

Thin Shock-Layer Theory Revisited

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During the hypersonics activities of the 1960s, thin shock-layer theory was developed as a first-order correction to Newtonian theory, in the hope of gaining insight into efficient lift at very high speeds. The wings are limited to conical geometry, but can have arbitrary transverse camber. Good predictions were made of the forces generated by delta and caret wings, but no optimization studies were carried out. This paper collects together some old results, including some that were not widely published, establishes a parameter that measures hypersonic lifting efficiency, and presents a wing design that achieves a very high value of that parameter.

I. Introduction

This paper is concerned with efficient flight at hypersonic speeds. Only the inviscid forces on the lower surface of the wing are considered, and those are obtained approximately using Thin Shock-Layer Theory (TSLT), which is a first-order correction to Newtonian Theory. Newtonian Theory itself idealises the flow by collisionless molecules bouncing off the surface to arrive at the simple expression $2 \sin^2 \alpha$ for the surface pressure coefficient, where α is the deflection angle of the local surface. Within this theory, the lifting surface that gives the best ratio of lift to drag is a flat plate. This remains true for any theory that assumes a purely local relationship between pressure and deflection. Prior to the coming of Computational Fluid Dynamics, local pressure-deflection laws were used with some success to predict the aerodynamics of simple shapes at hypersonic speeds.¹² This probably contributed to a widespread belief that flat surfaces are indeed optimal under hypersonic conditions.

A configuration that does produce lift by interference was proposed by Eggers and Syvertson,² who considered a simple configuration consisting of half a cone placed below a thin delta wing. At zero incidence, the full cone would produce a conical shock wave. If the wing is sized so that it would just span that shock, then the half-cone will produce the same flow. The wing would produce no lift by itself, but picks up some interference lift from the cone body. Unfortunately, this attractive concept is not efficient. It turned out that when the Eggers-Syvertson configuration was tested in a windtunnel over a range of incidences, it achieved a significantly better $(L/D)_{max}$ when placed upside down, with the flat surface underneath! Although this restored some faith in the 'flat-is-best' idea, such a conclusion is a little naive, because the Eggers-Syvertson configuration is merely a simple application of the interference principle, and not necessarily a good one.

In this paper we will reexamine the issue, but we will not attempt a full computational treatment of the three-dimensional problem. Rather, we approach the task by limiting consideration to conical wings, formed from straight generating rays that all pass through a common apex, and using the simplifications of Thin Shock-Layer Theory, as formulated by Messiter.⁶ In this theory, which is valid for flows at high Mach number and also high incidence, the high incidence reduces the post-shock Mach number to a value such that the flow in the cross-plane is transonic. Therefore the pressure along a given ray depends on shape of the surface within a certain domain of dependence. Therefore we can investigate whether lifting

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efficiency can be influenced by so-called *conical camber*. This question has been examined in a different limit (Hypersonic Small Disturbance Theory, which relates to high Mach number and low angle of attack) in¹¹ and in a preliminary numerical fashion for the Euler equations in.⁵

Because much of the material on TSLT is rather old, we begin by reviewing it, sometimes from a more up-to-date perspective. The material in Section IIA is standard to all of the references. The first part of Section IIB was given in work of the author⁷ that had very limited circulation, and the integral equation relating shock and body shape appears in,¹⁰ derived differently. The divergence form of TSLT is new, and leads to two nice integral theorems. These were given in,^{7,8} although with different proofs. All the subsequent material, which deals with devising an appropriate measure of lifting efficiency in this flight regime, and devising wings that enhance it, is new.

II. Presentation of Thin Shock-layer Theory

Thin Shock-Layer Theory is applicable to flows with large values of the hypersonic similarity parameter $M_\infty \sin \alpha$, where α measures a typical flow deflection angle, so that the density behind the shock is large, and the distance from the shock to the surface is small. Examples of the theory are to be found in the classic text by Hayes and Probstein,⁴ applied to flows having two-dimensional, axisymmetric, or conical symmetry. It is this last application that is considered here. It allows the study of delta wings having ‘conical camber’; such wings are composed of straight generators passing through their apex, but their surfaces may not be planar.

