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by
N. X. VINH
The University of Michigan
Ann Arbor, Michigan

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INTEGRALS OF THE MOTION FOR OPTIMAL TRAJECTORIES IN ATMOSPHERIC FLIGHT*

N. X. Vinh**
The University of Michigan
Ann Arbor, Michigan 48104

Abstract

The problem of optimizing the flight trajectory of a rocket vehicle moving in a resisting medium, in a general gravitational force field, is considered. Optimal laws for the modulation of the aerodynamic and propulsive forces are formulated in terms of the primer vector, the vector adjoint to the velocity. Relations for flight along an intermediate-thrust arc, and integrals of the motion for several cases of practical interest are derived.

I. Introduction

The problem of determining optimal trajectories for a rocket powered lifting vehicle flying inside the atmosphere of a planet has received considerable attention in recent years. General control laws for the lift and the bank angle, and for the thrusting program have been obtained in terms of the primer vector, the adjoint vector associated to the velocity vector(1). In this paper we modify the results in Ref. 1 to apply to the case when the thrust direction is constrained to align with the velocity vector. This is the case when the engine is mounted fixed with respect to the vehicle and the flight is effectuated in the dense layer of the atmosphere and at small angle of attack to validate the assumption(2). From the general theory, solutions for optimal maneuvers for flight in a uniform gravitational field will be obtained by canonical transformations.

II. The Optimal Controls

Consider the motion of a powered, lifting vehicle in a general gravitational force field. At the time t, the state of the vehicle is defined by

$\vec{r}(t)$ = position vector

$\vec{V}(t)$ = velocity vector

$m(t)$ = instantaneous mass.

The flight is controlled by a thrusting force $\vec{T}$ and the aerodynamic force $\vec{A}$ (Fig. 1). It is assumed that $\vec{T}$ is aligned with the velocity, and its magnitude is bounded by

$$0 \leqq T \leqq T_{max} \tag{1}$$

Furthermore, we assume that the vehicle has a plane of symmetry, both the thrust and the aerodynamic force are applied at the center of mass and, in coordinated flight, the aerodynamic force and the velocity are contained in that plane of symmetry. It is customary to decompose $\vec{A}$ into a drag force $\vec{D}$, always opposite to $\vec{V}$ and a lift force $\vec{L}$ orthogonal to it. We shall use the usual assumption

$$\begin{aligned} L &= \tfrac{1}{2}\rho S V^2 C_L \\ D &= \tfrac{1}{2}\rho S V^2 C_D \end{aligned} \tag{2}$$

where ρ is the atmospheric mass density, function of $\vec{r}$, S is a reference area. In hypersonic flight, the lift coefficient C_L, and the drag coefficient C_D are assumed independent of the Mach number and the Reynolds number. For simplicity, we shall assume a parabolic drag polar, defined by

$$C_D = C_{D_0} + kC_L^2 \tag{3}$$

where C_{D_0} and k are constants. It is convenient to define a lift control parameter λ such that

$$\lambda = \sqrt{\frac{k}{C_{D_0}}}\; C_L \tag{4}$$

Then, when $\lambda = 1$, the flight is at maximum lift-to-drag ratio.

The motion of the vehicle, flying in a general gravitational force field, and subject to aerodynamic force and thrusting force, is governed by the equations

$$\begin{aligned} \frac{d\vec{r}}{dt} &= \vec{V} \\ \frac{d\vec{V}}{dt} &= \frac{T}{m}\frac{\vec{V}}{V} + \frac{\vec{A}}{m} + \vec{g}(\vec{r}, t) \\ \frac{dm}{dt} &= -\frac{T}{c} \end{aligned} \tag{5}$$

where c is the constant exhaust velocity of the gas ejected from the engine and $\vec{g}$ is the acceleration of the gravitational field. The optimal transfer problem is defined as follows.

At the initial time, $t = 0$, $\vec{r} = \vec{r}_0$, $\vec{V} = \vec{V}_0$, $m = m_0$. The vectors $\vec{r}_0$, $\vec{V}_0$ and the scalar m_0 are prescribed. At the final time, $t = t_f$, $\vec{r} = \vec{r}_f$, $\vec{V} = \vec{V}_f$ and $m = m_f$. The problem is to find the time history of $\vec{A}(t)$ and $T(t)$ such that some scalar function of the final state is a minimum.

