Revisiting the functions of proof in mathematics classrooms:

A view from a theory of instructional exchanges

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Abstract

We consider the functions of proof in mathematics from the perspective of the work of the mathematics teacher. The work of proving and the time spent on proving, what can a teacher account it to? How can he or she justify it? We frame that problem in a descriptive theory of teaching and place within that frame the work of scholars who have inquired on the function of proof in mathematics. We argue that the multiple functions that proof plays in mathematics are resources that a teacher could use to account for the work of proving. We describe how the functions of proof identified in the literature can assist the work of the teacher and illustrate the role these functions of proof can play using classroom scenarios that showcase the work of proving. Since the teacher is not only accountable to mathematics but also accountable to students' learning of that mathematics some times work is valuable because it helps represent important mathematical knowledge, sometimes because it helps students acquire, or demonstrate they have, knowledge. The existence of these different sources of value is not only a resource for the teacher to value diverse work but also permits to anticipate management dilemmas concerning the different ways of accounting for the work of proving.

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The purpose of this paper is to provide an instruction-based frame to the question, often asked in mathematics education, of what the functions of proof are in a classroom. We take aim at a problem of teaching: The work of proving and the time spent on proving, what can a teacher account it to? How can he or she justify it? We frame that problem in a descriptive theory of teaching and place within that frame the work of scholars who have inquired on the function of proof in mathematics (Bell, 1976; Hanna, 1990; de Villiers, 1990). We argue that the multiple functions that proof plays in mathematics are resources that a teacher could use to account for the work of proving. We describe how the functions of proof identified in the literature can assist the work of the teacher. Yet the teacher is not only accountable to mathematics when accounting for classroom work; she is also accountable to students' learning of that mathematics. Those two demands for accountability provide different figurative currencies with which to value what is done in the classroom: Some times work is valuable because it helps represent important mathematical knowledge, sometimes because it helps students acquire, or demonstrate they have, knowledge. The existence of these two currencies is not only a resource for the teacher to value diverse work but also permits to anticipate management dilemmas concerning the different ways

of accounting for the work of proving. We illustrate how a teacher could account for the work of proving using scenarios of geometry instruction produced as animations of cartoon characters. We use a reconsideration of functions of proof in mathematics to show how these could provide currency for the teacher to value different kinds of mathematical work. The scenarios also permit to anticipate the different dilemmas a teacher might need to contend with by virtue of being accountable not only to mathematics but also to students' learning. In particular we illustrate how these dilemmas could undercut the possibility that certain kinds of mathematical work be done in the classroom. While this article presents a conceptual discussion and illustrations, a subsequent submission examines empirical data from teachers' responses to those scenarios; such data informs about teachers' perceptions of those dilemmas and tensions.

Several have written about the function of proof in mathematics, taking as background the usual logical (or philosophical) reason, according to which a proof confers truth to a statement. That "verification" function of proof is also, often interpreted in subjective terms, equating the truth of a statement with an individual's belief in the truth of a statement and thus allocating proof a role in the subjective acquisition of such belief. Along those lines proof has been described as ascertaining for oneself or persuading others, convincing oneself, a friend, or an enemy, about the truth of a statement (Harel & Sowder, 1998; Inglis & Mejía Ramos, 2009; Mason, 1982). Duval (2002) has alluded to these two ways of reading the relationship between proof and knowledge, by saying that a proof can change the logical value as well as epistemic value of a statement: that is a proof may logically validate a statement, but it can also affect the belief of the cognizing subject on the truth of the statement.

Those two functions of proof—to convince individuals and to establish results in the field—while apt to justify why proof deserves a place in mathematics classrooms, are by no

means the only functions of proof in mathematical activity. Mathematics education scholars have contributed to such elaboration on the functions of proof both by reflecting on its many roles in the discipline of mathematics and by identifying its roles in human understanding. From these roles researchers have derived recommendations for the direction of teaching in classrooms. Thus Hanna, (2000), Hersh, (1993), and (Knuth, 2002) have argued that proof has a role as a tool for promoting mathematical understanding in students. On the other hand, historically and at least in selected places in the curriculum (such as in the teaching of Euclidean geometry), proof has had a natural place in the communication of mathematical knowledge: The mathematical text to be communicated included proofs along with the statements of theorems. At least in the United States but also in other countries, the study of geometry has been a place where students have been charged with producing proofs (Herbst, 2002). The implementation of calls, such as NCTM's (2000), to extend the place of proof to more of the curriculum run into the issue of how a teacher may account for attention to the work of proving. While the argument has traditionally been that proof is central to mathematical work, this paper attempts to unpack that argument by inquiring into the different roles that proof plays in mathematical work and how those might apply to the mathematics classroom. We take as our charge to organize this field by embedding those various roles within an explicit treatment of the teacher's role as manager of instruction.

