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Introduction

Preconditioning, in the numerical solution of PDEs, is the attempt to accelerate convergence to the steady state by removing sources of stiffness from the numerical procedure. Stiffness is due to different kinds of error being removed at different rates; it can be construed as a large spread in the eigenvalues of the iteration operator. Some kinds of preconditioning are applied at the level of the discrete algebraic problem; the one that we are concerned with here comes at an earlier stage of the analysis, where we select the particular transient equation that the code will model. In particular, we may aim for the transient solution to follow a path that does not mimic the true physical transients. The simplest example, which was introduced in the 1960s, is 'local timestepping'. The time derivative in the PDE is multiplied by a scalar factor that depends on both the discrete grid and the current solution, in such a way that the local value of the Courant number is kept roughly constant. Different parts of the solution are then marched at different rates, and the local rate of adjustment is made as fast as possible. This simple idea can save orders of magnitude in computing time, and quickly became standard practice.

However, it is not guaranteed that all aspects of the solution will converge equally fast. The process of convergence is twofold. Solution errors are propagated to the boundaries (where they must be expelled by effective boundary conditions) and they are also damped out. Propagation to the boundary takes place at different speeds for different types of error, and can follow any of the wave paths that are present in the physics (acoustic, convective, etc.) If these wavespeeds are very different, there will be error modes that take many more iterations than others to reach the boundary. The idea of *local preconditioning* is to multiply the time derivative in the PDE, not just by a scalar, but by a matrix, such that we equalize, as much as possible, not just the local Courant numbers based on the fastest waves, but the Courant numbers based on any type of wave. Ideally, all error components that propagate as waves should reach the boundary simultaneously. This particular form of preconditioning can be achieved by analysis at the PDE level, and is the form considered here. Other forms of preconditioning are important, but those that seek to improve the ways in which errors are damped usually must deal more directly with discrete aspects of the computation, such as the grid aspect ratio and the precise discretization employed.

Because local preconditioning of a system with m unknowns involves $m^2 - 1$ degrees of freedom (a free matrix, up to a scaling factor), blind search for an optimum is confusing and unproductive, but a general theory has yet to emerge. Successful preconditioners have evolved along largely empirical lines. A few facts are known. For example, if the natural system is

$$\mathbf{u}_t + \mathbf{A}\mathbf{u}_x + \mathbf{B}\mathbf{u}_y = 0, \tag{1}$$

with $\mathbf{A}, \mathbf{B}$ symmetric matrices, and the preconditioned system is

$$\mathbf{u}_t + \mathbf{P}[\mathbf{A}\mathbf{u}_x + \mathbf{B}\mathbf{u}_y] = 0, \tag{2}$$

then replacing $\mathbf{P}$ with its transpose $\mathbf{P}^T$ gives the same set of modified wavespeeds. However, it does not lead to a method with the same convergence rate, and this shows that wavespeeds (or, equivalently, domains of dependence) are not the only significant properties of a preconditioner. Darmofal[1] has drawn attention to the importance of the modified eigenvectors. A known constraint on $\mathbf{P}$ is that it be positive, in order that the modified IBVP be stable under the same set of boundary conditions as the natural IBVP. A

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well-known byproduct of local preconditioning is that discretizations errors can be much reduced, especially for almost incompressible flows.[2,4,6]

Here, we develop a systematic procedure that assigns a unique preconditioner to any given set of first-order PDEs in two dimensions. The preconditioner that is produced is positive and yields a probably optimal set of wavespeeds. Applied to the Euler equations it is the preconditioner of Van Leer, Lee and Roe (VLLR).[5] The payoff for developing a systematic procedure should be the possibility of applying local preconditioning to more complicated systems of equations, such as those of magnetohydrodynamics, and in this paper we give the first analysis of such a preconditioner. Actually, for the equations of ideal magnetohydrodynamics the preconditioner that is obtained is rather complicated, and perhaps too complicated to be useful in practice, and for that reason we restrict ourselves in this paper to the case of aligned flow, in which the magnetic field is aligned with the streamlines. This is enough to show that a very large reduction of stiffness is possible. It is likely that the general case can be simplified, because, for the Euler equations, preconditioners much simpler than VLLR are known to be very useful.[7,8]

The idea behind the construction is to decompose the unknowns into either the Riemann invariants, in the hyperbolic case, or, in the elliptic case, pairs of variables satisfying Cauchy-Riemann-like systems. We may speak of an elliptic-hyperbolic splitting of the system. In another paper presented at this meeting, Nishikawa and Van Leer demonstrate that this same splitting allows effective multigrid strategies to be applied separately to each component, achieving true textbook convergence in a very natural way for the Euler equations.[3] The next section will describe the splitting strategy from a perspective that is perhaps simpler than previous accounts. Then, in Section 3, we explain our strategy using the Euler equations as an illustrative example. In Section 4, we apply it to the MHD equations, and show that the stiffness due to disparate propagation speeds is greatly reduced.

The present paper gives a relatively informal outline of the procedure. A more detailed and rigorous presentation is being prepared.

Complex characteristic equations

Consider the steady form of (1),(2),

$$\mathbf{A}\mathbf{u}_x + \mathbf{B}\mathbf{u}_y = 0. \tag{3}$$

Multiply this from the left by a vector $\boldsymbol{\ell}$ that solves the generalized left eigenvalue problem

$$\boldsymbol{\ell}(\mathbf{B} - \lambda\mathbf{A}) = 0. \tag{4}$$

This gives

$$\boldsymbol{\ell}\mathbf{A}(\mathbf{u}_x + \lambda\mathbf{u}_y) = 0, \tag{5}$$

which is a characteristic (scalar) equation holding along the line $dy/dx = \lambda$. It asserts that along that line

$$\boldsymbol{\ell}\mathbf{A}d\mathbf{u} = \boldsymbol{\ell}\mathbf{B}d\mathbf{u}/\lambda = 0$$

In the linear case, with $\mathbf{A}, \mathbf{B}$ constant matrices, we can also write

$$d(\boldsymbol{\ell}\mathbf{A}\mathbf{u}) = d(\boldsymbol{\ell}\mathbf{B}\mathbf{u}) = 0$$

and the scalar quantity $R = \boldsymbol{\ell}\mathbf{A}\mathbf{u} = \boldsymbol{\ell}\mathbf{B}\mathbf{u}/\lambda$ is called a Riemann invariant, constant along the characteristics. Note that $\mathbf{A}, \mathbf{B}$ play interchangable roles in the analysis unless it happens that $\lambda = 0$.

