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ITERATIVE EXPLICIT GUIDANCE FOR LOW THRUST
SPACECRAFT

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ITERATIVE EXPLICIT GUIDANCE FOR LOW THRUST SPACECRAFT

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Abstract

A retargeting procedure is developed for use as a nonlinear low thrust guidance scheme. The selection of a control program composed of a sequence of inertially fixed thrust-acceleration vectors permits all trajectory computations to be made with closed form expressions, and allows the controls to be represented by constant parameters, thrust-acceleration vectors and thrusting times. By requiring each trajectory to be time optimal, the guidance problem is transformed into a parameter optimization problem which is solved by the conjugate gradient method. The scheme is applied to a low thrust capture mission, and the results of computer simulations are presented.

I. Introduction

In the near future the capability for un-manned exploration of the solar system will be expanded by the introduction of low thrust inter-planetary spacecraft. To date, proposed methods of guidance for these vehicles have been based on linear perturbation theory. Namely, the spacecraft's equations of motion were linearized with respect to a numerically generated reference trajectory; and a guidance scheme was designed to control the resultant linear system. References (1-6) describe a number of these schemes.

In general, linear guidance laws continuously modify a continuously varying reference control program, and consequently, lack the simplicity desired for reliability and ease of implementation. A reduction in complexity can be obtained by taking the nominal control to be a sequence of thrust vectors fixed in some celestial reference frame for specified time periods, and by guiding with constant corrections to each fixed thrust vector. The corrections may be computed by linear means, but increased guidance flexibility is possible if nonlinear methods can be em-ployed.

Nonlinear guidance, which is basically a retargeting of the spacecraft, normally requires a large computing effort involving the propagation of long powered trajectories

coupled with an iterative process to update the the control program. This computational complexity is one of the reasons for the prevalence of the linear approach in low thrust guidance to date. However, if the fixed thrust vector concept is employed with the additional assumptions that the celestial reference frame is inertial and the thrust-acceleration is constant, a significant reduction in computing effort is realized. Under those assumptions, a closed form solution for the motion of a vehicle in a central force field can be obtained, and the need for numerical trajectory propagation is eliminated. In addition, the representation of the control program by constants simplifies the iterative update process by permitting the use of parameter optimization rather than calculus of variations techniques to compute control changes. In the following sections, the closed form trajectory solution is discussed, and a nonlinear guidance scheme employing it and a parameter optimization method are presented and evaluated. Although only planar flight is considered, all results generalize to the three dimensional case.

II. The Constant Thrust-Acceleration

Closed Form Solution

Beletskii [7] has developed a solution for the three dimensional motion of a vehicle with a constant thrust-acceleration vector in an inverse square gravitational field. The equations of planar motion for such a vehicle may be written as

$$\ddot{r} - r\dot{\theta}^2 = -1/r^2 + \epsilon \cos{(\alpha-\theta)} \qquad (1)$$

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} = \epsilon \sin{(\alpha-\theta)} \qquad (2)$$

where

r = radial distance
θ = polar angle
ε = constant thrust-acceleration magnitude
α = constant thrust-acceleration direction angle
(˙) = first time derivative
(¨) = second time derivative

The variables in (1-2) have been nondimensionalized

with respect to conditions on a circular orbit of arbitrary radius.

Two integrals for (1) and (2) may be obtained from work-energy and moment of momentum considerations. They are

$$\tfrac{1}{2}(\dot{r}^2 + r^2\dot{\theta}^2) - 1/r - \epsilon r \cos(\theta-\alpha) = E \qquad (3)$$

$$r^2\dot{r}\dot{\theta}\sin(\theta-\alpha) + (r^3\dot{\theta}^2-1)\cos(\theta-\alpha) +$$

$$\tfrac{1}{2}\epsilon r^2 \sin^2(\theta-\alpha) = c \qquad (4)$$

By the introduction of the new dependent and independent variables

$$u = r[1-\cos(\theta-\alpha)], \quad v = r[1+\cos(\theta-\alpha)] \qquad (5)$$

$$\tau = \int_{t_o}^{t} dt/r \qquad (6)$$

where t = nondimensional time and t_o = initial value of t, integrals (3) and (4) may be written as

