AIAA/AAS Astrodynamics Specialist Conference and Exhibit
5-8 August 2002, Monterey, California

AIAA 2002-4639

FORMATION FLIGHT WITH GENERATING FUNCTIONS: SOLVING THE RELATIVE BOUNDARY VALUE PROBLEM*

V.M. Guibout† D.J.Scheeres ‡

ABSTRACT

The relative motion of a formation of spacecraft is studied. Using the Hamiltonian description $\mathbf{q}, \mathbf{p}, H(\mathbf{q}, \mathbf{p}, t)$ and solving for the generating functions for canonical transformations allows us to describe motion in the vicinity of a nominal trajectory as the solution to a two-point boundary value problem. We put some properties of such motion into relief before identifying an algorithm to solve this problem. The technique we are developing is suitable for use in designing and optimizing the non-linear relative motion of a formation of spacecraft within a very general orbital setting.

Introduction

We are developing concepts that allow us to describe the relative non-linear motion of a formation of spacecraft as the solution to a two-point boundary value problem. By doing so, we can specify the initial and final positions (or velocities) and immediately know what the corresponding velocities (or positions) of an arbitrarily large collection of spacecraft must be. Having such descriptions developed analytically will allow for the efficient design and optimization of relative motion for large collections of spacecraft flying in formation. The technique we are developing is suitable for use in very general orbital settings, and essentially consists of finding a non-linear solution to the Hamilton-Jacobi partial differential equation for the generating functions of canonical transformations.

Our approach is distinct from the classical use of the generating functions as we are using them to solve the boundary value problem, whereas in previous applications they were used to solve the initial value problem instead.[3]

Hamilton's principle and Classical Dynamics

The Hamiltonian

A holonomic system can be described by the standard form of the Lagrange Equations:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0 \tag{1}$$

where the q_i's are generalized coordinates and $i = 1, 2, \cdots, n$. We define the generalized momentum conjugate to q_i as:

$$p_i = \frac{\partial L}{\partial \dot{q}_i} \tag{2}$$

Equation 1 becomes:

$$\dot{p}_i = \frac{\partial L}{\partial q_i} \tag{3}$$

Now define the Hamiltonian function as:

$$H(q, p, t) = \sum_{i=1}^{n} p_i \dot{q}_i - L(q, \dot{q}, t) \tag{4}$$

Then one can easily verify from Eq. 4 that:

$$\begin{cases} \dot{q}_i &= \dfrac{\partial H}{\partial p_i} \\ \dot{p}_i &= -\dfrac{\partial H}{\partial q_i} \end{cases} \tag{5}$$

and

$$\frac{\partial L}{\partial t} = -\frac{\partial H}{\partial t} \tag{6}$$

*Paper AIAA 2002-4639.

†Graduate Research Assistant, PhD Candidate, Aerospace Engineering Department, The University of Michigan, guibout@umich.edu

‡Associate Professor, Senior member AIAA, Aerospace Engineering Department, The University of Michigan, scheeres@umich.edu

These $2n$ first-order equations given in Eq. 5 are known as Hamilton's canonical equations of motion and provide a full description of the motion.

In our applications we are considering holonomic systems. Nevertheless, one can also derive a set of canonical equations of motion for a non-conservative system, where the generalized forces are not all derivable from a potential function.

Canonical transformation

Now consider transformations in phase space that preserve the Hamiltonian form of the equations of motion in the new variables. Such transformations are called canonical transformations. Let us consider a transformation from one set of coordinates (q_i, p_i) to a new set (Q_i, P_i) defined by equations of the form:

$$\begin{aligned} Q_i &= Q_i(q,p,t) && (7)\\ P_i &= P_i(q,p,t) && (8) \end{aligned}$$

Such transformations are often referred to as contact transformations in the literature.

Since we require the Hamiltonian form of the equations of motion in the new variables to be preserved, the following equations must be satisfied:

$$\begin{cases} \dot{Q}_i = \dfrac{\partial K}{\partial P_i} \\ \dot{P}_i = -\dfrac{\partial K}{\partial Q_i} \end{cases} \qquad (9)$$

where $K = K(Q,P,t)$ is the Hamiltonian of the system in the new set of coordinates. To link K and H, one must recall Hamilton's principle,[2] which can be stated as:

$$\delta I = \delta \int_{t_0}^{t_1} L dt = 0 \qquad (10)$$

where L is the Lagrangian of the system. Equation 10 can be expanded using the Hamiltonian in both set of coordinates as:

$$\delta \int_{t_0}^{t_1} \left(\sum_i p_i \dot{q}_i - H(q,p,t) \right) dt = 0 \qquad (11)$$

$$\delta \int_{t_0}^{t_1} \left(\sum_i P_i \dot{Q}_i - K(Q,P,t) \right) dt = 0 \qquad (12)$$

This means that the integrands of the two integrals differ at most by a total time derivative of an arbitrary function F. Such a function is called a generating function of the transformation and must be a function of both the old and the new variables. But since the two sets of coordinates are connected by the $2n$ transformations in Eqs. 7 and 8, F depends only on n old coordinates and n new coordinates.

Therefore the generating functions can be written as one of the functions:

$$F_1(q,Q,t), F_2(q,P,t), F_3(p,Q,t), F_4(p,P,t) \qquad (13)$$

and must satisfy a relation of the form:

$$\sum_i p_i \dot{q}_i - H = \sum_j P_j \dot{Q}_j - K + \frac{d}{dt} F_i \qquad (14)$$

For example, expanding the total time derivative of F_1 yields:

$$\frac{d}{dt} F_1(q,Q,t) = \sum_i \frac{\partial F_1}{\partial q_i} \dot{q}_i + \sum_i \frac{\partial F_1}{\partial Q_i} \dot{Q}_i + \frac{\partial F_1}{\partial t} \qquad (15)$$

We now assume the new and old coordinates q and Q to be independent and rewrite Eq. 14 in the following form:

$$\begin{aligned} p_i &= \frac{\partial F_1}{\partial q_i} \\ P_i &= -\frac{\partial F_1}{\partial Q_i} \\ K &= H + \frac{\partial F_1}{\partial t} && (16) \end{aligned}$$