A Theoretical Perspective on Instruction and the Work of Teaching

We propose that mathematics instruction proceeds as a sequence of exchanges or transactions between, on the one hand, the moment-to-moment, possibly interactive work that students do with their teacher and on the other hand, the discrete claims a teacher can lay on what has been accomplished. Central to this theory of instructional exchanges is the notion of

didactical contract (Brousseau, 1997): The hypothesis that a bond exists that makes teacher and students mutually responsible vis-à-vis their relationships with knowledge. In particular, a contract exists that makes the teacher responsible to attend not only to the student as learner of mathematics but also to mathematics as the discipline to be represented so that it can be learned. A descriptive, general hypothesis of our theory is that one such contract exists in any instance of institutionalized mathematics instruction. Particular classrooms may have specific customary ways of negotiating and enacting that contract and those may vary quite a bit, but we expect they will always include specific ways in which the teacher is held accountable not only to the students' insofar as learners of mathematics but also to mathematics as the discipline to be represented for its learning.

A second, related hypothesis helps us observe the work of the teacher in instruction. This hypothesis derives from the observation that classroom activity takes place in multiple timescales: While meaningful classroom interactions can be detected at a timescale of the fraction of a second, progress in the syllabus and consequential examinations take place in a larger timescale, of months and marking periods in a year. Thus, the second key hypothesis is that the work of the teacher includes managing activities and objects in two different timescales: the work done moment-to-moment (at the scale of the utterance) and the mathematical objects of knowledge, which exist at the larger scales of the month, semester- or year-long curriculum (Lemke, 2000, p. 277). In other words, we hypothesize that a teacher needs to operate symbolic transactions or exchanges between activities in one timescale and objects of knowledge in the other: activities at the scale of moment-to-moment interaction serve the teacher to deploy or instantiate mathematical objects of knowledge and reciprocally objects of knowledge serve to account for the activities done moment-to-moment. Further, we posit that this exchange is

complicated because, after the first hypothesis, the expression "mathematical objects of knowledge" contains two kinds of contractual implications: on the one hand objects of knowledge need to be *represented* so as to be available for study and learning by students, and on the other hand students need to learn them, eventually *come to know* them. While the relationship between those two elements is often seen as linear—namely knowledge representations are objects of study and learning, eventually being known by students—the notion that a didactical contract exists underscores the bidirectional nature of that relationship: Since students *have to* learn, the process of instruction includes making representations of knowledge that *can* be learnt by those students. Transformations and transpositions of the knowledge to be learned happen partly in response to the need for that knowledge to be learned (Chevallard, 1991).

A consequence of those hypotheses is that as the mathematics teacher manages exchanges between the mathematical work done and the objects of knowledge to lay claim on, she needs to attend to two complementary sets of values or two currencies. The work done needs to be accounted for as *representation or embodiment of mathematical ideas (or practices) at stake* in the contract.[5] And the work needs to be accounted for as *students' actual learning* of the ideas that they have contracted upon. The value of the work done can be assessed in each of those currencies. Of course we don't mean to say that every bit of interactive work done necessarily has to be valuable in both currencies; what we say is that the teacher has two currencies with which they may give value to classroom work: One currency consists of claims

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[5] This latter exchange contains two assumptions. One of them, likely to be unproblematic among mathematics educators is that the teacher is accountable to the discipline of mathematics for representing the subject matter. The other assumption is that such representation is achieved in and through interactive work: Knowledge is not only represented in the texts that students use or in what the teacher says but more generally in the discourse that is publicly created by humans, through language and with artifacts. The notion that a piece of knowledge may be represented through the coordination of multiple players is not only apparent in the traditional philosophical dialogue (as in Plato's works) but also in orchestral music, dance, theater, and film.

on the representation of knowledge and the other currency consists of claims on students'

learning of that knowledge. A particular chunk of classroom work may be of high exchange

value on one and low (or perhaps also high) on the other of those two currencies.