$$(du/d\tau)^2 = 2(1+c)u + 2Eu^2 - \epsilon u^3 \qquad (7)$$

$$(dv/d\tau)^2 = 2(1-c)v + 2Ev^2 + \epsilon v^3 \qquad (8)$$

which can be put into standard elliptic form

$$\pm\sqrt{\epsilon}\,\tau = \int_{u_o}^{u} \frac{du}{\sqrt{u(u_1-u)(u-u_2)}} \qquad (9)$$

$$\pm\sqrt{\epsilon}\,\tau = \int_{v_o}^{v} \frac{dv}{\sqrt{v(v_1-v)(v_2-v)}} \qquad (10)$$

with

$$u_1, u_2 = (1/\epsilon)[E \pm \sqrt{E^2 + 2\epsilon(1+c)}]\ ,$$

$$u_1 > u_2$$

$$v_1, v_2 = (1/\epsilon)[E \pm \sqrt{E^2 - 2\epsilon(1-c)}]\ ,$$

$$v_1 > v_2$$

$$u_o, v_o = \text{values of } u, v \text{ at } t_o$$

The signs for (9) and (10) are defined by the initial conditions and the relations

$$du/d\tau = \pm\sqrt{\epsilon\,u\,(u_1-u)(u-u_2)} =$$

$$r\dot{r}[1 - \cos(\theta - \alpha)] + r^2\dot{\theta}\sin(\theta - \alpha) \qquad (11)$$

$$dv/d\tau = \pm\sqrt{\epsilon\,v\,(v_1-v)(v_2-v)} =$$

$$r\dot{r}[1 + \cos(\theta - \alpha)] - r^2\dot{\theta}\sin(\theta - \alpha) \qquad (12)$$

Since for real motion the integrals must exist and u and v must be non-negative, only a few cases occur. The ones considered by Beletskii include

$$0 < u_2 \le u < u_1\ ,\quad u_2 < 0 \le u < u_1 \qquad (13)$$

$$v_2 < 0 < v_1 < v\ ,\quad v_2 < v_1 < 0 < v,$$

$$0 < v_2 < v_1 < v\ ,\quad 0 < v < v_2 < v_1, \qquad (14)$$

$$0 < v \text{ with } v_1\ ,\ v_2 \text{ complex}$$

For various sets of initial conditions most of these cases arise in low thrust trajectories; and as examples, two of the solutions are presented below. From (9) with $u_2 < 0 \le u < u_1$,

$$u = u_1\,cn^2\,w \qquad (15)$$

where $w = \mp\tfrac{1}{2}\sqrt{\epsilon(u_1 - u_2)}\,\tau + F(\varphi_o, k)$

$$\sin\varphi_o = \sqrt{(u_1 - u_o)/u_1}\ ,\quad k = \sqrt{u_1/(u_1 - u_2)}$$

From (10) with $0 < v$, v_1, v_2 complex,

$$v = A(1 - cn\,w)/(1 + cn\,w) \qquad (16)$$

where $w = \pm\sqrt{\epsilon A}\,\tau + F(\varphi_o, k)$

$$\cos\varphi_o = (A - v_o)/(A + v_o)$$

$$k = \sqrt{(A + b)/2A}$$

$$A = \sqrt{2(1-c)/\epsilon}\ ,\quad b = -E/\epsilon$$

Solutions for the other cases are similar to (15) and (16) and are presented in Reference 7.

To complete the description of a trajectory, the following transformation between u, v and physical variables is required

$$r = \tfrac{1}{2}(u+v) \qquad (17)$$

$$dr/dt = (1/2r)(du/d\tau + dv/d\tau) \qquad (18)$$

$$\cos(\theta-\alpha) = (1/2r)(v-u) \qquad (19)$$

$$d/dt\,[\cos(\theta-\alpha)] = (1/2r^3)\cdot$$

$$(u\,dv/d\tau - v\,du/d\tau) \qquad (20)$$

$$t - t_o = \int_0^\tau \tfrac{1}{2}(u+v)\,d\tau \qquad (21)$$

Since $\dot{\theta}$ normally does not change sign, the quadrant of θ can be found by using

$$\text{sign}[\sin(\theta-\alpha)] = \qquad (22)$$

$$\text{sign}[v\,du/d\tau - u\,dv/d\tau]$$

which was obtained from equation (20). The integral in (21) may also be written in closed form. As an example, for solutions (15) and (16) the indefinite integrals

$$\int cn^2\ w\ dw = (1/k^2)\ [E(\omega,k) - (1-k^2)w] \quad (23)$$

$$\int [(1-cn\ w)/(1+cn\ w)]\ dw = w - \quad (24)$$

$$2\ E\ (\omega,\ k) - 2\ (sn\ w\ cn\ w)/(1+cn\ w)$$

with $tn\ w = \tan \omega$ can be employed to compute t.

