tion that h_L is independent of the choice of reference temperature, indeed that it is actually independent of temperature, is reasonable. For ease of use, the resultant simplified expression for h_L, given by Eq. (5), can be reduced to the form of a nomograph by standard methods. It is interesting to note that if the frequently used expressions for the variation of viscosity and conductivity with temperature, namely Pr = constant, $\mu \sim K \sim T$, are applied to to the function F one would obtain: $F(T)$ = constant. Experimental verification of the above has been obtained from reference 3 where the laminar expressions of Pohlhausen, Chapman and Rubesin, and Eq. (5) are compared with test data. Further verification has been obtained from existing flight-test data on cones of various sizes covering a wide range of Mach Number, Reynolds' Number, and ratio of wall temperature to temperature outside the boundary layer. The results of this correlation indicate that Eq. (5) predicts the test data reasonably well. The results of Eq. (5) are in excellent agreement with the values predicted by Pohlhausen's expression, as would be expected, and are in good agreement with the values predicted by the expression of Chapman and Rubesin, and Van Driest. The data include an appreciable amount of scatter at times in addition to several points lying in the transition region, which accounts for some of the disparity with theory. As a result of the above discussion, it may be stated that h_L is independent of temperature for all practical purposes and the resultant simplified expression for h_L can be conveniently represented by a simple nomograph.

A similar reasoning can be applied to the case of turbulent compressible flow in which case the results serve to justify the analysis of Eckert and present a calculation simplifying expression based on his equations. We will use the analysis of Eckert outlined in reference 2. The experimental calculations examined in the present investigation have generally been in the range $Re_\delta \leq 40 \times 10^6$ in which case the constant value $S = 1.18$ suggested in the literature is used. A theoretical correlation with experimental data using Blasius' incompressible expression gave good results in this range. As a result, this expression is used instead of the several expressions suggested by Eckert for turbulent incompressible flow. In order to simplify the calculation of the heat-transfer quantities from Eckert's analysis, the temperature dependent terms have once again been collected into a single term in the expression for the local turbulent heat-transfer coefficient. The resultant expression is given by

$$\left.\begin{aligned} h_T &= 0.0349\ [(\rho u x)_\delta{}^{0.8}/xR^{0.8}]G(T^*) \\ G(T^*) &= [k\ Pr/(\mu T^*)^{0.8}]\ (\text{B.t.u./sec. ft. °F.}) \times \\ &\qquad (\text{ft.}^2/\text{lb. sec. °R.})^{0.8} \end{aligned}\right\} \quad (6)$$

where T^* is Eckert's reference temperature. Using the aforementioned variation as a guide, namely, $\mu \sim K \sim T$, one obtains $G(T^*) \sim T^{*-0.6}$ and applying the previously mentioned data one obtains, with adequate accuracy,

$$G(T^*) = 0.113\ T^{*-0.6}, \quad 150 \leq T \leq 2000°\text{R}.$$

such that

$$h_T = 10.25 \times 10^{-6}[(\rho u x)_\delta{}^{0.8}/xT^{*0.6}] \quad (7)$$

Eq. (7) can also be conveniently represented by a nomograph. Various experimental results of flight-test data which were described above indicate that Eq. (7) predicts the test data generally within 10 per cent, and, in several instances, this figure rises to 15–20 per cent. It is of interest to note that the results of Eq. (7) are conservative. For comparison purposes the well-known expression of Van Driest was used to predict the same test data. The results of this correlation were similar to those obtained using Eq. (7) except that the results of Van Driest were nonconservative. Eq. (7) generally overpredicted the test data while Van Driest's expression gave values which were generally low. As a result, the analysis of Eckert seems justified.

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On the Generalized Prandtl Relation

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Consider a four-dimensional Euclidean space-time manifold, E_4. The time variable will be denoted by t, and the space variables by x^λ, $\lambda = 1, 2, 3$. In particular, we assume that the space variables are Cartesian and define ∞^1 Euclidean three spaces E_3 in E_4.

The tensor formulation of the hydrodynamical equations in these coordinate systems has the form:

$$(\partial/\partial t)\ (\rho v_\lambda) + \nabla_\mu(\rho v^\mu v_\lambda + p\delta_\lambda^\mu) = 0 \quad (1)$$

$$(\partial \rho/\partial t) + \nabla_\mu(\rho v^\mu) = 0 \quad (2)$$

$$(\partial/\partial t)\ \{\rho[(q^2/2) + e]\} + \nabla_\mu\{\rho v^\mu[(q^2/2) + h]\} = 0 \quad (3)$$

δ_α^β is the usual notation for the Kronecker delta, and ρ, v_λ, q, e, p, and h denote the density, component of velocity, magnitude of the velocity, internal energy, pressure, and enthalpy, respectively.

Eq. (1) is equivalent to the Eulerian equations of motion for compressible nonviscous fluids in the absence of body forces; Eq. (2) is the equation of continuity; and Eq. (3) is the energy relation.