In describing the work of the teacher as one of managing exchanges between work done

in interaction and objects contracted upon we therefore allude to the need for the teacher to

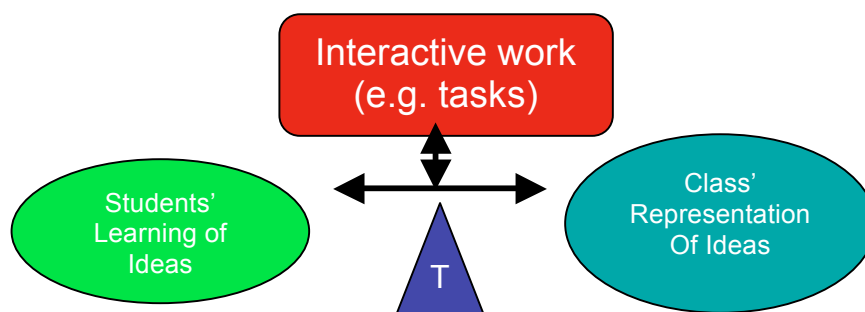attend to these two kinds of currency. Figure 1 represents those two.

Figure 1. Instructional Exchanges.

In what follows we apply this framework to examine the functions of proof in the

classroom. With this we mean to say we examine how a teacher might account for the work of

proof. In this paper we leave the expression "the work of proof" somewhat vague, to mean any

classroom performance that an observer (including the teacher) might describe by an appeal to

the label "proof" (but see Herbst & Balacheff, 2009, for a more precise way of describing the

work of proof). As noted above many kinds of work observable in the classroom might be

described with the label "proof;" from the reproduction of what Euclid wrote after stating the

Pythagorean proposition to the reasoning followed by a student as he derives a plausible

conclusion from a statement. Our argument is that these varying kinds of work may fulfill one or

more from a list of diverse functions, as these functions have been described in prior scholarship

on proof, and thus a teacher may account for them, or value them using different value tokens. Our general claim is that the various "functions" or "roles" of proof in mathematics constitute a set of tokens that a teacher may use to allocate value to classroom performances. Those tokens could be available for a teacher to value classroom performances, though the extent to which teachers perceive and appreciate those values is an empirical question, which we take on in the companion piece to this article.

Functions of Proof and their Role in Instructional Exchanges

In this section we revisit each of the functions of proof that has been identified in the literature and unpack the theory of instructional exchanges to explain how each of these functions of proof might play a role in the practice of instruction. We exemplify how each of these functions of proof could be used to put a value on classroom performances by using them to inspect classroom scenarios in a common course of studies, the traditional high school geometry course in the United States. These scenarios have been represented in animated movies that were created by project ThEMaT (Thought Experiments in Mathematics Teaching) to facilitate thought experiments with teachers about what could happen in this course (see Herbst & Miyakawa, 2008; Herbst, Nachlieli, and Chazan, in revision, for more information on those animations). Like video records of classroom episodes, animations can be used to spur conversations among practitioners about practice; unlike classroom episodes, animated scenarios can be designed to showcase practices that might be rare in actual classrooms. The animations don't mean to claim that such practices exist; rather, they mean to showcase what the practices could look like so as to ground their consideration by the theory of instructional exchanges.

The work of proving may count as verification of or conviction about the truth of a statement

Mathematical statements don't go without saying in the discipline of mathematics as they might, for example, in revealed or mystical knowledge. They are claimed to be true and their truth can be verified or refuted. Thus in the classroom the work of proving could be accounted for as accomplishing the verification of a statement. The management work of the teacher, effecting a transaction of the work done for the claim that the truth of a statement is known, includes not only attesting that the claim has been verified but also that students are convinced of the truth of the statement.

Philosophy of science has traditionally described the difference between mathematics and the sciences in terms of their means of verification of claims. While claims to scientific truths are verified by experimentation, claims to logical and mathematical truth are verified by proof. Scholars like Bell (1976), de Villiers (1990), Hanna (1982), and Hersh (1993) have alluded to this function of proof. Bell (1976, p. 24) indicates that the role of proof is verification or justification of the truth of a proposition. Hanna (1990, p. 9) indicates that 'proofs that (only) prove' show that a theorem is true. In mathematics education scholarship, the verification function of proof has often, also been given a subjective interpretation according to which proofs are done to convince or persuade humans of the truth of a statement. Thus Mason (1982) spoke of proving as convincing oneself, convincing a friend, and convincing an enemy. Harel & Sowder (1998) conceived of "proof schemes" as oriented to ascertaining the truth of a statement for oneself and to persuading others of the truth of a statement.

Thus one function of proof is to show that a proposition is demonstrably true in mathematics. The truth of the statement being what matters, it might be just as good to know one proof than to know another one, or perhaps to know only that a proof exists (Bass, 2009). But

classroom work serves not only to represent knowledge but also to promote and demonstrate students' learning. In appraising the work done, a teacher needs to attend not just to whether the truth of a statement has been established but also to whether the audience stood the chance of being, and perhaps was, convinced of the truth of that statement. This presents a set of contingencies associated with a teacher's chance to effect a transaction between the proving work done and the claim that the statement is known as true. Note how these contingencies play out in the following scenario.

