \\ 0 & \Lambda \end{bmatrix}$
Note that we have the freedom to introduce different error statistics for δx_0 and δx_1 but we keep them equal for simplicity.

Optimization of $\overline{|\Delta V|} = \overline{|\Delta V_1 + \Delta V_2|}$ when $t_2 - t_1 = k(t_1 - t_0)$

Let us now plot the cost of the maneuvers when the maneuver to null the position and the maneuver to null the velocity are made at the same time. Figure [2] shows the curves corresponding to the cost of the simultaneous maneuvers, the cost of the maneuver to null the position alone, ΔV_1, the cost to null the velocity alone, ΔV_2 and the sum of these two maneuvers if they were done separately. As expected, the cost of the maneuvers done separately is higher than the cost of the maneuvers done simultaneously.
A minimum of 53.3 is achieved when the maneuvers are made separately and $t_2 - t_1 = t_1 - t_0 = 0.54$ (this is worse than the minimum obtained by varying $t_2 - t_1$). When the maneuvers are made simultaneously, the new optimal time interval is $T = 0.54$ and the new minimum is 45.1, reducing the cost of the maneuvers by 2.4%.

Let us now compare the values of the period and of the minimum when $t_2 - t_1 = k(t_1 - t_0)$. In this case, maneuvers are made every time interval τ targeting the equilibrium point a time $k\tau$ later. Figure [3] shows that a new problem arises when k varies. When $t_2 - t_1$ is such that ϕ_{rv} is singular, the linear corrections become infinite.
For $k = 3$, the new optimal cost is 37.5 for a period $T = 0.4$. Evaluating higher values of k give other minima around 38, not better than the previous one. As a consequence, a new optimal cost is found for simultaneous maneuvers and $k = 3$. In comparison with the separated maneuvers, it improves the cost by 18.8%.

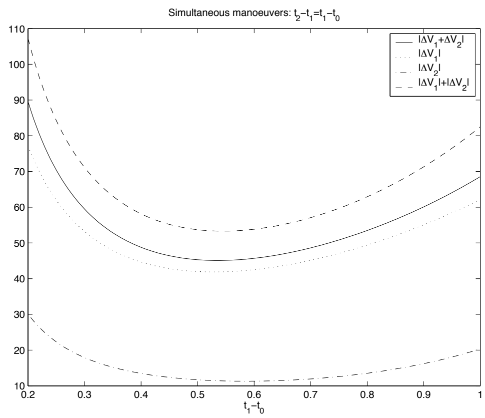

Figure 2: Comparison of simultaneous and separated maneuvers: k=1

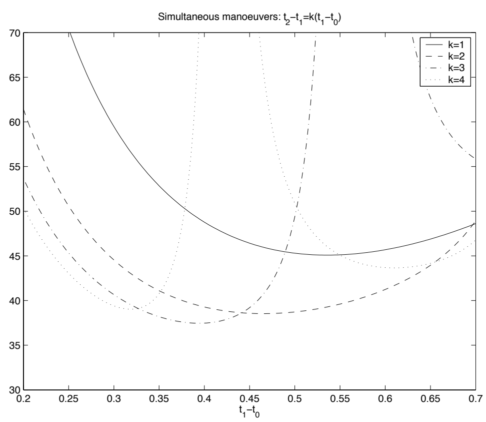

Figure 3: Simultaneous Maneuvers: Optimal Cost

Stable Manifold Targeting

Another approach to control is to make corrections to target the spacecraft to a state vector corresponding to the stable manifold of the equilibrium point. We hope to decrease the cost of the manoeuvers by targeting a point in state space closer than the origin. Let us write the new ΔV_1 and ΔV_2 using u_s as the standard direction on which we want to target the spacecraft, where $[u_s \quad -\lambda_1 u_s]$ is the eigenvector corresponding to the stable manifold. As a result, the final position of the spacecraft at t_2, after the second maneuver, will be $\eta[u_s \quad -\lambda_1 u_s]$ and the spacecraft should approach the equilibrium point along the stable manifold. Then with $t = t_2 - t_1$, if δx_1 and δv_1 specify the position and velocity at the time t_1 of the first maneuver:

$$\Delta V_1 = \phi_{rv}(t)^{-1}(\eta_1 u_s - \phi_{rr}(t)\delta x_1) - \delta v_1 \qquad (37)$$

$$\Delta V_2 = (-\lambda_1 I - \phi_{vv}\phi_{rv}^{-1})\eta_2 u_s + (\phi_{vv}\phi_{rv}^{-1}\phi_{rr} - \phi_{vr})\delta x_1 \qquad (38)$$

Using the stable manifold targeting method, we tried several strategies to minimize the cost. The minimization of $|\Delta V_1| + |\Delta V_2|$ gives similar results to the minimization of $|\Delta V_1|^2 + |\Delta V_2|^2$, which has the advantage of having an analytical solution for η. This approach removes the singularities found above, but does not reduce the minimal cost.

LQR Control Algorithm

LQR method

Now let us formulate the problem in terms of digital control:
$x(k+1) = Gx(k) + Hu(k) + w(k)$
where $G = \Phi$, $H = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$, $u(k) = \Delta V(k)$, and w is a gaussian noise representing the orbit determination error such that:
$E[w(k)] = 0$, $E[w(j)w^T(k)] = \Lambda^{-1}\delta_{jk} = Q_w\delta_{jk}$