If the independent variables are actually q and P instead of q and Q, then the generating function is of the type F_2. Using a Legendre transformation, we find F_2 as a function of F_1:

$$F_2(q,P,t) = F_1(q,Q,t) + \sum_i P_i Q_i \qquad (17)$$

and then Eq. 14 becomes for F_2:

$$\sum_i p_i \dot{q}_i - H = -\sum_j Q_j \dot{P}_j - K + \frac{d}{dt} F_2 \qquad (18)$$

which yields:

$$\begin{aligned} p_i &= \frac{\partial F_2}{\partial q_i} \\ Q_i &= \frac{\partial F_2}{\partial P_i} \\ K &= H + \frac{\partial F_2}{\partial t} && (19) \end{aligned}$$

The same reasoning can be applied for F_3 and F_4 if the independent variables are (p,Q) or (p,P) and

we find:

$$\begin{aligned} q_i &= -\frac{\partial F_3}{\partial p_i} \\ P_i &= -\frac{\partial F_3}{\partial Q_i} \\ K &= H + \frac{\partial F_3}{\partial t} \end{aligned} \tag{20}$$

$$\begin{aligned} q_i &= -\frac{\partial F_4}{\partial p_i} \\ Q_i &= +\frac{\partial F_4}{\partial P_i} \\ K &= H + \frac{\partial F_4}{\partial t} \end{aligned} \tag{21}$$

Thus, we have shown that there exist four main types of canonical transformations which allow us to switch from one coordinate set to another. Moreover, these canonical transformations must satisfy a set of equations of the form $K = H + \frac{\partial F_i}{\partial t}$.

The Hamilton-Jacobi equation

One can choose the new variables to be constants of motion by requiring that the new Hamiltonian K be identically zero. Indeed, Hamilton's canonical equations of motion then become:

$$\frac{\partial K}{\partial P_i} = \dot{Q}_i = 0 \tag{22}$$

$$-\frac{\partial K}{\partial Q_i} = \dot{P}_i = 0 \tag{23}$$

and the new and old Hamiltonian are now related by:

$$\frac{\partial F_i}{\partial t} + H(q, p, t) = 0 \tag{24}$$

For $i = 1$, we find

$$\frac{\partial F_1}{\partial t} + H(q, \frac{\partial F_1}{\partial q}, t) = 0 \tag{25}$$

Eq. 25 is often referred to as the Hamilton-Jacobi equation.[2]

One can also expand Eq. 24 for $i = 2, 3, 4$:

$$\frac{\partial F_2}{\partial t} + H(q, \frac{\partial F_2}{\partial q}, t) = 0 \tag{26}$$

$$\frac{\partial F_3}{\partial t} + H(-\frac{\partial F_3}{\partial p}, p, t) = 0 \tag{27}$$

$$\frac{\partial F_4}{\partial t} + H(-\frac{\partial F_4}{\partial p}, p, t) = 0 \tag{28}$$

It appears that F_1 (F_3) and F_2 (F_4) verify the same differential equation. Nevertheless, they differ in their initial conditions. From Eqns. 16 and 21 we deduce that F_1 and F_4 are undefined at time $t = 0$, whereas Eqns. 19 and 20 give well-defined initial conditions for F_2 and F_3. Indeed, at the initial time the generating functions must define the identity transformation. F_2 and F_3 can generate such a transformation whereas F_1 and F_4 cannot.

Given the Hamiltonian of a system, one is able to completely solve for F_2 and F_3 starting from an initial time using the Hamilton-Jacobi equation. F_1 and F_4 can only be solved if they are known at some epoch $t \neq 0$.

In our application of the Hamilton-Jacobi equation we take the initial conditions of the trajectory to be the constants of the motion, and use the Hamilton-Jacobi equation to develop a local solution for the motion as a function of the initial conditions.

Resolution of the boundary value problem using the Hamilton-Jacobi equation

In our study, we investigate the Hamilton-Jacobi equation in order to solve the boundary value problem: Given n initial conditions and n conditions at $t = t_f$, solve for the n remaining initial conditions and the n remaining conditions at $t = t_f$. For instance, given an initial position and a final position, what velocities should the spacecraft have at $t = t_0$ and at $t = t_f$ to effectively reach the specified position. Such a problem is called the targeting problem and corresponds to finding the $F_1(q, Q)$ function between time t_0 and t_f.

The boundary value problem for a one-dimensional harmonic oscillator

We will first explore a simple example, which consists of solving the boundary value problem for a one-dimensional harmonic oscillator. Given an initial momentum P and final position q, we solve for the initial position Q and final momentum p. For such a problem, it is clear that F_2 is the relevant generating function.

The Hamiltonian is :

$$H(q, p) = \frac{1}{2m}p^2 + \frac{k}{2}q^2 \tag{29}$$

Thus, one finds F_2 as a solution of Eq. 26:

$$\frac{\partial F_2}{\partial t} + \frac{1}{2m}\frac{\partial F_2}{\partial q}^2 + \frac{k}{2}q^2 = 0 \tag{30}$$

Since H is a second order polynomial in q and p, it is reasonable to look for F_2 as a polynomial in q and P:

$$F_2(q,P,t) = \sum_{i,j} a_{i,j}(t) q^i P^j \tag{31}$$

Plugging Eq. 31 into Eq. 30 and solving for each coefficient of F_2 using the following initial conditions[1]: $a_{11}(0) = 1$, $a_{i,j}(0) = 0$ for all $(i,j) \neq (1,1)$ yields:

$$\begin{aligned} F_2 = &-\frac{1}{2}\sqrt{km}\tan(\sqrt{\frac{k}{m}}t)q^2 + \sec(\sqrt{\frac{k}{m}}t)qP \\ &- \frac{1}{2}\frac{1}{\sqrt{km}}\tan(\sqrt{\frac{k}{m}}t)P^2 \end{aligned} \tag{32}$$

Then, Eqs. 19 allow us to solve the boundary problem:

$$\begin{aligned} p &= \frac{\partial F_2}{\partial q} \\ &= -\sqrt{km}\tan(\sqrt{\frac{k}{m}}t)q + \sec(\sqrt{\frac{k}{m}}t)P \\ Q &= \frac{\partial F_2}{\partial P} \\ &= \sec(\sqrt{\frac{k}{m}}t)q - \frac{1}{\sqrt{km}}\tan(\sqrt{\frac{k}{m}}t)P \end{aligned} \tag{33}$$

