$$y'' + 2(a/a)y' + (g/a)y = 0 \tag{6}$$

where g is the gravitational constant.

Claim: Assume that $a(t)$ is a positive bounded above function. If $a(t)$ satisfies one of the following conditions for all $t > t_0$

1) $a' \geq 0$
2) $a' \leq 0$
3) $a' + t^{-\beta}a \geq 0$, for some $\beta > 1$

then (6) is stable.

Proof: Criteria 1 and 2 can be proved by Corollary 3 with $M = 1, D = 2a'/a, K = g/a$, and $\alpha(t) = \max\{-a'/a, -4a'/a\}$. If $a' \geq 0$, then $\alpha(t) = -a'/a$. If $a' \leq 0$, then $\alpha(t) = -4a'/a$. This shows that $\int^{\infty}\alpha(s)\,ds < c$, which implies (6) is stable.

Criterion 3 can be obtained by Corollary 4 with $A = 0$, $M = 1, D + G = 2a'/a, K = g/a$, and $\alpha(t) = t^{-\beta}, \beta > 1$. The claim is then proved.

Remark 7. Criteria 1 and 2 can also be seen in Hsu and Wu.[3]

VI. Conclusion

One stability criterion and two instability criteria for the first-order linear time-varying system are given in this Note. These criteria are extensions and/or consolidations of the results in Refs. 3 and 4. These conditions, though not intuitive, can be checked easily for a given system.

A general necessary and sufficient condition on stability is very difficult to derive. However, it is interesting to know if such conditions can be obtained in some forms of linear systems.

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Maximum Entropy Controller Synthesis for Colocated and Noncolocated Systems

Jonathan H. Friedman* and Dennis S. Bernstein†
University of Michigan, Ann Arbor, Michigan 48109

I. Introduction

MAXIMUM entropy controller synthesis was developed specifically for the robust control of flexible structures.[1–4] The goal of this paper is to provide well-documented numerical examples that illustrate the characteristics of the method. The examples we consider in this note were chosen to contrast the properties of maximum entropy controllers in two key cases, namely, colocation and noncolocation. Our results confirm previous observations, namely, that maximum entropy controllers employ positive real phase stabilization in the colocated case and wider and deeper notch gain stabilization in the noncolocated case. The computations were performed using a standard quasi-Newton technique in conjunction with the appropriate cost gradient expressions.

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*Graduate Student, Department of Aerospace Engineering.
†Associate Professor, Department of Aerospace Engineering.

II. Maximum Entropy Controller Synthesis

Consider the structural model

$$\dot{x} = \left(A + \sum_{i=1}^{r} \sigma_i A_i\right) + Bu + D_1 w \tag{1}$$

$$y = Cx + D_2 w \tag{2}$$

with feedback controller

$$\dot{x}_c = A_c x_c + B_c y \tag{3}$$

$$u = C_c x_c \tag{4}$$

performance variables

$$z = E_1 x + E_2 u \tag{5}$$

and performance measure

$$J(A_c, B_c, C_c) = \lim_{t\to\infty} \mathscr{E}\left\{\frac{1}{t}\int_0^t z^T(s)z(s)\mathrm{d}s\right\} \tag{6}$$

where $x \in \mathcal{R}^n$, $u \in \mathcal{R}^m$, $y \in \mathcal{R}^\ell$, $w \in \mathcal{R}^d$, $z \in \mathcal{R}^q$, $x_c \in \mathcal{R}^{n_c}$, σ_i is an uncertain parameter representing uncertainty in ω_{di}, and

$$A_i = \operatorname{diag}\left\{0, \ldots, 0, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, 0, \ldots, 0\right\} \tag{7}$$

so that the ith 2×2 diagonal block is the only nonzero entry in A_i. The disturbance w is a standard white noise signal and $\mathscr{E}$ denotes expectation. The matrix A is assumed to be in real normal coordinates, that is,

$$A = \operatorname{diag}\left\{\begin{bmatrix} -\eta_1 & \omega_{d1} \\ -\omega_{d1} & -\eta_1 \end{bmatrix}, \ldots, \begin{bmatrix} -\eta_r & \omega_{dr} \\ -\omega_{dr} & -\eta_r \end{bmatrix}\right\} \tag{8}$$