If the steady problem is purely hyperbolic, with m variables, there will be m such Riemann invariants. Collectively they provide an alternative description of the flow. However, the number of characteristic equations may be fewer than m. This happens if the determinant equation

$$\det(\mathbf{B} - \lambda\mathbf{A}) = 0 \tag{6}$$

has fewer than m real roots. In that case, some of the roots will be complex conjugate, and the characteristic equation (5) will read

$$(\boldsymbol{\ell}_R \pm i\boldsymbol{\ell}_I)\mathbf{A}(\mathbf{u}_x + (\lambda_R \pm i\lambda_I)\mathbf{u}_y) = 0 \tag{7}$$

The real and imaginary parts of this equation form a pair of equations describing the behaviour of the real and imaginary parts of the Riemann invariant. Specifically, if we write the Riemann invariant as $R = R_R + iR_I$, we obtain

$$(R_R)_x + \lambda_R(R_R)y - \lambda_I(R_I)_y = 0 \tag{8}$$
$$(R_I)_x + \lambda_R(R_I)_y + \lambda_I(R_R)_y = 0 \tag{9}$$

which is easily verified to be an elliptic system. As an example, in supersonic inviscid flow, the characteristic equations can be written, in natural coordinates s, n respectively along and normal to the streamlines, as

$$[\sqrt{M^2-1}p_s + p_n] + \frac{\rho q^2}{\sqrt{M^2-1}}[\sqrt{M^2-1}\theta_s + \theta_n] = 0 \tag{10}$$

where either sign may be taken for the radical. For subsonic Mach numbers, the square root is imaginary, and the real and imaginary parts of the characteristic equation form the elliptic system.

$$(M^2-1)p_s + \rho q^2\theta_n = 0 \tag{11}$$
$$\rho q^2\theta_s + p_n = 0 \tag{12}$$

It is still true that the Riemann invariants provide an alternative way to describe the flow, but it is no longer true, even in the linear case, that each flow variable behaves independently. Some of them are now coupled in pairs, but each pair behaves independently of any other variable. The solution space of $\mathbf{u}$ can be partitioned into one-dimensional subspaces corresponding to the hyperbolic parts, and two-dimensional subspaces corresponding to the elliptic parts. For linear problems, the subspaces communicate only at boundaries.

Numerical applications of this idea are various. In fluctuation-splitting schemes the different components of the residual can be distributed differently, using upwind-biased strategies for the hyperbolic components and central strategies for the elliptic components; this results in improved accuracy, especially at low speeds. In a multigrid scheme, the different components of the fine-grid residual can be restricted onto different coarse grids, using full coarsening for the elliptic components and semi-coarsening in an appropriate direction for the hyperbolic components. In the present paper, each component of the residual is allowed to evolve independently in time, such that all components evolve at rates that are as nearly equal as possible. The preconditioner that results from this approach is of the form

$$\mathbf{P} = \sum_k a_k \boldsymbol{\ell}_k^T \boldsymbol{\ell}_k \tag{13}$$

with each $\boldsymbol{\ell}$ defined as above, and with k running over the set of characteristic equations. The factors a_k are scalar factors controlling how fast each error component will propagate. If the $\{a_k\}$ are properly chosen, this preconditioner yields the optimum domain of dependence for any given set of two-dimensional first-order equations.

Euler Preconditioning Revisited

We consider only the linear problem, because nonlinear problems will be handled numerically by a local linearization. We choose variables

$$\mathbf{u} = (\rho/\rho_0, u/a_0, v/a_0, dp/(\rho_0 a_0^2) - d\rho/\rho_0)^t,$$

which are perturbations of a flow parallel to the x-axis with density ρ_0 and sound speed a_0, at a Mach number M. The governing equations are of the form (1) with

$$\mathbf{A} = a_0 \begin{bmatrix} M & 1 & 0 & 0 \\ 1 & M & 0 & 0 \\ 0 & 0 & M & 0 \\ 0 & 0 & 0 & M \end{bmatrix}, \tag{14}$$

$$\mathbf{B} = a_0 \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{15}$$

The Natural Domain of Influence

The domain of influence of a point disturbance at the origin is obtained by finding all possible plane

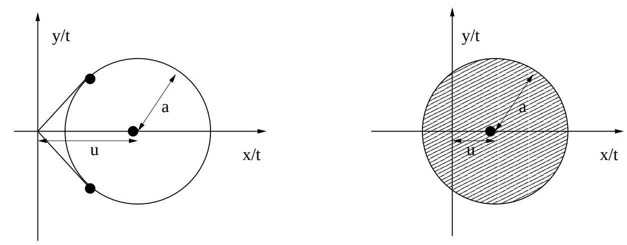

Fig. 1 Wave diagrams for the Euler equations. Left: Supersonic , Right: Subsonic

wave solutions satisfying $\mathbf{u} = f(x\cos\theta + y\sin\theta - \lambda(\theta)t)$ where $\lambda(\theta)$ is found from

$$\det(\mathbf{A}\cos\theta + \mathbf{B}\sin\theta - \lambda(\theta)\mathbf{I}) = 0. \tag{16}$$

Then the envelope of the wavefronts $x\cos\theta + y\sin\theta - \lambda(\theta)t = 0$ contains the domain of influence. For the Euler equations it is well known that this gives, in the plane $(x/t, y/t)$ a circle of radius M centered at $x/t = 1, y/t = 0$ (the Mach cone), together with two points at $x/t = 1, y/t = 0$ (the streamline). This can be obtained in another way by first setting the fluid speed to zero ($M = 0$), which gives a circular domain centered at the origin, and then offsetting the origin by $(-M, 0)$. The point, on the boundary of the domain of influence, that lies farthest from the origin gives the fastest speed with which an error mode may travel. The point that lies nearest gives the slowest such speed. The ratio k of these extreme distances gives a kind of condition number. For the Euler equations this is

$$k = \frac{1+M}{\min(M, |M-1|)}$$

Characteristic Equations and Reduced Equations

The characteristic equations are, in the steady hyperbolic case,

$$\begin{aligned} \pm\beta du_1 + M du_3 &= 0 \quad \mathit{on} \quad dy/dx = 1/\beta && (17)\\ du_1 + M du_2 &= 0 \quad \mathit{on} \quad dy/dx = 0, && (18)\\ du_4 &= 0 \quad \mathit{on} \quad dy/dx = 0 && (19) \end{aligned}$$

where $\beta = \sqrt{M^2 - 1}$. These can be produced from the original system through multiplication by $\boldsymbol{\ell}_k$ where $\boldsymbol{\ell}_k$ is in the left nullspace of

$$\mathbf{B} - \lambda_k \mathbf{A} = \begin{bmatrix} -\lambda_k M & -\lambda_k & 1 & 0 \\ -\lambda_k & -\lambda_k M & 0 & 0 \\ 1 & 0 & -\lambda_k M & 0 \\ 0 & 0 & 0 & -\lambda_k M \end{bmatrix} \quad (20)$$