Using the maximum principle, we introduce the adjoint elements $\vec{q}$, $\vec{p}$ and p_m to form the Hamiltonian.

$$H = \vec{q}\cdot\vec{V} + \vec{p}\cdot\left(\frac{\vec{A}}{m} + \vec{g}\right) + \frac{T}{m}\left[\frac{(\vec{p}\cdot\vec{V})}{V} - \frac{mp_m}{c}\right] \tag{6}$$

where $\vec{q}$, $\vec{p}$ and p_m are defined by the adjoint equations

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** Professor, Department of Aerospace Engineering, The University of Michigan, Member AIAA.

$$\frac{d\vec{q}}{dt} = -\nabla(\vec{p}\cdot\vec{g}) - \frac{1}{m}\nabla(\vec{p}\cdot\vec{A})$$

$$\frac{d\vec{p}}{dt} = -\vec{q} - \frac{2(\vec{p}\cdot\vec{A})}{mV^2}\vec{V} - \frac{T}{mV}[\vec{p} - \frac{(\vec{p}\cdot\vec{V})}{V^2}\vec{V}] \tag{7}$$

$$\frac{dp_m}{dt} = \frac{T}{m^2}\frac{(\vec{p}\cdot\vec{V})}{V} + \frac{1}{m^2}(\vec{p}\cdot\vec{A})$$

The following general results for optimal trajectories have been obtained in Ref. 1.

1. If the direction of the thrust can be taken arbitrarily, whenever the engine is operating, the thrust must be directed along the vector $\vec{p}$, called the primer vector. This is the extension of the classical result obtained by Lawden for transfer in a vacuum.

2. The thrusting program is governed by the switching function, which, for the case considered in this paper when the thrust is always aligned with the velocity, is defined by

$$K = \frac{(\vec{p}\cdot\vec{V})}{V} - \frac{mp_m}{c} \tag{8}$$

If $K > 0$, we select $T = T_{max}$ (boosting phase)
$K < 0$, we select $T = 0$ (coasting phase)
$K = 0$, for a finite time interval, we select T = variable (sustaining phase)

3. The optimal lift modulation is such that $C_L = C_{L_{max}}$, or, for variable lift program

$$\tan\epsilon = \frac{\partial C_D}{\partial C_L} \tag{9}$$

where ϵ is the angle between $\vec{V}$ and $\vec{p}$.

4. The optimal bank angle is such that the vectors $\vec{V}$, $\vec{p}$ and $\vec{A}$ are coplanar, that is

$$(\vec{V}\times\vec{p})\cdot\vec{A} = 0 \tag{10}$$

5. For optimal flight constantly at maximum lift-to-drag ratio, $\vec{p}$ is orthogonal to $\vec{A}$, that is

$$\vec{p}\cdot\vec{A} = 0 \tag{11}$$

6. Along a variable-thrust arc, we have

$$\frac{(\vec{p}\cdot\vec{V})}{V} - \frac{mp_m}{c} = 0 \tag{12}$$

By taking the derivative of this equation, and using (5), (6) and (7) together with (12) and rearranging, we have

$$H + \frac{(\vec{p}\cdot\vec{A})}{m}\left(\frac{V}{c}\right) + \frac{(\vec{p}\cdot\vec{V})}{V^2}\left[\frac{(\vec{V}\cdot\vec{A})}{m} + (\vec{V}\cdot\vec{g})\right] = 2(\vec{p}\cdot\vec{g}) \tag{13}$$

The variable thrust magnitude control will be obtained upon taking the derivative of this equation.

These results are valid for a general gravitational force field. In particular, when $\vec{g}$ is time invariant, the Hamiltonian given by (6) is constant.

