THE INTERPLAY AMONG GESTURES, DISCOURSE AND DIAGRAMS IN STUDENTS' GEOMETRICAL REASONING

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Abstract

This study identifies forms of interactions with diagrams that are involved in conjecturing; more specifically, how students display their thinking publicly through using multimodal representations. We describe how students interact with diagrams in both gestural and verbal forms, and examine how such multimodal interactions with diagrams reveal their reasoning about diagrams. We hypothesize that when limited information is given in a diagram, students make use of gestural and verbal expressions to compensate for those limitations as they engage in making conjectures. As a byproduct, the study also proposes a set of graphical representations of gestures that have been identified as important for geometrical reasoning. These can be employed to codify the gestural interactions and to depict the practices of teaching and learning in geometry classrooms.

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"Mathematical reasoning is not auxiliary to basic goals of mathematics education but rather is fundamental to knowing and being proficient with mathematics-- that mathematical reasoning is itself basic." (Ball & Bass, 2003)

Principles and Standards for School Mathematics (NCTM, 2000) establish the expectation that students develop reasoning skills and be able to formulate and prove conjectures. Since diagrams help to state geometric problems and retrieve related geometric concepts geometric diagrams are key resources in students' geometrical reasoning. An investigation of students' interactions with diagrams may help us understand how students reason when making and proving conjectures about geometric objects.

Research has discussed the gap between the physical properties of a diagram and the geometrical representations of a figure (Duval, 1995; Fischbein, 1993; Laborde, 2005; Mariotti, 1995). Duval (1995) has argued that diagrams demand different kinds of grasps[i]. For example, students may need to grasp the figure operationally, that is, they may need to be able to modify the diagram mentally or physically. Students may need to grasp the figure perceptually to be able to recognize the properties of the figure by its shape, size or sub-figures. Students may need to demonstrate sequential grasp when constructing or describing a figure. And students may need to have a discursive grasp to identify the mathematical properties represented in the figure. Herbst (2004) has argued that students engage in different interactions with the diagrams, which are arguably tied to the instructional situations that frame the mathematical work they are called to do (Herbst, 2006). Some interactions with diagrams involve proximal contact with diagrams, such as in constructing or measuring. Other interactions involve using the diagram as referent to illustrate verbal statements that could be made without the diagram's existence. Yet other kinds of interaction use the visual inspection of a diagram as the source of verbal descriptions while they keep contact distal. These different kinds of interactions with diagrams may engage students in particular ways of thinking. We argue that some interactions may help advance students' reasoning and conjecturing. We are interested in how interactions with diagrams can support students in the work of figuring out whether a conjecture is reasonable.

The building of mathematical knowledge of geometric objects requires that one go beyond the making of empirical statements about figures. But since students' knowledge of mathematical objects depends on their representations, their building of knowledge is likely to require more than simple engagement in deductions from definitions and axioms. Herbst (2004) has argued that building geometric knowledge also requires students to make "*reasoned conjectures,*" statements about figures that arise through deduction from the possibilities of a geometric figure instantiated in a diagram. Herbst (2004) further proposes that, in order to engage in making “reasoned conjectures,” students may have to act on a diagram, creating representations for new geometric objects.

To effectively examine students’ reasoning through interactions with diagrams, both gestural and verbal expressions need to be observed. Gestures and words create a “multimodal representation” (McNeill, 1998) of students’ understanding (Kelly, Singer, Hicks, & Goldin-Meadow, 2002). The importance of language as representation of mathematical understanding has been addressed profusely. This literature has contributed to establish the notion that language not only expresses thought but also generates it (O' Connor, 1998; Sfard, 2001). We propose that gestures are another communication modality that can also be used to generate ideas rather than just express them. McNeill (1992) notes that gestures are “parts of the discourse” that can be seen as a mode of communication, especially in explanation and description (Roth & Welzel, 2001). When students present their conjectures, the use of gestures help them to develop and communicate complex explanations without the need to use formal mathematical language; thus gestures may enable students to engage in arguments about geometric objects before all those objects have been conceptualized formally and represented in formal language. With both gestural and verbal expressions, students can communicate more of their reasoning and thinking to their peers and teachers.

This study identifies forms of interactions with diagrams that are involved in conjecturing, particularly ways in which students display their thinking to the public through multimodal representations. The purpose of this study is to understand how students interact with diagrams in both gestural and verbal forms, and how such multimodal interactions with diagrams may advance their reasoning. Thus this study is led by the following questions:

- How is students' reasoning revealed through verbal and gestural forms?
- How are students' interactions with diagrams in the situations of making reasoned conjectures differ from those interactions typically found when students do proofs in customary geometry classes?

In this paper, we examine the role of students' interaction with diagrams, their use of gestures, and their use of language in their making of conjectures.

Conceptual Framework

Learning as participating in situated contexts

According to Lave and Wenger (1991), "learning is an integral and inseparable aspect of social practice" (p. 29). Full participation in the socio-cultural practice of a community contributes to successful learning. The learning-as-participation metaphor (Sfard, 1998) explains that learning is constructed through active engagement in a community of practice. That is, when participating in a situated activity, learners are embedded within the culture, and are expected to interact with the resources within that setting. Becoming active participants in particular situations is equated with learning (Brown, Collins, & Duguid, 1989).

From this situated learning point of view, the learning of mathematical practices requires becoming a participant of situations where those practices are done (Lave, 1988). Consider the case of the geometry class: To engage in geometrical thinking and learning, students have to engage in situations that involve interactions with diagrams. Such interactions may include working with diagrams verbally (i.e. describing), physically (i.e. drawing), or with gestures. Through the participation in the work with diagrams, students may make conjectures and justify them.

Intertextual meaning making

Intertextuality is generally defined as the juxtaposition of different texts, and refers to the construction of meaning among texts in different occasions (Bloome & Egan-Robertson, 1993; Johnstone, 2002; Lemke, 1992, 1995; Short, 1992). The meanings made intertextually are context or culture dependent (Lemke, 1995). Therefore, the intertextual relationships constructed among particular texts can reflect certain cultural practices in that particular community.

Short (1992) notes that learning and understanding are built upon intertextual connections. In a classroom setting, students need to identify, make meaning of, and correlate various texts, in order to construct meaning. Lemke (2002) argues that, to promote students' mathematics learning, language and visual representations should be seen as "an integral component of a larger sense-making resource system". That is, students have to make connections in the teacher's explanation through the teacher's speech, gestures and diagrams drawn on the board (Lemke, 1992). Or when students create diagrams, they have to refer to different resources, such as the mathematical concepts, the definitions, or teacher's previous explanations, and then make connections among these texts to create the new diagrams. Research on the intertextuality of the geometry classroom will enable us to see what meanings students can make and how they do so (Lemke, 2002).

In this study, two layers of intertextual relationships will be examined. First, intertextual meanings will be connected among students' speech, their gestures, and the diagrams drawn physically or virtually (through gesture). The second layer will be investigated in the employments of diagrams, specifically in how the diagrams students create and refer to evolve through students' work with them. For example, students may first have to identify the properties of a figure given the diagram in their worksheet. To do that students may need to refer to their prior knowledge regarding definitions and theorems and use those to make conjectures about the figure under consideration. Afterwards, they may have to alter the diagrams (e.g. marking or labeling), or create their own diagrams physically on the board or virtually through gestures, in order to justify their conjectures. Finally, they may need to apply the conjectures to the original figure.

Interactions with diagrams

Properties of figures

We use the words diagram and figure to mean different things. We save the word diagram to refer to the sign used in communication, and we save the word figure to the mathematical object that sign purports to refer to. The interrelationship between diagrams and geometric figures has been addressed in the mathematics education literature on visual perception and geometrical reasoning. This literature shows, among other things, that the figure a diagram points to is problematic. For example, Fischbein (1993) speaks of conceptual and figural properties of a figure. When students are working on geometric problems with diagrams they can access the visualized (perceived) image of those geometric objects as well as the concept of those objects. The relationship between these two can be complicated: the capacity to perceive a figure (through its diagram) has been identified as an obstacle to understanding a figure conceptually (Duval, 1995).

Laborde (2005) proposes two kinds of properties of a figure—spatio-graphical (SG) and theoretical (T) that may be revealed when students are working on geometric problems with diagrams. Theoretical properties are those necessitated by the definition of the figure while spatio-graphical are those which are contingent to specific cases of the figure, as eventuated in choices made when constructing a diagram (e.g., orientation, specific angle values, specific side lengths, etc.). According to Laborde, geometry beginners' identification and interpretation of figures tend to be based on spatio-graphical properties represented in diagrams. For example, students may determine that an angle is 90 degrees by actually measuring its representation in the diagram with a protractor. To advance students' interactive relationship with a diagram to a theoretical level, according to Laborde, requires further mathematical knowledge, exploration, and justification.

Students' interactions with diagrams

In his discussion of students' interactions with diagrams in geometry, Herbst (2004) proposes four modes of interactions between the actor, the diagram and the geometrical object (the figure). It is an *empirical* interaction when an actor relies on physical features of a diagram to make a statement about a figure. Within this mode, components of a diagram are identified with components of a figure (e.g., a dot *is* a point, a stroke *is* a segment), as if there was no semiotic mediation or as if this was iconic. On the contrary, *representational* interaction refers to when an actor uses the theoretical properties of a figure to make a statement about a diagram (e.g., to say what the diagram is meant to show; this is often aided by a markup convention that includes hash marks, arcs, arrows, etc.). Within this mode of interaction, components of a diagram are seen as indices or symbols for geometrical objects (components of a figure). While those two modes of interaction describe polar opposite ways of treating the relationship between diagram and figure, Herbst (2004) also identifies other modes of interaction.