Using this equation, let us find the optimal control law $u(k)$ that will minimize:

$$J = E\left[S_N + \sum_{k=0}^{N-1} x^T(k)Qx(k) + u^T(k)Ru(k)\right] \qquad (39)$$

with: $S_N = x^T(N)P(N)x(N)$
Let:

$$\begin{aligned} P(k) &= Q + G^T P(k+1)G - G^T P(k+1)H \\ &\quad [R + H^T P(k+1)H]^{-1} H^T P(k+1)G \end{aligned} \qquad (40)$$

$$K(k) = [R + H^T P(k+1)H]^{-1} H^T P(k+1)G \qquad (41)$$

Using these parameters, it can be shown [2] that the control vector that minimizes J is given by the equation: $u(k) = -K(k)x(k)$.
Finally:

$$\begin{aligned} J_{\min} &= x^T(0)P(0)x(0) \\ &\quad + E\left[\sum_{k=0}^{N-1} \mathrm{trace}[P(k+1)Q_w]\right] \end{aligned} \qquad (42)$$

Let us now consider the steady-state quadratic optimal

control. Taking the limit $N \to \infty$, the optimal control solution becomes a steady state solution and the time varying gain matrix $K(k)$ becomes a constant gain matrix K.
For $N = \infty$, the performance index may be modified to:

$$J = E\left[\sum_{k=0}^{\infty} x^T(k)Qx(k) + u^T(k)Ru(k)\right] \quad (43)$$

Where the steady state matrix, P is solution of the Algebraic Riccati Equation:
$P = Q + G^T PG - G^T PH[R + H^T PH]^{-1}H^T PG$
$K = [R + H^T PH]^{-1}H^T PG$
A problem arises with the cost function J because as $\mathrm{trace}[PQ_w]$ is finite and not necessarily equal to zero, J is infinite. However, as it is shown in the following, if:

$$J = \lim_{k\to\infty} E\left[x^T(k)Qx(k) + u^T(k)Ru(k)\right] \quad (44)$$

Then: $J = \mathrm{trace}[PQ_w]$
Taking $R = I$, we have:

$$\overline{|\Delta V|^2} = \lim_{k\to\infty} u^T(k)Ru(k) \quad (45)$$

And:

$$J = \overline{|\Delta V|^2} + \lim_{k\to\infty} \overline{x^T(k)Qx(k)} \quad (46)$$

Let us now find $\overline{|\Delta V|^2}$:
Let $P_X(k+1) = E[x(k+1)x^T(k+1)]$, then:
$P_X(k+1) = (G - HK)P_X(k)(G - HK)^T + Q_w$
As the characteristic roots of $G - HK$ are inside the unit circle, the effects of the initial condition $P_X(0)$ gradually diminish and P_X approaches a stationary value. This value is given by the solution of the Lyapunov equation:

$$P_X = (G - HK)P_X(G - HK)^T + Q_w \quad (47)$$

Finally:

$$\lim_{k\to\infty} E[x^T(k)Qx(k)] = \mathrm{trace}[P_X Q] \quad (48)$$

$$\lim_{k\to\infty} E[u^T(k)Ru(k)] = \mathrm{trace}[KP_X K^T] \quad (49)$$

As a result:

$$\begin{aligned}\overline{|\Delta V|^2} &= \mathrm{trace}[PQ_w] - \mathrm{trace}[P_X Q] \\ &= \mathrm{trace}[KP_X K^T]\end{aligned} \quad (50)$$

The parameter studied in the previous sections is $\overline{|\Delta V|}$. As a consequence, we need to estimate this parameter with the LQR method in order to have a comparison between the two controls. First of all:
$\sigma^2_{\Delta V} = \overline{|\Delta V|^2} - \overline{|\Delta V|}^2 \geq 0$ implies that $\overline{|\Delta V|} \leq \sqrt{\mathrm{trace}[KP_X K^T]}$.

Then, using:

$$x(k+1) = \sum_{j=0}^{k} (G - HK)^j w(k-j) \quad (51)$$

we find:

$$\Delta V(k+1) = \sum_{j=0}^{k} \phi(j)w(k-j) \quad (52)$$

with: $\phi(j) = -K(G - HK)^j$
Still using the method of computation of $\overline{|\Delta V|}$ described in the appendix, this gives:

$$\overline{|\Delta V(k)|} = \sqrt{\frac{2}{\pi}}\sqrt{\lambda + \mu} \quad \mathrm{EllipticE}\left(\sqrt{\frac{2\mu}{\lambda + \mu}}\right) \quad (53)$$

where λ and μ are finite series which converge rapidly to a solution that does not depend on k, which is consistent with the existence of a limit for $P_X(k)$.

Using this method, it becomes necessary to check the average spacecraft distance from the origin. Indeed if the spacecraft drifts too far from the equilibrium point, the linearization approximation cannot be kept. The average distance from the origin can also be studied using the same method as above with a series for λ and μ.
With $S = \mathrm{diag}(1, 1, 0, 0)$, $T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$ and $\phi_d(j) = T(G - HK)^j$:

$$d(k+1) = \left|\sum_{j=0}^{k} \phi_d(j)w(k-j)\right| \quad (54)$$

$$\lim_{k\to\infty} \overline{d(k)^2} = \mathrm{trace}[P_X S] \quad (55)$$

Implementation of LQR Method

In this part, the performance of the methods are compared (Figure [4]). For the average distance from the origin using the first method (labeled d_p in Figure [4]), we compute the average distance in the case where $t_1 - t_0 = t_2 - t_1$. Three different matrices Q have been considered. $Q = 0.05S$ implies that there is nearly no requirement on the position of the spacecraft. As can be seen in the next graph, the achievement in the cost is good, 30.6 instead of the previous 37.5, but the tradeoff is that the position is more than twice as far from the origin. $Q = 500$ implies that the requirement on the position is strong and