In maximum entropy theory, the performance $J(A_c, B_c, C_c)$ is given by

$$J(A_c, B_c, C_c) = \operatorname{tr} \tilde{Q}\tilde{E}^T\tilde{E} \tag{9}$$

where $\tilde{Q}$ satisfies the maximum entropy covariance equation

$$0 = \tilde{A}\tilde{Q} + \tilde{Q}\tilde{A}^T + \sum_{i=1}^{r} \delta_i^2\left[\frac{1}{2}\tilde{A}_i^2\tilde{Q} + \tilde{A}_i\tilde{Q}\tilde{A}_i^T + \frac{1}{2}\tilde{Q}\tilde{A}_i^{2T}\right] + \tilde{D}\tilde{D}^T \tag{10}$$

and where $\tilde{A}$, $\tilde{A}_i$, $\tilde{D}$, and $\tilde{E}$ are defined by

$$\tilde{A} \triangleq \begin{bmatrix} A & BC_c \\ B_cC & A_c \end{bmatrix}, \quad \tilde{A}_i \triangleq \begin{bmatrix} A_i & 0 \\ 0 & 0 \end{bmatrix}, \quad \tilde{D} \triangleq \begin{bmatrix} D_1 \\ B_cD_2 \end{bmatrix}$$
$$\tilde{E} \triangleq [E_1 E_2 C_c] \tag{11}$$

and δ_i is a measure of the magnitude of the uncertainty σ_i.

To minimize $J(A_c, B_c, C_c)$ given by Eq. (9) where $\tilde{Q}$ satisfies Eq. (10), we define a Lagrangian function

$$\mathscr{L}(A_c, B_c, C_c, \tilde{Q}) \triangleq \operatorname{tr} \tilde{Q}\tilde{E}^T\tilde{E} + \operatorname{tr} \tilde{P}\left(\tilde{A}\tilde{Q} + \tilde{Q}\tilde{A}^T + \sum_{i=1}^{r} \delta_i^2\left[\frac{1}{2}\tilde{A}_i^2\tilde{Q} + \tilde{A}_i\tilde{Q}\tilde{A}_i^T + \frac{1}{2}\tilde{Q}\tilde{A}_i^{2T}\right] + \tilde{D}\tilde{D}^T\right) \tag{12}$$

where $\tilde{P}$ is a nonzero Lagrange multiplier. Now by partitioning $\tilde{Q}$ and $\tilde{P}$ as

$$\tilde{Q} = \begin{bmatrix} Q_1 & Q_{12} \\ Q_{12}^T & Q_2 \end{bmatrix}, \qquad \tilde{P} = \begin{bmatrix} P_1 & P_{12} \\ P_{12}^T & P_2 \end{bmatrix}$$

and assuming for simplicity that $E_1^T E_2 = 0$ and $D_1 D_2^T = 0$, it can be shown that $\tilde{P}$ satisfies

$$0 = \tilde{A}^T \tilde{P} + \tilde{P}\tilde{A} + \sum_{i=1}^{r} \delta_i^2 \left[\frac{1}{2}\tilde{A}_i^{2T} \tilde{P} + \tilde{A}_i^T \tilde{P} \tilde{A}_i + \frac{1}{2} \tilde{P} \tilde{A}_i^2 \right] + \tilde{E}^T \tilde{E} \tag{13}$$

and the cost gradients are given by

$$\frac{\partial J(A_c, B_c, C_c)}{\partial A_c} = 2(Q_{12}^T P_{12} + Q_2 P_2) \tag{14}$$

$$\frac{\partial J(A_c, B_c, C_c)}{\partial B_c} = 2C(Q_1 P_{12} + Q_{12} P_2) + 2D_2 D_2^T B_c^T P_2 \tag{15}$$

$$\frac{\partial J(A_c, B_c, C_c)}{\partial C_c} = 2(Q_{12}^T P_1 + Q_2 P_{12}^T)B + 2Q_2 C_c^T E_2^T E_2 \tag{16}$$

The expressions (14–16) follow from the fact that the cost gradients are equal to the gradients of the Lagrangian.[5]

To perform the optimization, we used the MATLAB subroutine *fminu*, which implements the BFGS quasi-Newton algorithm. The search algorithm was modified to ensure closed-loop stability within the line search subroutine. As in Ref. 5 and the homotopy methods[3,6] we initialized the optimization routine with the standard LQG solution. In addition to the optimization routine we used the algorithm developed in Ref. 6 for solving Eqs. (10) and (13).