III. Application to the Case of a Uniform Gravitational Field

Consider the case where $\vec{g}$ is a constant vector. In Cartesian rectangular coordinates, let

$$\vec{r} = (x_1, x_2, x_3) \text{ with } x_3 \text{ along } \vec{g}$$

$$\vec{V} = (v_1, v_2, v_3)$$
$$\vec{q} = (q_1, q_2, q_3)$$
$$\vec{p} = (p_1, p_2, p_3)$$
$$\vec{A} = (A_1, A_2, A_3)$$

We shall use the transformation

$$\begin{aligned} v_1 &= V\cos\gamma\cos\psi \\ v_2 &= V\cos\gamma\sin\psi \\ v_3 &= V\sin\gamma \end{aligned} \tag{14}$$

It is seen that γ is the flight path angle, and ψ the heading. We consider an exponential atmosphere of the form

$$\rho = \rho_0 \exp(-\beta x_3) \tag{15}$$

where β is the constant height scale. For the transformation (14) to be canonical, while conserving the same Hamiltonian

$$p_1 dv_1 + p_2 dv_2 + p_3 dv_3 = p_V dV + p_\gamma d\gamma + p_\psi d\psi$$

This gives the linear transformation for the adjoint components of the primer vector

$$\begin{bmatrix} \cos\gamma\cos\psi & \cos\gamma\sin\psi & \sin\gamma \\ -\sin\gamma\cos\psi & -\sin\gamma\sin\psi & \cos\gamma \\ -\cos\gamma\sin\psi & \cos\gamma\cos\psi & 0 \end{bmatrix}\begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} p_V \\ p_\gamma/V \\ p_\psi/V \end{bmatrix} \tag{16}$$

Reversing the matrix equation

$$\begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} \cos\gamma\cos\psi & -\sin\gamma\cos\psi & -\frac{\sin\psi}{\cos\gamma} \\ \cos\gamma\sin\psi & -\sin\gamma\sin\psi & \frac{\cos\psi}{\cos\gamma} \\ \sin\gamma & \cos\gamma & 0 \end{bmatrix}\begin{bmatrix} p_V \\ p_\gamma/V \\ p_\psi/V \end{bmatrix} \tag{17}$$

In particular, we can verify the following relations

$$p^2 = p_1^2 + p_2^2 + p_3^2 = p_V^2 + \frac{p_\gamma^2}{V^2} + \frac{p_\psi^2}{V^2\cos^2\gamma} \tag{18}$$

and

$$\vec{p}\cdot\vec{V} = p_1 v_1 + p_2 v_2 + p_3 v_3 = V p_V \tag{19}$$

For the transformation of the aerodynamic forces, we define the bank angle ϕ as the angle between the plane of symmetry of the airplane and the vertical plane passing through the velocity $\vec{V}$ (Fig. 2). Then, from the geometry of the figure

$$\begin{aligned} (A_1\cos\psi + A_2\sin\psi)\cos\gamma + A_3\sin\gamma &= -D \\ (A_1\cos\psi + A_2\sin\psi)\sin\gamma - A_3\cos\gamma &= -L\cos\phi \\ A_1\sin\psi - A_2\cos\psi &= -L\sin\phi \end{aligned} \tag{20}$$

Therefore, we have for the Cartesian components of the aerodynamic force

$$\begin{aligned} A_1 &= -D\cos\psi\cos\gamma - L(\cos\phi\cos\psi\sin\gamma + \sin\phi\sin\psi) \\ A_2 &= -D\sin\psi\cos\gamma - L(\cos\phi\sin\psi\sin\gamma - \sin\phi\cos\psi) \\ A_3 &= -D\sin\gamma + L\cos\phi\cos\gamma. \end{aligned} \tag{21}$$

The aerodynamic optimal controls can be obtained directly from the general theory. Using formula