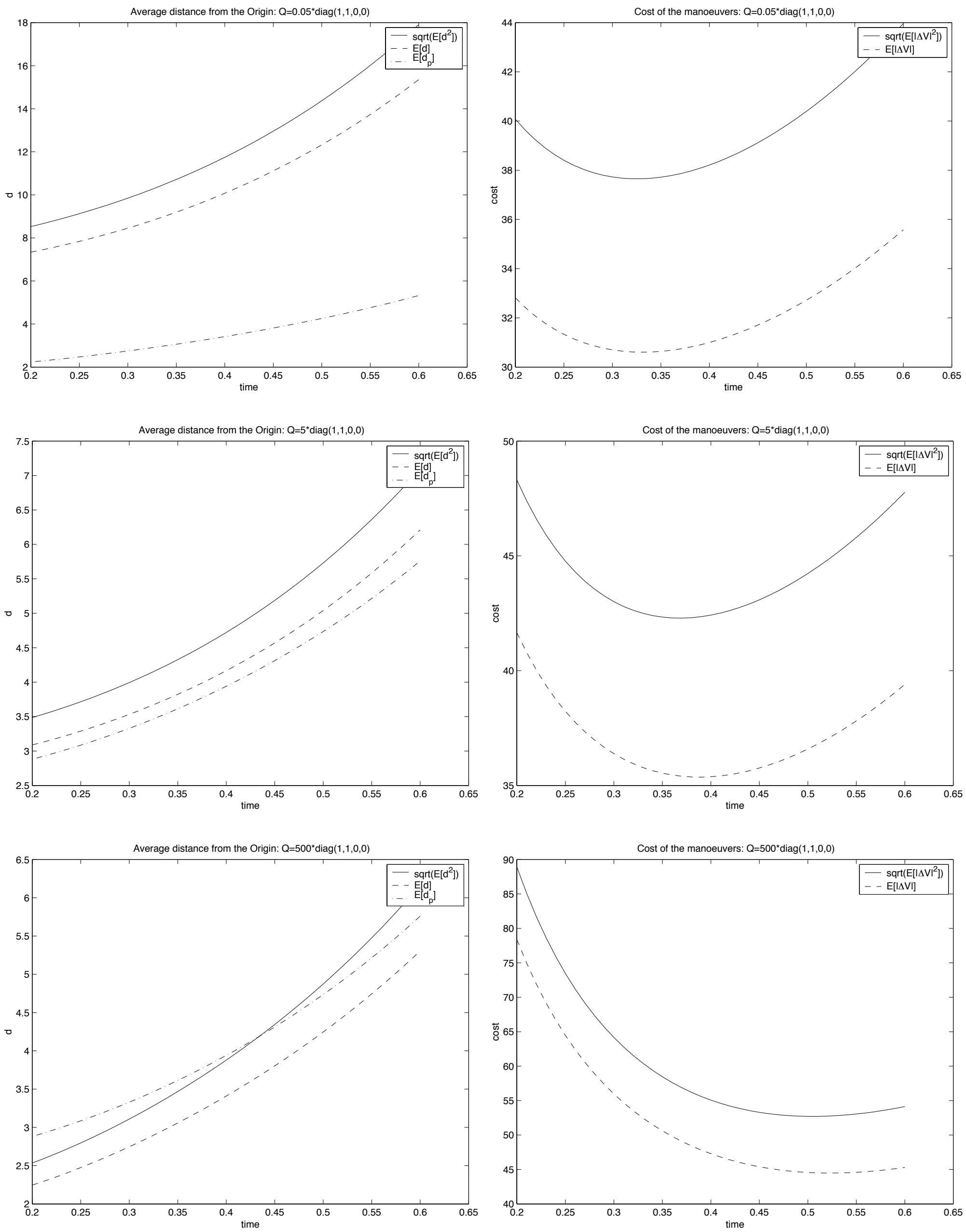

Figure 4: LQR Method: Optimal Cost

therefore the spacecraft will stay close to the origin. As expected, the best cost is now worse than 37.5: 44.5. Considering the case $Q = 5S$, we find that the average distances from the origin are nearly the same for the two control methods. However, a new optimal cost is obtained of about 35.3. It seems that the LQR method improves the optimal cost without deterioring the average distance from the equilibrium point of the spacecraft.

Note that if we take the ratio $\sigma_r/\sigma_v = \omega/\sigma_{vd}$ with ω equal to the rotation rate of the Moon around the Earth, we find that targeting the origin gives the optimal cost of 19.9 for $k = 3$ whereas the optimal costs for the LQR method are 16.5, 18 and 22.4 for $Q = 0.05S$, $Q = 5S$ and $Q = 500S$ respectively. Therefore, we can conclude that the LQR method gives better results than the method of targeting the equilibrium point in this case as well.

Comparison and Discussion of Optimal Approach

Testing the validity of the model

Our linear control has also been implemented using the non-linear model. One must be careful in the implementation of the controller to take into account the fact that the LQR method uses the previous state error to correct the trajectory whereas the method targeting the origin uses the previous state error minus what the state would be if the model was behaving linearly.

Procedure of computation: LQR method

- from $x(k)$, the non-linear equations are integrated over the period T to obtain $x(k+1)'$.

- $x(k+1) = x(k+1)' - HK(x(k) - x_{eq}) + w(k)$

Procedure of computation: Targeting the Equilibrium Point $t_2 - t_1 = k(t_1 - t_0)$

- from $x(k)$, the non-linear equations are integrated over the period T to obtain $x(k+1)'$.

- The case of the k first maneuvers, where only the correction ΔV_1 is performed, has to be considered separately. As a result, in this case: $x(n+1) = x(n+1)' + \phi_1 x_{\text{corr}}(n) + w(n)$ whereas after kT, $x(n+1) = x(n+1)' + \phi_1 x_{\text{corr}}(n) + \phi_2 x_{\text{corr}}(n-k) + w(n)$

- the computation of $x_{\text{corr}}(n)$ requires us to compute what the state would be if the model were behaving linearly, using $K_1 = -\phi_1$: $x_{\text{corr}}(n) = x(k) - x_{eq} - \sum_{j=1}^{n} G^{n-j}(G - HK_1)x_{\text{corr}}(j-1)$ for the k first steps and $x_{\text{corr}}(n) = x(k) - x_{eq} - \sum_{j=1}^{k} G^{k-j}(G - HK_1)x_{\text{corr}}(n+j-k-1)$ for the following steps.