III. Illustrative Example: Colocated Case

The first example is a two-mass system with a colocated sensor/actuator pair as shown in Fig. 1, where the measured output y_c is the velocity of mass M_1 (y_c and y_{nc} denote the outputs for the colocated and noncolocated cases). The dynamics of the system are given by

$$M_1\ddot{q}_1 + C_1\dot{q}_1 + K_1 q_1 = u + C_2(\dot{q}_2 - \dot{q}_1) + K_2(q_2 - q_1) \tag{17}$$

$$M_2\ddot{q}_2 + C_2(\dot{q}_2 - \dot{q}_1) + K_2(q_2 - q_1) = 0 \tag{18}$$

$$y_c = \dot{q}_1 \tag{19}$$

with the parameter values given in Fig. 1. After transforming to real normal coordinates the following A, B, and C are obtained:

$$A = \begin{bmatrix} -0.0002 & 0.2208 & 0 & 0 \\ -0.2208 & -0.0002 & 0 & 0 \\ 0 & 0 & -0.0103 & 1.4322 \\ 0 & 0 & -1.4322 & -0.0103 \end{bmatrix}$$

$$B = \begin{bmatrix} -0.1439 \\ 0.2168 \\ -0.0426 \\ 1.1892 \end{bmatrix} \tag{20}$$

$$C = [-0.0545 \quad 0.0819 \quad -0.0352 \quad 0.8181] \tag{21}$$

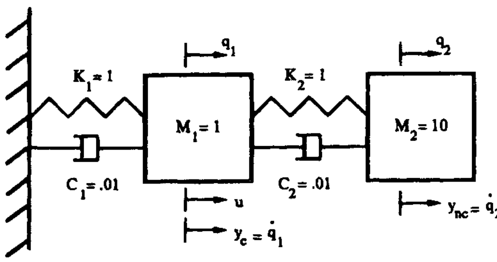

Fig. 1 Two-mass system.

The performance criterion was chosen so that LQG synthesis would place a notch at the second mode. This is accomplished when

$$D_1 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad D_2 = [0 \ 1]$$

$$E_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} E_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{22}$$

This matrix E_1 weights the amplitude of the first mode but does not penalize the amplitude or velocity of the second mode. In practice this performance criterion reflects the situation in which the closed-loop performance depends primarily upon the lower frequency modes, while the higher frequency modes are uncertain.

Note that in Eq. (21) the nominal damped natural frequencies and damping ratios are $\omega_{d1} = 0.2208$, $\omega_{d2} = 1.4322$, $\zeta_1 = 0.0011$, and $\zeta_2 = 0.0072$, and that the plant is positive real. In Table 1 and Fig. 2 we compare the standard LQG design to three maximum entropy designs, where the only parameter that is varied is δ_2, which is a measure of the magnitude of the uncertainty σ_2 in the second damped natural frequency. When compared to the LQG compensator, the maximum entropy method with a small measure of uncertainty δ_2 first adjusts the phase so that the controller is stable. Then the method continues to alter the phase as δ_2 increases, yielding positive real controllers as δ_2 becomes large. The increase in robustness obtained by increasing δ_2 was also assessed by determining the range of values of σ_2 for which the closed-loop system remains stable. This range of values, which increases with δ_2, is given by the last column of Table 1 and is illustrated by the performance/robustness tradeoff curves shown in Fig. 3.

It is important to stress that, although the use of positive real controllers in the colocated case is standard practice to achieve robustness, the maximum entropy method is the only technique we know of that yields such controllers as a direct consequence of uncertainty.