By means of the various expressions derived above, it is possible to analytically describe the motion of a vehicle with a constant thrust-acceleration vector. Given a set of initial conditions, E and c can be evaluated, and the correct cases and sign conventions identified. Then for a specified time t, τ can be found from equation (21), u and v computed, and the state determined from (17-20).

III. Guidance Philosophy

A vehicle employing inertially fixed thrust-acceleration vector control is limited in its ability to attain a desired terminal condition because only three parameters may be selected to define the trajectory, i.e., thrust-acceleration magnitude and direction, and powered flight time. In order to provide the flexibility necessary for guidance, the admissible class of control programs is therefore extended to include those which may be represented as a sequence of fixed thrust-acceleration vectors. Such programs are described by the set of elements

$$[\epsilon_i,\ \alpha_i,\ \tau_i]\ ,\ i = 1\ to\ N \quad (25)$$

where

ϵ_i = thrust acceleration magnitude during i'th time interval

α_i = thrust direction angle during the i'th time interval

τ_i = duration of the i'th time interval

N = number of powered flight intervals in the control program

By selection of N and the values of $(\epsilon_i,\ \alpha_i,\ \tau_i)$, a wide variety of trajectories may be generated, and the closed form expressions of the previous section may be used to find the vehicle's state at any time.

The objective of the guidance scheme is to find a trajectory, defined by a set $[\epsilon_i,\ \alpha_i,\ \tau_i]$ for a given N, which satisfies the condition

$$J_1 = \tfrac{1}{2}\ \Delta\ x_f^T\ S\ \Delta\ x_f \le \sigma_f^2 \quad (26)$$

where

Δx_f = vector of terminal state errors

S = weighting matrix

σ_f = error tolerance

A straight forward method of obtaining the required trajectory is to find one which minimizes J_1. However, because the minimum of J_1 may be non-unique, a better approach is to adjoin a performance index J_2 to (26) and seek to minimize the quantity

$$J = J_1 + K\ J_2 \quad (27)$$

where K is a factor selected to guarantee satisfaction of (26).

Since a given set of parameters defines a trajectory which then gives a value to J, it may be considered a function of several parameters

$$J = J\ (\epsilon,\ \alpha,\ \tau) \quad (28)$$

where

$\epsilon,\ \alpha,\ \tau$ = N - dimensional parameter vectors with components $\epsilon_i,\ \alpha_i,\ \tau_i$

and its minimization becomes a parameter optimization problem. A number of computational algorithms exist for the solution of such problems; but for onboard guidance purposes, the conjugate gradient (C. G.) method [8] with numerical partial derivatives appears to be the most attractive. It is a simple first order technique which requires less storage than more sophisticated methods such as Davidon's algorithm [11]; and it exhibits good convergence properties for this problem. The use of numerical partials is dictated by the extreme complexity of the analytic partials; however, as Johnson and Kamm (References 9 and 10) have shown, numerical differentiation can be employed effectively with accelerated gradient methods. Moreover, because J is evaluated quite accurately by means of closed form expressions, finite differences yield good approximate derivatives.

In order to generate a new control during a guidance cycle, the C. G. iterator requires an initial guess, and the logical choice is the current control program. This choice has an additional advantage since the first step in C. G. algorithm predicts the final state reached using the initial control estimate. Consequently, if the predicted final state satisfies (26), no control changes are necessary and the update operation can be bypassed. Every time a new control is determined, it will contain a number of elements $(\epsilon_i,\ \alpha_i,\ \tau_i)$ equal to the number unused in the previous control program. Therefore, as the vehicle proceeds to its target, the number of parameters in each updated control decreases, resulting in a loss of flexibility which may make the satisfaction of condition (26) difficult. A remedy for this problem is to subdivide the current control in order to introduce additional parameters. For example, if only two elements