The theory was developed during the 1960s and 1970s, largely prior to developments in Computational Fluid Dynamics. The published literature^{3,4,6,9,10,13} consists of some general results, together with some special solutions that can be obtained analytically. TSLT achieves some drastic and elegant simplifications of the governing equations, but at the cost of admitting certain anomalies. Perhaps the most serious of these is that a correct distinction is not made between those regions of the flow that are ‘conically hyperbolic’ and ‘conically elliptic’. In TSLT the entire flow is conically hyperbolic.¹³ Nevertheless, TSLT makes qualitatively correct predictions of shock detachment, and the flows are in generally good agreement with more exact calculations and with experiment.^{7,9,10,13}

A. Similarity Variables, Governing Equations and Characteristics

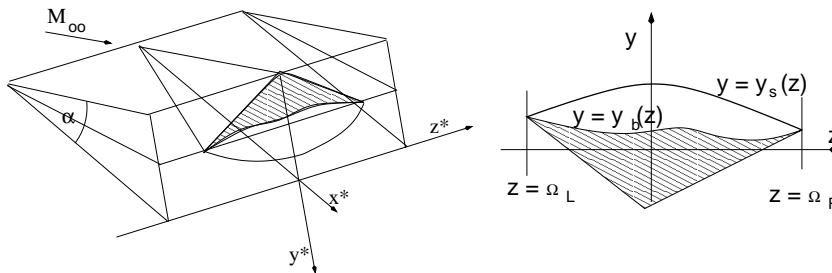


Figure 1. (left) The flow past a two-dimensional wedge, and, superposed on that, a perturbation representing the flow over a conically cambered delta wing. (right) The flow over the wing displayed in scaled conical coordinates.

We summarise here the derivation given in.⁶ Consider a wing with conical symmetry, that is to say generated by straight lines passing through a fixed point, and whose surface lies close to the plane $z = 0$.

This plane makes an angle α relative to a uniform flow whose Mach number is M_∞ . The assumption of Thin Shock-Layer Theory is that the wing produces a flow that is a small perturbation of the known flow that results from a shockwave lying in that plane. A perturbation parameter is defined as the inverse of the density ratio across that shockwave.

$$\epsilon = \frac{\gamma - 1}{\gamma + 1} + \frac{2}{(\gamma + 1)M_\infty^2 \sin^2 \alpha} \quad (1)$$

Conical coordinates are defined as

$$y = \frac{y^*}{x^* \epsilon \tan \alpha}, \quad z = \frac{z^*}{x^* \epsilon^{\frac{1}{2}} \tan \alpha}$$

and the flow variables are expanded as

$$\begin{aligned} \frac{u^*}{U_\infty} &= \cos \alpha + \epsilon \sin^2 \alpha u + \mathcal{O}(\epsilon^2) \\ \frac{v^*}{U_\infty} &= \epsilon \sin \alpha v + \mathcal{O}(\epsilon^2) \\ \frac{w^*}{U_\infty} &= \epsilon^{\frac{1}{2}} \sin \alpha w + \mathcal{O}(\epsilon^2) \\ \frac{p^* - p_\infty}{\rho_\infty U_\infty^2} &= \sin^2 \alpha + \epsilon \sin^2 \alpha p + \mathcal{O}(\epsilon^2) \\ \frac{\rho_\infty}{\rho^*} &= \epsilon + \epsilon^2(1 + p) - \frac{\gamma - 1}{2} \epsilon(2u + w^2) + \mathcal{O}(\epsilon^3) \end{aligned} \quad (2)$$

Here, all of the physical quantities are denoted by stars, and the scaled dimensionless quantities, to leading order, by unadorned symbols. The choice of $\epsilon^{\frac{1}{2}}$ as the spanwise scaling variable is somewhat arbitrary, but is chosen so that the Mach angle within the shock layer will be of order unity. This means that the equations are properly scaled to study conically transonic behavior, and the possibility of interference lift.