IV. Illustrative Example: Noncolocated Case

In the second example we examine the same two-mass system as in Sec. III. However, in this example the sensor/actuator pair is noncolocated with measured output $y_{nc} = \dot{q}_2 = Cx$, where

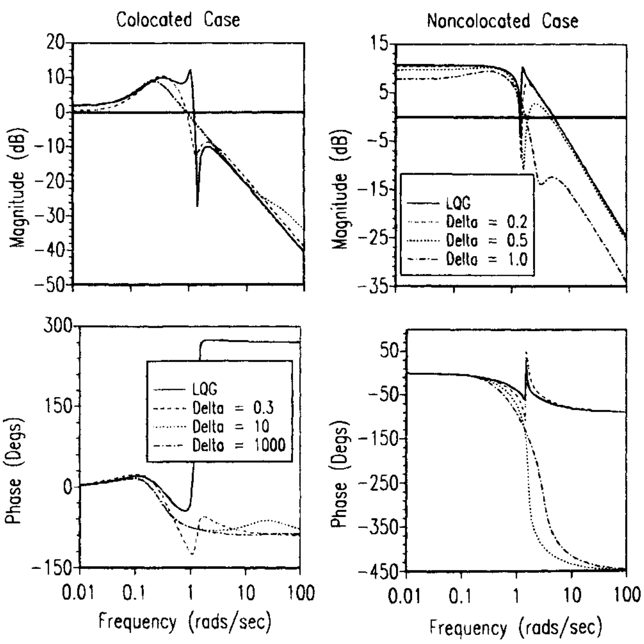

Fig. 2 Compensator transfer functions.

Table 1 Compensator comparison—colocated case

δ_2	Stable?	Minimum phase?	Positive real?	$\mathcal{H}_2$ Nominal state cost	$\mathcal{H}_2$ Nominal control cost	Stability boundary	
0 (LQG)	No	Yes	No	13.8522	1.9189	1.4318	1.4655
0.3	Yes	Yes	No	14.2884	1.7801	1.1533	10^{15}
10	Yes	Yes	Yes	15.2425	1.7937	-10^{15}	10^{14}
1000	Yes	Yes	Yes	15.1942	1.8463	-10^{13}	10^{14}

Table 2 Compensator comparison—noncolocated case

δ_2	Stable?	Minimum phase?	Positive real?	$\mathcal{H}_2$ Nominal state cost	$\mathcal{H}_2$ Nominal control cost	Stability boundary	
0 (LQG)	Yes	Yes	Yes	772.9009	11.0468	1.4245	1.4341
0.2	Yes	Yes	No	776.1827	10.4267	1.3242	1.4887
0.5	Yes	No	No	786.9195	8.5317	1.1482	1.7400
1.0	Yes	No	No	816.7371	5.4372	1.0300	10^{16}

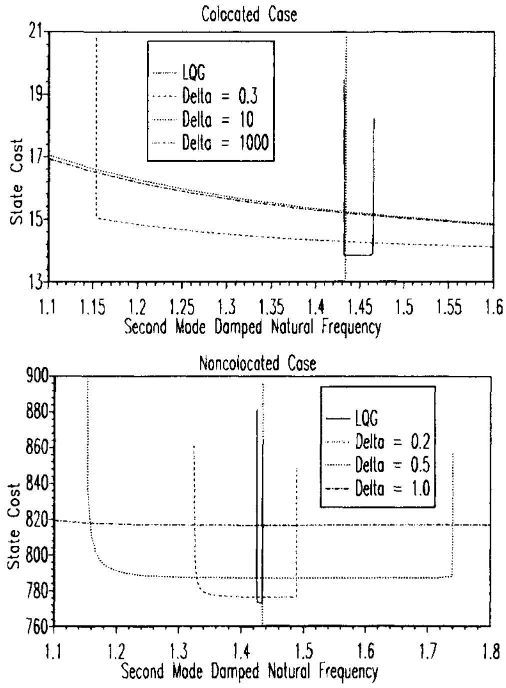

Fig. 3 H_2 state cost for the perturbed system.

$$C = [-0.1063 \quad 0.1597 \quad 0.0018 \quad -0.0419] \qquad (23)$$

Also the matrix E_1 in Eq. (22) is increased by a factor of 10, so as to enhance the notching characteristics of the LQG compensator and to better demonstrate the properties of the maximum entropy controllers. As will be seen, this increase also led to the use of lower values of δ_2 to achieve levels of robustification comparable to those obtained in the colocated case.