Thus,

$$\begin{aligned} \boldsymbol{\ell}_1 &= (M, -1, \beta, 0) \quad \text{on} \quad dy/dx = 1/\beta && (21)\\ \boldsymbol{\ell}_2 &= (M, -1, -\beta, 0) \quad \text{on} \quad dy/dx = -1/\beta && (22)\\ \boldsymbol{\ell}_3 &= (0, 1, 0, 0) \quad \text{on} \quad dy/dx = 0 && (23)\\ \boldsymbol{\ell}_4 &= (0, 0, 0, 1) \quad \text{on} \quad dy/dx = 0 && (24) \end{aligned}$$

(For the moment, the magnitudes of these vectors do not matter)

A less usual way of writing the characteristic equations is to form a product such as

$$(\boldsymbol{\ell}_k^T \boldsymbol{\ell}_k)(\mathbf{A}\mathbf{u}_x + \mathbf{B}\mathbf{u}_y) = 0 \quad (25)$$

Because we have multiplied by the matrix $\boldsymbol{\ell}_k^T \boldsymbol{\ell}^k$, rather than by the vector $\boldsymbol{\ell}_k$, this is now still a system of four equations, although a highly degenerate one. We will call it the reduced equation. Defining $\mathbf{A}_k = \boldsymbol{\ell}_k^T \boldsymbol{\ell}_k \mathbf{A}$ and $\mathbf{B}_k = \boldsymbol{\ell}_k^T \boldsymbol{\ell}_k \mathbf{B}$, and taking $k = 1$ as an example, we have

$$\mathbf{A}_1 = \begin{bmatrix} M\beta^2 & 0 & M^2\beta & 0 \\ -\beta^2 & 0 & -M\beta & 0 \\ \beta^3 & 0 & M\beta^2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad (26)$$

$$\mathbf{B}_1 = \begin{bmatrix} M\beta & 0 & M^2 & 0 \\ -\beta & 0 & -M & 0 \\ \beta^2 & 0 & M\beta & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \quad (27)$$

On examination, we find four multiples (one of them a zero multiple) of the scalar characteristic equation. The wavespeeds $\mu(\theta)$ supported by

$$\mathbf{u}_t + \mathbf{A}_1 \mathbf{u}_x + \mathbf{B}_1 \mathbf{u}_y = 0 \quad (28)$$

satisfy

$$\mu^3(\mu - 2M\beta^2 \cos\theta - 2M\beta \sin\theta),$$

The envelope of these waves is the origin (counted three times) and the single ray

$$x/t = 2M\beta^2, \qquad y/t = 2M\beta$$

meaning that this set of equations propagates one set of errors along that ray and leaves all other errors untouched. The speed with which those errors propagate is $2\sqrt{M^2\beta^4 + M^2\beta^2} = 2M^2\beta$.

A general explanation in terms of linear algebra is that because we have $\lambda_k \boldsymbol{\ell}_k \mathbf{A} = \boldsymbol{\ell}_k \mathbf{B} = \boldsymbol{\ell}_k(\mathbf{A}\mathbf{A}^{-1})\mathbf{B} = \boldsymbol{\ell}_k \mathbf{A}(\mathbf{A}^{-1}\mathbf{B})$, then $(\boldsymbol{\ell}_k \mathbf{A})$ is a left eigenvector of $\mathbf{A}^{-1}\mathbf{B}$. Also, because $\mathbf{A}, \mathbf{B}$ are symmetric, we have $\lambda_k \mathbf{A} \boldsymbol{\ell}_k^T = \mathbf{B}\boldsymbol{\ell}_k^T$, so that $\boldsymbol{\ell}_k^T$ is a right eigenvector of $\mathbf{A}^{-1}\mathbf{B}$. It follows that the two sets of vectors $\{\boldsymbol{\ell}_k \mathbf{A}\}$ and $\{\boldsymbol{\ell}_k^T\}$ are orthogonal, and therefore both $\mathbf{A}_k$ and $\mathbf{B}_k$, when multiplying $\mathbf{u}_x$ or $\mathbf{u}_y$, project them into the subspace spanned by $\boldsymbol{\ell}_k^T$. The equation

$$\mathbf{u}_t + \mathbf{A}_k \mathbf{u}_x + \mathbf{B}_k \mathbf{u}_y = 0 \quad (29)$$

only updates the part of the solution lying in that subspace. If we replace (29) with

$$\mathbf{u}_t + a_k[\mathbf{A}_k \mathbf{u}_x + \mathbf{B}_k \mathbf{u}_y] = 0 \quad (30)$$

we still expel only a single error mode from the domain, but we move it faster by a factor a_k. We can now see that a preconditioner of the form (13) will move each error mode independently (because they are orthogonal) and that the propagation rates can be adjusted, and made equal.