(9), with relation (3), we have for the optimal variable lift control

$$\tan \epsilon = 2kC_L \tag{22}$$

For the bank angle, by expanding the determinant (10), using (14) and (17) for the components of $\vec{V}$ and $\vec{p}$, we have

$$p_\gamma [A_1 \sin\psi - A_2 \cos\psi] - \frac{p_\psi}{\cos\gamma}$$

$$\cdot [(A_1 \cos\psi + A_2 \sin\psi)\sin\gamma - A_3 \cos\gamma] = 0$$

By relations (20), we have for the optimal variable bank angle

$$\tan\phi = \frac{p_\psi}{p_\gamma \cos\gamma} \tag{23}$$

We can rewrite relation (22) by observing that

$$\vec{p}\cdot\vec{V} = pV\cos\epsilon = Vp_V$$

Therefore

$$p^2 = \frac{p_V^2}{\cos^2\epsilon} \tag{24}$$

On the other hand, from (18), and using the optimal law (23)

$$p^2 = p_V^2 + \frac{p_\gamma^2}{V^2} + \frac{p_\gamma^2 \tan^2\phi}{V^2}$$

or

$$p^2 = p_V^2 + \frac{p_\gamma^2}{V^2\cos^2\phi} \tag{25}$$

By eliminating p^2 between (24) and (25), we have

$$\tan\epsilon = \frac{p_\gamma}{Vp_V\cos\phi} \tag{26}$$

The optimal lift control, expressed in terms of the adjoint variables, is then

$$2kC_L = \frac{p_\gamma}{Vp_V\cos\phi} \tag{27}$$

We see that the aerodynamic controls are governed by the adjoint components p_V, p_γ and p_ψ

IV. The Integrals of the Motion

A number of the integrals of the motion, mostly associated with a flight at maximum lift-to-drag ratio, has been displayed in Ref. 1. Here we shall derive them directly, and furthermore obtain some additional integrals. The equations of the motion, written in components form, and using the velocity transformation (14), are

$$\frac{dx_1}{dt} = V\cos\gamma\cos\psi$$

$$\frac{dx_2}{dt} = V\cos\gamma\sin\psi$$

$$\frac{dx_3}{dt} = V\sin\gamma$$

$$\frac{dV}{dt} = \frac{(T-D)}{m} - g\sin\gamma$$

$$\frac{d\gamma}{dt} = \frac{L\cos\phi}{mV} - \frac{g}{V}\cos\gamma \tag{28}$$

$$\frac{d\psi}{dt} = \frac{L\sin\phi}{mV\cos\gamma}$$

$$\frac{dm}{dt} = -\frac{T}{c}$$

The Hamiltonian (6) becomes, with new variables, in a uniform gravitational field

$$H = (q_1\cos\psi + q_2\sin\psi)V\cos\gamma + q_3 V\sin\gamma - p_V\left(\frac{D}{m} + g\sin\gamma\right) + p_\gamma\left(\frac{L\cos\phi}{mV} - \frac{g}{V}\cos\gamma\right) + p_\psi\frac{L\sin\phi}{mV\cos\gamma} + \frac{T}{m}\left(p_V - \frac{mp_m}{c}\right) \tag{29}$$

Along an optimal trajectory, with variable lift and bank controls given by (23) and (27) we have

$$H = (q_1\cos\psi + q_2\sin\psi)V\cos\gamma + q_3 V\sin\gamma - g\left(p_V\sin\gamma + \frac{p_\gamma}{V}\cos\gamma\right) + \frac{\rho SV^2}{2m}\left(kC_L^2 - C_{D_0}\right)p_V + \frac{T}{m}\left(p_V - \frac{mp_m}{c}\right) \tag{30}$$

We notice that, if the flight is at maximum lift-to-drag ratio $kC_L^2 - C_{D_0} = 0$, and the constant Hamiltonian is free of aerodynamic components. This result has been shown to be valid for a general gravitational force field in Ref. 1. The switching function is seen to be

$$K = p_V - \frac{mp_m}{c} \tag{31}$$

The adjoint equations, with optimal variable lift and bank controls, are

$$\frac{dq_1}{dt} = 0$$

$$\frac{dq_2}{dt} = 0$$

$$\frac{dq_3}{dt} = \frac{\beta\rho SV^2}{2m}(kC_L^2 - C_{D_0})p_V$$

$$\frac{dp_V}{dt} = -(q_1\cos\psi + q_2\sin\psi)\cos\gamma - q_3\sin\gamma - \frac{gp_\gamma}{V^2}\cos\gamma + \frac{\rho SVC_{D_0}}{m}p_V \tag{32}$$