To solve the boundary value problem in position (given initial and final positions, find the initial and final velocities), we use the Legendre transformation defined in Eq. 17 to get F_1 knowing F_2. First, using Eq.33 we find P as a function of (Q,q):

$$P = \sqrt{km}\left(\sec(\sqrt{\frac{k}{m}}t)\cot(\sqrt{\frac{k}{m}}t)q - \cot(\sqrt{\frac{k}{m}}t)Q\right) \tag{34}$$

then replacing P in Eq. 17 yields:

$$\begin{aligned} F_1 = &\frac{1}{2}\sqrt{km}\csc(\sqrt{\frac{k}{m}}t) \\ &\left(-2qQ + (q^2+Q^2)\cos(\sqrt{\frac{k}{m}}t)\right) \end{aligned} \tag{35}$$

We notice that F_1 is not defined at time $t = 0$.

[1] at $t = 0$ F_2 is the identity transformation $F_2(q,P,0) = \sum_i q_i P_i$

The boundary value problem for a formation of spacecraft

Formulation of the problem in terms of Hamiltonian equations

We have seen in the previous section how to solve the boundary value problem using canonical transformation theory. Let us now apply this method to a more interesting case: a collection of spacecraft.

Given a Hamiltonian system, such as a spacecraft moving under gravitational forces, and some initial conditions (Q_0, P_0), we can solve the Hamilton equations using a numerical integration scheme (at least):

$$\dot{X}_0 = J\nabla H_0 \tag{36}$$

where $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$, $X_0 = \begin{pmatrix} q(Q_0,P_0,t) \\ p(Q_0,P_0,t) \end{pmatrix}$, and $\nabla H_0 = \begin{pmatrix} \frac{\partial H}{\partial q} \\ \frac{\partial H}{\partial p} \end{pmatrix}(q(Q_0,P_0,t), p(Q_0,P_0,t), t)$

Suppose different initial conditions (Q_1, P_1) are chosen such that:

$$Q_1 = Q_0 + \Delta Q \tag{37}$$
$$P_1 = P_0 + \Delta P \tag{38}$$

Then equation 36 becomes:

$$\dot{X}_1 = J\nabla H_1 \tag{39}$$

where p and q are now functions of (Q_1, P_1, t). Define two parameters Δq and Δp such that:

$$\begin{aligned} q_1 &= q_0 + \Delta q \\ p_1 &= p_0 + \Delta p \end{aligned} \tag{40}$$

where

$$q_1 = q(Q_1,P_1,t) \qquad p_1 = p(Q_1,P_1,t) \tag{41}$$
$$q_0 = q(Q_0,P_0,t) \qquad p_0 = p(Q_0,P_0,t) \tag{42}$$

Equation 39 becomes:

$$\dot{X}_0 + \dot{X}_h = J\nabla H_1 \tag{43}$$

where $X_h = \begin{pmatrix} \Delta q \\ \Delta p \end{pmatrix}$

Using 40, we can expand the right hand side of Eq. 43 assuming $(\Delta q, \Delta p)$ small enough for convergence of the serie:

$$\begin{aligned} &\nabla H(q_1,p_1,t) = \nabla H(q_0,p_0,t) \\ &+ \begin{pmatrix} \frac{\partial^2 H}{\partial q^2}(q_0,p_0,t)\Delta q + \frac{\partial^2 H}{\partial q \partial p}(q_0,p_0,t)\Delta p \\ \frac{\partial^2 H}{\partial q \partial p}(q_0,p_0,t)\Delta q + \frac{\partial^2 H}{\partial p^2}(q_0,p_0,t)\Delta p \end{pmatrix} \\ &+ \cdots \end{aligned} \tag{44}$$

Replacing into Eq. 43 yields

$$\dot{X}_0 + \dot{X}_h = J\nabla H_0 + J\begin{pmatrix} \frac{\partial^2 H}{\partial q^2}(q_0,p_0,t)\Delta q + \frac{\partial^2 H}{\partial q \partial p}(q_0,p_0,t)\Delta p \\ \frac{\partial^2 H}{\partial q \partial p}(q_0,p_0,t)\Delta q + \frac{\partial^2 H}{\partial p^2}(q_0,p_0,t)\Delta p \end{pmatrix} + \cdots \tag{45}$$

which simplifies using Equation 36 and ignoring higher order terms:

$$\dot{X}_h = J\begin{pmatrix} \frac{\partial^2 H}{\partial q^2}(q_0,p_0,t)\Delta q + \frac{\partial^2 H}{\partial q \partial p}(q_0,p_0,t)\Delta p \\ \frac{\partial^2 H}{\partial q \partial p}(q_0,p_0,t)\Delta q + \frac{\partial^2 H}{\partial p^2}(q_0,p_0,t)\Delta p \end{pmatrix} \tag{46}$$

This is a Hamiltonian system if and only if the right hand side can be written as the partial of a function H_h with respect to X_h. Let

$$H_h = \frac{1}{2} X_h \begin{pmatrix} \frac{\partial^2 H}{\partial q^2}(q_0,p_0,t) & \frac{\partial^2 H}{\partial q \partial p}(q_0,p_0,t) \\ \frac{\partial^2 H}{\partial q \partial p}(q_0,p_0,t) & \frac{\partial^2 H}{\partial q^2}(q_0,p_0,t) \end{pmatrix} X_h \tag{47}$$

We can check that:

$$\dot{X}_h = J\nabla H_h \tag{48}$$

The expansion of the right hand side of Eq. 43 at higher order leads to:

$$H_h = \sum_{\substack{i,j=0 \\ i+j\geq 2}}^{n} \frac{\Delta q^i \Delta p^j}{i!j!} \frac{\partial^{i+j} H}{\partial q^i \partial p^j}(q_0,p_0,t) \tag{49}$$

We conclude that the Hamiltonian property of the system is preserved by a Taylor series expansion. We are then free to use generating functions to solve the boundary problem[2].