The flow is studied in any crossflow plane ($x^* = \text{const}$) and is governed by the following dimensionless equations.

$$\partial_y v + \partial_z w = 0 \quad (3)$$

$$(v - y)\partial_y v + (w - z)\partial_z v + \partial_y p = 0 \quad (4)$$

$$(v - y)\partial_y w + (w - z)\partial_z w = 0 \quad (5)$$

One of the main simplifications in TSLT is the decoupling of the equations. Here, the first and third equations are enough to solve for v and w , after which the second equation gives p . Additional uncoupled equations give u and ρ , should they be needed. The governing equations can be put into the quasilinear form

$$\mathbf{A}\mathbf{u}_y + \mathbf{B}\mathbf{u}_z = 0$$

with $\mathbf{u} = (p, v, w)^T$ and

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & v - y & 0 \\ 0 & 0 & v - y \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & w - z & 0 \\ 0 & 0 & w - z \end{bmatrix}$$

The characteristic directions $dz/dy = \lambda$ are found from $\det(\mathbf{B} - \lambda\mathbf{A}) = 0$, which gives $\lambda = 0$ (the vertical direction, twice) and $\lambda = (w - z)/(v - y)$ (the conical streamlines)

The third of the equations given is already in characteristic form, stating that the spanwise velocity w is constant along the ‘conical streamlines’ $dy/dz = (v - y)/(w - z)^a$. We get a second characteristic equation by using the first equation to substitute for $\partial_z w$ in the third equation;

$$(v - y)\partial_y w - (w - z)\partial_y v = 0 \quad (6)$$

which hold along the vertical characteristic. However, the system of equations is degenerate, and does not have a complete set of eigenvectors corresponding to the vertical characteristic. This anomaly results from a distortion of the geometry. In the full conical Euler equations, the characteristics are the conical streamlines (faithfully reproduced in the simplification) and a pair of conical Mach lines, which define the domains of influence of an event located along a conical ray. Such lines should exist only where the flow is conically supersonic. In the simplification, these Mach lines coincide along the vertical direction, and exist whether the flow is conically subsonic or supersonic. Although there is no third characteristic equation, the second of the above equations matches the derivative of w along one characteristic with the derivative of p along another, so that characteristic coordinates are still useful.

Since the body $y = y_b(z)$ is a conical streamline, the boundary condition on the body is

$$(w - z)\frac{dy_b}{dz} = v - y \quad (7)$$

and the boundary conditions on the shock surface $y = y_s(z)$ can be found as

$$p_s = -(y'_s)^2 - 1 + 2y_s - 2zy'_s, \quad (8)$$

$$v_s = -(y'_s)^2 - 1 + y_s - zy'_s, \quad (9)$$

$$w_s = -y'_s. \quad (10)$$

Remark 1 *Given any solution to the TSLT equations, the following transformation of that solution will also be a solution.*

$$z \rightarrow z, \quad y \rightarrow y + \delta y, \quad w \rightarrow w, \quad v \rightarrow v + \delta y, \quad p \rightarrow p + 2\delta y \quad (11)$$

This transformation represents a change of incidence amounting to $\epsilon \tan \alpha \delta y$.

B. Streamline Geometry

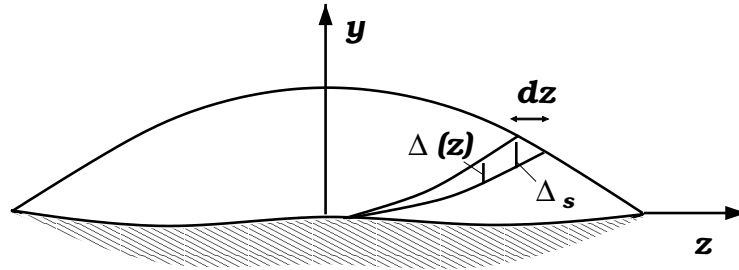


Figure 2. Cross-sectional view of flow. The shock layer is bounded by the shockwave and the body. Two neighboring conical streamlines are shown.

Let $\Delta(z)$ be the (small) vertical spacing between any pair of conical streamlines, and $S = (v - y)/(w - z) = \lambda^{-1}$ be the slope of a streamline. If $D\Delta/Dz$ denotes the derivative of Δ along the streamline, it is easy to

