Periodicity of σ_θ requires that the bracketed quantity in the preceding equation be zero. Thus, integration gives

$$F = C + (D/r) - I(r) \quad (6)$$

where

$$I(r) = \frac{1}{r}\iint rh(r)\, a_0(r)dr\, dr \quad (7)$$

The first of Eqs. (1) and periodicity of σ_r require that

$$rV_r = \text{periodic terms} - \theta\frac{d}{dr}(rF) - \left(G + \frac{d^2G}{d\theta^2}\right) \quad (8)$$

Combining Eqs. (2) and (5) and integrating yield

$$rV_r = \theta\int rha_0dr + \text{periodic terms} + J(\theta) \quad (9)$$

In view of the first of Eqs. (1) and the periodicity of σ_r, $J(\theta)$ must be periodic. Then, combining Eqs. (6), (8), and (9) yields

$$(d^2G/d\theta^2) + G = \text{periodic terms} - C\theta$$

which integrates to give

$$G(\theta) = C_1\theta\sin\theta + C_2\theta\cos\theta + \text{periodic terms} - C\theta$$

Finally, the most general stress function guaranteeing periodicity of all stresses is

$$\phi(r,\theta) = C_0\theta + C_1r\theta\sin\theta + C_2r\theta\cos\theta - r\theta I(r) + \psi(r,\theta) \quad (10)$$

where the C's are constants, $I(r)$ is defined by Eq. (7), and ψ is periodic in θ and has continuous second partial derivatives.

It is interesting to note the simplicity of Eq. (10). Except for the presence of the $r\theta I(r)$ term, it is identical to the Michell[5] solution of the biharmonic equation, which governs the isotropic, homogeneous, uniform-thickness elasticity problem without body forces. The results of this note have been used by the author in the solution of several plane elasticity problems involving varying thickness and polar orthotropy.

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Comment On the Shear Stiffness of Fabrics

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We who are interested in fabric structures are much indebted to R. W. Leonard for an interesting and rigorous generalization of the analysis of Ref. 2. I take exception, however, to the statement that the limitations of the theory of Ref. 2 appear not to have been fully recognized, for they were explicitly pointed out in the derivation. Some liberties were taken, however, in applying the theory to a pressurized cylinder having its thread sets originally at 45° with the axis of the cylinder, and this is admittedly not entirely clear from Ref. 2, although the difficulties are discussed.

In Ref. 2, the relation $G_{12} = \sigma_{11}$ was obtained for the case of small strains and threads initially at right angles, where however, σ_{11} was the stress in the diagonal threads assuming no deformation (the effect of the Poisson strain on stress is neglected in any small-strain theory) as distinguished from Leonard's definition of σ_{11}.

This has been overlooked by Leonard, and as a result, the error as compared with the exact theory is less than he indicates. The theoretically correct value of G_{12} found by him is 0.714 pr. The value used in Ref. 2 is 0.75 pr, only 5 percent more. But since $-1/3 < \sin\gamma < 0$ in the actual test cylinder owing to the coating of neoprene and friction between thread sets, the error in application was probably considerably smaller than this and surely within the limits of engineering accuracy, aside from the direct contribution of the elastomer to shear stiffness.

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Author's Reply

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In adopting the result $\sigma_{11} = 0.75\ pr$ in his treatment of the second problem, Dr. Topping has assumed that the stresses due to pressure are not influenced by the resulting strains. This is not a good assumption, since the radius of the cylinder increases approximately 15 percent due to large shear strains in the fabric. However, his use of $\sigma_{11} = 0.75\ pr$ does indeed largely compensate for the use of the incorrect formula $G_{12} = \sigma_{11}$. The important points to be made are: (1) large shear strains must often be taken into account if fabric behavior is to be understood, and (2) the effective minimum shear stiffness is dependent on the geometry and loading and a single simple formula—such as $G_{12} = \sigma_{11}$—should not be expected to represent all situations.

Shroud Design for Simulating Hypersonic Flow Over the Nose of a Hemisphere†

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Following is an analytical method for designing a shroud[1] which will generate the hypersonic pressure distribution on a hemisphere. The method was found to be successful throughout the region of subsonic flow ($0 \leq \eta \lesssim 44°$, Fig. 1). This shroud was designed as part of a low-turbulence wind tunnel used for investigating the effects of cooling on boundary-layer transition on a hemisphere.[2]

The design of the shroud contour was carried out in two steps. First, an approximate solution for the incompressible, irrotational flow field was found in the region $0 \leq \eta \leq 45°$, and, second, the resulting contour was corrected for compressibility near the sonic region, assuming one-dimensional flow.

Proceeding to the incompressible problem, a stream function, $\psi(r, \eta)$, can be defined by

$$\left.\begin{aligned}\partial\psi/\partial r &= -ru\sin\eta \\ \partial\psi/\partial\eta &= r^2v\sin\eta\end{aligned}\right\} \quad (1)$$

where u and v are the velocities in the η and r directions, respec-

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tively (Fig. 1). The irrotationality condition expressed in terms of ψ leads to the following linear equation:

$$r^2(\partial^2\psi/\partial r^2) + (\partial^2\psi/\partial\eta^2) - \cot\eta(\partial\psi/\partial\eta) = 0 \qquad (2)$$

The hypersonic velocity distribution can be very accurately expressed,[2] for $\eta < \pi/4$, by the linear relation

$$u(R, \eta) = a_s\sqrt{\frac{2}{\gamma}}\,\eta, \quad (\eta \leq \pi/4) \qquad (3)$$

where a_s is the speed of sound at the stagnation point and γ is the ratio of specific heats. Thus, the boundary conditions on the zero streamline, for $0 \leq \eta \leq \pi/4$, become

$$\left.\begin{aligned} \psi(R, \eta) &= 0 \\ (\partial\psi/\partial r)(R, \eta) &= -Ra_s\sqrt{2/\gamma}\,\eta\sin\eta \\ \psi(r, 0) &= 0, \quad (r \geq R) \end{aligned}\right\} \qquad (4)$$

The usual method of solving an elliptic differential equation is to specify either the dependent variable or its normal derivative (or a linear combination of both) at every point on the boundary of a closed region. However, in the present application the problem is to determine the shape of one boundary (shroud streamline) such that both ψ and $\partial\psi/\partial r$ have the specified values on the known boundary (hemisphere).