$$(\epsilon_1,\ \alpha_1,\ \tau_1)\ ,\ (\epsilon_2,\ \alpha_2,\ \tau_2) \quad (29)$$

remain, a three element program can be computed

using as initial guesses

$$(\epsilon_1, \alpha_1, \tau_1) \ , \ (\epsilon_2, \alpha_2, \tfrac{1}{2}\tau_2) \ , \ (\epsilon_2, \alpha_2, \tfrac{1}{2}\tau_2)$$

$$\text{for } \tau_2 \geq \tau_1$$

or

$$(\epsilon_1, \alpha_1, \tfrac{1}{2}\tau_1) \ , \ (\epsilon_1, \alpha_1, \tfrac{1}{2}\tau_1), (\epsilon_2, \alpha_2, \tau_2)$$

$$\text{for } \tau_2 < \tau_1$$

This procedure, which can provide nine parameters, should be sufficient to define a trajectory with up to four terminal constraints, i.e., a fully constrained planar trajectory. It must be noted, however, that as the spacecraft nears its target point, even the subdivision procedure may not give acceptable results. The reason for this failure is the inherent problem of controllability in low thrust flight due to the vehicle's inability to perform large maneuvers in short time periods. Consequently, in the final portion of the trajectory when the time-to-go drops below a specified threshold, it is best to terminate guidance and use the current control program to completion.

IV. Simulation Results

In order to evaluate the performance of the guidance scheme, it was applied to the difficult mission of low thrust planetary approach. Specifically, the spacecraft was required to proceed from escape conditions to a specified final radial position, radial velocity, and angular velocity which corresponded to a point on an inward tangential thrust spiral trajectory terminating on a circular orbit. Throughout the flight, the gravitational and thrust forces are of the same order of magnitude, and guidance is essential for successful mission completion. The nominal trajectory was obtained from a piecewise approximation to the last half revolution of a constant thrust-acceleration tangential thrust trajectory. Ten elements were found to give a reasonable approximation; and since ϵ is constant along the tangential trajectory, the nominal control program contains only 20 parameters, $\alpha_i, \tau_i, i = 1$ to 10. Figure 1 shows the nominal trajectory, and Tables 1 and 2 give its end conditions and control program.

In order to determine the ability of the guidance scheme to null out errors due to off nominal conditions, a set of trials was made to find the minimum time trajectory to the nominal final state under the following perturbed conditions

(1) - 10% error in thrust-acceleration magnitude
(2) $\pm$ 10% errors in initial radial velocity
(3) $\pm$ 10% errors in initial angular velocity

The results of the trials are presented in Figures (2-4). A tolerance $\sigma_f^2 = 10^{-5}$

guarantees position errors of less than 10 km and velocity errors of the order of meters/second. The weighting parameters in all of the cases were 1, 10^5, 10^5 on final position, radial velocity, and angular velocity errors, respectively. Rather than iterate for all 20 parameters until an acceptable trajectory was obtained, an alternate strategy was employed. Fifteen iterations were carried out varying all 20 parameters; then the initial 10 were held constant, and only the last 10 were varied until convergence was obtained. This two phase procedure is, in effect, a coarse trajectory adjustment involving the full control program, followed by a fine adjustment with the controls of the 5 trajectory intervals nearest the target. The primary reason for this strategy is a reduction in computation time, since for the "fine adjustment" a problem of reduced dimension is solved by the C.G. algorithm. A secondary reason is to demonstrate that a fully converged trajectory need not be obtained before control implementation. That is, after a number of iterations an improved but unconverged control program will be obtained. This program can be implemented, and then at a later point in the trajectory, additional corrections can be made to direct the vehicle to the target. Of course, it must be observed that due to low thrust controllability problems, sufficient time must be allotted for those additional corrections to be effective.

Since the speed of the iterative procedure is important for real time guidance, the iteration times for the five cases were noted. When 20 parameters were being varied, each iteration required an average of about 6 seconds on an IBM 360/67 computer. For the 10 parameter computations, the time dropped to only about 2 seconds per iteration. Since the thrust acceleration vector is nominally held fixed for periods of the order of hours, guidance cycle times of the order of minutes are not unreasonable; and such time intervals are more than adequate to permit computation of an updated control program.