The supersonic preconditioner

Define the potential part of the solution to be the part carried by the acoustic waves $\boldsymbol{\ell}_{1,2}$ and the convective part to be the entropy and enthalpy disturbances carried by the streamwise characteristics $\boldsymbol{\ell}_{3,4}$. The potential part of the preconditioner is

$$\mathbf{P}_\phi = \boldsymbol{\ell}_1 \boldsymbol{\ell}_1^T + \boldsymbol{\ell}_2 \boldsymbol{\ell}_2^T = a_\phi \begin{bmatrix} 2M^2 & -2M & 0 & 0 \\ -2M & 2 & 0 & 0 \\ 0 & 0 & 2\beta^2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (31)$$

and the advective part is

$$\mathbf{P}_a = \boldsymbol{\ell}_3 \boldsymbol{\ell}_3^T + \boldsymbol{\ell}_4 \boldsymbol{\ell}_4^T = a_a \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (32)$$

With $a_\phi = a_c = 1$, the potential part of the error propagates with speed $2M^2\beta$ (see above) and the advective part with speed M. To make these equal, we need to take

$$\frac{a_\phi}{a_a} = \frac{1}{2M\beta} \quad (33)$$

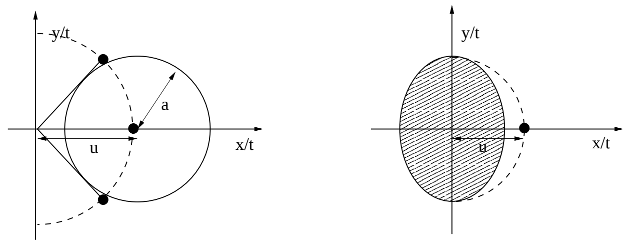

Fig. 2 Wave diagrams for the preconditioned Euler equations. Left: Supersonic , Right: Subsonic

If we take $a_a = 1, a_\phi = 1/(2M\beta)$ we arrive at the preconditioner of Van Leer, Lee and Roe, in the supersonic case;

$$\mathbf{P}_{VLLR} = \begin{bmatrix} M/\beta & -1/\beta & 0 & 0 \\ -1/\beta & 1/(M\beta)+1 & 0 & 0 \\ 0 & 0 & \beta/M & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (34)$$

The rays along which the errors propagate are now $x/t = \beta, y/t = 1$, (see Fig 2) and the condition number is now unity. In fact, if any given set of equations of the form 1 is purely hyperbolic, the condition number can be made equal to unity, because all of the propagation speeds can be controlled independently (of course, in the purely hyperbolic case, we should not be preconditioning at all, but marching in space!).

The subsonic preconditioner

The potential part of the solution now lies in the subspace spanned by the real and imaginary parts of $\boldsymbol{\ell}_{1,2}$;

$$\boldsymbol{\ell}_R = (M, -1, 0, 0) \qquad \boldsymbol{\ell}_I = (0, 0, \beta^*, 0) \quad (35)$$

where $\beta_*^2 = 1 - M^2$. The potential part of the preconditioner is then

$$\mathbf{P}_\phi = \boldsymbol{\ell}_R \boldsymbol{\ell}_R^T + \boldsymbol{\ell}_I \boldsymbol{\ell}_I^T \quad (36)$$

which evaluates to

$$\mathbf{P}_\phi = \begin{bmatrix} 2M^2 & -2M & 0 & 0 \\ -2M & 2 & 0 & 0 \\ 0 & 0 & 2(1-M^2) & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (37)$$

This is identical with the supersonic case except for the $(3,3)$ element, which has changed sign to remain positive. We can form $\mathbf{A}_\phi = \mathbf{P}_\phi \mathbf{A}$ and $\mathbf{B}_\phi = \mathbf{P}_\phi \mathbf{B}$;-

$$\mathbf{A}_\phi = \begin{bmatrix} -2M\beta_*^2 & 0 & 0 & 0 \\ 2\beta_*^2 & 0 & 0 & 0 \\ 0 & 0 & 2M\beta_*^2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (38)$$

$$\mathbf{B}_\phi = \begin{bmatrix} 0 & 0 & 2M^2 & 0 \\ 0 & 0 & -2M & 0 \\ 2\beta*^2 & 0 & 0 & \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (39)$$

The wavespeeds supported by the system

$$\mathbf{u}_t + a_\phi[\mathbf{A}_\phi \mathbf{u}_x + \mathbf{B}_\phi \mathbf{u}_y] = 0 \quad (40)$$

are now given by

$$\mu^2(\theta) = 4a_\phi^2 M^2 \beta_*^2 (1 - M^2 \cos^2\theta) \quad (41)$$

and they correspond to a domain of influence bounded by the ellipse

$$\frac{x^2}{4a_\phi^2 M^2 \beta^4} + \frac{y^2}{4a_\phi^2 M^2 \beta_*^2} = 1 \quad (42)$$

The fact that the ellipse is centred on the origin, rather than on the point $(M, 0)$ representing the advection speed, has always seemed very odd from a physical perspective. It becomes understandable when we realize that this part of the solution has been completely decoupled from the advected part.

The fastest propagation in this part of the solution is now perpendicular to the the flow, with speed $2M\beta_*$, and the slowest is parallel to the flow, with speed $M\beta_*^2$. The ratio of these two speeds is $1/\beta_*$ which is the condition number for the potential part of the flow. Because the evolution of the potential part has been reduced to a 2×2 system it is feasible to investigate whether any further preconditioning would reduce the condition number. It turns out that this is not possible, and that in fact for any two-dimensional problem the elliptic preconditioner is given by (36). We may exercise the only remaining degree of freedom by choosing that the fastest signals in the potential and advective errors are equal; in that case $2a_\phi M\beta_* = a_c M$. By taking $a_\phi = 1/(2\beta_*), a_c = 1$ we obtain the subsonic version of the VLLR preconditioner;

$$\mathbf{P}_{VLLR} = \begin{bmatrix} M^2/\beta_* & -M/\beta_* & 0 & 0 \\ -M/\beta_* & 1/\beta_* + 1 & 0 & 0 \\ 0 & 0 & \beta_* & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (43)$$

and the condition number of the Euler equations is improved in the subsonic case to $1/\beta_*$

More generally, if any of the characteristic equations are complex, there is a 2×2 subsystem whose condition number cannot be made equal to unity. The condition number of the total system cannot be reduced below that of the most badly conditioned subsystem.