The hypersurface (or lower-dimensional manifold) in space time along which discontinuities occur will be denoted by

$$\overset{j}{\phi}\ (t, x^\lambda) = {}^jk \quad \text{where the } {}^jk \text{ are constants} \quad (4)$$

If Eq. (4) consists of only one equation ($j = 0$), then the discontinuity manifold defines a hypersurface. If the system consists of more than one equation, then the equations define a manifold of lower dimension. In any case, the vector fields for the various types of j, $\partial\overset{j}{\phi}/\partial t$, $\partial\overset{j}{\phi}/\partial x^\lambda$, determine vectors normal to the discontinuity manifold. Furthermore, the unit normal vectors of this manifold are determined by

$$\left.\begin{aligned} n_t &= (\partial\phi/\partial t)/[(\partial\phi/\partial t)^2 + g^{\lambda\mu}(\partial\phi/\partial x^\lambda)\ (\partial\phi/\partial x^\mu)] \\ n_\lambda &= (\partial\phi/\partial x^\lambda)/[(\partial\phi/\partial t)^2 + g^{\lambda\mu}(\partial\phi/\partial x^\lambda)\ (\partial\phi/\partial x^\mu)] \end{aligned}\right\} \quad (5)$$

where $g^{\alpha\beta}$ is the metric tensor.

In the case where the discontinuity manifold is a shock wave, the values of ρ, v_λ, e, p, and h and their space-time derivatives are assumed to be continuous, while in crossing the discontinuity manifold these values are discontinuous.

Let $\overline{\rho v_\lambda}$ denote the jump in the value of ρv_λ across the shock discontinuity, $\overline{\rho v^\mu v_\lambda}$ the jump in the value of $\rho v^\mu v_\lambda$, etc. Then by means of the integral forms of Eqs. (1), (2), and (3), the following jump relations corresponding to Eqs. (1), (2), and (3) may be obtained[1]

$$(\partial\phi/\partial t)\ \overline{\rho v_\lambda} + (\partial\phi/\partial x^\mu)\ (\overline{\rho v^\mu v_\lambda} + \delta_\lambda{}^\mu\bar{p}) = 0 \quad (6)$$

$$(\partial\phi/\partial t)\bar{\rho} + (\partial\phi/\partial x^\mu)\overline{\rho v^\mu} = 0 \quad (7)$$

$$(\partial\phi/\partial t)\ [(\overline{p q^2}/2) + \overline{\rho e}] + (\partial\phi/\partial x^\mu)\ [(\overline{\rho q^2 v^\mu}/2) + \overline{\rho h v^\mu}] = 0 \quad (8)$$

Let the subscripts 1 and 2 indicate values on the sides of the shock wave, then

$$\overline{\alpha\beta} = (\alpha\beta)_2 - (\alpha\beta)_1 = (\alpha_2 - \alpha_1)(\beta_2 - \beta_1) + \alpha_1\beta_2 + \alpha_2\beta_1 - 2\alpha_1\beta_1$$

Upon simplifying the right hand side of the above, the jump may be expressed in the form

$$\overline{\alpha\beta} = \bar{\alpha}\bar{\beta} + \alpha_1\bar{\beta} + \beta_1\bar{\alpha}$$

Hence,

$$\overline{\rho v_\lambda} = \bar{\rho}\overline{v_\lambda} + \rho_1\overline{v_\lambda} + v_{\lambda_1}\bar{\rho}$$

$$\overline{\rho v^\mu v_\lambda} = \overline{\rho v^\mu}\,\overline{v_\lambda} + \rho_1 v_1^\mu \overline{v_\lambda} + v_{\lambda_1}\overline{\rho v^\mu} \quad \text{etc.}$$

Expanding Eq. (6) in this manner the following equation is obtained:

$$(\partial\phi/\partial t)(\bar{\rho}\overline{v_\lambda} + \rho_1\overline{v_\lambda} + \bar{\rho}v_{\lambda_1}) + (\partial\phi/\partial x^\mu)[\overline{\rho v^\mu}\,\overline{v_\lambda} + (\rho v^\mu)_1\overline{v_\lambda} + \overline{\rho v^\mu}\, v_{\lambda_1}] + \bar{p}\,\partial\phi/\partial x^\lambda = 0$$

Assume now that $\bar{p} = c^2\bar{\rho}$ where c^2 is the "velocity of sound." Then using Eq. (7), the above equation may be reduced to

$$\rho_1\overline{v_\lambda}[(\partial\phi/\partial t) + v_1^\mu(\partial\phi/\partial x^\mu) + c^2\bar{\rho}(\partial\phi/\partial x^\lambda)] = 0$$

Solving for $\bar{\rho}$,

$$\bar{\rho} = -\{\rho_1\overline{v_\lambda}[(\partial\phi/\partial t) + v_1^\mu(\partial\phi/\partial x^\mu)]/c^2(\partial\phi/\partial x^\lambda)\} \qquad (9)$$