The Hamilton Jacobi equations

Since we have complete knowledge of $H = H_h$ as a function of time, we can solve for the generating functions F_1, F_2, F_3 and F_4 from equations 25- 28 (as mentioned earlier, we are missing initial conditions at $t = 0$ to solve for F_1 and F_4).

It is now reasonable to assume that F_i is analytic. Then when solving for F_i, we can use an expansion of F_i of the same order as the expansion of H_h

[2]One must be aware that we are solving the boundary problem in terms of the relative motion and not absolute motion.

First we note that at the first order, F_i is equal to zero. Indeed, suppose for instance that $F_2 = a(t)\Delta q + b(t)\Delta P$, then $\Delta q = \frac{\partial F_2}{\partial \Delta p} = a(t)$ and $\Delta Q = b(t)$. But $(\Delta q, \Delta p, \Delta Q, \Delta P)$ are supposed to be of the first order at least. We conclude that $a(t) = b(t) = 0$.

At order 2 F_i is then a quadratic form in its two variables, without any linear term, and H_h satisfies Eq 47.

Define H_h and F_2 using block matrices:

$$H_h = \frac{1}{2} X_h^T \begin{pmatrix} H_{qq} & H_{qp} \\ H_{pq} & H_{pp} \end{pmatrix} X_h \tag{50}$$

and

$$F_2 = \frac{1}{2} Y^T \begin{pmatrix} F_{qq} & F_{qp} \\ F_{pq} & F_{pp} \end{pmatrix} Y \tag{51}$$

where $Y = \begin{pmatrix} \Delta q \\ \Delta P \end{pmatrix}$ and the matrix defining H_h and F_2 are both symmetric. Then, using Eqs 19:

$$\begin{aligned} \Delta p &= \frac{\partial F_2}{\partial \Delta q} \\ &= \begin{pmatrix} F_{qq} & F_{qp} \end{pmatrix} Y \end{aligned} \tag{52}$$

Substituting into Eq. 26 yields:

$$0 = Y^T \left\{ \begin{pmatrix} \dot{F}_{qq} & \dot{F}_{qp} \\ \dot{F}_{qp}^T & \dot{F}_{pp} \end{pmatrix} + \frac{1}{2} \begin{pmatrix} I & F_{qq}^T \\ 0 & F_{qp}^T \end{pmatrix} \begin{pmatrix} H_{qq} & H_{qp} \\ H_{pq} & H_{pp} \end{pmatrix} \begin{pmatrix} I & 0 \\ F_{qq} & F_{qp} \end{pmatrix} \right\} Y \tag{53}$$

This equation is equivalent to the following 4 matrix equations:

$$\begin{aligned} &\dot{F}_{qq} + \frac{1}{2}(H_{qq} + H_{qp}F_{qq} + F_{qq}H_{pq} + F_{qq}H_{pp}F_{qq}) = 0 \\ &\dot{F}_{qp} + \frac{1}{2}(H_{qp}F_{qp} + F_{qq}H_{pp}F_{qp}) = 0 \\ &\dot{F}_{pq} + \frac{1}{2}(F_{pq}H_{pq} + F_{pq}H_{pp}F_{qq}) = 0 \\ &\dot{F}_{pp} + \frac{1}{2}(F_{pq}H_{pp}F_{qp}) = 0 \end{aligned} \tag{54}$$

where we recall that $F_{pq} = F_{qp}^T$ These equations allow us to solve for either F_2 or F_1 at the second order. A similar set of equations can be derived for

F_3 and F_4:

$$\begin{aligned}
&\dot{F}_{qq} + \frac{1}{2}(H_{pp} - H_{pq}F_{qq} - F_{qq}H_{qp} \\
&\qquad\qquad + F_{qq}H_{qq}F_{qq}) = 0 \\
&\dot{F}_{qp} + \frac{1}{2}(-H_{pq}F_{qp} + F_{qq}H_{qq}F_{qp}) = 0 \\
&\dot{F}_{pq} + \frac{1}{2}(-F_{pq}H_{qp} + F_{pq}H_{qq}F_{qq}) = 0 \\
&\dot{F}_{pp} + \frac{1}{2}(F_{pq}H_{qq}F_{qp}) = 0
\end{aligned} \tag{55}$$

Note that the first of Eqs 54 and 55 are Ricatti equations, the second and third are non-homogeneous, time varying, linear equations and are equivalent to each other, and the last are quadrature.

If we want to generalize this method to higher order, tensor notation is required. Following are the equations derived for F_2 for a system of dimension n.

The Taylor expansion is now written as:

$$f(x) = f^0 + f^1\cdot\vec{x} + (f^2\cdot\vec{x})\cdot\vec{x} + ((f^3\cdot\vec{x})\cdot\vec{x})\cdot\vec{x} + \cdots \tag{56}$$

Applying this formula to $H(\vec{x})$ and to the canonical transformation $F_2 = F(\vec{y})$ yields:

$$H(\vec{x}) = h_{i,j}x_ix_j + h_{i,j,k}x_ix_jx_k + \cdots \tag{57}$$

$$F(\vec{y}) = f_{i,j}y_iy_j + f_{i,j,k}y_iy_jy_k + \cdots \tag{58}$$

where we assume the summation convention. Let us now express $\vec{x} = (\Delta q, \Delta p)$ as a function of $\vec{y} = (\Delta q, \Delta P)$.

For all $a \le n$ and $j = n + a$

$$x_a = y_a \tag{59}$$

$$\begin{aligned}
x_j &= \frac{\partial F}{\partial y_a} \\
&= f_{a,k}y_k + f_{k,a}y_k \\
&+ f_{a,k,l}y_ky_l + f_{k,a,l}y_ky_l + f_{k,l,a}y_ky_l \\
&+ \cdots
\end{aligned} \tag{60, 61}$$

where n is the dimension of the space.