Two-dimensional ideal MHD

The equations governing linearized two-dimensional magnetohydrodynamics (MHD) are of the form (1) with variables

$$\mathbf{u} = \left(\frac{p}{\rho_0 a_0^2}, \frac{u}{a_0}, \frac{v}{a_0}, \frac{b_x}{\sqrt{\rho_0} a_0}, \frac{b_y}{\sqrt{\rho_0} a_0}, \frac{dp}{\rho_0 a_0^2} - \frac{d\rho}{\rho_0}\right)^T, \tag{44}$$

and

$$\mathbf{A} = \begin{bmatrix} M & 1 & 0 & 0 & 0 & 0 \\ 1 & M & 0 & 0 & -b\sin\alpha & 0 \\ 0 & 0 & M & 0 & b\cos\alpha & 0 \\ 0 & 0 & 0 & M & 0 & 0 \\ 0 & -b\sin\alpha & b\cos\alpha & 0 & M & 0 \\ 0 & 0 & 0 & 0 & 0 & M \end{bmatrix} \tag{45}$$

$$\mathbf{B} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -b\sin\alpha & 0 & 0 \\ 1 & 0 & 0 & b\cos\alpha & 0 & 0 \\ 0 & -b\sin\alpha & b\cos\alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{46}$$

where $b = \sqrt{b_x^2 + b_y^2}$ and α is the angle made by the magnetic field vector with the flow direction. Analysing even this linear system leads to very complex algebra, and we present here only the results for the special case of aligned flow, where the flow direction and magnetic field direction are parallel ($\alpha = 0$). This simplifies the algebra considerably, but still seems to include all of the relevent physical regimes.

Domain of influence

We begin by considering a point disturbance in a stationary fluid; we take $M = 0$, and with no loss of generality at this stage, $\alpha = 0$. The wavespeeds satisfy

$$\mu^2(\theta)[\mu^4(\theta) - (1+b^2)\mu^2(\theta) + b^2\cos^2\theta] \tag{47}$$

The four roots of the quartic factor are magnetoacoustic waves. The two nonpropagating roots are the entropy wave, and a nonphysical wave that would transport the divergence of the magnetic field if such a divergence were present in the data. In this two-dimensional model there are no Alfén waves. They

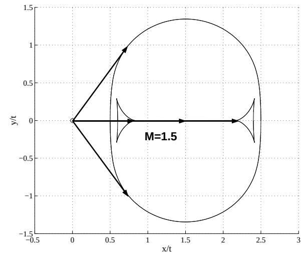

Fig. 3 Wave Diagram. $M = 1.5$ **and** $b = 0.9$

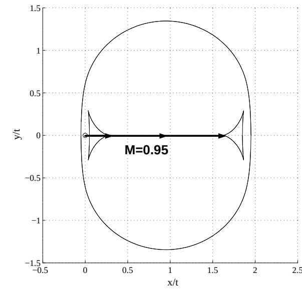

Fig. 4 Wave Diagram. $M = 0.95$ **and** $b = 0.9$

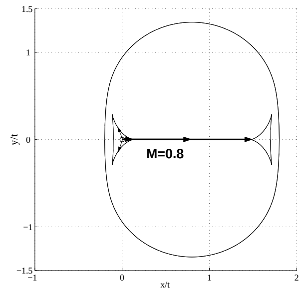

Fig. 5 Wave Diagram. $M = 0.8$ **and** $b = 0.9$

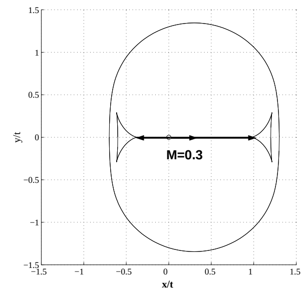

Fig. 6 Wave Diagram. $M = 0.3$ **and** $b = 0.9$

should not complicate the preconditioning because they are always hyperbolic.

It is worth making the simple observation that λ has the functional form

$$\lambda = \sqrt{b}\ \mathrm{fn}((b + 1/b), \theta) \tag{48}$$

It will turn out that the condition number also has a simple functional form, and this will allow a more economical representation of our results.

Finding the envelope of the magnetoacoustic waves is a tricky piece of algebra, but the result is given in many MHD texts. The envelope has three parts. There is an oval whose major axis is vertical, with speed $\sqrt{1+b^2}$ and whose minor (horizontal) axis is $\max(1, b)$. This is the envelope of the fast magnetoacoustic waves. There are two cuspoid shapes, extending between $b/\sqrt{1+b^2}$ and $\min(1, b)$ on the horizontal axis.

If the velocity of the fluid is (u, v) the diagram should be offset by that vector relative to the origin. Alternatively, we can keep the diagram fixed, and put a 'virtual origin' at $(-u, -v)$. At this stage we still have freedom to put $v \neq 0$, meaning that we are in cordinates oriented with the magnetic field, and the flow is not aligned. To find steady waves, we attempt to draw tangents to the envelope from the virtual origin. There are three cases to consider.

1. The virtual origin is outside the oval. Two tangents can be drawn to the oval, and one tangent to each cuspoid. The magnetoacoustic part of the flow is completely hyperbolic. (See Figure 3)

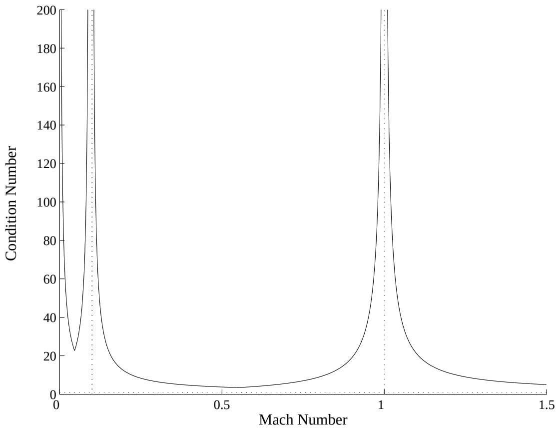

Fig. 7 Original MHD. $B_s = 0.1$

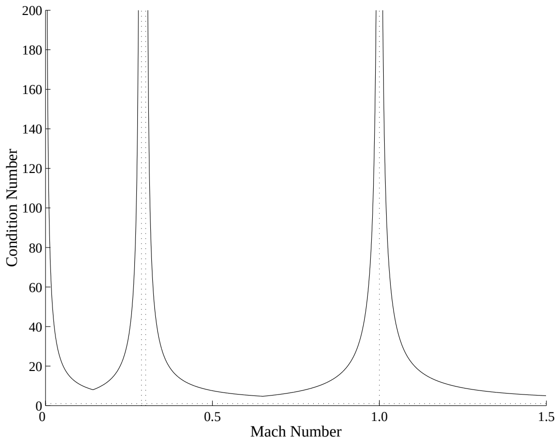

Fig. 8 Original MHD. $B_s = 0.3$

2. The virtual origin is inside the oval, but outside both cuspoids. No tangent can be drawn to the oval, but one tangent can still be drawn to each cuspoid. The magnetoacoustic part of the flow is mixed. (See Figures 4 and 6)