The animated story "The Midpoint Quadrilateral[6]" provides grounds to consider the role that this verification function of proof in mathematics can play in mathematics instruction. In the story, the teacher has tasked the class with proving that the midpoint quadrilateral, the dual, of an isosceles trapezoid is a rhombus (see Figure 2). A student, Kappa, attempts an argument that uses two observations made by other students. One observation, made earlier that day by Beta, asserted that the midpoint quadrilateral of an isosceles trapezoid has congruent consecutive sides. The other, a conjecture made the day before by Lambda, asserted that the dual of any quadrilateral is always a parallelogram. Kappa added that since the dual is at least a parallelogram, "opposite sides are congruent" and since also "sides next to each other are congruent" then "all [sides are] congruent. So it has to be a rhombus." In the story, the teacher denies Kappa the opportunity to use Lambda's conjecture to prove this claim, since Lambda's conjecture had not been proved yet. Then Kappa spends most of his time looking for a proof that will verify Lambda's conjecture. The actions of the teacher in the story show one way in which a teacher could be responsive to the notion that to accept a statement as true, the existence of a proof is needed. This notion compels the teacher to reject the application of the statement that

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[6] This animated story can be seen in ThEMaT's Researchers' Hub, http://grip.umich.edu/themat

any midpoint quadrilateral is a parallelogram because this statement has not been proved yet. Additionally, the notion propels Kappa to search for a proof for the statement, whose existence would eventually enable him to use the statement that Lambda had conjectured. The function of proof as verification could assist the teacher in allocating value to two episodes: (1) his rejection of Kappa's first argument and (2) Kappa's later work searching for a proof of Lambda's conjecture. The story also exemplifies how a teacher may be caught in between the possibility that an argument that convinces a student might not quite satisfy mathematical standards to be accepted as a verification of a truth.
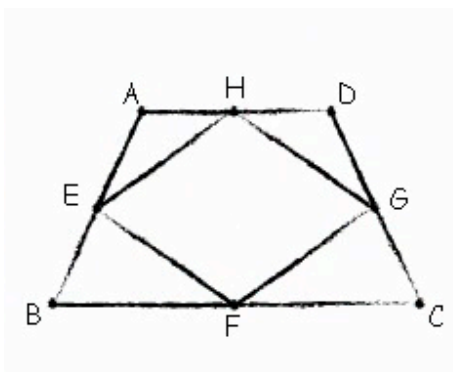

Figure 2: The dual of an isosceles trapezoid is a rhombus

In terms of the theory of instructional exchanges, the "Midpoint quadrilateral" scenario illustrates that a teacher may be confronted with the need to exchange two unequal terms. On the one hand there are chunks of mathematical work such as the work made by Kappa to prove that the midpoint quadrilateral of an isosceles trapezoid is a rhombus. On the other hand there are (1) the status that the teacher can attribute to that assertion as representations of knowledge and (2) what the teacher can acknowledge about the students' state of conviction about that assertion. As regards the statement, that the midpoint quadrilateral of an isosceles trapezoid is a rhombus, to

the extent that its proof hinges on a statement whose theorem status is yet unclear, the teacher does not feel entitled to give to it the status of proof that that Kappa's work seeks. And yet, Kappa himself appeared to be convinced by his argument, later seeking through a "proof" of Lambda's conjecture to convince the teacher. The various contingencies associated with whether the work done amounts to verification, conviction, or both of them span a possible tension in teaching: if the work done to verify mathematically does not convince students, is its value sufficient? Likewise if the work done to convince students does not amount to a mathematical verification, is it worth enough?

But conviction and verification are not the only functions of proof that could be used to allocate value to classroom work. In fact, De Villiers (1990) has critiqued this reduction of proof to verification or conviction adducing that, in mathematical work, a mathematician's inner conviction of the truth of a statement often *precedes* his decision to prove. Yet the existence and importance of proof in mathematics is warranted on other functions that it serves.

The work of proving may count as explanation or understanding of a statement

Mathematical statements are connected with other statements by way of the concepts they predicate about. A second set of stakes of the work of proving is associated with the contingencies of on the one hand explaining mathematically why a statement is true and on the other hand, of attesting to students' understanding of what the statement means.