Once these models were implemented, we tested them to find what the maximum allowable variance is. For the LQR method, taking the values of variance computed before, we find that these values can be multiplied by 40000. This means that the model can correct position errors of 40000 km and velocity errors of 40ms^{-1} in velocity! For the method of targeting the equilibrium, the values can be respectively multiplied by 32000, 32000, 24000 and 900 for k equal to 1, 2, 3 and 4. The lower value of 900 corresponds to the period $T = 0.4$ for $k = 4$. Indeed for this particular value, we have seen that the cost increases very rapidly due to the fact that the matrix ϕ_{rv} is becoming singular.

Comparison between the results of the simulations and the analytical conclusions

To compare the results of the simulations and the results of the analytical conclusions, we took the scaling values of the Earth-Sun system. Figures [5] and [6] give the accelerations necessary to control the spacecraft in the Earth-Sun system in km/s/year, Figure [5] for the LQR method with $Q = 500S$, $Q = 5S$ and $Q = 0.05S$ and Figure [6] for the method of targeting the equilibrium with $k = 1$, $k = 2$, $k = 3$ and $k = 4$. These figures have been generated with $N = 400$ maneuvers for the non linear model. As can be seen, the comparisons give good results, indicating that the nonlinear model will be close to the analytical computations made using the linear model. The non-linear model seems to deviate more from the analytical solution for the LQR method, mainly due to the different scaling of the plots.

Application to the Restricted 3-Body Problem

Now let us write the nondimensional equations of the Restricted 3 Body Problem in two dimensions:

$$\ddot{x} - 2\dot{y} = x - \frac{(1-\mu)(x+\mu)}{[(x+\mu)^2 + y^2]^{3/2}} - \frac{\mu(x-1+\mu)}{[(x-1+\mu)^2 + y^2]^{3/2}} \quad (56)$$

$$\ddot{y} + 2\dot{x} = y - \frac{(1-\mu)y}{[(x+\mu)^2 + y^2]^{3/2}} - \frac{\mu y}{[(x-1+\mu)^2 + y^2]^{3/2}} \quad (57)$$

There are five equilibrium points for this system. Two of them, called L1 and L2, correspond to the two equilibrium points of the Hill problem, μ corresponds to the mass ratio of the smallest body mass to the the sum of the two masses. For the Earth-Sun system, μ is small and the problem can be approximated by the Hill problem. However, for the Earth-Moon system, $\mu = 0.0122$, and the Hill approximation gives quite different results from the Restricted 3-Body Problem, as will be shown.

The linearization around the equilibrium points $[x_{1,2} \quad 0 \quad 0]$, where $x_1 = 0.99$, $x_2 = 1.01$ for the Earth-Sun system and $x_1 = 0.837$, $x_2 = 1.156$ for the

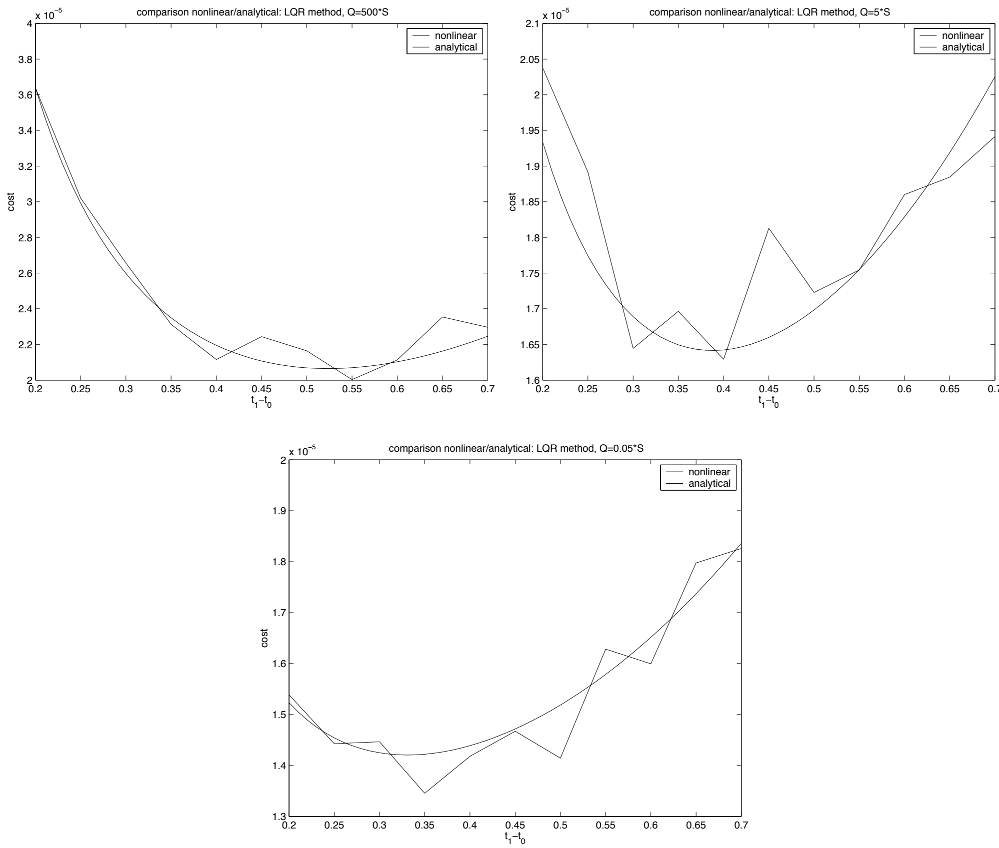

Figure 5: LQR Method: Cost of the Nonlinear Model vs Cost of the Linear Model

Earth-Moon system, gives:

$$\delta\dot{\boldsymbol{x}} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1+2a & 0 & 0 & 2 \\ 0 & 1-a & -2 & 0 \end{bmatrix} \delta\boldsymbol{x} \qquad (58)$$

with: $a_i = \frac{1-\mu}{|x_i+\mu|^3} + \frac{\mu}{|x_i-1+\mu|^3}$

Solving the charateristic polynomial of this new matrix for these libration points gives two opposite real eigenvalues $\pm\lambda_1$ and two complex conjugate eigenvalues $\pm j\lambda_2$ as in the Hill problem. To compute $\Phi(t, t_0, \boldsymbol{x}_0)$, the polynomial decomposition of part one can still be used with the α_i (22).