^aThe conical streamlines are conical surfaces (composed of straight generators passing through the apex) that are everywhere tangential to the local flow vector

show that

$$\frac{1}{\Delta} \frac{D\Delta}{Dz} = \frac{\partial S}{\partial y} \quad (12)$$

$$= \frac{(w-z)(\partial_y v - 1) - (v-y)}{\partial_y w (w-z)^2} \quad (13)$$

$$= \frac{-1}{(w-z)} \quad (14)$$

Since w is constant along a streamline, this equation can be integrated to yield

$$\Delta = C(w-z). \quad (15)$$

The spacing vanishes when the two streamlines merge, on the body. Its magnitude elsewhere is determined by its value at the shock. From Fig 2 we see that two streamlines originating at the shock with locations $z, z+dz$ will have an initial vertical separation

$$\Delta_s = dz(-y'_s + S_s) \quad (16)$$

$$= dz \left(-y'_s + \frac{v_s - y_s}{w_s - z_s} \right) \quad (17)$$

$$= dz \left(-y'_s + \frac{-(y'_s)^2 - 1 - z_s y'_s}{-y'_s - z_s} \right) \quad (18)$$

$$= \frac{-dz}{w - z_s} \quad (19)$$

It follows that the area of the (z, y) plane occupied by the streamtube in question is therefore equal to that of a triangle whose vertical base is $dz/(z_s - w(z_s))$ and whose horizontal 'height' is $z_s - w(z_s)$. This area is simply $dz/2$. Integrating this result gives

Theorem 1 *First Integral Theorem*

$$\int_{-\Omega}^{\Omega} (y_s - y_b) dz = \Omega. \quad (20)$$

Therefore, in these dimensionless variables, the average thickness of the shock layer is $\frac{1}{2}$. In these same variables, for comparison, the thickness of the shock layer on a plane wedge is unity. This factor of 2.0 reflects the difference between conical and slab symmetry.

The streamline geometry also leads to an integral equation connecting the shapes of the shock $y = y_s(z)$ and the body $y = y_b(z)$. Consider a general point P within the flow, parameterized by its spanwise coordinate z , and the spanwise location where the associated streamline crosses the shock z_s . The two quantities z, z_s parameterize the point unambiguously. We will calculate the vertical coordinate $y(z, z_s)$ of the point by subtracting from $y_s(z)$ the height of each streamtube that originates in the interval $z \leq \zeta \leq z_s$. This is, from (15),(19).

$$\Delta(\zeta) = -\frac{(w(\zeta) - z)d\zeta}{(w(\zeta) - \zeta)^2}. \quad (21)$$

so that

$$y = y_s(z) + \int_z^{z_s} \frac{(w(\zeta) - z)d\zeta}{(w(\zeta) - \zeta)^2} \quad (22)$$

If P lies on the body, there are two possibilities. If it lies on the outer part of the body, swept by the streamline that originates at the leading edge, then the upper limit of the integral is the span Ω . If P lies on the inner part of the body, where the boundary condition is $w = z$, the upper limit must be chosen as the root of the equation $w(z_s) = z$. Thus, if $z_s(w)$ is the function inverse to $w(z_s)$, we can write

$$y_b(z) = y_s(z) + \int_z^{\min z_s(z), \Omega} \frac{(w(\zeta) - z)d\zeta}{(w(\zeta) - \zeta)^2} \quad (23)$$

This formula was derived by Messiter⁶ using more purely algebraic arguments. It allows a few special solutions to be found inversely,³ by specifying shock shapes such that the integral evaluates to something simple. Messiter found some numerical solutions, as did Squire.^{9,10} A more efficient numerical procedure is outlined below, based more directly on the geometry. Among the special solutions we note a mathematically trivial, but practically significant, example.

This is defined by the shock shape $y_s = 1$, for which the flow field is simply

$$v(z, y) \equiv 0, \quad w(z, y) = 0, \quad p(z, y) \equiv 1 \quad (24)$$

There is no perturbation at all to the basic flow over the wedge, and the conical streamlines are $y/z = \text{const}$. If we choose to regard the pair of streamlines $y/z = \pm 1/\Omega$ as comprising the surface of the body, we have the so-called ‘caret wing’, sometimes taken as the prototype of a hypersonic lifting body.

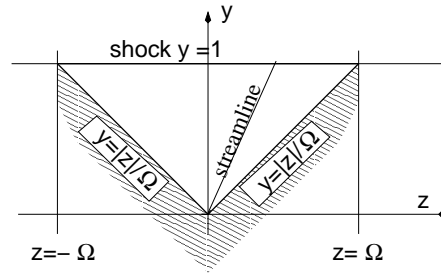


Figure 3. A V-shaped wing, extending over the region between the shock and the body, supports precisely the same flow as the basic wedge. This is called a *caret wing*.