3. The virtual origin is inside one cuspoid. Three tangents can now be drawn to the cuspoid, another tangent to the other, and the magnetoacoustic part of the flow is again completely hyperbolic. (See Figure 5)

The condition number is the ratio between the largest and smallest distances from the virtual origin of any points on the envelope. This is almost impossible to determine analytically, but can be found numerically. We have done so for the special case of aligned flow where the virtual origin is at $x/t = -M, y/t = 0$. We need to remember that the actual origin is also a propagation path (for advected disturbances) and include it with the envelope. The condition number is singular in four cases, when $M = 0, b/\sqrt{1+b^2}, \min(b,1)$ and $\max(b,1)$, these being the four conditions for which the virtual origin crosses the envelope. No new singularities are introduced by having the flow non-aligned, because if the virtual origin lies off the horizontal axis,

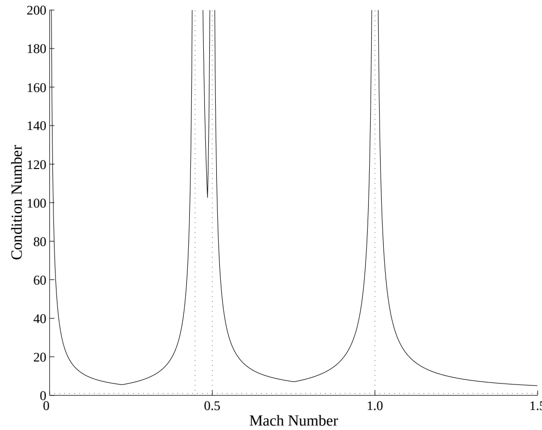

Fig. 9 Original MHD. $B_s = 0.5$

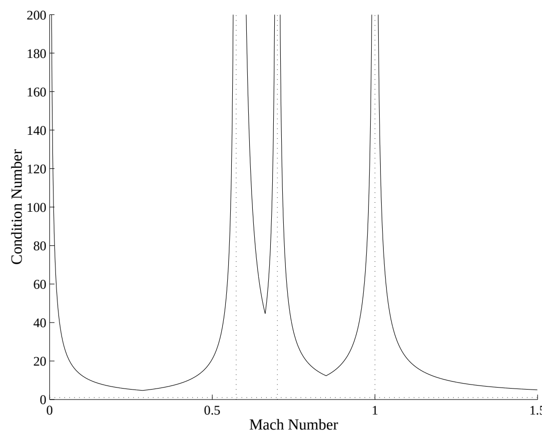

Fig. 10 Original MHD. $B_s = 0.7$.

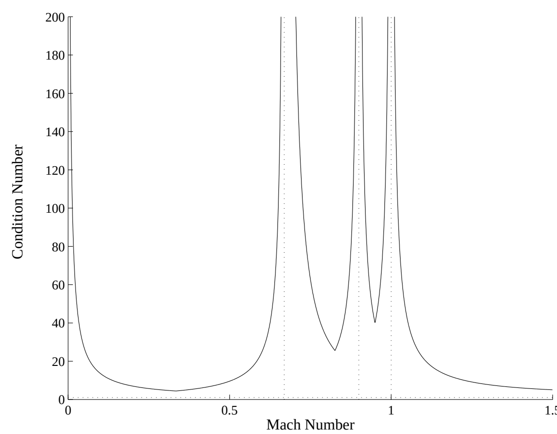

Fig. 11 Original MHD. $B_s = 0.9$.

there are no new crossings. In fact, the singularities introduced by crossing the cuspoids may disappear. The condition number for aligned flow is plotted, as a function of M, for various values of B, in Figures 7-11. As a result of the functional form for λ noted earlier, it can be shown that k depends only on the parameters $M/\sqrt{b}, (b + 1/b)$, and accordingly we present our results only for the cases $b < 1.0$

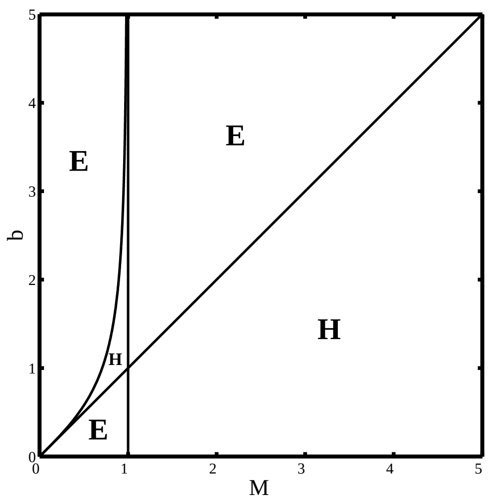

Fig. 12 Regions of hyp/ell for the magnetoacoustic waves. E: Elliptic, H: Hyperbolic. Borders are the lines $M = 1$ and $M = b$, and the curve $M = b/\sqrt{1+b^2}$.

Preconditioning the MHD equations

Again, we restrict attention to aligned flow. The first step is to identify the steady waves, and there are always at least four. The entropy and divergence waves propagate with speed M and their left eigenvectors are

$$\boldsymbol{\ell}_E = (0,0,0,0,0,1) \quad \text{on} \quad dy/dx = 0 \tag{49}$$
$$\boldsymbol{\ell}_D = (0,0,0,0,1,0) \quad \text{on} \quad dy/dx = 0 \tag{50}$$

The two slow waves propagate with speeds M and their left eigenvectors are

$$\boldsymbol{\ell}_{S+} = \left(\frac{-b}{\sqrt{1+b^2}}, \quad 1, 0, \frac{1}{\sqrt{1+b^2}}, 0, 0\right) \quad \text{on} \quad dy/dx = 0$$
$$\boldsymbol{\ell}_{S-} = \left(\frac{-b}{\sqrt{1+b^2}}, -1, 0, \frac{1}{\sqrt{1+b^2}}, 0, 0\right) \quad \text{on} \quad dy/dx = 0$$

The steady fast waves, if they exist, have slopes given by

$$\lambda^2 = \frac{M^2(1+b^2) - b^2}{(M^2-1)(M^2-b^2)} \tag{51}$$

Figure 12 shows, in an (M, b) plane, when these waves exist, so that the flow is purely hyperbolic, and when they are imaginary, so that the flow is partly elliptic. If they do exist, the left eigenvectors can be found as

$$\boldsymbol{\ell}_F^{\pm} = \left(M, -1, \pm\lambda\beta^2, \frac{\beta^2 b}{M}, -\frac{\lambda\beta^2 b}{M}, 0\right)$$

