Equation (16) then becomes

$$\frac{d}{dt}\begin{bmatrix} \delta v \\ \delta r \\ \delta m \end{bmatrix} = A \begin{bmatrix} \delta v \\ \delta r \\ \delta m \end{bmatrix} + B\delta F \tag{19}$$

where the new coefficient matrices A and B are changed accordingly from Eq. (17). Specifically, A is increased by a seventh row and column and may be partitioned as

$$A = \left[\begin{array}{c:c:c} O & M & a \\ \hdashline I & O & O \\ \hdashline O & O & O \end{array}\right] \qquad a = -\frac{1}{m^2}\begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix} \tag{20}$$

Correspondingly, the fundamental matrix Λ is increased by a seventh row and column. Since both velocity and position-state transition depend upon mass, whereas mass-state transition depends only upon mass in a one-to-one fashion, Λ may be partitioned as

$$\Lambda = \left[\begin{array}{c:c:c} \Lambda_1 & \Lambda_2 & \lambda_1 \\ \hdashline \Lambda_3 & \Lambda_4 & \lambda_2 \\ \hdashline 0 & 0 & 1 \end{array}\right] \tag{21}$$

$$\Lambda_{\text{boundary}} = I$$

where λ_1 and λ_2 are each three-dimensional vectors. Proceeding as in the previous section, it can be shown that Λ^{-1} is given by

$$\Lambda^{-1} = \left[\begin{array}{c:c:c} \Lambda_4{}^T & -\Lambda_2{}^T & p_1 \\ \hdashline -\Lambda_3{}^T & \Lambda_1{}^T & p_2 \\ \hdashline 0 & 0 & 1 \end{array}\right] \tag{22}$$

where

$$p_1 = -\Lambda_4{}^T\lambda_1 + \Lambda_2{}^T\lambda_2$$

$$p_2 = \Lambda_3{}^T\lambda_1 - \Lambda_1{}^T\lambda_2$$

Thus, the modified inversion property retains the major characteristic of simple term rearrangement, although some algebra is required to obtain the elements of p_1 and p_2.

References

[1] Swanlund, G., "Analysis techniques to determine guidance computation requirements for space vehicles," ARS **J. 32,** 755–761 (1962).

[2] McLean, J. D., Schmidt, S. F., and McGee, L. A., "Optimal filtering and linear prediction applied to a midcourse navigation system for the circumlunar mission," NASA TN 1208 (1962).

[3] Friedlander, A. L., "A study of guidance sensitivity for various low-thrust transfers from earth to Mars," NASA TN 1183 (1962).

Flow Field in Hypersonic Re-Entry

Herschel Weil*
University of Michigan, Ann Arbor, Mich.

The subject of this note is the sphere moving at a uniform hypersonic velocity in free molecule flow. Three formulas, each different because of different simplifying assumptions, have appeared in the literature to approximate the number density of diffusely reflected particles along the symmetry axis in the direction of the sphere's motion relative to the fluid. The purposes of this note are 1) to compare these formulas for number density and point out the causes for their differences, and 2) to present formulas and numerical results for the mean velocity field of the diffusely reflected molecules.

The three formulas for number density n of reflected particles normalized by the unperturbed number density are given here in terms of distance x in units of sphere radii and the sphere velocity V relative to the fluid and normalized by the most probable thermal speed in a Maxwellian distribution, $(2kT/m)^{1/2}$, where

k = Boltzmann's constant
T = temperature of the unperturbed gas
m = mass of a molecule of the gas

Formulas are given by Gurevich[1] as

$$n(x) = (1/2x)\ln[(x+1)/(x-1)] \tag{1}$$

and by Probstein[2] as

$$n(x) = 2(\pi)^{1/2}V\,F_0(x) \tag{2}$$

where

$$F_0(x) = [(2x^3+1) - (2x^2+1)(x^2-1)^{1/2}]/(3x^2)$$

A formula that holds not only on the symmetry axis but also for all angles θ was given by Dolph and Weil[3]† as

$$n(r,\theta) = \tfrac{1}{2}(T_\infty/T_b)^{1/2}\{[1-(1-\rho^{-2})^{1/2}] \times \exp(-V^2r^2\cos^2\theta) - \tfrac{1}{2}(\pi)^{1/2}V\cos\theta\,\mathrm{erfc}(Vr\cos\theta)F_0(\rho)\} \tag{3}$$

where erfc( ) is the complementary error function, ρ is the radial distance normalized by the sphere radius, T_∞ and T_b are, respectively, the temperatures of the unperturbed gas and of the sphere, and $\theta = \pi$ corresponds to the x axis used in Eqs. (1) and (2). Equation (3) reduces along this axis to

$$n(x) = \tfrac{1}{2}(T_\infty/T_b)^{1/2}\{[1-(1-x^{-2})^{1/2}]\exp(-V^2x^2) + (\pi)^{1/2}V\,\mathrm{erfc}(-Vx)F_0(x)\} \tag{4}$$