The formula (23) can be differentiated twice to provide information on the shock curvature. Differentiating once

$$\frac{dy_b}{dz} = \frac{dy_s}{dz} - \int_z^{z_u} \frac{d\zeta}{(z - w(\zeta))^2} + \frac{1}{z - w_s(z)} - \frac{z - w(z_u)}{(z_u - w(z_u))^2} \frac{dz_u}{dz} \quad (25)$$

where $z_u = \min z_s(z), \Omega$. The last term can be struck out, because one of its factors always vanishes. Differentiating again,

$$\frac{d^2 y_b}{dz^2} = \frac{d^2 y_s}{dz^2} - \frac{1 - w'(z)}{(z - w_s(z))^2} + \frac{1}{(z - w_s(z))^2} - \frac{dz_u/dz}{(z_u - w(z_u))^2} \quad (26)$$

Recognizing that $w'(z) = -d^2 y_s/dz^2$, this can be rewritten as

$$\frac{d^2 y_b}{dz^2} = \frac{d^2 y_s}{dz^2} \left[1 - \frac{1}{(z - w_s(z))^2} \right] - \frac{dz_u/dz}{(z_u - w(z_u))^2} \quad (27)$$

In this expression, the second term arises from the way in which streamlines carrying different values of w terminate at different locations on the surface. Away from the surface, there is no such effect, and all streamlines at any given spanwise station have the same curvature, given by the first term only. This curvature can be related to the vertical pressure gradient by writing

$$\frac{D^2 y}{Dz^2} = \partial_z S + S \partial_y S. \quad (28)$$

Expanding this and using the governing equations gives

$$\frac{D^2 y}{Dz^2} = -\frac{\partial_y p}{(w-z)^2} \quad (29)$$

Hence

$$\partial_y p = \frac{d^2 y_s}{dz^2} \left[\frac{(z-w)^2}{(z-w_s)^2} - (z-w)^2 \right] \quad (30)$$

There are two cases worth noting. If we consider a point on the inner portion of the body, where $w = z$, then $p_y = 0$. If we consider a point just behind the shock, then

$$\partial_y p = \frac{d^2 y_s}{dz^2} [1 - (w_s - z)^2] \quad (31)$$

The relationship between the signs of the shock curvature and the pressure gradient therefore depends on whether the flow is conically subsonic or supersonic.

C. A marching procedure for attached shocks

Begin at the leading edge, where the body slope is known to be, say S . As above, we have

$$S = \frac{-(y'_s)^2 - 1 - zy'_s}{-y'_s - z}$$

Treating this as a quadratic equation for y'_s yields

$$y'_s = \frac{1}{2} \left[(S - z) \pm \sqrt{(S + z)^2 - 4} \right] \quad (32)$$

The positive root corresponds to a shockwave which is ‘weak’ in the crossflow plane, and the negative root to the ‘strong’ alternative. It will usually be correct to begin by choosing the weak root.^{7,13} Then a short section of the shockwave can be drawn in, and also the streamtube behind it. Taking the upper edge of this streamtube as a new ‘displacement surface’, the process can be repeated.

At some point it will be necessary to switch to choosing the strong root. This can be seen by considering the situation close to the centerline of a symmetric wing, when z becomes small but S is large, and symmetry requires that y'_s tends to zero. Only the negative (strong) root delivers this solution. By picking the value of z at which the switch is made, we can enforce the boundary condition that $y'_s(0) = 0$.

It is interesting to note that at each stage of the construction there will be two values of the spanwise velocity, $w_{1,2}$ corresponding to the two roots, and that $(w_1 - z)(w_2 - z) = 1$, so the two roots give spanwise velocities for which $w - z$ is, in one case greater than unity, and in the other case less. Therefore the switching point is a conical sonic point. Within thin shock-layer theory, this gives a vestige of the distinction between conically subsonic and supersonic flow.

To treat an asymmetric wing, one would march independently from each tip inward. There would be a sonic switching point on each branch, to be determined jointly by the two conditions that the continuations of the two branches must, at the point where they cross, have the same ordinate and slope.

The marching procedure can be extended to second order by using the relationship between the curvature of the shock wave and the curvature of the streamline just behind it. However, it is difficult to implement either version for general wing shapes. As we approach the centerline, both families of characteristics become vertical, so that any marching procedure becomes very slow. Additionally, the shock shape is usually discontinuous at the ‘sonic point’, and this discontinuity is propagated inward along the streamline from that point. Finally, the location of the sonic point is tricky to determine. At the time when TSLT was originally developed, these issues would have strongly deterred the production of general numerical solutions. Today, the programming is feasible, and far faster than attempting the full governing equations.