Let us compare the characteristic exponent λ_1 given by the two problems for the Earth-Moon system:
For $x_1 = 0.837$: $\lambda_1 = 2.93$
For $x_2 = 1.156$: $\lambda_1 = 2.15$
Whereas for the Hill problem: $\lambda_1 = 2.51$
Using the results from the previous part, these values imply that the optimal control period will be smaller for $x_1 = 0.837$ than the optimal period given by the Hill problem and greater for $x_2 = 1.156$.

Tables (1) and (2) compare the results given by the Restricted Three Body Problem with the results of the Hill Problem. They show that the values predicted using the Hill problem are close to the values predicted by the Restricted 3-Body Problem for the Earth-Sun system. For the Earth-Moon system, the values predicted by the two models are quite different, however the results of the Hill problem are close to the average of the L1 and L2 results and thus can be used to estimate results. The errors made in the two methods are proportional, and the Hill Problem gives an expected cost between 22% and 26% higher than the real cost for $x_2 = 1.156$ and gives an expected cost between 20% and 23% lower than the real cost for $x_1 = 0.837$. The optimal maneuver periods also change, as discussed above.
Figures (7) and (8) show a comparison of these costs for the two systems and the two methods, taking $x_1 = 0.99$ for the Earth-Sun system and $x_1 = 0.837$ for the Earth-Moon system. The results in the Earth-Moon system are contracted in time because of the error made in the eigenvalues.

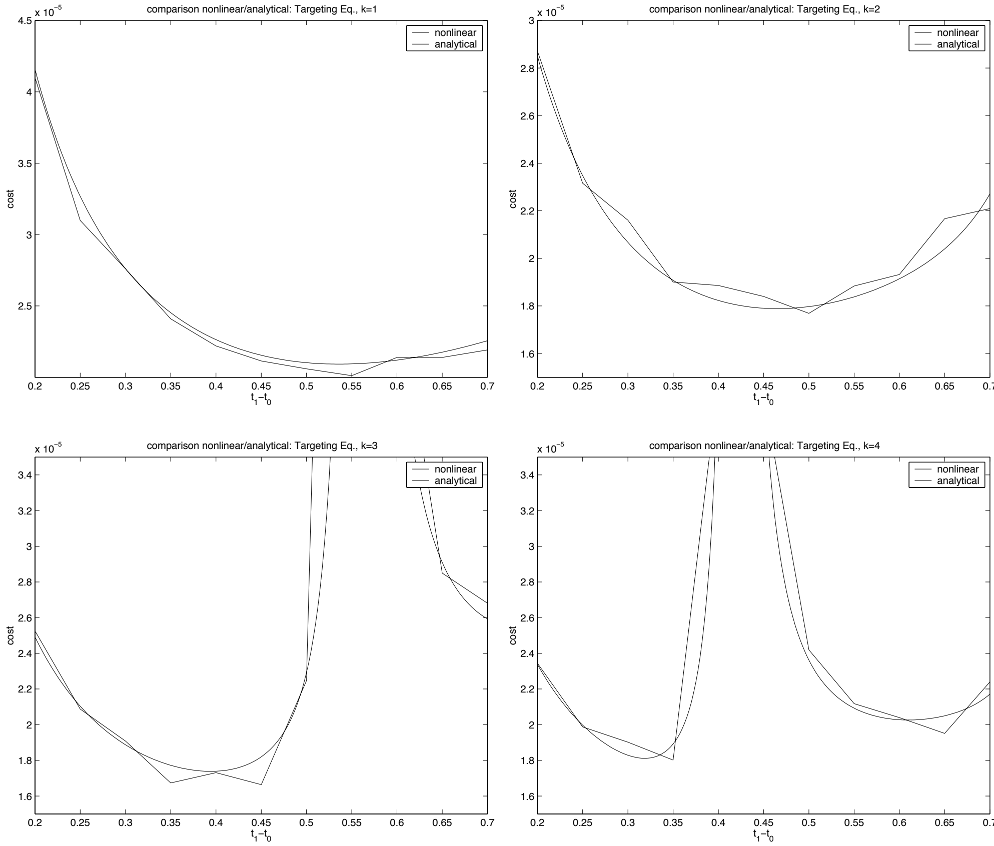

Figure 6: Targeting the Eq.: Cost of the Nonlinear Model vs Cost of the Linear Model

Main bodies	Problem		cost km/s/period	period
Earth-Sun	Hill	S=100	5.35e-5	28.5 days
		S=0.01	3.78e-5	18.6 days
	3-Body	S=100	5.30e-5	28.5 days
	$x_0 = 1.01$	S=0.01	3.74e-5	19.2 days
	3-Body	S=100	5.39e-5	27.9 days
	$x_0 = 0.99$	S=0.01	3.82e-5	18.6 days
Earth-Moon	Hill	S=100	3.60e-4	45.42 hours
		S=0.01	2.75e-4	33.03 hours
	3-Body	S=100	2.95e-4	52.6 hours
	$x_0 = 1.156$	S=0.01	2.17e-4	38.2 hours
	3-Body	S=100	4.89e-4	39.2 hours
	$x_0 = 0.837$	S=0.01	3.53e-4	27.9 hours