Herbst (2004) identifies a *descriptive* mode of interaction and proposes it as characteristic of the role that diagrams play in the situation of “doing proofs” (Herbst & Brach, 2006) in high school geometry classrooms in the United States. Within this mode of interaction, diagrams include two layers: on the one hand, they *represent* the givens of the problem and contain other elements that can represent properties justified through the proof; on the other hand, they rather accurately embody properties that could be read off the diagram, suggesting to the user what could be asserted about the figure. When they are “doing proofs,” students use visual perception to hypothesize what could be true (thus interacting with the diagram in the empirical mode). But students are also expected to rely on diagrams only as symbols (using the markings to detect which elements of a diagram signify elements of the figure) at the time of justifying the statements they make (thus interacting in the representational mode part of the time). The descriptive mode alludes to this hybrid mode of interaction.

In order to have students make “reasoned conjectures” and construct mathematical knowledge, Herbst (2004) suggests that students have to interact with diagrams *generatively*. The work of making reasoned conjectures involves students in making hypotheses and predicting what could be true about a figure. Generative interactions with a diagram that might support such work include creating objects in the diagram that were not originally given and attributing status of geometric objects to them and prescribing hypothetical (possible) properties of diagrams that rely on those objects. Mathematical arguments can be assisted by those generative actions (e.g., if I slide a vertex of a triangle on a line parallel to the opposite side, the height and the base will be constant, so the area will be the same). An important distinction between the generative and the descriptive modes of interaction is that generative interactions put the agent in proximal contact with the diagram, altering it, unlike the descriptive mode in which contact is distal and limited to perception.

Gesture as a meaning making symbol system

Gestures are like "symbols" (McNeill, 1992) of the "visible action as utterance" (Kendon, 2004). Gestures may represent certain meanings which language may not convey properly. Kendon (2004) notes about gestures that, "at times they are used in conjunction with spoken expressions, at other times as complements, supplements, substitutes or as alternatives to them" (p. 1). Due to the nature of speaking and gesturing, gestures and utterance may occur at different time scales. However, Kendon states that gestures and utterance should be seen as a unit of spoken communication.

McNeill (1992) categorizes gestures into four types: iconic, metaphoric, deictic, and beat. *Iconic* gestures are hand movements that represent the meaning conveyed through speech. *Metaphoric* gestures are similar to iconic ones in that they all present ideas, but metaphoric ones display abstract concepts or relationships. Both iconic and metaphoric gestures require imagistic thinking to certain extent. Unlike the above two kinds of gestures, d*eictic* gestures are also known as pointing movements, which include both abstract and concrete pointing. Finally, *beats* are small movements that have little meaning by themselves, but rather complement discourse, for example by marking its pace.

In a classroom setting, gestural and verbal expressions can provide a multimodal representation of students' thoughts (Kelly, et al., 2002; McNeill, 1998). More importantly, gestures can be taken as part of students' explanation and communication, especially when their ideas or concepts are not yet well developed (Goldin-Meadow & Singer, 2003; Roth & Lawless, 2002a; Roth & Welzel, 2001).

Studies on gestures in science and mathematics learning suggest that gestures are ways to express explicitly students' imagistic thoughts and spatial reasoning (Cook & Goldin-Meadow, 2006; Nemirovsky & Noble, 1997; Nemirovsky & Tierney, 2001; Nemirovsky, Tierney, & Wright, 1998; Noble, Nemirovsky, Wagoner, Solomon, & Cook, 1996; Roth & Lawless, 2002b; Roth & Welzel, 2001). By using verbal and gestural expressions, students can show their visualization of diagrams more explicitly (Presmeg, 2001).

As Kendon (2004) suggests, the interpretation of gestural and verbal expressions should be contextualized. To investigate the role of gestures in geometric reasoning, it is important to examine how gestures are employed and involved in the interactions with diagrams.

When students need to make conjectures on a diagram, gestures are visible resources with which they can describe what they are seeing in diagrams. Gestures can be used as tools to prescribe what could or should be true about a figure by depicting a diagram in a particular way. In this study we are particularly interested in how students utilize gestures to further their reasoning, and what gestures represent students' geometrical thinking.

The modality system of language as resource for making statements

Modality is a subsystem of language which is used to encode the various degrees of uncertainty that lie between polarities (Halliday, 1985; Halliday & Matthiessen, 2004; Martin & Rose, 2003). It contains resources to express degrees in a spectrum between the two poles of positivity and negativity for various kinds of attitudinal meanings. Comparing to modality, polarity indicates the two poles of positivity (yes) and negativity (no). Modality and polarity allow additional meaning to the statements. Modality is a way for speakers to express their opinions and make statements that are less determinate than those stated as "yes" or "no."

Modality can be seen as a tool for identifying the degrees between the positivity and negativity. For example, between the positive statement of “this is a triangle” and the negative statement of “this is not a triangle”, there may be different degrees of possibility about the figure, from very possible to less possible—“it must be a triangle”, “it would probably be a triangle”, “it might possibly be a triangle”. The modality system of language provides semiotic resources to encode that range of possibilities.

According to Halliday (1985, 2004), there are four types of modality: probability, usuality, obligation, and inclination: probability and usuality state the intermediate degrees of propositions (what is the case), whereas obligation and inclination show intermediate degrees of proposals (what should be the case). Within each category, modality can be expressed in various degrees of values and these can be realized with lexical choices. For example, degrees of probability, from high to low, can be conveyed with words such as “certainly/ probably/ possibly/ unlikely.” Likewise, usuality can be realized with words such as always/ usually/ sometimes/ never. Students’ use of the modality system is a crucial observable in assessing the nature of their interaction with diagrams.

Data Collection and Method

Data selection

The data in this study comes from a corpus of video recordings of classroom practices in a project that studied mathematical work in high school geometry classrooms. An intervention was developed by the second author to observe students' interactions with diagrams. The lesson was based on a problem in which students had a context for making conjectures about angles formed by parallel and intersecting lines, a topic that had not yet been taught in the class. Given the diagram shown in Figure 1, students were asked to determine the measures of all angles formed by the given lines, but challenged to measure the least number of them. No information was given about the relationship between lines (e.g., while two pairs of lines appear to be parallel, nothing was said about them being parallel or not). Two teachers, Megan Keating and Lucille Vance [ii], implemented the intervention in five of their classes.

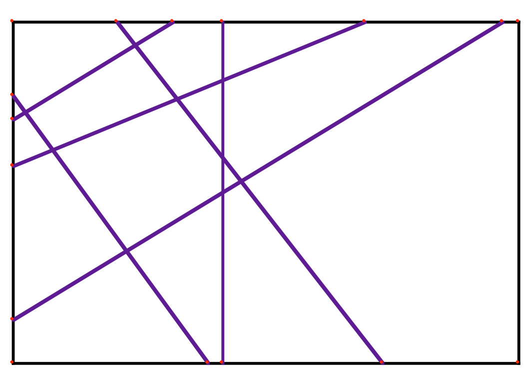


Figure 1. Parallel and intersecting lines

To help highlight students' interactions with diagrams in the intervention lesson, an intact lesson was selected from the corpus to examine; we compared how the uses of gestures and language in both settings. An intact lesson refers to the daily customary teaching practice in high school geometry classroom. Records of intact lessons had been collected in both Megan's and Lucille's classes on a weekly basis through the school year.

Video segments from five intervention lessons and one intact lesson were identified in which students interacted with diagrams in public. To understand how students' reasoning and justification of conjectures were communicated to the teacher and their peers, we attended to their gesture and discourse in students' public interactions with diagrams. Students' public interactions with diagrams could appear in different guises: For example, students might talk at the board and interact with the diagrams drawn on the board or shown on an overhead projector. Or students might talk at their seat referring to the diagram presented on the board. However, among the intact lessons gathered in this study, students rarely had public interactions with diagrams. In most of the cases, students were called up to the board to write their solutions to homework problems and then present the solutions. The intact lesson selected in this study represents a typical case in which students interact with diagrams as they present their homework publicly.

Method

In order to better illustrate how students express their thinking in public, it is important that the video data are analyzed through transcripts and video images (Zack & Graves, 2001). First, along with transcripts, “mind reading” (McNeill, 1992) gestures was used in interpreting gestures from video images. According to McNeill, mind reading is “noticing the gestures with which speakers unwittingly reveal aspects of their inner mental processes and points of view toward events when these are not articulated in speech” (p. 109). As gestures are seen as an “imagistic form” of speakers’ utterance, it is useful to mind-read the gestures that are not explicitly expressed in the speech. Thus, mind reading gestures can help us identify students’ reasoning that is absent in utterances, the gestures that compensate the constraints of diagrams, and the thinking in the geometrical setting that may be depicted by the gestures.

To capture the authenticity of students’ uses of gestures and their interactions with diagrams, we represent those gestures graphically. Various graphic representations for gestures were developed based on the gestures identified in actual classrooms. This graphic representation of gestures can be seen as a tool for transcribing other gestural expressions in classroom interaction. Integrating transcripts and gestures will show us what diagrams students are referring to, what diagrams they are drawing virtually or on the board, what specific marks they are adding to the present diagrams, and what gestures they are making when talking about the diagrams.

Second, we further analyze students' discourse by identifying the markers of modality that students used to express meanings about the diagrams. As noted above, modality differs from polarity in that modality contains degrees of uncertainty and allows space for negotiation. When students make conjectures and justify them, their use of modality in the discourse indicates that they are not stating facts known as true. Instead, they propose ideas that they are uncertain yet, and as they provide reasons to justify them, they express stronger degrees of modality.

To handle the discourse in which students interacted with diagrams in public, we parsed the transcripts into clauses, and then identified tokens of modality in each clause. We look at the following indicators: (1) finite modal operators, for example, "must/ should/ might" show the degrees of obligation or inclination from high to low; (2) modal adjuncts, for example, degrees of usuality, from high to low, can be expressed with the words as "always/ usually/ sometimes/ never." In the context of interacting with diagrams, students might say how likely it is that the figure would be what they think or what they think should be true. In particular, utterances that prescribe that figures *have to be* or *should be* in certain ways point to an interaction with a diagram aimed at generating necessary geometrical properties of a figure (Herbst, 2004). This event may show that students' reasoning focuses on more than "spatio-graphical" properties (Laborde, 2005) of the diagrams, and reason with diagrams at the theoretical level.