D. Divergence form

The thin-shock layer equations can be put into divergence form

$$\mathbf{F}_y + \mathbf{G}_z = \mathbf{s}, \quad (33)$$

where

$$\mathbf{F} = \begin{pmatrix} v \\ v(v-y) + p \\ w(v-y) \end{pmatrix}, \quad \mathbf{G} = \begin{pmatrix} w \\ v(w-z) \\ w(w-z) \end{pmatrix}, \quad \mathbf{s} = \begin{pmatrix} 0 \\ -2v \\ -2w \end{pmatrix}$$

By integrating these around an arbitrary domain D we obtain

$$\oint_{\partial D} (w dy - v dz) = 0 \quad (34)$$

$$\oint_{\partial D} w[(w-z)dy - (v-y)dz] = -2 \iint_D w dy dz \quad (35)$$

$$\oint_{\partial D} v[(w-z)dy - (v-y)dz] - p dz = -2 \iint_D v dy dz \quad (36)$$

These integrals are valid for D chosen within the shock layer, but not extending across the shock. The first integral can be made nicely analogous to the others;

$$\oint_{\partial D} [(w-z)dy - (v-y)dz] = \oint (y dz - z dy) = -2 \iint_D dy dz \quad (37)$$

Let us take D to be the region between the shock and the body. Consider the contribution made to the contour integrals by the body. Since the body surface is always either a streamline $(w-z)dy - (v-y)dz = 0$ or a conical stagnation region $w-z = v-y = 0$, the integrand vanishes everywhere and the contribution is zero. Now consider the contribution made by the shock. Just behind the shock, we find from the boundary conditions (8,9,10) that

$$(w-z)dy - (v-y)dz = dz.$$

We therefore have the three results, remembering that the anticlockwise integral goes from right to left along the shock,

$$\int_{\Omega_L}^{\Omega_R} dz = 2 \iint_D dy dz, \quad (38)$$

$$\int_{\Omega_L}^{\Omega_R} w_s(z) dz = 2 \iint_D w(y, z) dy dz, \quad (39)$$

$$\int_{\Omega_L}^{\Omega_R} (v_s(z) - p_s(z) + p_b(z)) dz = 2 \iint_D v(y, z) dy dz. \quad (40)$$

The first of these results leads to an alternative proof of Theorem 1. The second does not seem to lead anywhere. The third gives an important formula for the total normal force developed. We have

$$\begin{aligned}
2 \int \int_D v \, dydz &= 2 \int \int_D \left[y + (w - z) \frac{Dy}{Dz} \right] dydz \\
&= 2 \int \int_D \left[3y + (w - z) \frac{Dy}{Dz} - 2y \right] dydz \\
&= 3 \int (y_s^2 - y_b^2) dz + 2 \int \int_D \left[(w - z) \frac{Dy}{Dz} - 2y \right] dydz
\end{aligned} \tag{41}$$

where $\frac{Dy}{Dz}$ is the slope of the conical streamline characteristic. (And more generally we will use D/Dz to denote a derivative along the streamline characteristic) To evaluate the second term I_2 , consider the contribution made to it by the region between two adjacent streamlines, between which the sidewash w is a constant $w = w_s$. At a spanwise location z the depth of the streamtube is

$$dy = -\frac{(w_s - z)dz_s}{(w_s - z_s)^2}$$

and so

$$\begin{aligned}
dI_2 &= -\frac{dz_s}{(w_s - z_s)^2} \int_{z=w_s}^{z=z_s} \left[\frac{Dy}{Dz} (w_s - z)^2 - 2y(w_s - z) \right] dz \\
&= -\frac{dz_s}{(w_s - z_s)^2} \int_{z=w_s}^{z=z_s} \left[\frac{D}{Dz} \{y(w_s - z)^2\} \right] dz \\
&= -\frac{dz_s}{(w_s - z_s)^2} y_s (w_s - z_s)^2 \\
&= -y_s dz_s
\end{aligned} \tag{42}$$