$$\frac{dp_\gamma}{dt} = (q_1\cos\psi + q_2\sin\psi)V\sin\gamma - q_3 V\cos\gamma + g\left(p_V\cos\gamma - \frac{p_\gamma}{V}\sin\gamma\right) - \frac{p_\psi L\sin\phi}{mV\cos^2\gamma}\sin\gamma$$

$$\frac{dp_\psi}{dt} = (q_1\sin\psi - q_2\cos\psi)V\cos\gamma$$

$$\frac{dp_m}{dt} = \frac{\rho SV^2}{2m^2}(kC_L^2 - C_{D_0})p_V + \frac{Tp_V}{m^2}$$

We first have the integrals

$$q_1 = a_1 = \text{constant}$$
$$q_2 = a_2 = \text{constant} \tag{33}$$

If the longitudinal range and the lateral range are free, *these constants are zero. It is seen that* q_3 is also constant if the flight is at maximum lift-to-drag ratio. Next, using (33) we have

$$\frac{d}{dt}(q_1 x_2 - q_2 x_1) = \frac{dp_\psi}{dt}$$

Hence

$$q_1 x_2 - q_2 x_1 = p_\psi + \text{constant} \tag{34}$$

For free longitudinal and lateral ranges

$$p_\psi = a_6 = \text{constant} \tag{35}$$

If the final heading is not prescribed, this constant is zero, and from (23), $\phi = 0$. The flight is effectuated in a vertical plane containing the initial velocity.

Now consider

$$\frac{d}{dt}\left(mp_m\right) = \frac{\rho S V^2}{2m}\left(kC_L^2 - C_{D_0}\right)p_V + \frac{T}{m}\left(p_V - \frac{mp_m}{c}\right) \tag{36}$$

First, along a coasting arc, $T = 0$, *or along a sustaining arc,* $K = 0$, and if the flight is at maximum lift-to-drag ratio

$$mp_m = \text{constant} \tag{37}$$

For flight at variable lift coefficient, and along a coasting arc or a sustaining arc, by eliminating the time between (36) and the equation for q_3, we have

$$dq_3 = \beta d(mp_m)$$

By integrating, we have

$$q_3 = \beta mp_m + a_3 \tag{38}$$

where a_3 is a new constant of integration.

Consider the derivative of Vp_V

$$\frac{d}{dt}\left(Vp_V\right) = -H - 2g\left(p_V \sin\gamma + \frac{p_\gamma}{V}\cos\gamma\right) + \frac{T}{m}\left(2p_V - \frac{mp_m}{c}\right) \tag{39}$$

Along a coasting arc, $T = 0$, and using Eggers assumption by *neglecting the gravity compared to* aerodynamic force[3], an assumption generally valid for a skip or pull-up maneuver, we have, upon integration

$$Vp_V = -Ht + a_4 \tag{40}$$

where a_4 is a constant of integration. In particular, when the final time is free, $H \equiv 0$, and

$$Vp_V = a_4 \tag{41}$$

We write the Hamiltonian for the case of free longitudinal and lateral range, and free time, along a coasting arc or a sustaining arc. Relation (30) becomes

$$\frac{\rho S V^2}{2m}\left(kC_L^2 - C_{D_0}\right)p_V = -q_3 V \sin\gamma + g\left(p_V \sin\gamma + \frac{p_\gamma}{V}\cos\gamma\right) \tag{42}$$

With this integral we can delete the differential equation for p_γ. Using (42) we write the equation for q_3

$$\frac{dq_3}{dt} = -\beta q_3 V \sin\gamma + \beta g\left(p_V \sin\gamma + \frac{p_\gamma}{V}\cos\gamma\right)$$

The equation can be integrated if skip trajectory assumption is used. We have

$$\frac{dq_3}{dt} = -\beta q_3 V \sin\gamma = -\beta q_3 \frac{dx_3}{dt}$$

Upon integrating

$$q_3 = \text{const.} \times \exp(-\beta x_3) = \text{const.} \times \rho \tag{43}$$

By the change of variable from x_3 to ρ, using the exponential law (15), a canonical transformation requires

$$q_3\, dx_3 = q_\rho d\rho = -\beta\rho q_\rho dx_3$$

Hence

$$q_3 = -\beta\rho q_\rho$$

and therefore

$$q_\rho = \text{constant} \tag{44}$$

This shows the advantage of using the atmospheric mass density as the altitude variable.