However, a high degree of modal expression, such as ‘always,’ is still less certain than the polar expression: Polarity includes positive (such as “it is so”) and negative (such as “it is not so”) statements. Unlike modality, polarity does not allow space for negotiation. Polarity statements in students’ discourse may indicate that students are referring to the facts they have known as true in diagrams. We hypothesize that students use polarity when stating facts about the diagrams, and they show modal expressions when making and justifying conjectures.

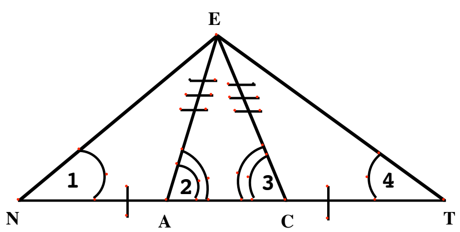
Data Analysis

In this section, we compare students’ interactions in the intact and experimental lessons from three perspectives: First, we identify common actions, such as labeling and marking, on the diagrams in both settings. Second, we distinguish students’ uses of gestures when interacting with diagrams. Finally, we attend to the modality expressions in students’ discourse.

Interactions with diagrams through labeling and marking

Marking the diagram was commonly observed in students’ presentations of their work on diagrams in both intact and experimental lessons. The selected intact lesson is from Megan Keating’s class. At the beginning of the lesson, four students were asked to present their solutions to the homework on the board, including drawing diagrams, writing two-column proofs, and then presenting them to the whole class. Marcus presented his diagram (see Figure 2a) and proof (see Figure 2b) without being asked to correct them, which tacitly implies the teacher approved of them. In the intact lesson, students would not ordinarily label a mathematical object that had not been labeled by the textbook.

a.



b.

Statements	*Reasons*
1. $\angle 1 \cong \angle 4$, $\overline{NA} \cong \overline{TC}$	*1. Given*
2. $\overline{EN} \cong \overline{ET}$	*2.* $2 \cong base\Delta \rightarrow 2 \cong opp.$ *sides*
3. $\Delta ENA \cong \Delta ETC$	*3. SAS pst*[iii]
4. $\overline{EA} \cong \overline{EC}$	*4. CPCTC*
5. $\angle 2 \cong \angle 3$	*5.* $2 \cong$ *opp. sides* $\rightarrow 2 \cong$ *base* $\angle$*'s*

Figure 2. Marcus's boardwork

To get a better sense of how Marcus interacted with the diagram, the diagram given in the textbook (Boyd, et al., 1998, p.226) is provided below (see Figure 3a-3b). As it shows, the labels were given in the original diagram in the textbook, and what Marcus did was to add different marks to highlight the angles and segments that were mentioned in the proof. Specifically, he made the same number of hash marks to indicate the congruence of a pair of segments, and the same number of arcs to show the congruence of two angles.

a.

Given: $\angle 1 \cong \angle 4$

$\overline{NA} \cong \overline{TC}$

Prove: $\angle 3 \cong \angle 2$

b.

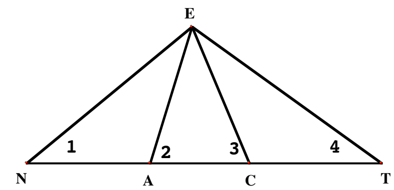
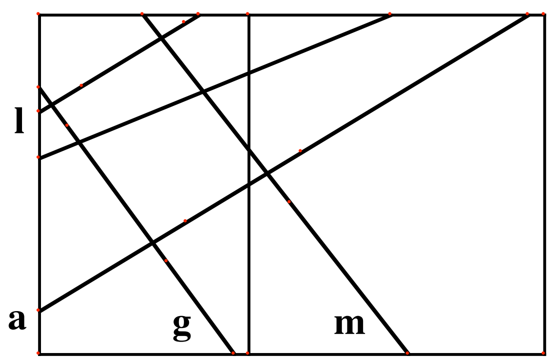


Figure 3. The book problem Marcus was working from
(drawn according to Boyd, et al., 1998, p.226)

In the intervention lesson, in addition to marking a diagram, students also labeled objects in the diagrams when they presented their findings at the board. For example, in Megan's second period class, the first group that was called up to the board, labeled the four lines as *l, a, g, m* before their presentation to the class (see Figure 4a). They also made the assumption that lines *l* and *a*, and lines *g* and *m* are two pairs of parallel lines. This action supported students making references to the diagram when they stated and explained their conjectures. Later, when they showed the measurements of the angles, they drew arcs on the angles with equal measurements that had been obtained by measuring one and deducing others with the assistance of their conjectures about the parallelogram formed by the two pairs of parallel lines (see Figure 4b).

a



b.

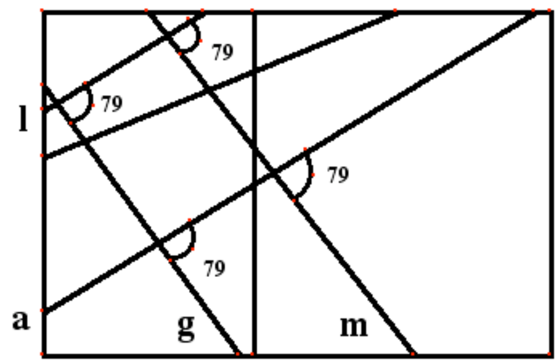


Figure 4. (a) Two pairs of lines were labeled to be parallel in the group's conjecture (b) Angles were marked with arcs with measurements

Interactions with diagrams through gestures

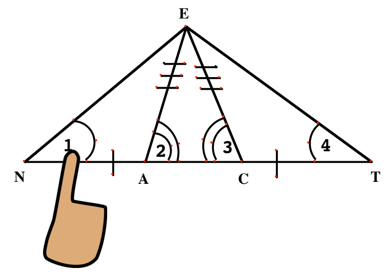
In this section, we examine the different uses of gestures in the intact and intervention lessons. Particularly, we focus on how students used gestures to represent various geometrical ideas that were not visually available.

The intact lesson: Making references by pointing

In Marcus's short presentation of the solution to a homework problem, he drew the diagram and wrote the complete proof before the oral presentation:

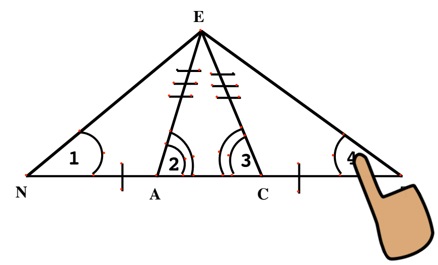
Marcus: Alright! The given for 38 was, uh, angle 1 is congruent to angle 4, which is uh, two base angles (points to the angle 1 and angle 4 on the board respectively, see Figure 5a-5b)[iv] of the triangle NET,

a.



Marcus points to angle 1

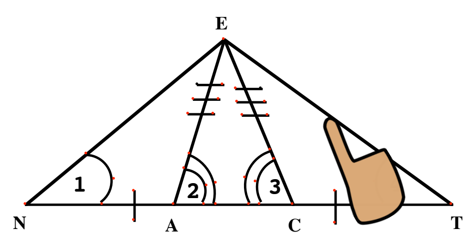
b.



Marcus points to angle 4

and then $\overline{NA}$ is congruent to $\overline{TC}$, so from there, I put $\overline{EN}$ is equal to $\overline{ET}$, because two congruent base angles give you two congruent opposite sides,

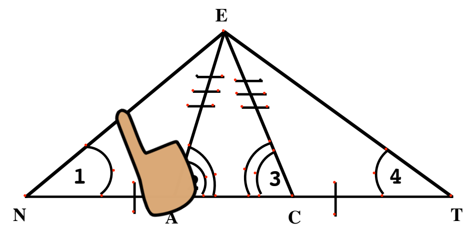
c.



Marcus points to segment $\overline{ET}$

which would be these two (points to segment $\overline{ET}$ and $\overline{EN}$ respectively, see Figure 5c-5d),

d.

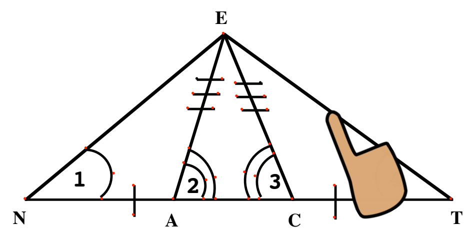


Marcus points to segment $\overline{EN}$

and then from there,

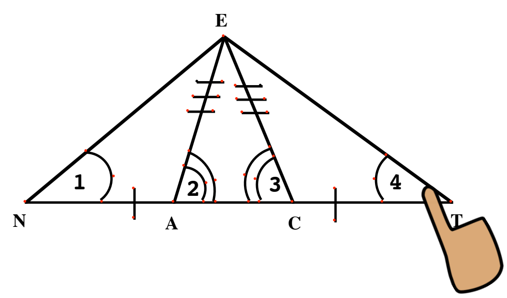
e.

you can say, triangle ENA is congruent to triangle ETC,



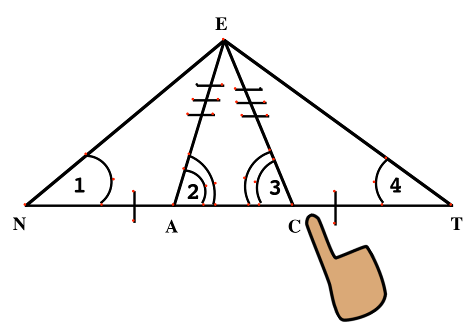
Marcus traces segment $\overline{ET}$

f.



Marcus traces angle ETC

g.

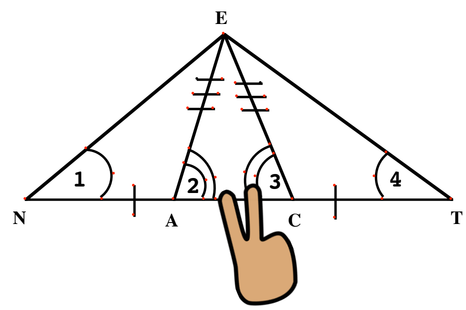


Marcus points to vertex C

because of the side-angle-side postulate,

right there (traces angle ETC from
segment $\overline{ET}$, to vertex T and then to
segment $\overline{TC}$, see Figure 5e-5g),
and then from there,
you can say,
$\overline{EA}$ is equal to $\overline{EC}$,
because of CPCTC,
and then, uh,
you can say, these two angles (points to
angle 2 and angle 3 simultaneously, see
Figure 5h),

h.