We write the Hamilton-Jacobi equation:

$$\frac{\partial F}{\partial t} + H = 0 \tag{62}$$

$$\begin{aligned}
&\dot{f}_{i,j}y_iy_j + \dot{f}_{i,j,k}y_iy_jy_k \\
&\quad + h_{i,j}x_ix_j + h_{i,j,k}x_ix_jx_k + \cdots = 0
\end{aligned} \tag{63}$$

Replacing $\vec{x}$ by $\vec{y}$ in equation 63 using 61, and keeping only terms of order less than 3 yields:

$$\begin{aligned}
0 = &\dot{f}_{i,j}y_iy_j + \dot{f}_{i,j,k}y_iy_jy_k \\
&+ h_{a,b}y_ay_b + h_{a,b,c}y_ay_by_c \\
&+ (h_{a,n+b} + h_{n+b,a})y_a(f_{b,k}y_k + f_{k,b}y_k \\
&\quad + f_{b,k,l}y_ky_l + f_{l,b,k}y_ky_l + f_{k,l,b}y_ky_l) \\
&+ h_{n+a,n+b}(f_{a,k}y_k + f_{k,a}y_k \\
&\quad + f_{a,k,l}y_ky_l + f_{l,a,k}y_ky_l + f_{k,l,a}y_ky_l) \\
&\quad (f_{b,m}y_m + f_{m,b}y_m + f_{b,m,p}y_my_p \\
&\qquad + f_{p,b,m}y_my_p + f_{m,p,b}y_my_p) \\
&+ (h_{n+a,b,c} + h_{c,n+a,b} + h_{b,c,n+a})y_by_c \\
&\quad (f_{a,k}y_k + f_{k,a}y_k) \\
&+ (h_{n+a,n+b,c} + h_{n+b,c,n+a} + h_{c,n+a,n+b}) \\
&y_c(f_{a,k}y_k + f_{k,a}y_k)(f_{b,l}y_l + f_{l,b}y_l) \\
&+ h_{n+a,n+b,n+c}(f_{a,k}y_k + f_{k,a}y_k) \\
&(f_{b,l}y_l + f_{l,b}y_l)(f_{c,m}y_m + f_{m,c}y_m)
\end{aligned} \tag{64}$$

We notice that the equations of order 2 are the same as the ones found previously. The equations of order 3 are given explicitly in the appendix. The process of deriving equations for F can be continued to arbitrarily high order using a symbolic manipulation program (we have implemented and solved the expansion to order 8 using Mathematica).

The initial value problem

The state-transition matrix

Another approach to the study of relative motion of spacecraft is to use the State Transition Matrix. This method is developed in[1] for the case of a spacecraft moving in a point mass gravity field. R. Battin derives differential equations which allow one to compute the state transition matrix $\Phi(t, t_0)$. Then we have:

$$\begin{pmatrix} \Delta q \\ \Delta p \end{pmatrix} = \Phi \begin{pmatrix} \Delta Q \\ \Delta P \end{pmatrix} \tag{65}$$

Thus, inverting some blocks of the Φ matrix allows us to solve the boundary value problem. For instance, if $\Phi = \begin{pmatrix} \Phi_{rr} & \Phi_{rv} \\ \Phi_{vr} & \Phi_{vv} \end{pmatrix}$, then,

$$\begin{aligned}
\Delta p &= \Phi_{vr}\Phi_{rr}^{-1}\Delta q + (\Phi_{vv} - \Phi_{vr}\Phi_{rr}^{-1}\Phi_{rv})\Delta P \\
\Delta Q &= \Phi_{rr}^{-1}\Delta q - \Phi_{rr}^{-1}\Phi_{rv}\Delta P
\end{aligned} \tag{66}$$

This method assumes we have computed the state transition matrix Φ with sufficient accuracy to invert

it. Moreover, it only provides an approximation of order 2 for the solution of the boundary problem.

Battin[1] derived some interesting properties of this boundary value problem, in the next section, we will prove that these properties are actually specific to all Hamiltonian systems.

Some properties of the boundary value problem

In,[1] Battin shows that one can define the matrices C and $\tilde{C}$ as:

$$\begin{aligned} \tilde{C} &= \Phi_{vr}\Phi_{rr}^{-1} \\ C &= \Phi_{vv}\Phi_{rv}^{-1} \end{aligned} \tag{67}$$

Then, given $\Delta P = 0$, $\tilde{C}\Delta q = \Delta p$ and given $\Delta Q = 0$, $C\Delta q = \Delta p$.

He also proves that these matrices verify the Ricatti equation, and are therefore symmetric.

Using our generating function approach, we show that these matrices are just coefficients of the Taylor expansion of F_1, F_2, F_3 and F_4.

From Eqs. 61[3]:

$$\begin{aligned} \Delta p_a &= x_{n+a} \\ &= \frac{\partial F}{\partial y_a} \\ &= (f_{a,k} + f_{k,a})y_k \qquad (68) \\ &= 2f_{a,k}y_k \end{aligned}$$

$$\Delta p = F_{qq}\Delta q + F_{qp}\Delta P \tag{69}$$

$$\begin{aligned} \Delta Q_a &= \frac{\partial F}{\partial y_{n+a}} \\ &= (f_{n+a,k} + f_{k,n+a})y_k \qquad (70) \\ &= 2f_{n+a,k}y_k \end{aligned}$$

$$\Delta Q = F_{pq}\Delta q + F_{pp}\Delta P \tag{71}$$

where we recall that

$$F = f_{i,j}y_iy_j = \tfrac{1}{2}Y^T \begin{pmatrix} F_{qq} & F_{qp} \\ F_{pq} & F_{pp} \end{pmatrix} Y,\ F_{qp}^T = F_{pq}$$

and F_{qq} and F_{pp} are symmetric.

Solving for $(\Delta q, \Delta p)$ yields:

$$\Delta q = F_{qp}^{-T}\Delta Q - F_{qp}^{-T}F_{pp}\Delta P \tag{72}$$

$$\Delta p = F_{qq}F_{qp}^{-T}\Delta Q + (F_{qp} - F_{qq}F_{pq}^{-T}F_{pp})\Delta P \tag{73}$$

We deduce the expression of Φ:

$$\begin{cases} \Phi_{rv} &= -F_{qp}^{-T}F_{pp} \\ \Phi_{rr} &= F_{qp}^{-T} \\ \Phi_{vv} &= F_{qp} - F_{qq}F_{qp}^{-T}F_{pp} \\ \Phi_{vr} &= F_{qq}F_{qp}^{-T} \end{cases}$$

[3] keeping only the second order terms

We conclude that

$$\tilde{C} = \Phi_{vr}\Phi_{rr}^{-1} = F_{qq} \tag{74}$$

In the same manner, but using F_1, we show that:

$$C = \Phi_{vv}\Phi_{rv}^{-1} = F_{qq} \tag{75}$$

where F_{qq} is the first block of F_1.