Because the plant is not positive real, the maximum entropy method can no longer guarantee closed-loop stability by adjusting the phase of the compensator. Instead, the method robustifies the LQG design by widening and deepening the notch at the second mode. In addition, it can be seen that the center notch frequency moves to the right, which makes the stability region asymmetric for larger δ_2. On the other hand, we found that the notch can be centered at the nominal damped natural frequency by decreasing the nominal design frequency. However, experience shows that this approach also does not necessarily lead to a symmetric stability region. We suspect that the notch center frequency moves to the right to avoid possible overlap with the lower modal frequency. For these designs the performance/robustness tradeoff curves are shown in Fig. 3 and the stability boundaries are given in Table 2.

Despite the fact that phase no longer appears to be the principal means of robustification in the noncolocated case, the maximum entropy synthesis method does adjust the phase of the compensator. In particular, as δ_2 increases, the compensator transitions from minimum phase to nonminimum phase, as seen in Fig. 2. This change in phase increases the phase margin near the second mode as in the colocated case.

V. Conclusions

The purpose of this note was to contrast the robustness of maximum entropy controllers in the colocated and noncolocated cases, and to demonstrate a new computational technique for maximum entropy controller synthesis. Based on these examples, we can conclude that maximum entropy controllers achieve robustness by tending toward phase stabilization in the colocated case and employing robustified notch filters in the noncolocated case. The starting point for these designs was LQG theory, which, in this case, yielded rather sensitive controllers. There exist, of course, alternative methods for robustifying LQG designs, such as loop shaping, frequency weighting, and $\mathcal{H}_\infty$ theory. A comparison of these techniques with maximum entropy controllers remains a topic for future investigation.

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Multiobjective Controller Design Using Eigenstructure Assignment and the Method of Inequalities

R. J. Patton,* G. P. Liu,† and J. Chen†
University of York,
York YO1 5DD, England, United Kingdom

Introduction

In recent years, eigenstructure assignment has been an active topic of research in multivariable control theory. Since the degrees of freedom are available over and above pole assignment using state or output feedback,[1] respectively, numerous researchers have exercised those degrees of freedom to make the systems have good insensitivity to perturbations in the system parameter matrices via eigenstructure assignment.[1-5] Most eigenstructure assignment techniques in the last decade only pay attention to the optimal solutions for some special performance indices, e.g., $\|V_R\|_2\|V_R^{-1}\|_2$, where V_R is the right eigenvector matrix. However, many practical control systems are required to have the ability to satisfy simultaneously different and often conflicting performance criteria, for instance, closed-loop stability, low feedback gains, and insensitivity to model parameter variations.

In this Note, we provide a new approach to make the closed-loop system satisfy a set of required performance criteria with less conservatism, using eigenstructure assignment and the method of inequalities.[6]

Multiobjective Controller Design

Consider a linear multivariable time-invariant, completely controllable, state feedback system:

$$\dot{x} = Ax + Bu, \qquad u = Kx \tag{1}$$

where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^m$ is the control input vector, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, and $K \in \mathbb{R}^{m\times n}$. Then the closed-loop system representation is given by $\dot{x} = (A + BK)x$. Introducing an $n \times n$-dimensional eigenvector matrix $V_R = [V_{R1}, V_{R2}, \dots, V_{Rn}]$, where V_{Ri} $(i = 1, 2, \dots, n)$ is the right eigenvector corresponding to the eigenvalue λ_i, a general solution for this problem can be given in the form of a parametric expression, $K(\Lambda, V_R)$, for all feedback gain matrices K which assign the self-conjugate set of eigenvalues $\Lambda = [\lambda_1, \lambda_2, \dots, \lambda_n]$ to the closed-loop system. If both the vector Λ and the right eigenvector matrix V_R of $(A + BK)$ are specified, the controller K is determined.