Marcus points to angle 2 and angle 3 simultaneously

angle 2 and angle 3 are congruent,
because two congruent opposite sides
give you congruent base angles.

Figure 5. Marcus points to the diagram when talking through the proof

As is shown above, Marcus talked through his proof on the board step by step. When certain parts of the diagram were mentioned in the proof, he pointed to the angles and segments, or virtually highlighted the triangles.

The intervention lesson: Extending lines outside the given box

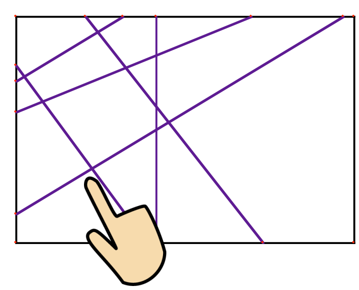
In the intervention lesson, in order to get more measurements of angles without actually measuring them, students had to apply known properties, such as the angle sum theorem of a triangle. Students used gestures that depicted a virtual diagram onto the given diagram in order to propose some hypothetical situations outside the given frame.

Conjecturing a triangle. In Megan's second period, Audrey proposed that two of the given lines would intersect outside the given frame, thus forming a triangle. She first identified a possible triangle by pointing to two vertices in that hypothesized triangle (see Figure 6a and 6b) and pointing to a spot outside the given frame (see Figure 6c). This virtual spot indicates the third vertex formed by the continuation of two line segments. Based on this virtual diagram, Audrey could get the measurement of the third angle in the virtual triangle by applying the angle sum theorem. This is how she explained it:

Audrey: If you…alright

You know that this (points to the up left vertex of the triangle, see Figure 6a) is eighty,

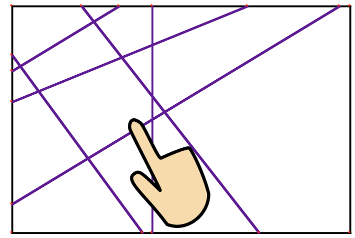
a.



Audrey points to the up left vertex of the triangle

and this (points to the up right vertex of the triangle, see Figure 6b) is eighty[v],

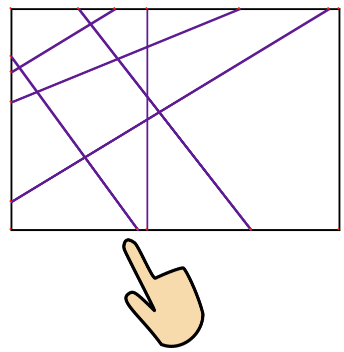
b.



Audrey points to the up right vertex of the triangle

So this (points to the virtual third vertex, outside the screen, see Figure 6c) has to equal a hundred…uh I mean one eighty

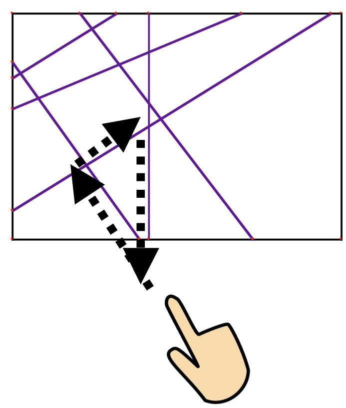
c.



Audrey points to a spot outside the screen

[Megan: the triangle] [vi] the triangle (traces loosely around the triangle, and stops at the spot outside the screen, see Figure 6d) has to equal one eighty

d.



Audrey traces loosely around the triangle, and stops at the spot outside the screen (Dotted lines added indicating the tracing path; arrows indicate the directions of tracing movements)

so eighty (points to the top left angle in the triangle) plus eighty[vii] (points to the top right angle in the triangle) and then you have to add forty in here (writes "40" at the virtual third angle)

Figure 6. Audrey visualizes a triangle

Although the intersecting point was not shown on the given frame, and whether the two lines would intersect outside the given frame was not stated in the given activity, Audrey and her group mates made the assumption that a virtual triangle existed. To justify this assumption, she virtually gestured a triangle and applied the angle sum theorem to show that if there was to be an intersecting point, formed by the two extended line segments, as the third vertex of a triangle, they would be able to know the measure of its angle.

Conjecturing two lines parallel. Students used their gestures to show that two lines were parallel, and to virtually indicate that the two parallel lines would extend outside the screen. In Lucille Vance's fourth period class, Reed pointed to the two lines that might be parallel from his seat. He had his thumb and index finger parted to show the constant distance between the two lines (see Figure 7a-b), and moved along with the two lines virtually outside the screen (see Figure 7c). The conception of parallelism was displayed by Reed's gestures from two perspectives. First, his open palm with some space between his thumb and index finger showed the equal distance between two parallel lines. Secondly, the movement of his hand—tracing and extending the two lines outside the screen—conveyed the sense that two parallel lines would not intersect as they were extended.

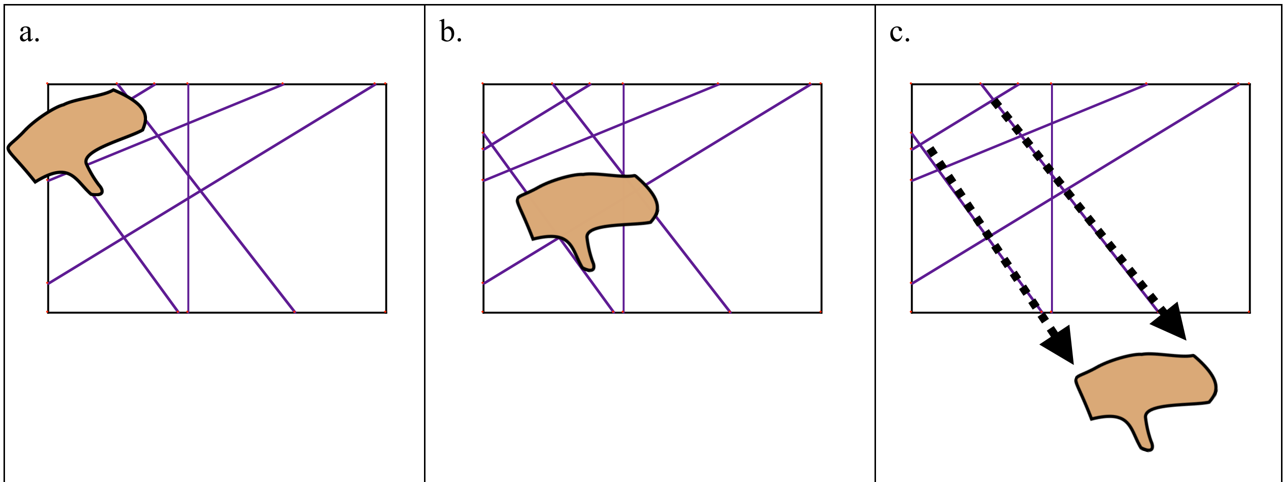
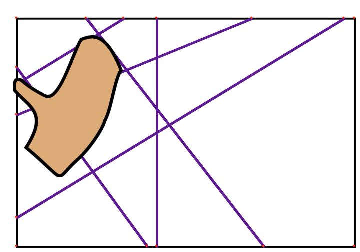


Figure 7. Reed traces the two lines conjecturing they are parallel (Dotted lines added, indicating the tracing path; arrows indicate the directions of tracing movements).

The conjecture of those two lines were parallel was further justified. In Megan's fourth period, a group of students came up with the same conjecture regarding two lines parallel. When asked about the reason of the two lines being parallel, Collin and Anthony pointed out that the two lines would not intersect anywhere, and would not form a triangle outside the screen.

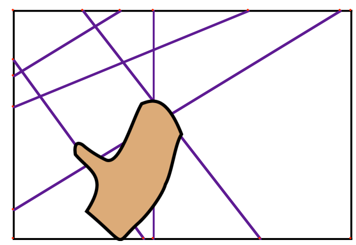
Collin: We found out that since
If you extended these two lines (use his thumb and right index finger to trace, and virtually extends the two lines outside the screen, see Figure 8a-8c)

a.



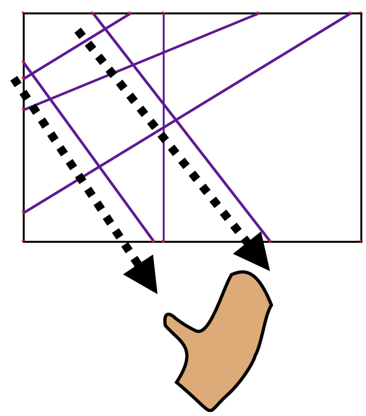
Collin traces the two lines from top

b.



Collin traces the two lines downward

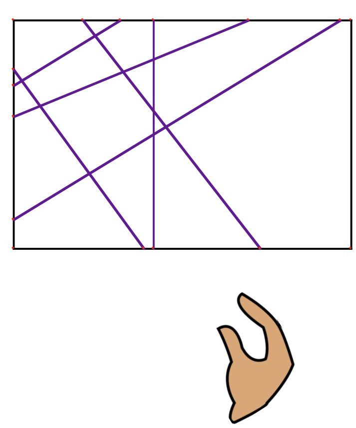
c.



Collin traces the two lines, and stops at a spot outside the screen

You'd eventually,
If they would not be parallel,
you get a triangle (hand rests on a spot that might be the intersection of the two lines, see Figure 8d)

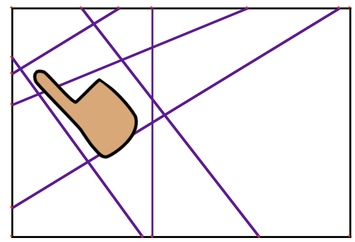
d.