Thus, we have shown that C and $\tilde{C}$ are symmetric by nature, and moreover that they verify the Ricatti equation given in Eq. 54[4]

Some computations

Using the theory we have developed in the previous section, we now numerically solve the boundary value problem to order 4. Before doing any computation, we should first note that:

1. F_1 and F_4 are undefined at $t = 0$.
2. F_2 and F_3 are well-defined at $t = 0$.
3. It is possible to transform from one generating function to another using the Legendre transformation in Eq. 17.
4. It can be proved that a maximal solution to the differential equations defining F_i exists, but there is no guarantee that the solution is defined for all time.

Numerical integration of Eqs. 54 provides a solution at order 2 to the relative boundary value problem. To get higher order terms, we use higher order expansion of H and F in Eqns 58.

We used the following procedure to check the validity of our computations. Given a nominal trajectory, we first solve for F_2. Then, we integrate another trajectory with different initial conditions (called the perturbed trajetory). Those two integrations gived full knowledge of $(\Delta Q, \Delta P)$ and $(\Delta q, \Delta p)$ at every instant. Then, using only $(\Delta q, \Delta P)$ we solve for $(\Delta p, \Delta Q)$ using F_2. Comparing the values of Δp and ΔQ found using both methods results in a set of plots for the error in final momentum Δp and a set of plots for the error in initial position ΔQ. The magnitude of this error represents the error in solving the boundary value problem using our approach.

The following initial conditions were used :

[4] In the particular case of a gravitational problem, $H_{qp} = 0$ and $H_{pp} = I$

- for the nominal trajectory:

$$Q_0 = (6500km, 0km), P_0 = (0, \sqrt{\mu/6500})$$

- for the perturbed trajectory:

$$Q_1 = (6550km, 0km), P_1 = (0, \sqrt{\mu/6500})$$

where $\mu = 4 \cdot 10^5 km^3 s^{-2}$.

The nominal and perturbed trajectories are plotted in figure 1. We note two features of our solution.

First, we experience a singularity around 1500 *sec* in solving for F_2 (Fig. 3). Eq. 66 shows that the singularity of F_2 is directly related to the non invertibility of ϕ_{rr} that is to:

$$\det(\Phi_{rr}) = 0$$

This corresponds to there being multiple solutions to the boundary value problem at this time. Figure 2 shows the determinant of the four block matrices of the matrix ϕ using the initial conditions stated above (In the particular case of a circular orbit, $\det(\phi_{rr}) = \det(\phi_{vv})$). In a similar way, one can relate the singularities of every F_i with the non invertibility of a block matrix within the state transition matrix. The singularities of F_1 correspond to $\det(\phi_{rv}) = 0$, the one of F_3 correspond to $\det(\phi_{vv}) = 0$ and the one of F_4 correspond to $\det(\phi_{vr}) = 0$.

One way to get around this difficulty is to solve for F_i until a singularity is approached, then evaluate the value of F_j at a time prior to the singularity and propagate F_j to a time where transformation back to F_i is needed. The issue of evaluating F_j at time t with the knowledge of F_i at the same instant reduces to a problem of inversion of series. For instance, given F_2 we find F_1 using the Legendre transformation Eq. 17 and substituting in it ΔP as a function of $(\Delta q, \Delta Q)$. To do so one must invert the following relation $\Delta Q = \frac{\partial F2(\Delta q, \Delta P)}{\partial \Delta P}$ as a series. Such transformations are well defined and have been dealt with by V.A. Brumberg.[5] This method is reliable only if the singularities of F_1, F_2, F_3 and F_4 do not occur at the same time. Figure 2 shows that there is always one block of the state transition matrix with a determinant greater than 1.

Second, the accuracy of our solution to the boundary value problem increases dramatically as we increase the order of our solution for F_2, giving a numerical indication that our approach is converging onto the true solution over its interval of definition. Figures 4-6 represent the error in solving the boundary problem using F_2 for $tf \in (2500, 3500)$. To obtain such plots, we needed to avoid the singularities discussed above. We integrate F_2 from $t = 0$ to $t = 100$ *sec*, then we used a Legendre transformation to find F_1 at $t = 100$ *sec*. We integrate F_1 up to $t = 2500$ *sec* (we recall that F_2 was singular at $t = 1500$ *sec*) and use again the Legendre transformation to find F_2 at that time $t = 2500$ *sec*. We finally integrate F_2 up to 3500 *sec*. These figures demonstrate the convergence of our algorithm across multiple singularities.

Conclusion

This paper derives a novel application of the Hamilton-Jacobi Equation to solve the problem of non-linear targeting for a formation of spacecraft. The application constructs direct solutions for the Taylor Series expansion of the generating function to arbitrarily high order in displacements from a nominal, numerically defined trajectory. By extending the procedure to high degree in powers of the specified boundary variables, we are able to immediately generate non-linear solutions to boundary value problems. By formulating the problem as a set of ordinary differential equations, we are able to generate time histories of the boundary value problem as well. In this paper we demonstrate the basic theory and provide an example that shows its convergence. We demonstrate the generation of high order solutions for the F_i across multiple singularities. In the future we will resolve the singularities with the aid of higher order expansions. Future work will also explore the application of this approach to the initial value problem.