In practice, it is usually intended to locate the eigenvalue vector Λ in a well-defined set to meet the requirements of the practical control system (e.g., stability, speed of response, etc.). This leads to eigenvalue constraints, for example of the form $\lambda_{L_i} \le \lambda_i \le \lambda_{U_i}$, where $\lambda_{L_i} \in \mathbb{R}$ and $\lambda_{U_i} \in \mathbb{R}$ are the lower bound vector and the upper bound vector, respectively. These constraints may be removed by considering the change of variables given by

$$\lambda_i(\nu_i) = \lambda_{L_i} + (\lambda_{U_i} - \lambda_{L_i})\sin^2(\nu_i), \tag{2}$$

with $\nu_i \in \mathbb{R}$. Since the system is assumed to be completely controllable, the ith closed right eigenvector V_{Ri} is given by

$$V_{Ri} = (\lambda_i I - A)^{-1}BW_i, \qquad i = 1, 2, \dots, n \tag{3}$$

where $W_i \in \mathbb{R}^{m\times 1}$ and the matrix $W = [W_1, W_2, \dots, W_n]$.

For the case when the matrix $(\lambda_i I - A)$ is not invertible, for example when one or more closed-loop eigenvalues are required to be identical to open-loop values, then the following alternative to Eq. (3) by Roppenecker[5] and Liu and Patton[7] can be used without loss of generality. Clearly, the right eigenvector matrix V_R is a function of $\Upsilon = [\nu_1, \nu_2, \dots, \nu_n]$ and W, i.e., $V_R(\Upsilon, W)$. Thus, the parametric formula of the controller matrix K can be described by $K(\Upsilon, W)$. A parametric representation of the control matrix K is given by[5]

$$K(\Upsilon, W) = WV_R^{-1}(\Upsilon, W) \tag{4}$$

In most parameter insensitive design methods using eigenstructure assignment, the performance indices are given on the basis of the right eigenvector matrix. For example, a very common performance index is given by

$$\phi(\Upsilon, W) = \|V_R\|_2\|V_R^{-1}\|_2 \tag{5}$$

where $\|V_R\|_2 =$ (maximum eigenvalue of $V_R^* V_R)^{1/2}$.

Though the performance index $\phi(\Upsilon, W)$ can be used to represent an upper bound of the eigenvalue sensitivities, it is often conservative because of the following relations:

$$\phi(\Upsilon, W) \ge \max\{\phi_i(\Upsilon, W) : i \in \{1, 2, \dots, n\}\} \tag{6}$$

where $\phi_i(\Upsilon, W)$ is the individual sensitivity of the eigenvalue λ_i to perturbations in any of the elements of the matrices A and B, defined by

$$\phi_i^2(\Upsilon, W) = \frac{(V_{Ri}^* V_{Ri})(V_{Li}^* V_{Li})}{(V_{Li}^T V_{Ri})^*(V_{Li}^T V_{Ri})}, \quad i = 1, 2, \dots, n \tag{7}$$

where the superscript * denotes "conjugate-transposed," V_{Li} is the ith closed-loop left eigenvector given by the relation $V_L^T = V_R^{-1}$ with the left eigenvector matrix $V_L = [V_{L1}, V_{L2}, \dots, V_{Ln}]$. Hence, to reduce the conservatism the problem becomes to find a pair (Υ, W) such that

$$\min_{\Upsilon, W}\{\phi_i(\Upsilon, W)\} \qquad \text{for } i = 1, 2, \dots, n \tag{8}$$

To give a feel for the usefulness of the multiobjective approach as opposed to single-objective design techniques, let us consider the minimization of the cost functions $\phi_i(\Upsilon, W)$ $(i = 1, 2, \dots, n)$. Let the minimum value of ϕ_i be given by ϕ_i^*, for $i = 1, 2, \dots, n$, respectively. For these optimal values ϕ_i^*, there exist corresponding values given by $\phi_j(\phi_i^*)$ $(j \ne i,$

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*Senior Lecturer, Department of Electronics. Senior Member AIAA.
†Research Fellow, Department of Electronics.