Collin narrows the space between his thumb and index finger, and rests the wrist at a spot outside the screen

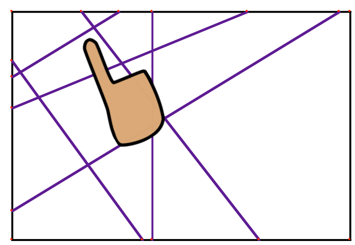
and then, to find the measure of that final angle,
You'd add those two together (points to the two interior angles formed by the two lines and an upper transversal, see Figure 8e-8f),

e.



Collin points to the left interior angle formed by the two lines and an upper transversal

f.

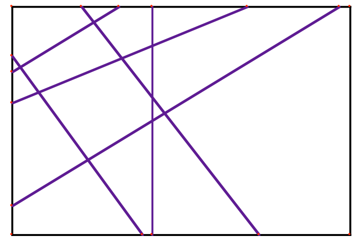


Collin points to the right interior angle formed by the two lines and an upper transversal

and subtract that from one eighty, and one ten plus seventy is equal to one eighty,
so…

Anthony: There can't be another point down here somewhere (points to a spot far from the screen, see Figure 8g)

g.



Anthony points to a spot far from the screen

Figure 8. Collin and Anthony prove two lines being parallel

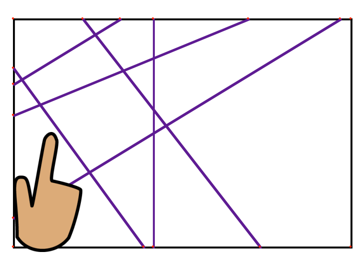
In this episode, Collin and Anthony justified their conjecture through an argument that resembles a proof by contradiction. Collin first stated that the two lines should be parallel by moving his open palm along outside the screen (see Figure 8a-8c) as Reed did in the previous example. Then, to prove that these two lines are parallel, he assumed that the lines would intersect at a certain point if the statement were false. Therefore, a hypothetical intersecting point was positioned outside the screen (see Figure 8d). Instead of gesturing with an open palm, Collin narrowed the space between his thumb and index finger. This variation of gestures indicates his differentiating notions between parallelism and incidence.

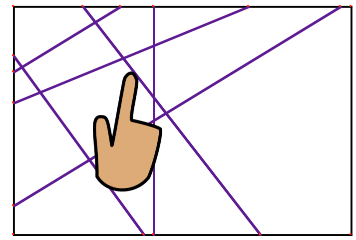
After making the assumption regarding two lines being intersected, he attempted to get the measurement of the third angle in the virtual triangle formed by the two lines. However, with the angle sum theorem and the known measurements of two other angles in the virtual triangle, it is impossible to have a third angle anywhere. Anthony's gesture, by pointing a spot (Figure 8g) farther than the one that Collin had, insisted on the impossibility of an intersecting point by two parallel lines.

The Intervention lesson: Reasoning about parallelism

Gestures play a dynamic role in students' reasoning about two lines parallel in the following segment. In Megan's third period class, two students, who were working in the same group, were asked to present their conjecture about two lines being parallel.

After working in groups for 22 minutes, Yakim was called up to the board to present his group's finding. He claimed that two lines were parallel and that a transversal would make alternate interior angles congruent. However, he could not provide further justification of this conjecture.

29	Yakim:	so, well, first, we figured out that these (points to the transparency projected on the board, see Figure 9a-9b) two lines are parallel,	a.  Yakim points to a line b.



Yakim points to another line

30 they weren't exactly parallel,
31 but, like, we figure that they pretty much meant them to be parallel,
32 so, we did, we made them parallel,
33 and um, since they are parallel,
34 we could find out angles across from each other,
35 because of the alternate interior angles postulate

Figure 9. Yakim claims that two lines are parallel

The diagram given in the task involves two major lines that are not said to be parallel. In order to work on the task—to measure the least angles to get all the measurements of the angles—Yakim and his group mates, decided to "make" (line 31, 32) the two lines to be parallel. And based on this assumption, they argued the congruency of alternate interior angles.

Later, Yuri, a member in Yakim's group, elaborated the conjecture from his seat. He used gestures to sketch a virtual diagram, and to explain that corresponding angles would be congruent if the two lines cut by a transversal were parallel.

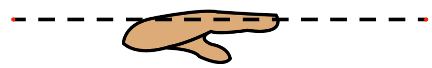
58 Yuri: I was just going to say that if the lines are parallel

59 Megan: yeah,

60 Yuri: then one minute you can figure out is you see that,

61 if you have one line (palm placed horizontally, see Figure 10a),

a.



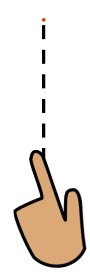
Yuri shows his right lower palm horizontally to virtually create a horizontal line (dotted line added indicating its virtual property)

62 I mean if you have one angle that is,

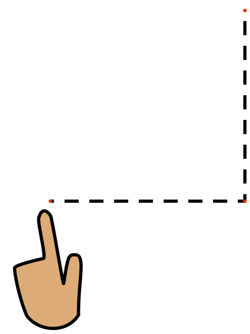
63 like, let's say 90 degrees (sketches virtually a 90-degree angle, see Figure 10b-10d),

b.

c.



d.



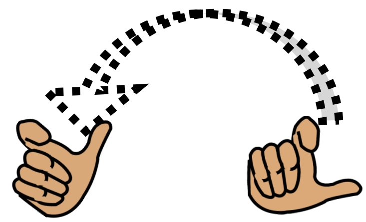
Yuri virtually creates a 90-degree angle

64	Megan:	yeah,	
65	Yuri:	then you have two lines,	
66		and if both of the lines are parallel,	
67		then you can tell that your…You have to have exact same angle,	
68		because they have to intersect [at the same point][2]	
69	Yakim:	[Yeah, because the] transversal is like...	
70	Yuri:	because both lines (uses his thumb and index finger to show the constant distance relationship between two lines. See Figure 10e), [both parallel lines,] (swings his right wrist, see Figure 10f)	e.

Yuri uses his thumb and index finger to show the constant distance relationship between two lines that are parallel

f.

[2] [ ] indicates overlapping speech



Yuri swings his wrist to show that the distance between two parallel lines remains the same even if they both are slanted at different angle

71 Megan: [go draw] what you are talking about, draw it up there!

Figure 10. Yuri virtually draws a diagram with gestures at the seat

To justify the conjecture—corresponding angles are congruent if two parallel lines are cut by a transversal—Yuri used various gestures to construct a specific case of diagram. First he placed his palm horizontally to virtually introduce a transversal (see Figure 10a). Secondly, he "drew" a 90-degree angle (see Figure 10b-10d), indicating a vertical line perpendicular to the previous line. Adding another line verbally ("then you have two lines", line 65), Yuri created a pair of lines intersecting with a transversal as an example to illustrate the conjecture. Specifically, this example consisted of two parallel lines perpendicular to the transversal (see Figure 11a). To show the parallel relationship between the two lines, similar to his counterparts, he used his right thumb and index finger to show the constant distance between the two parallel lines (see Figure 10e-10f), no matter at what angle they intersect with the transversal. Figure 11b shows Yuri's gesture, representing parallelism, was inscribed on the virtual diagram.

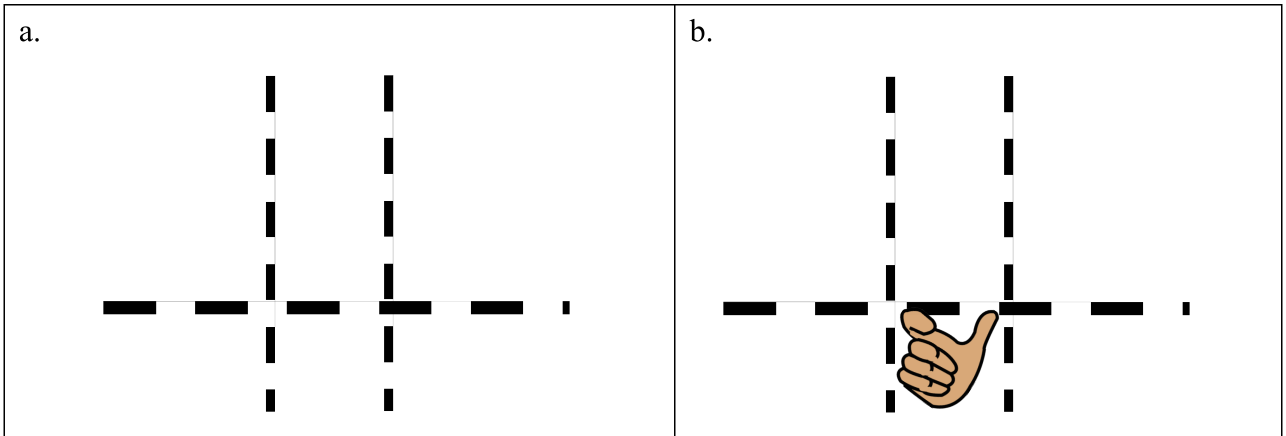
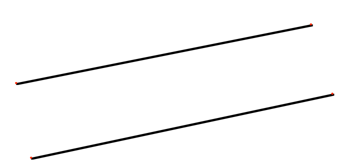


Figure 11. (a) This diagram is sketched based on Yuri's description with gestures. (To signify its virtual character, the diagram is drawn with dotted lines) (b) Yuri's gestures make his virtual diagram

However, Yuri was speaking from his seat and thus his example was not visually available to his peers and the teacher. The teacher then asked him to draw his virtual diagram on the board. He drew two pairs of parallel lines, and marked two pairs of corresponding angles on the board. He also used a variety of gestures to show that the parallel lines would be slanted in certain way that would make the corresponding angles congruent.