Note that Eqs. (1) and (4) both are finite at $x = 1$, whereas (1) becomes infinite as x approaches unity, which is not realistic. Both Eqs. (1) and (4) have slopes that become negatively infinite like

$$-\frac{\text{real positive constant}}{[2(x-1)]^{1/2}}$$

as $x \to 1$. Probstein points out that this singularity in slope is connected with molecules of grazing incidence.

For large values of V, $\mathrm{erfc}(-V\rho) \to 2$ and Eq. (4) reduces to

$$n(x) \sim (\pi)^{1/2}V(T_\infty/T_b)^{1/2}F_0(x)$$

When $T_\infty = T_b$, this is one half the value given by Eq. (2).

Both Eqs. (1) and (2) were derived assuming that the emitted particles are all traveling with the same speed and invoking the inverse square law for decay of flux issuing from each surface element. Equation (3) is the result of integrating, over velocity space, a close approximation to the distribution function determined by Wang Chang[4] for a Maxwellian distribution function with superimposed mean speed and perturbed by a diffusely reflecting sphere.

The differences between formulas (2) and (3) stem from two causes. To derive Eq. (2), a boundary condition for flux at the surface in terms of particle number density was used. To obtain this relation, the incident gas was assumed to have a Maxwellian distribution with a superimposed mean

Received by ARS December 10, 1962. This work was supported by NASA under Contract NSA 181-61.

* Associate Professor of Electrical Engineering, Radio Astronomy Observatory.

† In Ref. 3 it was assumed that $T_\infty = T_b$. In addition, the second term of n was written with a factor I_3 given separately. By error, I_3 was written as 2π times the correct expression. The correct expression was used in the subsequent numerical evaluation of Eq. (3) for $V = 5$ in Ref. 3 and also (with the temperature factor) in Ref. 5.

Table 1

ρ	Values of v_θ						Values of v_r				
	$\theta = 90°$	$\theta = 92.5°$	$\theta = 95°$	$\theta = 100°$	$\theta = 120°$	$\theta = 180°$	$\theta = 90°$	$\theta = 95°$	$\theta = 100°$	$\theta = 120°$	$\theta = 180°$
1.0	0	0	0	0	0	0	0.56419	0.56419	0.56419	0.56419	0.56419
1.01	0.09562	0.08240	0.07141	0.05238	0.01869	0	0.64339	0.64348	...	...	0.64354
1.04	0.26532	0.23165	0.20012	0.14678	0.05201	0	0.71916	0.71965	...	...	0.72000
1.06	0.34714	0.30448	0.26365	0.19339	0.06821	0	0.75131	0.75208	...	...	0.75262
1.08	0.41497	0.36569	0.31741	0.23284	0.08176	0	0.77729	0.77831	...	...	0.77903
1.10	0.47254	0.41841	0.36406	0.26706	0.09336	0	0.79923	0.80049	0.80110	0.80137	0.80137
1.20	0.66518	0.60347	0.53166	0.38971	0.13344	0	0.87606	0.87826	0.87930	0.87970	0.87970
1.5	0.85138	0.83190	0.76002	0.54871	0.17866	0	0.98471	0.98808	0.98961	0.98994	0.98994
2.0	0.82976	0.91637	0.87908	0.59393	0.18470	0	1.05279	1.05668	1.05755	1.05764	1.05764
3.0	0.66075	0.92772	0.90951	0.51801	0.15828	0	1.09611	1.09888	1.09929	...	1.09930
4.2	0.51094	0.94586	0.82421	0.42097	0.12857	0	1.11215	1.11408	1.11414	...	1.11414
5.0	0.44154	0.96712	0.74354	0.37165	0.11351	0	1.11699	1.11848	1.11849	...	1.11849
7.0	0.32808	0.98817	0.57581	0.28572	0.08726	0	1.12259	1.12346	...	...	1.12346
10.0	0.23604	0.84005	0.42575	0.21124	0.06452	0	1.12555	1.12601	...	...	1.12601

velocity, the gas flowing out directly at the surface, a Maxwellian distribution. A true Maxwellian distribution is isotropic and yields no net outward flux; rather, a Maxwellian distribution times the step function $S = 1$ for directions out of the surface and $S = 0$ for directions into the surface is appropriate. However, the normalization of such a distribution must be twice that of a Maxwellian distribution in order for the integral over all directions to equal the local number density. This factor of 2 in the normalization was not used in deriving the approximate boundary condition and hence not used in deriving Eq. (2). If it had been, the right-hand side of Eq. (2) would have been half as big.