Although no straightforward analytical methods appear to exist for solving Eq. (2) with conditions [Eqs. (4)], an approximate closed-form solution is obtained by making a few minor simplifications. Since in the range $0 \leq \eta \leq \pi/4$ both $\sin\eta$ and $\cot\eta$ are given to within 1 percent by the first two terms in their series expansions, Eq. (2) becomes, to a high degree of approximation,

$$r^2(\partial^2\psi/\partial r^2) + (\partial^2\psi/\partial\eta^2) - [(1/\eta) - (\eta/3)](\partial\psi/\partial\eta) = 0 \qquad (5)$$

with the boundary conditions

$$\left.\begin{aligned} \psi(R, \eta) &= 0, \quad (0 \leq \eta \leq \pi/4) \\ \frac{\partial\psi}{\partial r}(R, \eta) &= -Ra_s\sqrt{\frac{2}{\gamma}}\left(\eta^2 - \frac{\eta^4}{6}\right), \quad 0 \leq \eta \leq \pi/4 \\ \psi(r, 0) &= 0, \quad (r \geq R) \end{aligned}\right\} \qquad (6)$$

In attempting a power-series solution of the form

$$\psi(r, \eta) = \sum_{n=1}^{\infty} a_{2n}\eta^{2n} \qquad (7)$$

to the above problem it is found from the first and second boundary conditions that ψ can be written in the relatively simple form

$$\psi(r, \eta) = a_2(r)\eta^2 + a_4(r)\eta^4 \qquad (8)$$

By substituting Eq. (7) into Eq. (5) and applying the boundary conditions [Eqs. (6)], the following two simultaneous differential systems result for determining $a_2(r)$ and $a_4(r)$:

$$\left.\begin{aligned} r^2a_2'' + (2/3)a_2 &= -8a_4 \\ a_2'(R) &= -Ra_s\sqrt{2/\gamma} \\ a_2(R) &= 0 \end{aligned}\right\} \qquad (9)$$

and

$$\left.\begin{aligned} r^2a_4'' + (4/3)a_4 &= 0 \\ a_4'(R) &= (Ra_s/6)\sqrt{2/\gamma} \\ a_4(R) &= 0 \end{aligned}\right\} \qquad (10)$$

After solving these equations (Cauchy type) the stream function given by Eq. (8) is determined. The final result may be written

$$\psi(r, \eta) = -R^2a_s\sqrt{\frac{2}{\gamma}\frac{r}{R}}\left\{\left[\sqrt{\frac{108}{5}}\sin\left(\sqrt{\frac{5}{12}}\ln\frac{r}{R}\right) - \sqrt{\frac{48}{13}}\sin\left(\sqrt{\frac{13}{12}}\ln\frac{r}{R}\right)\right]\eta^2 - \left[\frac{1}{\sqrt{39}}\sin\left(\sqrt{\frac{13}{12}}\ln\frac{r}{R}\right)\right]\eta^4\right\} \qquad (11)$$

Thus, the (incompressible) shroud contour for any given volume flow is determined by setting ψ = constant in Eq. (11) and solving the quadratic in η^2 for $\eta = \eta(r)$.

The shroud shape, showing the compressibility corrections in the sonic region, is given in Fig. 1. The upstream contraction section was made to converge somewhat faster than the predictions of Eq. (11) for practical reasons. For $\eta > 45°$, the gap between the shroud and the sphere surface was made greater than that specified by theory so that boundary layer growth would not influence the position of the sonic point.

The measured pressure distribution on the shrouded sphere is compared with the desired hypersonic pressure distribution in Fig. 2. Excellent agreement is indicated nearly to the sonic point.

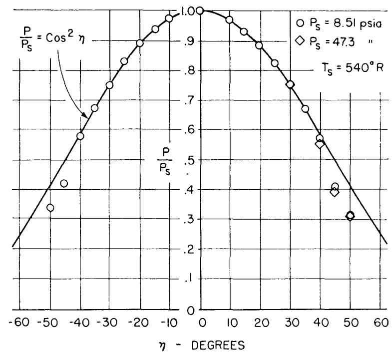

FIG. 2. Measured pressure distribution on shrouded sphere.

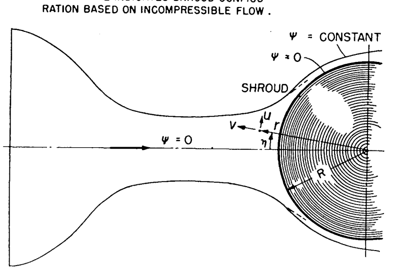

FIG. 1. Shrouded sphere.

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Neutral Burning Using an Interrupted-Tube Type of Solid Propellant Grain Geometry

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A SIMPLIFIED METHOD for rapidly determining the conditions for neutral burning for a solid-propellant interrupted-tube grain is presented in terms of the initial grain geometry, the burning rate, and time.

With the current interest in the segmented solid-propellant rocket motor, the interrupted-tube type of grain design has received considerable attention. In general, a rather neutral-burning propellant grain is desired, which is obviously not readily obtainable with the inherent progressivity associated with a