A The Hamilton-Jacobi equations at order 3

Define the following variables:

$$
\begin{aligned}
&A_{i,j,k} = \\
&h_{n+a,n+b,n+c}(f_{a,i}+f_{i,a})(f_{b,j}+f_{j,b})(f_{c,k}+f_{k,c}) \\
&B_{i,j,k} = \\
&h_{n+a,n+b}(f_{a,i}+f_{i,a})(f_{b,j,k}+f_{j,k,b}+f_{k,b,j}) \\
&C_{i,j,k} = \\
&h_{n+a,n+b}(f_{b,i}+f_{i,b})(f_{a,j,k}+f_{j,k,a}+f_{k,a,j}) \\
&D_{a,i,j} = (h_{a,n+b,n+c}+h_{n+c,a,n+b}+h_{n+b,n+c,a}) \\
&\qquad (f_{b,i}+f_{i,b})(f_{c,j}+f_{j,c}) \\
&E_{a,i,j} = \\
&(h_{a,n+b}+h_{n+b,a})(f_{b,i,j}+f_{j,b,i}+f_{i,j,b}) \\
&G_{a,b,i} = \\
&(h_{a,b,n+c}+h_{b,n+c,a}+h_{n+c,a,b})(f_{c,i}+f_{i,c}) \qquad (76)
\end{aligned}
$$

Keeping only third order terms in equation 64 yields:

$$
\begin{aligned}
&\dot{f}_{i,j,k}y_iy_jy_k + (A_{i,j,k}+B_{i,j,k}+C_{i,j,k})y_iy_jy_k \\
&+ (D_{a,i,j}+E_{a,i,j})y_ay_iy_j + G_{a,b,i}y_ay_by_i \\
&+ h_{a,b,c}y_ay_by_c = 0 \qquad (77)
\end{aligned}
$$

We deduce the coefficients of $y_iy_jy_k$:

- Coefficients of $y_{i\le n}^3$

$$
\begin{aligned}
&A_{i,i,i}+B_{i,i,i}+C_{i,i,i}+D_{i,i,i} \\
&\qquad +E_{i,i,i}+\dot{f}_{i,i,i}+G_{i,i,i}+h_{i,i,i} = 0
\end{aligned} \qquad (78)
$$

- Coefficients of $y_{i>n}^3$

$$
A_{i,i,i}+B_{i,i,i}+C_{i,i,i}+\dot{f}_{i,i,i} = 0 \qquad (79)
$$

- Coefficients of $y_{i\le n}^2y_{j\le n}$

$$
(A+B+C+D+E+\dot{f}+G+h)_{\tau(i,i,j)} = 0 \qquad (80)
$$

where $\tau(i,j,k)$ represents all the distinct permutations of (i,j,k), that is $A_{\tau(i,j,k),l} = A_{i,j,k,l}+A_{i,k,j,l}+A_{k,i,j,l}+A_{k,j,i,l}+A_{j,k,i,l}+A_{j,i,k,l}$ but $A_{\tau(i,i,j),l} = A_{i,i,j,l}+A_{i,j,i,l}+A_{j,i,i,l}$

- Coefficients of $y_{i\le n}^2y_{j>n}$

$$
\begin{aligned}
&(A+B+C+\dot{f})_{\tau(i,i,j)} \\
&+(D+E)_{i,\tau(i,j)}+G_{i,i,j} = 0
\end{aligned} \qquad (81)
$$

- Coefficients of $y_{i\le n}y_{j\le n}y_{k\le n}$:

$$
(A+B+C+D+E+\dot{f}+G+h)_{\tau(i,j,k)} = 0 \qquad (82)
$$

- Coefficients of $y_{i\le n}y_{j\le n}y_{k>n}$

$$
\begin{aligned}
&(A+B+C+\dot{f})_{\tau(i,j,k)}+(D+E)_{i,\tau(j,k)} \\
&+(D+E)_{j,\tau(i,k)}+G_{\tau(i,j),k} = 0
\end{aligned} \qquad (83)
$$

- Coefficients of $y_{i>n}^2y_{j\le n}$

$$
(A+B+C+\dot{f})_{\tau(i,i,j)}+(E+D)_{j,i,i} = 0 \qquad (84)
$$

- Coefficients of $y_{i>n}^2y_{j>n}$

$$
(A+B+C+\dot{f})_{\tau(i,i,j)} = 0 \qquad (85)
$$

- Coefficients of $y_{i\le n}y_{j>n}y_{k>n}$

$$
(A+B+C+\dot{f})_{\tau(i,j,k)}+(D+E)_{i,\tau(j,k)} = 0 \qquad (86)
$$

- Coefficients of $y_{i>n}y_{j>n}y_{k>n}$

$$
(A+B+C+\dot{f})_{\tau(i,j,k)} = 0 \qquad (87)
$$

Eqs 78-87 allow to solve for F_2 or F_1.

References

[1] R. Battin. *An introduction to the Mathematics and Methods of Astrodynamics*. Sec $9-5$, $9-6$. Revised Edition, AIAA, 1999.

[2] D. Greenwood. *Classical Dynamics*. Prentice-Hall, 1977.

[3] H. Goldstein. *Classical Mechanics*. Addison-Wesley, 1965.

[4] R. Moulton. *Differential equations*. Dover Publications, 1930.

[5] V. A. Brumberg. *Analytical techniques of Celestial Mechanics*. Sec 1. Springer,1995.

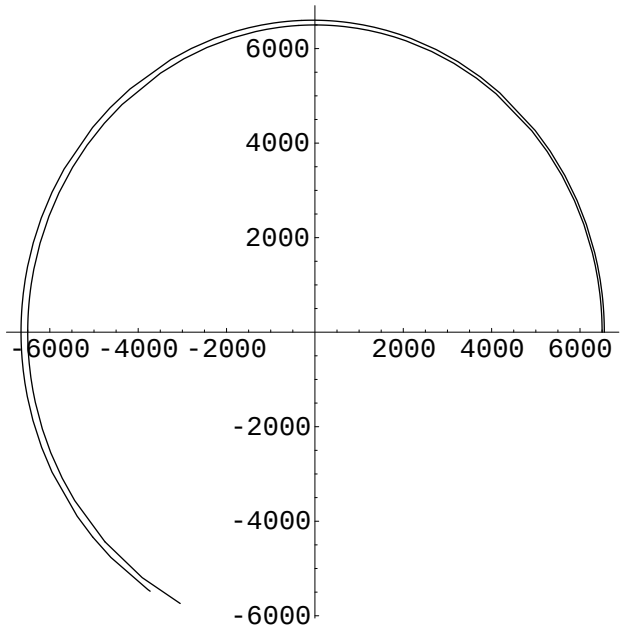
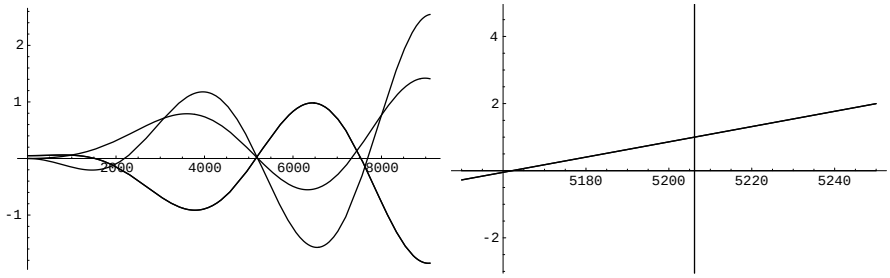
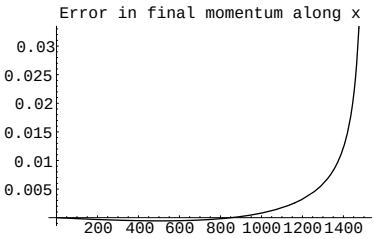