Expanding Eq. (7) the following equation is obtained:

$$(\partial\phi/\partial t)\bar{\rho} + (\partial\phi/\partial x^\mu)(\bar{\rho}\overline{v^\mu} + \rho_1\overline{v^\mu} + \bar{\rho}\, v_1^\mu) = 0$$

or $$\bar{\rho}[(\partial\phi/\partial t) + (\bar{v}^\mu + v_1^\mu)(\partial\phi/\partial x^\mu)] + \rho_1\overline{v^\mu}(\partial\phi/\partial x^\mu) = 0$$

Substituting the value of $\bar{\rho}$ from Eq. (9) into the last equation yields

$$-\frac{\rho_1\bar{v}_\lambda[(\partial\phi/\partial t) + v_1^\mu(\partial\phi/\partial x^\mu)][(\partial\phi/\partial t) + (\bar{v}^\mu + v_1^\mu)(\partial\phi/\partial x^\mu)]}{c^2(\partial\phi/\partial x^\lambda)} + \rho_1\bar{v}^\mu(\partial\phi/\partial x^\mu) = 0$$

or $$\bar{v}_\lambda[(\partial\phi/\partial t) + v^\alpha(\partial\phi/\partial x^\alpha)]_1[(\partial\phi/\partial t) + v^\beta(\partial\phi/\partial x^\beta)]_2 - c^2\overline{v^\mu}(\partial\phi/\partial x^\mu)(\partial\phi/\partial x^\lambda) = 0 \qquad (10)$$

Since the tangential component of velocity is continuous across a shock discontinuity,

$$\bar{v}_\lambda = \bar{v}^\mu(\partial\phi/\partial x^\mu)(\partial\phi/\partial x^\lambda)/g^{\alpha\beta}(\partial\phi/\partial x^\alpha)(\partial\phi/\partial x^\beta)$$

Thus Eq. (10) becomes

$$\bar{v}_\lambda\{[(\partial\phi/\partial t) + v^\alpha(\partial\phi/\partial x^\alpha)]_1[(\partial\phi/\partial t) + v^\beta(\partial\phi/\partial x^\beta)]_2 - c^2 g^{\alpha\beta}(\partial\phi/\partial x^\alpha)(\partial\phi/\partial x^\beta)\} = 0$$

Since by hypothesis $\bar{v}_\lambda \neq 0$,

$$[(\partial\phi/\partial t) + v^\alpha(\partial\phi/\partial x^\alpha)]_1[(\partial\phi/\partial t) + v^\beta(\partial\phi/\partial x^\beta)]_2 - c^2 g^{\alpha\beta}(\partial\phi/\partial x^\alpha)(\partial\phi/\partial x^\beta) = 0 \qquad (11)$$

Expressing this equation in terms of n_t, n_λ, as defined by Eqs. (5), yields the following equation:

$$(n_t + v^\lambda n_\lambda)_1(n_t + v^\mu n_\mu)_2 = c^2$$

This is the generalized Prandtl relation.

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Further Comments on Diffuser Instability

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I HAVE FOLLOWED, with much pleasure, the spirited polemics of Dailey[1] and Trimpi[2] on the subject of diffuser "buzz."

In view of the obvious satisfaction derived by the antagonists, it seems almost cruel to point out that the whole thing can be explained by a classical theorem on compressor surge which may be found in standard textbooks[3–7] on rotating machinery.

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A Test of the Uniqueness of Solutions for Problems of Nonsteady Flow Under Given Boundary Conditions*

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THE STEADY STATE which is approached by a flow system, from given space-like initial states and under given time-like periodic boundary conditions, is generally postulated, for lack of a rigorous proof,[1] to be uniquely determined by the boundary conditions and independent of the initial state—i.e., independent of the transient processes through which it is established. Even in the absence of viscosity, when the approach to periodicity can be expected to be asymptotic, the existence of a uniquely determined final steady state, regarded in this case as a limiting state, is usually taken for granted.

These postulates stem from physical intuition and have also derived at least qualitative support from the experimental observation of the behavior of periodic-flow devices under controlled boundary conditions.

This note concerns an investigation, the results of which provide further evidence of the uniqueness of the final steady state under given boundary conditions. The object of this investigation was to determine, by the method of characteristics,[2] the final steady state or limiting state which would be approached by a particularly simple, inviscid, and isentropic flow system under given boundary conditions from various randomly chosen initial states.

The system chosen for the analysis consisted of a tube filled with and surrounded by air, open at one end and closed at the other end by a piston oscillating with simple harmonic motion

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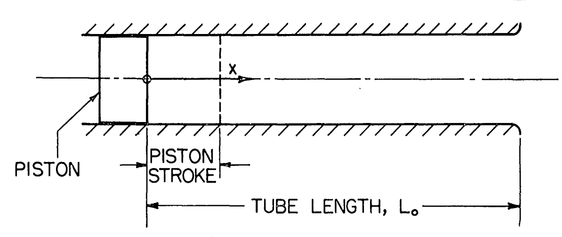

FIG. 1. System used for analysis.