Now go back to (40), insert the boundary conditions (8), (9), and collect the later results. We obtain

$$\int_{\Omega_L}^{\Omega_R} (-y_s + zy'_s) dz_s + \int_{\Omega_L}^{\Omega_R} p_b(z) dz = 3 \int_{\Omega_L}^{\Omega_R} (y_s^2 - y_b^2) dz - 2 \int_{\Omega_L}^{\Omega_R} y_s dz_s. \tag{43}$$

and so

$$\int_{\Omega_L}^{\Omega_R} p_b(z) dz = 3 \int_{\Omega_L}^{\Omega_R} (y_s^2 - y_b^2) dz - \int_{\Omega_L}^{\Omega_R} (y_s + zy'_s) dz_s. \tag{44}$$

Since the last integrand is an exact derivative, we obtain finally

Theorem 2 *Second Integral Theorem* The integrated normal force across the wing is given by

$$\int_{\Omega_L}^{\Omega_R} p_b(z) dz = 3 \int_{\Omega_L}^{\Omega_R} (y_s^2 - y_b^2) dz - [zy_s]_{\Omega_L}^{\Omega_R} \tag{45}$$

Note that this theorem is readily verified for the special case of a caret wing in its design condition, for which $y_b = |z|/\Omega$, $y_s = 1$, $p(y, z) \equiv 1$

The practical importance of this result is that the force acting on the wing can be found simply from the geometry of the shock and the body, without needing to make a detailed prediction of the pressure distribution.

III. A Measure of Lifting Efficiency

Our objective below will be to design lifting bodies that employ conical camber to increase their efficiency, but first a measure of lifting efficiency under the conditions of TSLT needs to be defined and this seems not to have been done previously.

Consider the limiting case of TSLT as $\epsilon \rightarrow 0$ or, equivalently, apply Newtonian theory to a flat wing. In either case the lift and drag coefficients are

$$C_L = K \sin^2 \alpha \cos \alpha \quad C_D = K \sin^3 \alpha \quad (46)$$

For TSLT we employ $K = 2$ but sometimes the formulas are used in practice with other values of K . As will be seen, the actual value is unimportant for our purposes. We can eliminate α from the equations to obtain the formula for the *lift-drag polar* (Figure 4)

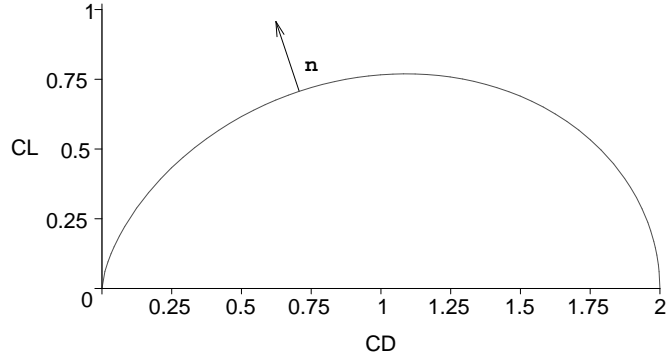


Figure 4. The lift-drag polar for flat wings in the Newtonian limit. The vector \mathbf{n} normal to this polar is the direction that measures progress.

$$C_L^2 + C_D^2 = K^{2/3} C_D^{4/3} \quad (47)$$

A perturbation that moves the force vector from one point on the polar to another has no advantage compared with simply changing incidence. A gain in lifting efficiency comes from a perturbation that moves the force vector normal to the polar. The direction of this normal is easily computed to be

$$\mathbf{n} = (\sin^2 \alpha - 2/3, \sin \alpha \cos \alpha) \quad (48)$$

A figure of merit for perturbations is therefore their component in this direction,

$$\delta M = \frac{(3 \sin^2 \alpha - 2) \delta C_D + 3 \sin \alpha \cos \alpha \delta C_L}{\sqrt{1 + 3 \cos^2 \alpha}}. \quad (49)$$

It is easier to work with the perturbations to the normal force (in the y -direction) and the axial force (in the x -direction). After a little algebra one obtains

$$\delta M = \frac{\sin \alpha \delta C_N + 2 \cos \alpha \delta C_X}{\sqrt{1 + 3 \cos^2 \alpha}}. \quad (50)$$