Figure 1: Nominal and perturbed trajectories, $t \in (0, 3500sec)$

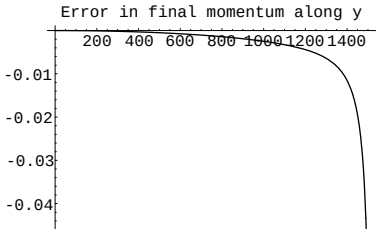


(a) For $t \in (0, 9000sec)$ (b) For $t \in (5150, 5250sec)$

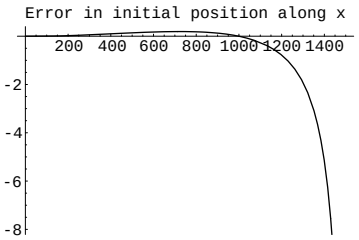
Figure 2: Determinant of Φ_{rr}, Φ_{rv}, Φ_{vr} and Φ_{vv}



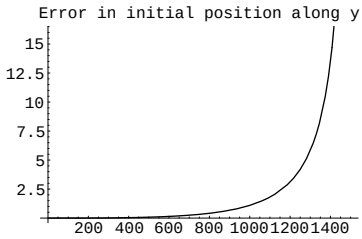
(a) Error in final momentum



(b) Error in final momentum

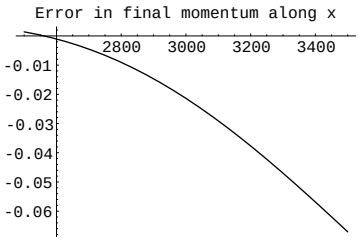


(c) Error in initial position

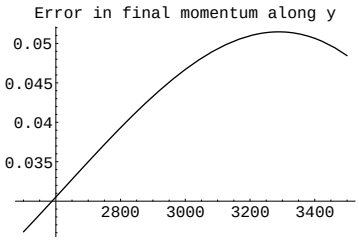


(d) Error in initial position

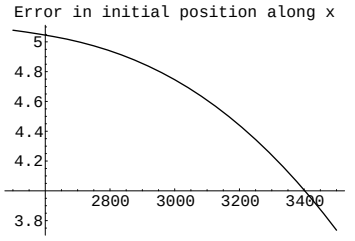
Figure 3: Order 2



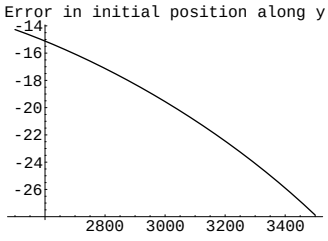
(a) Error in final momentum



(b) Error in final momentum

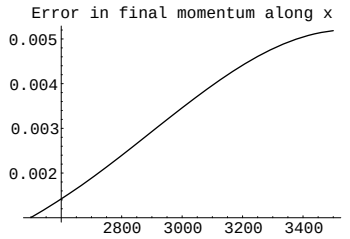


(c) Error in initial position

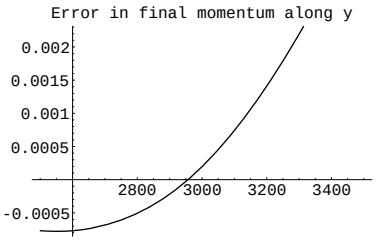


(d) Error in initial position

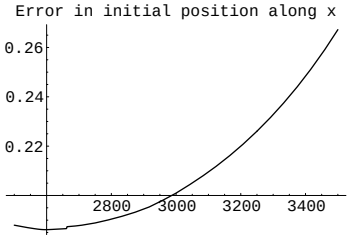
Figure 4: Order 2



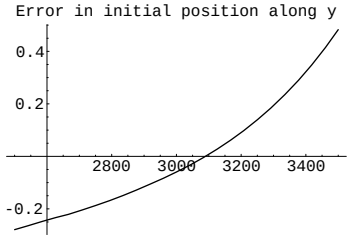
(a) Error in final momentum



(b) Error in final momentum

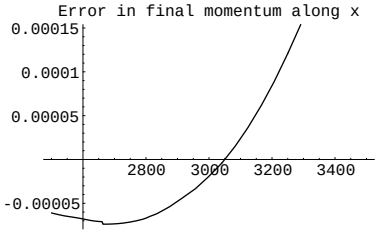


(c) Error in initial position

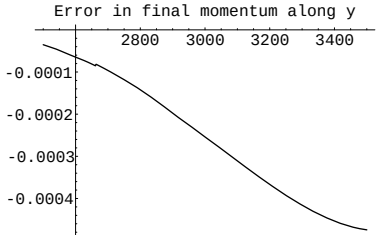


(d) Error in initial position

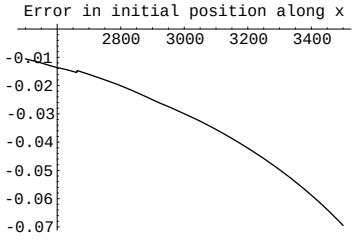
Figure 5: Order 3



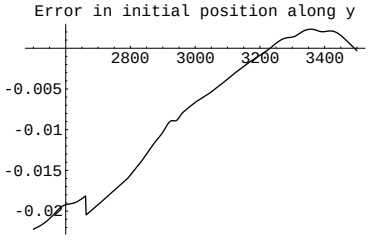
(a) Error in final momentum



(b) Error in final momentum



(c) Error in initial position



(d) Error in initial position

Figure 6: Order 4