Table 1: LQR method

Conclusions/Future Work

This paper compares different control strategies that can be implemented to control a spacecraft about an un-

Main bodies	Problem		cost km/s/period	period
Earth-Sun	Hill		4.70e-5	22.9 days
	3-Body	$x_0 = 1.01$	4.65e-5	23.1 days
	3-Body	$x_0 = 0.99$	4.75e-5	22.7 days
Earth-Moon	Hill		3.36e-4	38.2 hours
	3-Body	$x_0 = 1.156$	2.66e-4	44.4 hours
	3-Body	$x_0 = 0.837$	4.32e-4	32.8 hours

Table 2: Targeting the equilibrium

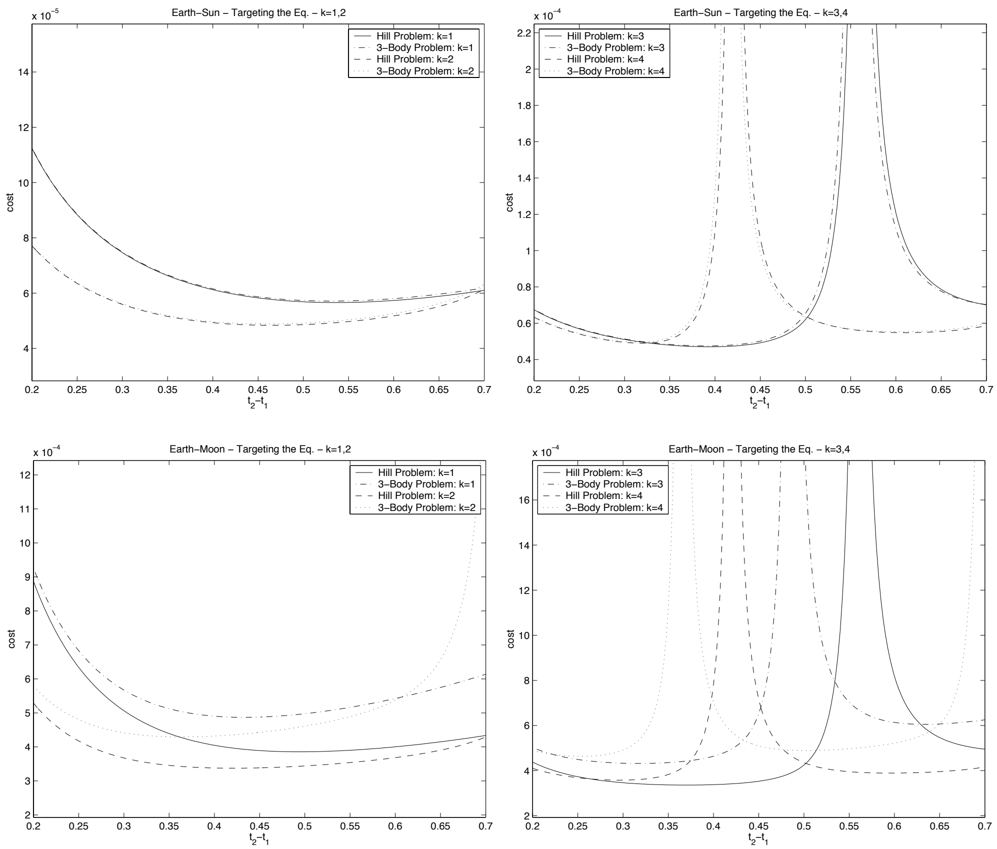

Figure 7: Targeting the Eq.: Comparison Hill Problem vs Restricted 3-Body Problem

stable equilibrium point. First an optimal statistical cost has been found for simultaneous maneuvers targeting the equilibrium point. This first control approach has been extended to the more general method of targeting the stable manifold. We have shown that targeting the stable manifold does not improve the results. The LQR method and its tradeoff has been investigated, leading to an improvements of the statistical cost. All these results have been checked by non-linear simulations. Finally the optimal costs of the two controls have been compared for the Hill problem and the Restricted 3 Body problem and are very similar for the Earth-Sun system. We have found that a maneuver spacing close to the characteristic time of the unstable eigenvalue is optimal in all cases.

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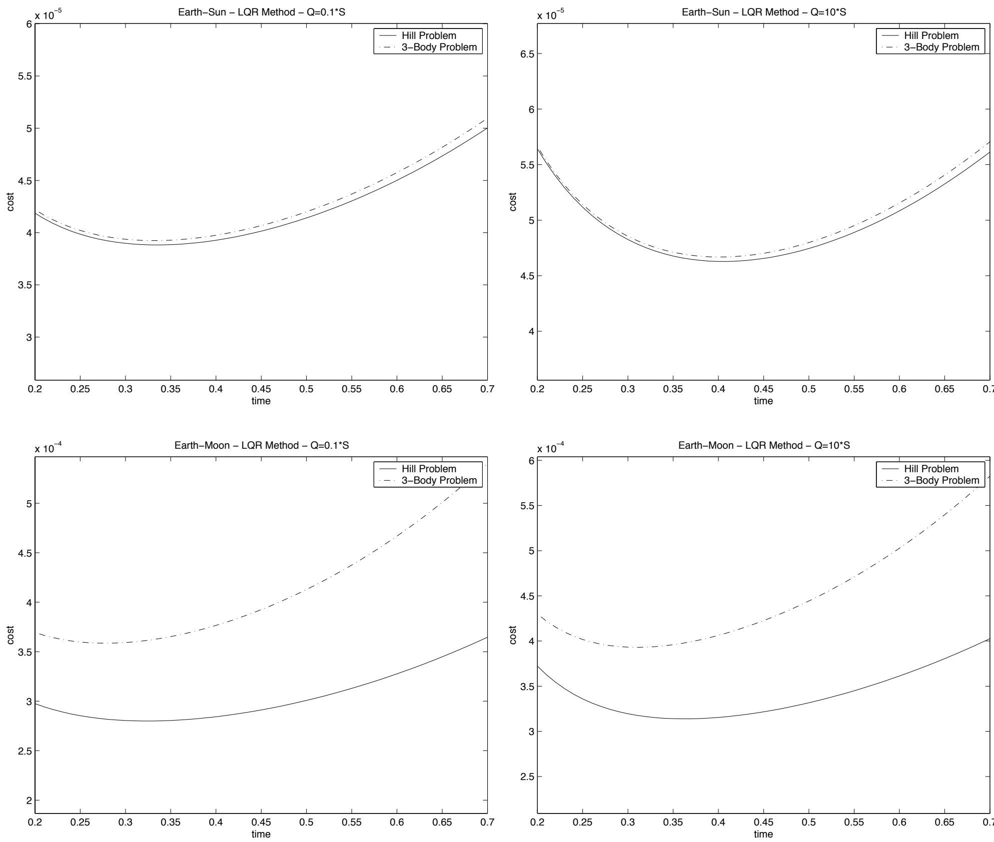