In terms of the TSLT variables it can be shown that

$$\delta C_N = \frac{\epsilon \sin^2 \alpha}{\Omega} \int_{-\Omega}^{\Omega} p \, dz \quad (51)$$

$$\delta C_X = \frac{\epsilon \sin^3 \alpha}{\Omega \cos \alpha} \int_{-\Omega}^{\Omega} (y_b - z y_b') \, dz \quad (52)$$

Notice that, to first order in ϵ , the change of axial force is produced only by the change of frontal projected area. The change in pressure distribution is a second-order effect. These results can be combined to yield

$$\delta M = \frac{\epsilon \sin^3 \alpha}{\sqrt{1 + 3 \cos^2 \alpha}} E \quad (53)$$

where

$$E = \frac{1}{\Omega} \int_{-\Omega}^{\Omega} (p_b - 2y_b + 2zy'_b) dz \quad (54)$$

It is easy to show that E is unchanged if the solution is translated in the y -direction. The problem of designing an optimum lifting wing with conical TSLT is therefore that of maximizing E . It may be observed that by using Theorems 1 and 2, E can be expressed in completely geometric terms as

$$E = 2y_T + \frac{1}{\Omega} \int_{-\Omega}^{\Omega} (3y_s^2 - 3y_b^2 - 4y_b) dz \quad (55)$$

For any caret wing, the value of E is exactly 2.0. For flat delta wings of very high aspect ratio (effectively unswept wedges) the value is also 2.0, but decreases as aspect ratio decreases, falling to about 1.8 when the shock detaches, and dropping rapidly afterward.

IV. Efficient Wings

Within TSLT we can pose two problems. The direct problem consists of proposing a wing shape and finding the shape of the associated shock. The inverse problem consists of proposing a shock shape and finding the shape of the associated wing. To find optimal solutions, that maximize E , it should not matter which we attempt. However, rather than attempt an optimisation procedure, an intuitively guided trial-and-error process was employed. This seemed to work best in conjunction with solving the direct problem. The integral formula for E , (55) was extremely useful. It avoided any computation of the detailed pressure distribution (which is where the anomalies of TSLT show up very annoyingly) and made it unnecessary to pursue the solution right to the centerline, since a simple extrapolation gave sufficient accuracy. Occasionally, the shock shape produced by a direct calculation was fed back into an inverse calculation as a check on the accuracy. The best wing to emerge is shown in Figure IV.

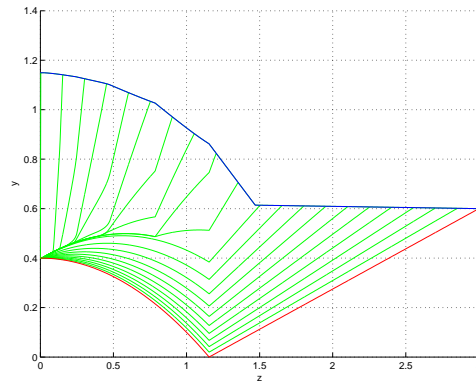


Figure 5. An efficient wing design; $E \simeq 2.4$

This has a value of E of about 2.4. In other words its performance exceeds that of a caret wing by a greater margin than the caret exceeds a flat delta. It combines three of the features that are commonly

regarded as beneficial for hypersonic lifting efficiency. It has a sharp leading edge with a attached shockwave (flow containment) although this was of course designed into the method. The leading edge is drooped, as on a caret wing, which avoids using the shock to impart sideways momentum (momentum principle) and there is also a central body that provides interference lift, as well as useful volume.

It is different from the optimum wings found by Triantafillou et al.¹¹ A careful reading of that paper reveals that the spanwise camber dy_b/dz was constrained to be of one sign in each half of the wing. Therefore the reflex shapes found here were implicitly excluded from the search. The shapes found by Kinsey⁵ from conical Euler solutions shared some features with those found here, but her code was unable to deal with embedded shockwaves, which would probably be produced by the wing in Fig IV in a more exact computation. Indeed, embedded shocks might be disadvantageous in practice also, producing local hot spots.

It is of course a major defect of the present study that it ignores viscous effects. It also ignores the possible importance of longitudinal camber. Adding volume near the apex of a delta wing creates additional interference lift in a swept-back region including the leading edge. Additionally, it creates resolves some of the trim problems associated with over-idealized configurations. But generally it is harder in hypersonics than in any other aerodynamic regime to bridge the gap between academic concepts and realistic design. Some of the reasons for this are reviewed by Bushnell.¹

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