As a final part of the evaluation, a simple test was conducted using the technique of subdividing intervals near the end of the flight. A representative unconverged trajectory from the $- \Delta\epsilon$ case was selected; and the state corresponding to the start of the final two control intervals was taken as the initial state. After subdividing the longest control interval, a search was carried out for a 3 element control program. Convergence was obtained in 7 iterations or about 14 seconds. Figure 2 gives the J_1 reduction with each iteration. This trial, of course, does not prove that interval splitting will always yield an acceptable control. However, it does permit some final refinement of the trajectory, and, therefore, appears to be a reasonable procedure.

As observed previously, the number of parameters affects the iteration time of the C.G. algorithm. In order to shorten that time, it is desirable to use as few parameters as possible

in defining the control program. Unfortunately, a reduction in parameter number decreases the flexibility of the trajectories, and a loss in performance can be expected. In order to determine the effect of a change in parameter number on the mission considered in this paper, several minimum time trajectories were computed; and the results appear in Figure 5 (for these trajectories $\epsilon = 9.7578 \times 10^{-4}$, not the nominal ϵ given in Table 2). It may be seen that a decrease from twenty to eight parameters causes less than one-half of one percent variation in flight time. Consequently, it appears that the use of only a few parameters is possible without a severe loss in performance.

V. Concluding Remarks

In this paper an explicit low thrust guidance technique which consists of a sequence of inertially fixed constant thrust-acceleration vectors for varying time intervals has been presented. Such a control program permits analytical description of the trajectory and simplicity in control implementation. The control updates are obtained from an optimization procedure which determines the sequence of inertial directions for the thrust vector and the time periods for which those directions are to be maintained. The guidance scheme is probably most useful in cases where gravitational and thrust forces are of the same order of magnitude, for example during planetary approach near zero energy conditions.

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Table 1   Nominal Trajectory Data

Variable	Initial Value	Final Value
Time	583.00	849.55
Radial Distance	28.083936	9.0841936
Polar Angle	1.8362218	-1.1741114
Radial Rate	- .17733447	- .060492352
Angular Rate	- .0071004979	- .036964070

Table 2   Nominal Control Program

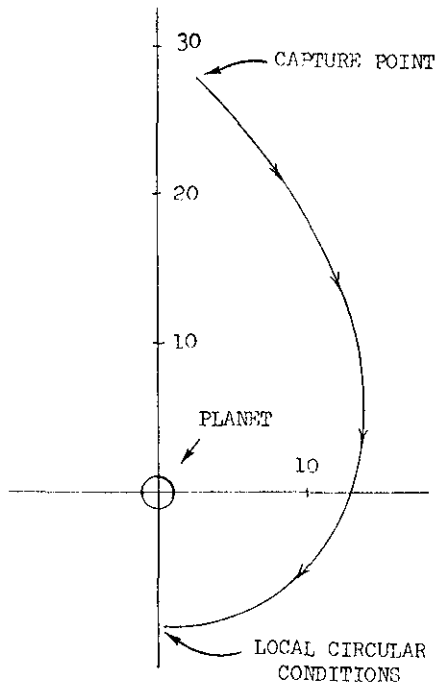| Initial Interval Time	Control Angle α (radians)	Interval Length Parameter τ
583.00	2.554370	1.148552
612.33	2.258907	2.000099
653.00	2.066465	1.000083
669.89	1.852573	1.000068
685.08	1.619306	1.000055
698.88	1.368575	1.000044
711.49	1.101983	1.000036
723.11	.8208626	1.000031
733.86	.5263267	1.000027
784.44	0.2193277	1.000027
Thrust Acceleration = .00108420		