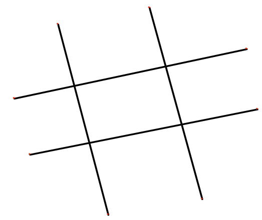
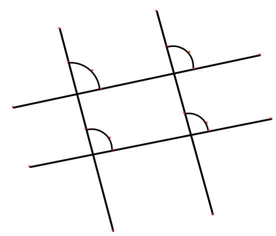
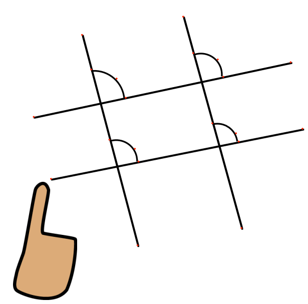
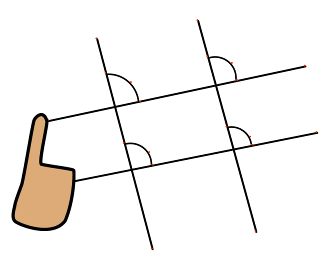
75 Yuri: so we meant these are parallel (draws two lines with a distance in between, see Figure 12a)

a.



Yuri draws two lines that have certain distance in between on the board

76 Megan: those are parallel, yeah.

77	Yuri:	and these are parallel (draws another pair of lines that intersect the previous pair, see Figure 12b)	b.  Yuri draws another two lines intersecting the previous pair
78		then and we can tell that all these (draws fours marks on the four intersecting angles formed by the two pairs of lines previously drawn, see Figure 12c) have to be equal,	c.  Yuri marks four angles formed by the two pairs of lines
79		because these, we know that both of these lines (points to the first pair of lines, see Figure 12d-12e) have to be slanted at same angle, right?	d.  Yuri points to the lower line of the first pair
			e. 

Yuri points to the upper line of the first pair

80 The entire line can either be, like, completely horizontal (places his right palm horizontally, see Figure 12f) or....

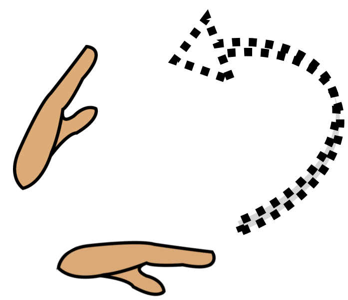
f.

Yuri shows his right palm horizontally

81 Megan: Ok,

82 Yuri: slanted. (swings his right palm, see Figure 12g)

g.

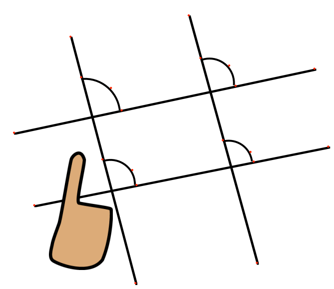


Yuri swings his right palm (dotted arrow refers to the direction of gesture movement)

83 Megan: Ok, I can go with that

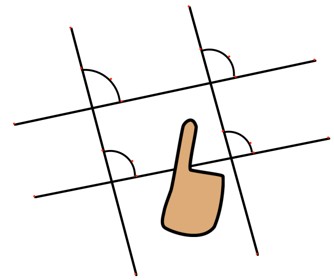
84 Yuri: and these lines (points to the second pair of lines, see Figure 12h-12i),

h.



Yuri points to the left line of the second pair

i.



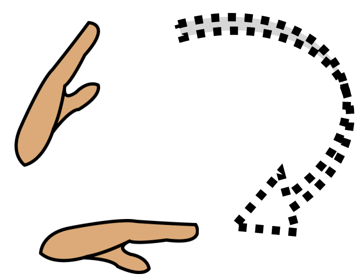
Yuri points to the right line of the second pair

85 and these lines have to be either, completely vertical (holds his right palm vertically, see Figure 12j) or slanted (swings his right palm, see Figure 12k),

j.

Yuri holds his right palm vertically

k.



Yuri swings his right palm (dotted arrow refers to the direction of gesture movement)

86 but they, they both have to be the same... (uses his thumb and index finger to show the constant distance relationship between two lines, see Figure 12l),

l.

Yuri uses his thumb and index finger to show the constant distance relationship between two lines

87 they both have to be slanted on

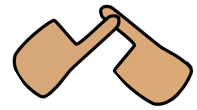
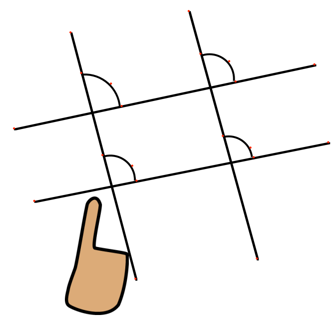
		the same angle [Megan: Okay]	
88		Therefore, when they intersect (crosses his two index fingers, see Figure 12m), you're always going to get the same angle	m.  Yuri crosses both his index fingers to show the intersecting relationship of two intersecting lines
89	Yakim:	Yeah, the opposite angles, but not…	
90	Yuri:	No, these…. (points to one of the intersecting angles he marks earlier, see Figure 12n)	n.  Yuri points to one of the intersecting angles he had marked earlier
91	Megan:	Those are corresponding	

Figure 12. Yuri gestures with the diagram to illustrate parallel lines make the corresponding angles congruent

Yuri first drew two sets of parallel lines intersecting with each other (see Figure 12a-c). He positioned his palm to show the orientation of each individual set of parallel lines: first, his palm was placed horizontally representing the horizontal pair of parallel lines (see Figure 12f), so he swung the palm upward to simulate the motion of parallel lines (see Figure 12g). Similarly, his palm latter represented the vertical pair of lines (see Figure 12j), and the swing of the palm was downward (see Figure 12k). With his palms swinging to represent the slanted orientations of sets of parallel lines, he showed that two sets of parallel lines should intersect at the exact same angles no matter how slanted each set of parallel lines were.

Interactions with diagrams through modality and polarity

In this section, we revisit the episodes in which students interacted with diagrams in public, and analyze the mathematical discourse using the notions of modality and polarity. We parse the transcripts into clauses, and examine the meanings that students made of the diagrams. We look for indicators of modality such as the finite modal operators and the modal adjuncts. For example, the finite modal operators *must* be, *will* be, and *might* be express degrees of probability from high to low; modal adjuncts such as *always*, *usually*, or *sometimes* show degrees of usuality from high to low. Unlike modality, polarity (positive vs. negative; yes vs. no) leaves no space for uncertainty or negotiation. Polarity refers to positive or negative statements, such as “it is so” and “it is not so.” We identify expressions of modality and polarity in students’ discourse when they make references to the diagrams. Then we compare how those expressions are different in intact and intervention lessons.

In the intact lesson, Marcus talked about his proof written on the board:

Marcus: Alright! The given for 38 was, uh, angle 1 **is** congruent to angle 4,
which **is** uh, two base angles of the triangle NET,
and then NA **is** congruent to TC,
so from there, I put EN **is** equal to ET,
because two congruent base angles **give** you two congruent opposite sides,
which would be these two
and then from there,
you can say**,** triangle ENA **is** congruent to triangle ETC,
because of the side-angle-side postulate,
right there
and then from there,
you can say**,** EA **is** equal to EC,
because of CPCTC,
and then, uh, **you can say**, these two angles, angle 2 and angle 3 **are** congruent,
because two congruent opposite sides **give** you congruent base angles.

Excerpt 1: Polarity indicators boldened and modality indicators underlined

As is shown above, when Marcus was referring to the properties of diagrams, he dominantly used present tense to show the positive statements (e.g. "is") about the diagram. For example, "$\overline{NA}$ is congruent to $\overline{TC}$", or "$\overline{EN}$ is equal to $\overline{ET}$" or "because two congruent base angles give you two congruent opposite sides". Some of the statements were given from the problem, e.g. $\overline{NA}$ *is* congruent to $\overline{TC}$*;* some statements were inferred from the given, e.g. $\overline{EN}$ is equal to $\overline{ET}$. These statements about diagrams are in the positive pole with no space for negotiation (Martin & Rose, 2003). Marcus also expressed modal meanings when he was generating new statements from previous ones. For example, the statement he made "you can say, angle 2 and angle 3 are congruent", is based upon the previous statement "EZ is equal to EC." Although the modal expression may show the inference he made has certain degree of uncertainty, it is in the positive polar form when he was referring to the property of the diagram ("angle 2 and angle 3 are congruent"). Therefore, when Marcus interacted with the diagram, he did not have to negotiate the information in the diagram. Instead, he stated the facts about the diagrams from the given. When he was inferring information from the given or postulates, he still stated them as facts. This observation coheres with observations about the "doing proof" situation in high school geometry classes (Herbst, et al., 2009) in which students know the statement to be true before they do the proof.

Unlike the dominant use of polar (positive) statements in the intact lesson, in the intervention lessons students used modality expressions when making conjectures and justifying them. In the lesson based on the intersecting lines activity, the diagram was presented with no further information regarding the properties of lines and the measurements of the angles. It was expected that students would come up with different conjectures and justify them. Through looking at the modality in the discourse, we can see what the students assume the diagrams "can be" and how the diagrams "should be" in certain ways according to their conjectures.

Audrey: If you…alright

You know that this **is** eighty,
and this **is** eighty,
So this <u>has to</u> equal a hundred…uh I mean one eighty,
[Megan: the triangle] the triangle <u>has to</u> equal one eighty
so eighty plus eighty and then you have to add forty in here

*Excerpt 2***:** Polarity indicators boldened and modality indicators underlined

In excerpt 2, Audrey claimed that the two lines would intersect by assuming that if one extended two lines outside the screen, then the lines would form a triangle. First, she pointed out that each of the known angles "is eighty." The present tense here suggests that she obtained the measurements of the two angles by measuring with protractor, so the angle "is" eighty in an empirical sense (she latter corrected that one of the angle measurement is sixty degree). Based on the two measurements, the third angle of the virtual triangle "has to" equal forty degrees, because the angle sum of a triangle "has to equal to one eighty." Therefore, the third angle is highly "obliged" (Halliday & Matthiessen, 2004) to be 40 degrees. She switches the verb from present tense "is" to "has to", indicating that she is no longer measuring but inferring. She inferred from the angle sum theorem, and concluded that the third angle of the triangle has to be in certain degree.

In Collin and Anthony's episode in which they claimed that two lines are parallel, they started by assuming the possibility of the intersection of the two lines. With the support of evidence, they concluded with a strong degree of certainty that the two lines would be parallel.