Figure 8: LQR Method: Comparison Hill Problem vs Restricted 3-Body Problem

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Appendix

Procedure to compute the mean and variance of $|\Delta V|$

The aim of this appendix is to give a general method for the computation of the mean and variance of $|\Delta V|$:
The mean is given by $\overline{|\Delta V|} = \int_\infty |\Delta V| f(\boldsymbol{x}) \mathrm{d}\boldsymbol{x}$
where f is the probability density function of a gaussian distribution. The variance is given by $\sigma^2_{\Delta V} = \int_\infty |\Delta V|^2 f(\boldsymbol{x}) \mathrm{d}\boldsymbol{x}$
More explicitly, one can write:

$$\overline{|\Delta V|} = \int_\infty |\Psi \boldsymbol{x}| \frac{\sqrt{|K|}}{(2\pi)^{N/2}} e^{-\frac{1}{2}\boldsymbol{x}^T K \boldsymbol{x}} \mathrm{d}\boldsymbol{x} \qquad (59)$$

The main problem comes from the integration of the norm of $\Psi \boldsymbol{x}$. Let us write:

$$\boldsymbol{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_N \end{bmatrix} = \begin{bmatrix} a_1^T \\ a_2^T \\ \vdots \\ a_N^T \end{bmatrix} \boldsymbol{x} = A\boldsymbol{x} \tag{60}$$

where: $\begin{bmatrix} a_1^T \\ a_2^T \end{bmatrix} = \Psi$

A is a $N \times N$ matrix. Let us assume $\mathrm{rank}(\Psi) = 2$ in order that A can be chosen to be nonsingular. Using $\boldsymbol{u}$ as the new variable, introduce $H = A^{-T} K A^{-1}$,with: $\mathrm{d}\boldsymbol{x} = \frac{\mathrm{d}\boldsymbol{u}}{|A|} = \left(\frac{|H|}{|K|}\right)^{1/2} \mathrm{d}\boldsymbol{u}$, where $|.|$ stands for the determinant of a matrix.

It is possible to write the new expression of the integral:

$$\overline{|\Delta V|} = \int_\infty \sqrt{{u_1}^2 + {u_2}^2} \frac{\sqrt{|H|}}{(2\pi)^{N/2}} e^{-\frac{1}{2}\boldsymbol{u}^T H \boldsymbol{u}} \mathrm{d}\boldsymbol{u} \tag{61}$$

Then if H is a diagonal matrix of the form $H = \mathrm{diag}(\alpha \quad \beta \quad 1 \quad 1)$, integrating over $u_3, \ldots, u_N$, the integral takes the new form:

$$\overline{|\Delta V|} = \frac{1}{2\pi} \int_\infty \sqrt{{u_1}^2 + {u_2}^2} e^{-\frac{\alpha {u_1}^2 + \beta {u_2}^2}{2}} \sqrt{\alpha\beta} \mathrm{d}u_1 \mathrm{d}u_2 \tag{62}$$

Let us now examine the hypothesis of $H = \mathrm{diag}(\alpha \quad \beta \quad 1 \quad 1)$ more closely, $A^T H A = K$ is equivalent to:

$$\alpha a_1 a_1^T + \beta a_2 a_2^T + a_3 a_3^T + \ldots + a_N a_N^T = K \tag{63}$$

Therefore taking the orthogonal x to $(a_2, \ldots, a_N)$ gives:
$\alpha a_1 a_1^T x = Kx \quad \Leftrightarrow \quad \alpha(a_1^T x) a_1 = Kx$
As a result, $x = K^{-1} a_1$ must be orthogonal to a_2. This is only true for a particular matrix K. As the integration must be performed for every K, a choice other than a_2 is needed in A. Let us introduce the new notation:

$$\Psi = \begin{bmatrix} a_1 \\ \overline{a_2} \end{bmatrix} \tag{64}$$

$$\gamma = \frac{a_1^T K^{-1} \overline{a_2}}{a_1^T K^{-1} a_1} \tag{65}$$

The last equality is possible for every $a_1 \neq 0$ as our information matrix and its inverse are positive definite. Let:

$$a_2 = \overline{a_2} - \gamma a_1 \tag{66}$$

Then: $a_2^T K^{-1} a_1 = 0$
Finally, as we assumed $\mathrm{rank}(\Psi) = 2$, a_1 and a_2 are independent.