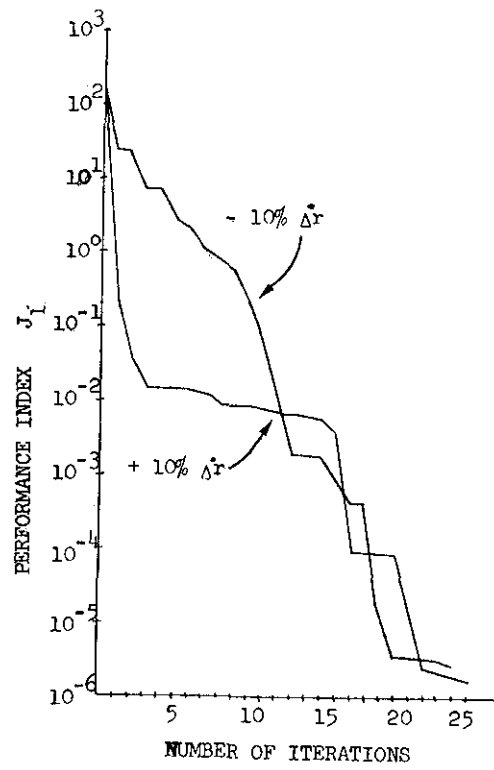Figure 1. Nominal Trajectory

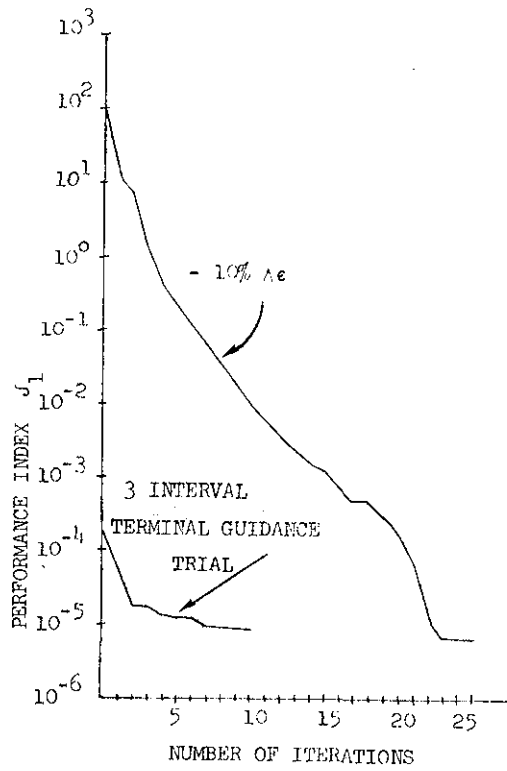Figure 3. Radial Velocity Perturbations

Figure 2. Thrust Acceleration
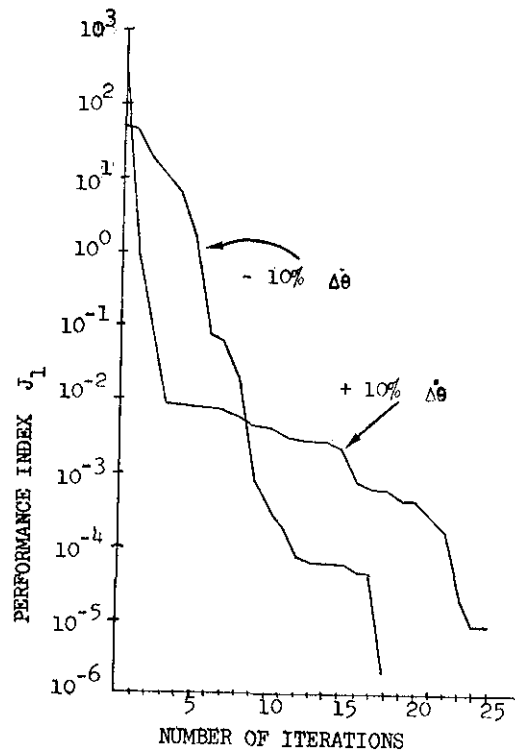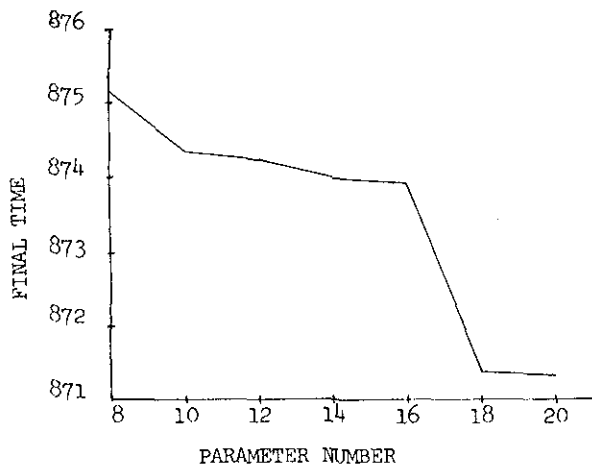Magnitude
Perturbations

Figure 4. Angular Velocity Perturbations

Figure 5. Variations of Final Time
With Parameter Number