Collin: We found out that since

If you extended these two lines
You'd eventually…
If they would not be parallel,
You'd get a triangle
and then, to find the measure of that final angle,
You'd add those two together
and subtract that from one eighty,
and one ten plus seventy **is** equal to one eighty,
so…
Anthony: There can't be another point down here somewhere.

*Excerpt **3***: Polarity indicators boldened and modality indicators underlined

At first, Collin proposed the possibility that the extended two lines "*would not* be parallel". Based on this assumption, these two lines would intersect at some point and then "you'd get a triangle." And since the two lines may intersect somewhere outside the screen, the measurement of the intersecting angle could be found. However, after adding up the known two angles to 180 degrees, Anthony proclaimed "there *can't be* another point down here", indicating that the two lines cannot be intersecting anywhere, and thus should be parallel. The measurements of two of the angles were obtained empirically, but then the students shifted from reporting to inferring, concluding with strong conviction that it was impossible for the lines to intersect one another.

Yuri presented an example to justify the conjecture about corresponding angles congruent if a pair of parallel lines is cut by a transversal. Yuri first proposed an example by sketching in the air a diagram with two vertical lines perpendicular to a horizontal line (see Figure 11a). He virtually sketched two parallel lines. Then he inferred that the two lines “have to” intersect with the horizontal line at the exact same angle (lines 67-68):

65	Yuri:	then you have two lines,
66		and if both of the lines are parallel,
		then you can tell that your…You have to have exact same
67		angle,
68		because they have to intersect [at the same point]
69	Yakim:	[Yeah, because the] transversal is like...
70	Yuri:	because both lines, [both parallel lines..]
71	Megan:	[go draw] what you are talking about, draw it up there!

Excerpt 5: With modality indicators underlined

After drawing the diagram on the board, Yuri used the drawing of the diagram and gestures to illustrate the justification. Since he had the visual evidence (the drawn diagram and gestures) to support his conjecture, he further expressed a high degree of certainty about the claim:

75	Yuri:	so we meant these **are** parallel
76	Megan:	those are parallel, yeah.
77	Yuri:	and these **are** parallel
78		then and we can tell that all these have to be equal,
		because these, we know that both of these lines have to be slanted
79		at same angle, right?
80		The entire line can either be, like, completely horizontal or....
81	Megan:	Ok,
82	Yuri:	slanted.
83	Megan:	Ok, I can go with that
84	Yuri:	and these lines
85		and these lines have to be either, completely vertical or slanted
86		but they, they both have to be the same...
87		they both have to be slanted on the same angle [Megan: okay]

Therefore, when they intersect, you're always going to get the same angle
Yakim: Yeah, the opposite angles, but not…
Yuri: No, these….
Megan: Those are corresponding

Excerpt 6: With polarity indicators boldened and modality indicators underlined

With the diagram on the board (see Figure 12b), he first put four arcs on the intersecting angles formed by two pairs of lines (see Figure 12c), and proposed that "all these *have to* be equal" (line 78). He then further explained in more detail: he pointed to the horizontal pair of lines and claimed, "both of these lines *have to* be slanted at same angle" (line 79), because they "*can* either be horizontal or slanted" (line 80-82). This shows Yuri perceived a pair of lines as a unit that "have to" be oriented in the same direction due to their parallelism. The same conception of parallel lines was also applied when he was talking about the second pair of parallel lines (line 85-87) that the set of parallel lines "have to" be slanted at the same angle. With two sets of lines staying at specific angles, Yuri further claimed that when the two pairs of lines intersected, "you're always going to get the same angle" (line 88). This word choice ("always") indicates that there is high degree of usuality in the situation that two pairs of intersecting parallel lines form at identical angles. The conjecture about the corresponding angles congruent in parallel lines cut by a transversal is made and justified with high certainty.

As the preceding description shows, in the intervention lessons, students were able to state different degrees of probability or usuality toward diagrams in their conjectures or justifications.

They made more frequent use of the resources of the modality system in order to use the given diagram to make assumptions and provide justifications for possible facts.

Discussion

In this study, students' interactions were observed from three different perspectives-- the use of diagrams, the use of gestures, and the use of language. In what follows we identify similarities and differences in the interactions with diagrams between the intact lesson and the intervention.

Use of diagrams

Students engaged in different interactions with diagrams in the different lessons. In the intact lesson, the diagram (see Figure 3) was given with labels for vertices and angles. These labels implicitly hints at what elements are likely to be needed when producing the proof (Herbst, 2004). Indeed, when Marcus did this proof he didn't need to do any additional labeling on the diagram (see Figure 2a).

The diagram given in the intervention lesson did include labels and consequently it did not hint at what elements to use. Students were expected to come up with conjectures and then justify them. They took responsibility in identifying which elements might be involved in a conjecture, and how the selected elements might feature in the conjecture. Students labeled the lines that were related to their conjectures and only those.

The “incompleteness” of the diagram given in the intervention lesson gave an opportunity for visualization. Beyond the given frame, students could visualize the lines extended to form a triangle, or extended indefinitely and not meeting. Therefore, constraints in the given diagram, such as the frame within which the lines were drawn, or the lack of labels, actually involved students in creating signs (e.g. virtual points made with joining fingers) to point to possible objects (e.g., intersection between two lines). The intervention lesson engaged students in interactions with diagrams that may not be seen in customary geometry class, an interaction that we would describe as *generative* (Herbst, 2004).

To justify the conjectures, students sketched virtual diagrams. These virtual diagrams were made based on students’ hypothetical claims, and pointed to as the geometric referents needed to help students prove those claims and conjectures. For example, Audrey conjectured that the angle between two lines would be 40 degrees and used for that the observation that two lines would form a triangle if extended. To prove her conjecture, she virtually drew a triangle and got the measurement of the third virtual angle by the angle sum theorem (see Figure 6). Collin and Anthony followed a similar approach to prove that two lines should be parallel, by showing that the virtual triangle could not exist (see Figure 8).

In his justification that corresponding angles are congruent if two parallel lines are cut by a transversal, Yuri demonstrated different uses of diagrams. In addition to drawing a virtual diagram in the air, he further "modified" the virtual version and presented the modified version on the board. First, he sketched a virtual diagram in the air (see Figure 11a) to elaborate the conjecture. Then he was called up to the board to actually draw the diagram (see Figure 12b) he was referring to. Considering the intertextual relationship between the virtual diagram created by gestures and the diagram drawn later on the board, these two diagrams are arguably different. The virtual diagram consisted of one pair of parallel lines intersected by a transversal, while the diagram on the board were two pairs of parallel lines intersecting with each other. The virtual diagram conveyed Yuri's initial justification of the conjecture, and then his modified version of diagram on the board helped him deepen and strengthen his justification.

The use of gestures

Gestures were employed in various ways in the selected episodes. Deictic gestures were commonly used when students were interacting with diagrams at the board in both intact and intervention lessons. This type of gesture is utilized to point to an object on the diagram and to draw the audiences' attention (McNeill, 1992). For example, when Marcus was presenting his proof in the homework, he pointed to the individual angle or segment with his index finger (see Figure 5a-5d), and pointed to the two angles at the same time with his right index and middle fingers (see Figure 5h). Besides pointing, Marcus also used his index finger to trace the sides of a triangle, a sub-unit of the diagram, to highlight a specific portion that calls for attention. The elements of the diagram that Marcus pointed to or traced were all visually available on the board.

However, in the intervention lessons, in addition to gesturing deictically, students utilized gestures to express ideas that involved imaginary objects, the elements of figures that were not visually available. These imagistic gestures can be described as iconic in McNeill's classification; they represent virtual mathematical objects or abstract concepts in the following five ways.

First, students pointed at imaginary things. Unlike Marcus who, in the intact lesson, had only pointed to objects that were visually available and labeled, two students in the intervention lesson pointed to spots outside the given frame where hypothetical objects would be located: Audrey positioned a spot as the third vertex of a virtual triangle (see Figure 6c); and similarly, Collin picked a spot as a possible intersecting point of two extended lines (see Figure 8d).

Second, students traced along the given lines and virtually extended the lines out of the given frame. For example, Audrey traced around three lines that might form a triangle (see Figure 6d). Two other students similarly moved the palm along the two lines outside the screen, indicating that those two lines might be parallel even after extension (see Figure 7; Figure 8a-8c). This kind of gestural movements can be classified as *iconic* (McNeill, 1992) in that it represents the extensional properties of lines and their potential directions outside the given frame.

Third, students created their own virtual diagrams through gestures. In Yuri's demonstration, he sketched a virtual diagram to justify the conjecture regarding corresponding angles congruent if two parallel lines cut by a transversal. He first placed his right palm horizontally to represent a transversal (see Figure 10a); then he sketched a 90-degree angle with his index finger (see Figure 10b-10d), indicating a line intersecting with the virtual transversal he just created. With the verbal description “then you have two lines” (Excerpt 5, line 65), the vertical line was duplicated. Therefore, the virtual diagram was constructed with a pair of parallel lines and a transversal (see Figure 11a).

Fourth, students used gestures to represent properties of diagrams. For example, a constant distance between the thumb and index finger was used to portray the parallel relationship between two lines. This kind of gesture was commonly adopted in the selected lessons (see Figure 7; Figure 8a-8c; Figure 10e; Figure 12l). Besides, Collin slightly narrowed the distance between his thumb and index finger to suggest that the distance between two lines would gradually decrease if the lines would intersect at some point (see Figure 8d). In addition, an open palm was employed to represent one set of parallel lines (see Yuri’s case in Figure 12f and 12j), indicating that the angle of the parallel lines with the horizon stays the same. Thus, parallel lines were seen as a set that has an identical orientation. Hence, the properties of figures can be symbolically expressed by various gestures that reveal students’ conceptions about figures. They revealed that parallel lines are equidistant, and they also revealed that the distance between two non-parallel lines would gradually decrease as the two lines would eventually meet.

Finally, students gestured to animate the dynamic movements of diagrams. By swinging his wrist back and forth or palm up and down, Yuri illustrated the consistent identical orientations of a set of parallel lines (see Figure 12g; Figure 12k). This kind of dynamical proposition of gestures displays the abstract concept of the parallel property of two lines; thus can be categorized as metaphoric in McNeill's classification.