Now let us prove that if a_k, $k \leq N$, is orthogonal to $(K^{-1}a_1, \ldots, K^{-1}a_{k-1})$, then a_k is independent of $(a_1, \ldots, a_{k-1})$:
Let $\lambda_1, \ldots, \lambda_{k-1}$ not all equal to 0 such that: $a_k = \sum_{j=1}^{k-1} \lambda_j a_j$
Then for all $i \in [1, k-1]$: $\lambda_i a_i^T K^{-1} a_i = 0$. As K^{-1} is positive definite, it gives $\lambda_i = 0$. Contradiction.

Finally taking a_3 in the orthogonal of $(K^{-1}a_1, K^{-1}\overline{a_2})$, normalizing a_3 such that $a_3^T K^{-1} a_3 = 1$, ... , a_N in the orthogonal of $(K^{-1}a_1, \ldots, K^{-1}a_{N-1})$, normalizing a_N such that $a_N^T K^{-1} a_N = 1$, $(a_1, \ldots, a_N)$ is a basis. As a result, A is nonsingular and:

$$AK^{-1}A^T = H^{-1} \quad \Leftrightarrow \quad H = A^{-T} K A^{-1} \tag{67}$$

α and β are given by:

$$\alpha^{-1} = a_1^T K^{-1} a_1 \tag{68}$$

$$\beta^{-1} = a_2^T K^{-1} a_2 = \overline{a_2}^T K^{-1} \overline{a_2} - \frac{(a_1^T K^{-1} \overline{a_2})^2}{a_1^T K^{-1} a_1} \tag{69}$$

It gives: $|\Psi x| = [(a_1^T x)^2 + (a_2^T x + \gamma a_1^T x)^2]^{1/2}$.
$|\Phi \boldsymbol{x_0}| = \sqrt{(1+\gamma^2){u_1}^2 + {u_2}^2 + 2\gamma u_1 u_2}$ with $u_1 = a_1^T x$, $u_2 = a_2^T x$.
As a consequence,

$$\overline{|\Delta V|} = |\Phi \boldsymbol{x_0}| e^{-\frac{\alpha {u_1}^2 + \beta {u_2}^2}{2}} \sqrt{\alpha\beta} \mathrm{d}u_1 \mathrm{d}u_2 \tag{70}$$

Taking for the new variables $(v_1, v_2) = (\sqrt{\alpha} u_1, \sqrt{\beta} u_2)$ and then changing to polar coordinates gives the following new expression for the integral:

$$\overline{|\Delta V|} = \frac{1}{2\pi} \int_\infty C_\theta \rho^2 e^{-\frac{\rho^2}{2}} \mathrm{d}\rho \mathrm{d}\theta \tag{71}$$

with:

$$C_\theta = \sqrt{\frac{(1+\gamma^2)\cos\theta^2}{\alpha} + \frac{\sin\theta^2}{\beta} + \frac{2\gamma\cos\theta\sin\theta}{\sqrt{\alpha\beta}}} \tag{72}$$

To compute the integral of C_θ, let us write:

$$C_\theta = \sqrt{\lambda + 0.5\left(\frac{1+\gamma^2}{\alpha} - \frac{1}{\beta}\right)\cos 2\theta + \frac{\gamma}{\sqrt{\alpha\beta}}\sin 2\theta} \tag{73}$$

with:
$\lambda = 0.5\left(\frac{1+\gamma^2}{\alpha} + \frac{1}{\beta}\right)$

Then, assuming $\mu \neq 0$, $\overline{|\Delta V|}$ is given by:

$$\overline{|\Delta V|} = \frac{1}{\sqrt{2\pi}} \int_0^{\pi} \sqrt{\lambda + \mu \cos(2\theta)} \mathrm{d}\theta \tag{74}$$

with:

$$\lambda = (a_1^T K^{-1} a_1 + \overline{a_2}^T K^{-1} \overline{a_2})/2 \tag{75}$$

$$\mu = \sqrt{(a_1^T K^{-1} a_1 - \overline{a_2}^T G^{-1} \overline{a_2})^2/4 + (a_1^T K^{-1} \overline{a_2})^2} \tag{76}$$

More explicitly:

$$\overline{\Delta V} = \sqrt{\frac{2}{\pi}} \sqrt{\lambda + \mu} \quad \mathrm{EllipticE}\left(\sqrt{\frac{2\mu}{\lambda + \mu}}\right) \tag{77}$$

where EllipticE is defined as:

$$\mathrm{EllipticE}(k) = \int_0^{\frac{\pi}{2}} (1 - k^2 \sin^2\theta)^{\frac{1}{2}} \mathrm{d}\theta \tag{78}$$

and is the complete elliptic integral of the second kind. It is easy to check that the integration is still valid when $\mu = 0$. The other particular case is when $\mathrm{rank}(\Psi) = 1$. In this case, it is trivial to check that $\overline{|\Delta V|} = \sqrt{\frac{2}{\pi}}\sqrt{\lambda}$. For this matrix Ψ, as $\lambda = \mu$ and $\mathrm{EllipticE}(1) = 1$, the formula above gives the same result.

The variance of the statistical cost of these maneuver is fairly easy to compute once this method has been developed for the mean:

$$\sigma^2_{\Delta V} = \int_{\infty} |\Delta V|^2 f(\boldsymbol{x_0}) \mathrm{d}\boldsymbol{x_0} - \overline{|\Delta V|}^2 \tag{79}$$

$$\sigma^2_{\Delta V} = \frac{1}{2\pi} \int_{\infty} C_\theta^2 \rho^3 e^{-\frac{\rho^2}{2}} \mathrm{d}\rho \mathrm{d}\theta - \overline{|\Delta V|}^2 \tag{80}$$

$$\sigma^2_{\Delta V} = 2\lambda - \frac{2}{\pi}(\lambda + \mu) \quad \mathrm{EllipticE}^2\left(\sqrt{\frac{2\mu}{\lambda + \mu}}\right) \tag{81}$$