The hyperbolic preconditioner is

$$\begin{aligned} \mathbf{P}^{hyp} &= \boldsymbol{\ell}_E^T\boldsymbol{\ell}_E + \boldsymbol{\ell}_D^T\boldsymbol{\ell}_D + a_S(\boldsymbol{\ell}_{S-}^T\boldsymbol{\ell}_{S-} - \boldsymbol{\ell}_{S+}^T\boldsymbol{\ell}_{S+}) \\ &\quad + a_F(\boldsymbol{\ell}_{F+}^T\boldsymbol{\ell}_{F+} + \boldsymbol{\ell}_{F-}^T\boldsymbol{\ell}_{F-}) \end{aligned} \tag{52}$$

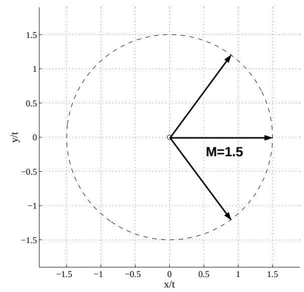

Fig. 13 Wave Diagram after preconditioning. $M = 1.5$ and $b = 0.9$

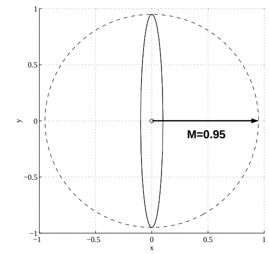

Fig. 14 Wave Diagram after preconditioning. $M = 0.95$ and $b = 0.9$

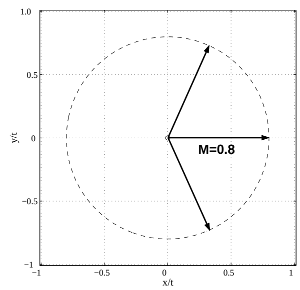

Fig. 15 Wave Diagram after preconditioning. $M = 0.8$ and $b = 0.9$

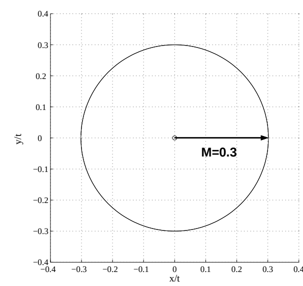

Fig. 16 Wave Diagram after preconditioning. $M = 0.3$ and $b = 0.9$

The fastest propagation speeds in every subsytem will be equal under the choices

$$a_S = \frac{1}{2}\frac{M(1+b^2)}{b\sqrt{1+b^2} - M - b^2} \tag{53}$$
$$a_F = \frac{1}{2}\frac{\sqrt{(M^2-1)(M^2-b^2)}}{(M^2-1)\{M^2(1+b^2)-b^2\}} \tag{54}$$

The preconditioner in the elliptic regions is given by

$$\begin{aligned} \mathbf{P}^{ell} &= \boldsymbol{\ell}_E^T\boldsymbol{\ell}_E + \boldsymbol{\ell}_D^T\boldsymbol{\ell}_D + a_{S+}\boldsymbol{\ell}_{S+}^T\boldsymbol{\ell}_{S+} + a_{S-}\boldsymbol{\ell}_{S-}^T\boldsymbol{\ell}_{S-} \\ &\quad + a_F(\boldsymbol{\ell}_{F+}^T\boldsymbol{\ell}_{F+} + \boldsymbol{\ell}_{F-}^T\boldsymbol{\ell}_{F-}) \end{aligned} \tag{55}$$

where

$$a_F = \frac{M^2\sqrt{(M^2-1)(M^2-b^2)}}{(M^2-1)\{M^2(1+b^2)-b^2\}^{3/2}}. \tag{56}$$

After some algebra, we can write

$$\mathbf{P} = \mathbf{P}_1 + a_2\mathbf{P}_2 \tag{57}$$

where with the notation $K = M^2(1+b^2) - b^2$

$$\mathbf{P}_1 = \begin{bmatrix} \frac{b^2M^2}{K} & -\frac{b^2M}{K} & 0 & -\frac{bM^2}{K} & 0 & 0 \\ -\frac{b^2M}{K} & -M^2(1+b^2) & 0 & \frac{bM}{K} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{bM^2}{K} & \frac{bM}{K} & 0 & \frac{M^2}{K} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

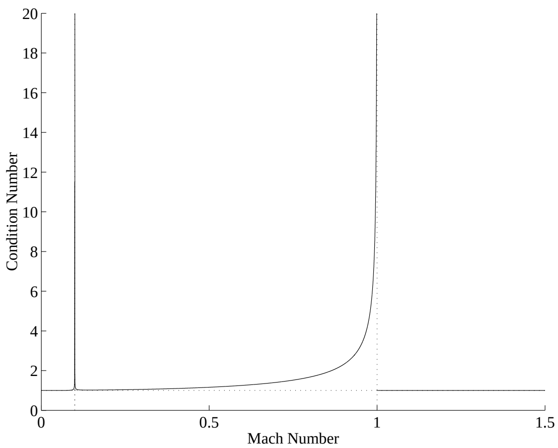

Fig. 17 Preconditioned MHD. $B_s = 0.1$

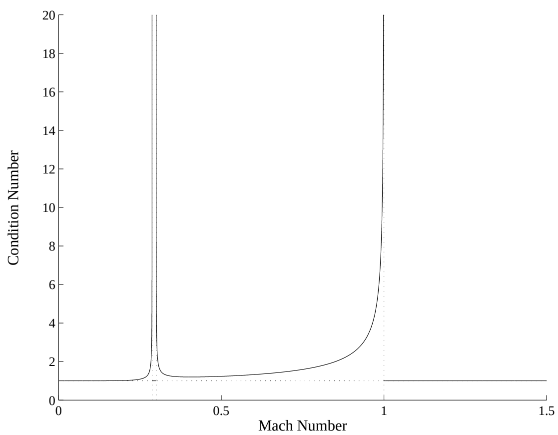

Fig. 18 Preconditioned MHD. $B_s = 0.3$

and

$$\mathbf{P}_2 = \begin{bmatrix} M^2 & -M & 0 & \beta^2 b & 0 & 0 \\ -M & 1 & 0 & -\frac{\beta^2 b}{M} & 0 & 0 \\ 0 & 0 & \lambda^2\beta^4 & 0 & \frac{\lambda^2\beta^4 b}{M} & 0 \\ b\beta^2 & -\frac{b\beta^2}{M} & 0 & \frac{\beta^4 b^2}{M^2} & 0 & 0 \\ 0 & 0 & \frac{\lambda^2\beta^4 b}{M} & 0 & \frac{\lambda^2\beta^4 b^2}{M^2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

with

$$a_2 = \frac{\sqrt{(M^2-1)(M^2-b^2)}}{(M^2-1)K} \tag{58}$$

in the hyperbolic case or

$$a_2 = \frac{M^2\sqrt{(M^2-1)(M^2-b^2)}}{(M^2-1)K^{3/2}} \tag{59}$$

in the elliptic case *.