The Eggers assumption, in the free range and *free time case, gives explicit laws for the modula*tion of the lift and the bank.

First, along a coasting arc, we write (43)

$$q_3 = b_3 \rho \tag{45}$$

where b_3 is a constant. The Hamiltonian relation (42) gives

$$Vp_V = \frac{2b_3 m \sin\gamma}{S(C_{D_0} - kC_L^2)}$$

Since in this case $Vp_V = a_4$, we have

$$\frac{kC_L^2}{C_{D_0}} = 1 - \frac{2b_3 m \sin\gamma}{S a_4 C_{D_0}}$$

That is, by observing that m is constant along a coasting arc

$$\lambda^2 = 1 + a \sin\gamma \tag{46}$$

where a is a constant. We have the classical result, first obtained by Contensou[4] and later extended to three-dimensional case by Griffin and Vinh[5] and Speyer and Womble[6]. We now rewrite the law for the bank angle

$$\tan\phi = \frac{p_\psi}{p_\gamma \cos\gamma} = \frac{a_6}{2ka_4 C_L \cos\phi\cos\gamma}$$

or

$$\sin\phi = \frac{b}{\lambda\cos\gamma} \tag{47}$$

where b is a constant. The constants a and b in the optimal lift and bank laws are determined by specifying terminal conditions at the ends of a coasting arc.

Next, along a sustaining arc, we have the relations (12) and (13), which for a constant gravity field, become

$$p_V - \frac{mp_m}{c} = 0 \tag{48}$$

and

$$\frac{\rho S V^2 p_V}{2m}\left[\left(\frac{V}{c} + 1\right) C_{D_0} - \left(\frac{V}{c} - 1\right) k C_L^2\right] = H + \frac{2gp_\gamma}{V}\cos\gamma + gp_V \sin\gamma \tag{49}$$

For a free time problem, $H \equiv 0$, and neglecting gravity, we have the following simple law for optimal variable lift control along a sustaining arc

$$\lambda^2 = \frac{(V/c) + 1}{(V/c) - 1} \tag{50}$$

An interesting fact is that variable lift control for sustaining flight is only optimal for high velocity, $V > c$, and for vehicle with high lift performance, $\lambda > 1$.

Finally for the boosting arc, by eliminating the time between the equations for V and m, we have

$$\frac{dV}{dm} = -\frac{c}{m} + \frac{c}{T_{max}}\left[\frac{D}{m} + g\sin\gamma\right]$$

with $T_{max} \to \infty$ (impulsive approximation) the equation can be integrated to give

$$m = \overline{m}\exp\left(-\frac{V}{c}\right) \tag{51}$$

where $\overline{m}$ is a constant mass.

V. Conclusions

General control laws for the modulation of the lift and the bank angle, and for the thrusting program along the optimal trajectory of a rocket powered, lifting vehicle, flying inside the atmosphere of a planet, with the thrust aligned along the velocity vector, have been obtained in terms of the primer vector, the adjoint vector associated to the velocity vector. From the general theory, solutions for optimal maneuvers for flight in a uniform gravitational field are obtained by canonical transformations. The problem is completely solved for the free range and free time case, using Eggers assumption for skip trajectory. It should be noted that this last stringent condition can be removed by using Loh's second order theory[7], as applied by Speyer and Womble for free flight trajectory[6]. Loh's second order theory which also includes the curvature of the flight path has also been applied successfully to the case of thrusting flight by Griffin and Vinh[8].