In the intervention lessons, gestures were extensively employed to represent the objects that had not been represented in the diagram, and so to externalize the students' conception of figures. Comparing the uses of gestures in both intact and intervention lessons, we argue that the different nature of the diagrams given call for different uses of gestures. Gestures were used only deictically in the intact lesson, since the given diagram from the homework provided all information (e.g. labels) that was needed in the proof. On the contrary, in the intervention lesson, the given diagram consisted of parts of several lines. The limited given information and visual constraints, however, allow for extensive and diverse uses of gestural expressions. Therefore, gestures can be seen as mediation tools to make up for the constraints and limitation of diagrams.

The use of language

Through modality, students' verbal interactions with diagrams are differentiated in the intact and intervention lessons. In the intact lesson, where the student presented his proof from homework, his dominant uses of present tense indicates that he was stating the facts about diagrams. The positive statements (e.g. is, are) show that no ambiguity or uncertainty exists in his proof statements or his perceptions of the diagram.

Unlike the polarity statements in the proof from the intact lesson, modality expressions were observed in students' presentations of diagrams in the intervention lessons. With the aids of gestures creating hypothetical diagrams, students were able to justify the conjectures with high degree of certainty (e.g. have to), or to state high value of usuality of a figural property (e.g. "you're always going to get the same angle").

Modality were also shown in the case of proof by contradiction. In the case of proving two lines being parallel, Collin first proposed that the two lines "would not" be parallel as an impossibility. After justifying with reasonable statements, Anthony concluded that there "can't" be any intersecting point formed by the two parallel lines (see Excerpt 3). "Can't" shows high certainty of nonoccurrence. The degree of impossibility increased as Collin and Anthony concluded the justification of their conjecture. These two modality expressions communicate degrees of impossibility, which reflects the characteristic of proof by contradiction.

The different uses of modality expressions and gestures in the intact and intervention lessons can be attributed to the different characteristics in the given diagrams. As mentioned before, the given diagram in the intact lesson presents the diagram as a perceptibly isosceles triangle, and includes the labels that are needed in the proof. Hence, in doing the proof, students are expected to read the object of reference (the isosceles triangle) using the provided signs (labels of vertices and angles), and no further alterations of diagrams are required. The sufficient information actually hinders students in making plausible claims about diagrams, thus limiting their uses of gestures to depict hypothetical diagrams. This kind of interaction can be identified as “descriptive”, along the lines of what Herbst (2004) states that is customary in high school geometry classes.

Herbst (2004) proposes a generative mode of interactions with diagrams that contrasts with the descriptive. In the generative mode students generate their mathematical arguments by actual interactions with diagrams. Generative interactions involve creating new signs to complement a diagram, so that students can “think with” (Herbst, 2004) diagrams and predict that the figure “should be” a particular way. This kind of interactions with diagrams was observed in the intervention lessons. Through gestural expressions, students virtually illustrated potential extension or orientations of diagrams. Thus, gestures depict certain “hypothetical phenomena” (Pozzer-Ardenghi & Roth, 2005) that could possibly be true to the figures represented by the diagram. Through modality expressions, students stated plausible claims with different degrees of certainty toward the diagrams. Therefore, modality can be seen as a tool to express the reasonableness of the statements.

Fischbein (1993) suggests that promoting conflicts between figural and conceptual aspects of diagrams could help students develop “figural concepts”. The constraints of the diagram (intersecting lines with no other information, and lines discontinued at the given frame) invited students’ gestural and verbal involvements in making hypothetical statements about diagrams and further justify their conjectures.

Hence, gestures complement the limitations of static diagrams and provide dynamic elements to support students’ reasoning. As McNeill (1992) states “gestures, together with the accompanying speech, offer a privileged view of thought. They are the closest look at the ideas of another person that we, the observers, can get” (p. 133), the coordination of speech and gestures could lead to effective communication of thoughts in classrooms (Roth & Welzel, 2001), especially in understanding students’ thinking and reasoning.

Conclusion

This study identifies forms of interactions with diagrams that are involved in conjecturing, particularly ways in which students display their thinking in public through multimodal representations. Although reform in mathematics education stresses the importance of conjecturing and proving, it has been argued that students in customary geometry classes usually have limited opportunities in making reasoned conjectures about figures (Herbst, 2004, 2006). This study shows that the nature of diagrams provided could play a role in what gestural and verbal interactions students develop with diagrams. The constraints of diagrams may enable students to use particular gestures and verbal expressions that rather than reporting on known facts permit them to make hypothetical claims about diagrams. It is to be expected that if a diagram did not include signs to represent all the objects that could be talked about, students' allusions to those objects, were they to occur, might be more conjectural than factual. Our report, however, shows that iconic and metaphorical gestures as well as modality expressions are mediation tools that are available to compensate the semiotic limitations of diagrams (e.g., their lack of elements drawn or labeled), and could be especially important in enabling students to engage in such conjecturing.

The analysis of gestures highlights the importance of multimodal representations in understanding students' thinking and learning. Using the four types of gestures that McNeill (1992) categorizes, we have identified contextualized uses of gestures representing geometrical ideas in different ways, especially representing ideas that are not visually available: First, deictic gestures can position a hypothetical point on a diagram. Second, gestures can be used metaphorically to "extend" existing lines. Third, gestures can be used to create signs for possible objects. Fourth, gestures may symbolize specific geometrical properties of diagrams (i.e. the constant distance between the thumb and index finger representing the parallelism between two lines). Last, gestures simulate the range of possibilities of a figure by displaying dynamic virtual diagrams.

Gestures could be conceived as mediation tools to observe students' thinking. The five different uses of gesture in the context of geometrical thinking identified in this study help examine students' conceptions of geometrical properties. More research on gestural uses in specific settings could contribute to understanding students' thinking and learning.

In addition, the graphic representations adopted in this study can be utilized as a tool to codify the gestural communication in mathematics classroom practice. Further developments of graphic gestural expressions that can represent the authentic interactions in classrooms could help in capturing the essences of students' interactions, and consequently, in understanding how students make sense of mathematical ideas.

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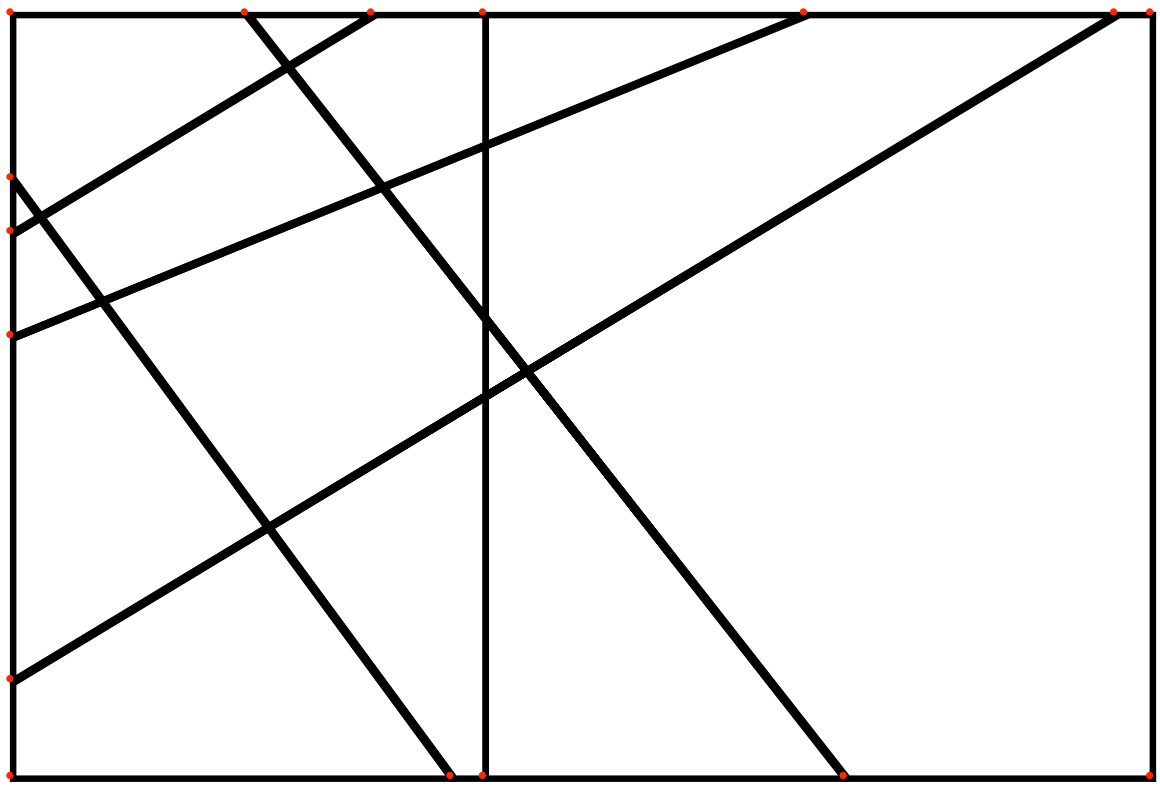
Appendix A: Activity Worksheet of the Intervention Lesson

Worksheet on angles formed by intersecting lines

Name:

Date:

Period:



There are six lines on the paper and some of their intersections are not visible.

1. Would it be possible for somebody to determine the measures of all the angles formed by those lines, considering that not all angles can be measured? Explain.
2. What is the total number of different angle measures that one would need to determine? Explain.
3. How many of those angle measures would be impossible to find unless one could extend the lines beyond the screen limits? Explain.
4. What is the minimum number of angles that one would have to measure before being able to say "I know all the angle measures"? Explain.

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[i] Duval calls these grasps “apprehensions,” in the sense of capture.

[ii] All the names of the teachers and students in this paper are pseudonyms.

[iii] “pst” refers to postulate.

[iv] The gestural actions are denoted with parenthesis.

[v] Audrey latter corrected that this angle is sixty degrees.

[vi] The overlapping speech is denoted in square brackets.

[vii] Audrey latter corrected that this angle is sixty degrees, so the third angle (the virtual angle) is 40 degrees.