The condition number for the preconditioned equations is easy to find. If $\lambda^2 > 0$ the system is purely hyperbolic and the condition number is unity. Otherwise the condition number is

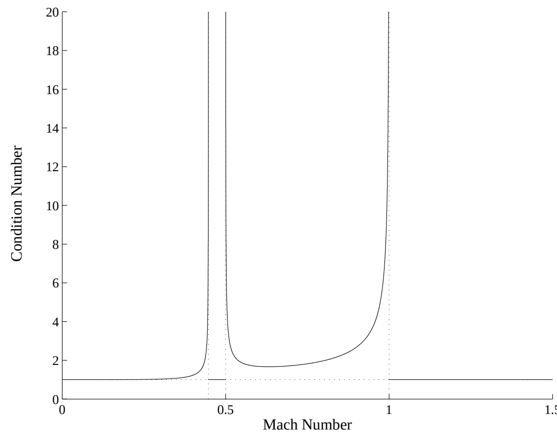

Fig. 19 Preconditioned MHD. $B_s = 0.5$

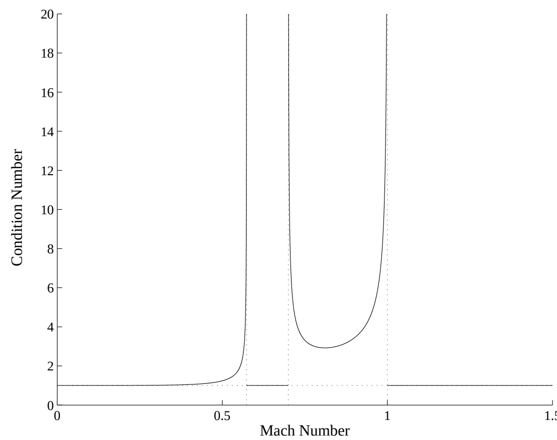

Fig. 20 Preconditioned MHD. $B_s = 0.7$.

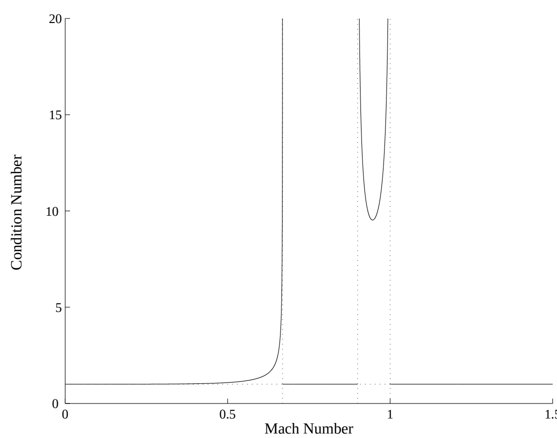

Fig. 21 Preconditioned MHD. $B_s = 0.9$.

$$k = \sqrt{\max(-\lambda^2, -\lambda^{-2})} \tag{60}$$

where $-\lambda^2$ can be written, from (51), in the form

$$-\lambda^2 = \frac{\frac{M^2}{b}\left(b + \frac{1}{b}\right) - 1}{1 + \frac{M^4}{b^2} - \frac{M^2}{b}\left(b + \frac{1}{b}\right)} \tag{61}$$

*$(M^2-1)(M^2-b^2)$ is always positive in the hyperbolic case; it can be negative in the elliptic case but then K is also negative, and therefore a_2 is real. See Figure 12.

This is plotted on Figs 17-21 (Wave diagrams are also given in Figs 13-16 to be compared with Figs 3-6). The condition number is considerably reduced everywhere. There are still singularities, but they are much weaker (note the considerable difference in vertical scales), and the singularity at the origin has been eliminated. This makes it probable that we can obtain fast solutions to flows that are slow compared with both the acoustic and Alfénic speeds. By analogy with the Euler equations we may expect that these solutions will be accurate also. We expect also that preconditioning the full MHD equations can be done with a matrix as simple as (57) by an appropriate low-speed approximation, and the work is currently underway.

Acknowledgement

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References

[1] D. Darmofal, P. Schmid, The Importance of Eigenvectors for Local Preconditioning of the Euelr Equations, *AIAA paper 95-1655,* June 1995.

[2] H. Guillard, C. Voizat, On the behaviour of upwind schemes in the low Mach number limit, *Compters and Fluids,* 28, 1999.

[3] H. Nishikawa, B. van Leer, Optimal Multigrid convergence by elliptic/hyperbolic splitting *AIAA paper 02-2951*

[4] H. Nishikawa, M. Rad, P. L, Roe, A third-order fluctuation-splitting scheme that preserves potential flow, *AIAA paper 01-2595,* 15th CFD Conference, Anaheim, CA, 2001,

[5] B. van Leer, W-T.Lee, P.L. Roe, Characteristic time-stepping, or, local preconditioning of the Euler equations, AIAA paper 91-1552, AIAA 10th Computational Fluid Dynamics Conference, Hawaii, 1991.

[6] M. Rad, A residual distribution approach to the Euler equations that preserves potential flow. *Ph.D. Thesis*, Aerospace Engineering, University of Michigan, 2001.

[7] E. Turkel, Preconditioning methods for solving the incompressible and low-speed compressible equations, *J. Comut. Phys.*, **72**, 1987.

[8] J. M. Weiss, W.A. Smith, Preconditioning applied to variable and constant density flows, *AIAA J.*, **33**, 1995.