The further differences between Eqs. (2) and (4) enter from approximations to the flux equality condition which were made in deriving the boundary condition for Eq. (2), essentially ignoring the dependance of the mathematical expression with position on the surface. The approximation to the Wang Chang distribution used to derive Eq. (3) amounted to using a more accurate approximation to the flux equality condition.

The mean velocity $\bar{v}$ of the reflected particles at any point can be found by averaging $\bar{v}$ using the Wang Chang distribution function. Using the same approximation to the distribution function, closed-form analytic results for the tangential and radial components v_θ and v_r are, respectively,

$$v_\theta = (V/8n)\ \mathrm{erfc}(V\rho\cos\theta)\ \sin\theta F_1(\rho)$$

where

$$F_1(\rho) = \frac{1}{\rho^3} - \left(1 + \frac{P^2}{2}\right) + P^2\rho\left(1 - \frac{P^2}{4}\right)\log_e\left(\frac{\rho+1}{\rho-1}\right) \tag{5}$$

and

$$v_r = [2(\pi)^{1/2}n]^{-1}\{\rho^{-2}\exp[-(V\cos\theta)^2] - \tfrac{1}{2}(\pi)^{1/2}V\cos\theta F_2(\rho)\ \mathrm{erfc}(V\rho\cos\theta)\}$$

where

$$F_2(\rho) = \frac{1}{\rho^3} + 1 - \frac{P^2}{2} - \frac{P^4\rho}{4}\log_e\left(\frac{\rho+1}{\rho-1}\right)$$

and

$$P = (1 - \rho^{-2})^{1/2}$$

The details of these integrations are given in Appendix B of Ref. 5, whereas numerical results for $V = 5$ are given here in Table 1. They indicate insensitivity of v_r to angle and in general indicate a monotone build-up of radial velocity with radial distance. Tangential velocity as a function of ρ builds up as would be expected from zero and then drops off. Further v_θ changes rapidly with decreasing θ near $\theta = 90°$, the radial position of the maximum dropping rapidly.

References

[1] Gurevich, A. V., "Perturbations in the ionosphere caused by a traveling body," *Artificial Earth Satellites* (Iskusstvennye Sputniki Zemli) **7**, 101–124 (1961); transl. by R. Matthews in Planetary Space Sci. **9**, 321–344 (1962).

[2] Probstein, R. F., "Shock wave and flow field development in hypersonic re-entry," ARS J. **31**, 185–194 (1961).

[3] Dolph, C. L. and Weil, H., "On the change in radar cross section of a spherical satellite caused by a plasma sheath," *Electromagnetic Effects of Re-entry* (Pergamon Press, London, 1961); reprinted in Planetary Space Sci. **6**, 123–132 (1961).

[4] Wang Chang, C. S., "Transport phenomena in very dilute gases, II," Univ. Mich. Eng. Research Inst. Rept. CM-654 (December 1950).

[5] Weil, H. and Barasch, M. L., "A theoretical lunar ionosphere," Icarus **1**, 346–356 (1963).

Interactions between a Hypersonic Wake and a Following Hypersonic Projectile

R. E. Slattery,* W. G. Clay,† and R. R. Stevens‡
Lincoln Laboratory,§ Massachusetts Institute of Technology, Lexington, Mass.

During a recent series of experiments at the Lincoln Laboratory Re-entry Simulating Range, it was discovered that the presence of the sabot in the wake of a spin-stabilized cone radically altered the flow around the cone. The base section of the sabot is partially split, so that normally it breaks upon leaving the gun muzzle, and the pieces are separated from the cone's flight path by centrifugal forces. When the sabot base fails to split, as sometimes occurs, it continues downrange along the same path as the cone, slowly falling behind it because of the differences in drag.

In Fig. 1 is shown a schlieren photograph of a 25° included-angle cone in normal, zero attitude flight at 5500 fps through 25 mm Hg of dry air. The techniques of stabilized flight,

Received January 3, 1963.

* Assistant Leader, Plasma Physics Group.
† Staff Member, Plasma Physics Group.
‡ Project Technician, Plasma Physics Group.

§ Operated with support from the U. S. Advanced Research Projects Agency.