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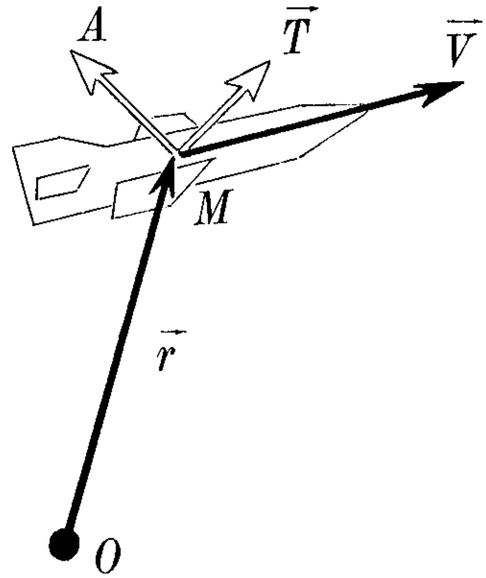

Figure 1. State and Control Variables

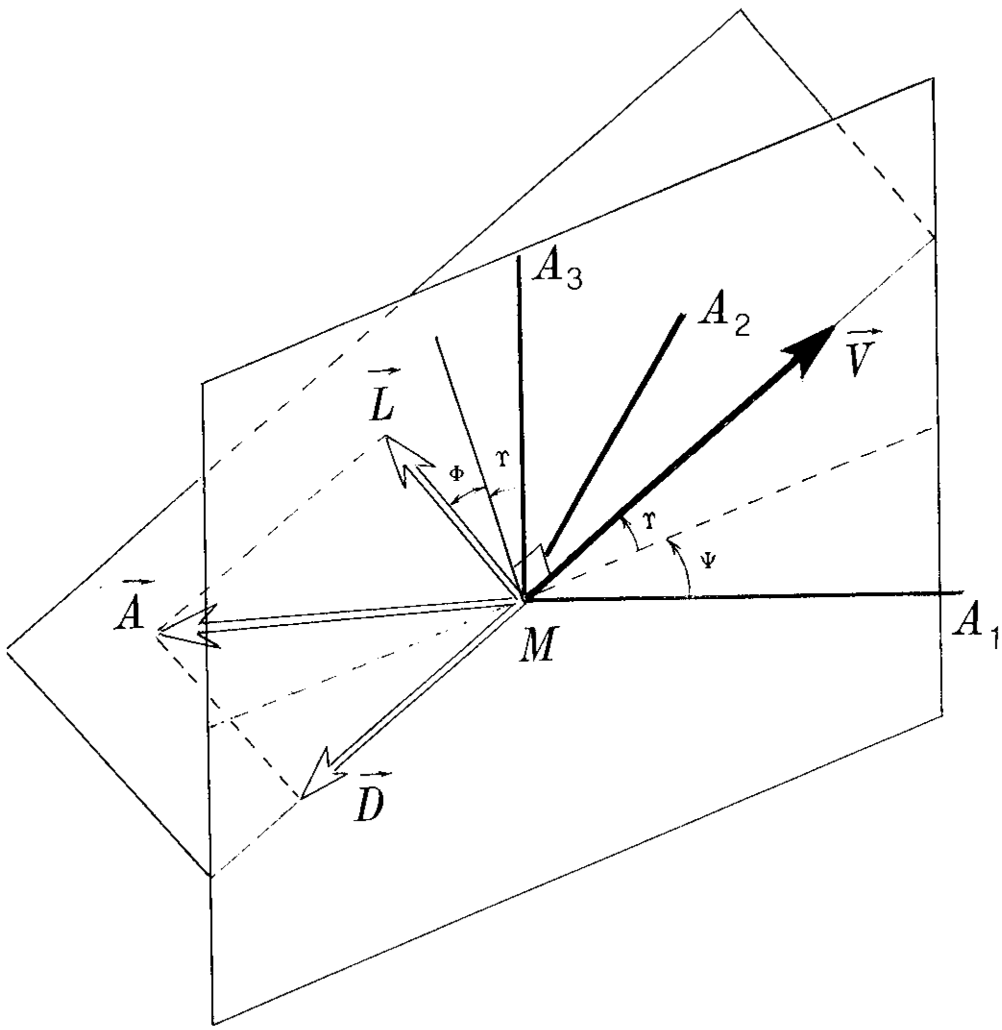

Figure 2. Aerodynamic Forces Transformation