Hanna (1983), building on the work of philosopher Mark Steiner (1978), has contrasted the function of proof as verification of a statement with the role of proof as explanation of a statement. The word explanation here alludes to showing how the asserted property of a mathematical concept coheres and connects with the known properties of that mathematical

concept. A proof can have the function of explaining why a theorem is a reasonable thing to say about a known concept by showing how the statement of a theorem coheres and connects with the key properties of the concepts involved in the proof. In this sense a proof can be an important tool for representing mathematical knowledge as made of connected, as opposed to disconnected, concepts and procedures. This explanatory function of proof has also been given a subjective interpretation in the notion that proofs can help students understand the meaning of mathematical ideas. Thus Hanna (2000) has expressed support for this function of proof in preferring proofs that explain over proofs that merely prove and in noting that the most important role of proof is to promote students' understanding of mathematics. Knuth (2002) has echoed it arguing to teachers that proofs are valuable because they can help students understand mathematics.

Figure 3. Steps of the construction of an angle bisector in the animation "Constructions..."
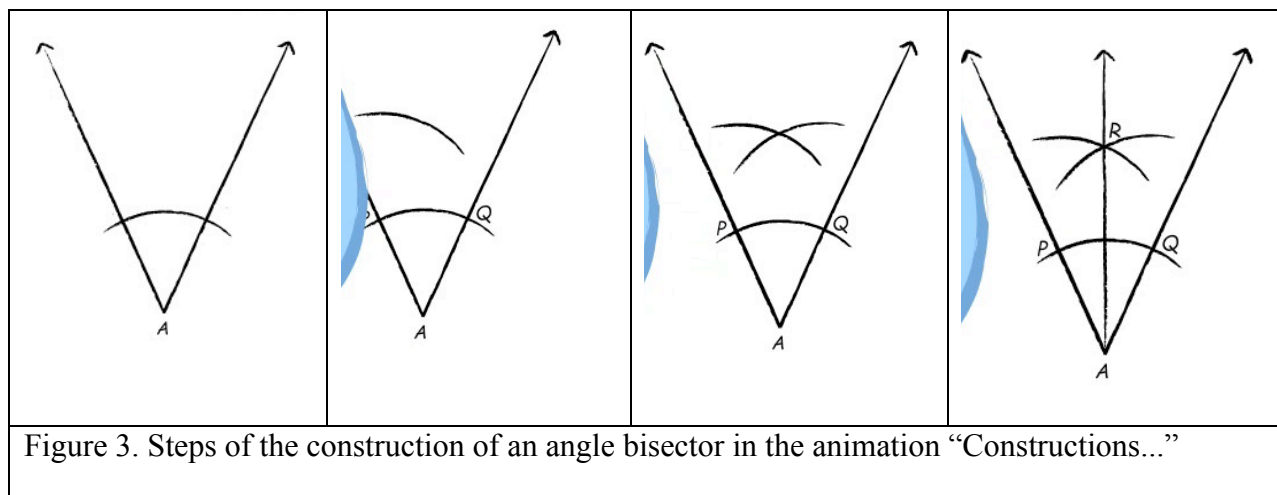
This function of proof as explanation is illustrated in the animated story "Constructions, Theorems, and Corollaries." The story shows a class of students who have learned a procedure to construct an angle bisector with compass and straightedge: They draw an arc of a circle centered at the angle's vertex and that intersects the two legs of the angle, then using the same arc they center the compass on each of those intersection points and draw two new arcs inside the angle;

where these two arcs intersect they make a point and draw a ray emanating from the angle's vertex and passing through this last point (see Figure 3). In the story, after reviewing that procedure, the teacher invites the class to develop the proof that the ray so constructed is indeed the angle bisector of the given angle. The proof revisits the steps of the construction. Segments are drawn to join the intersection of the last two arcs with each of the intersections of the first two arcs with the legs of the angle. In that way two triangles are formed that have as a common side a segment contained in the constructed, putative angle bisector (see Figure 4). To show that the ray is indeed an angle bisector, a proof is done that the two angles it makes with each of the original angle's sides are congruent. The fact that the same arc had been used originally to intersect the two legs of the angle is used to justify that one side in each of two triangles is congruent; a similar feature about the second set of arcs is used similarly. The congruence of the two triangles thus entails the congruence of corresponding angles. The proof thus uses congruent triangles to explain why the procedure of construction of an angle bisector produces two congruent angles. The proof relates the procedure of construction to some known properties such as equal distance and congruent triangles. It also creates connections with some more complex figures such as the rhombus. In fact the story shows how the proof of the construction naturally creates conditions for students to notice properties about the diagonals of the rhombus, such as the property that says that diagonals of a rhombus bisect opposite angles, which the teacher introduces as warranted by the same proof.

What is at stake in doing this proof? Certainly not how to do the construction procedure, which had been learned earlier and which could be performed without knowledge of its proof (in fact, it quite often is). We argue that there is more at stake than the verification that the construction provides the correct output. At stake also are the connections between this

procedure and other pieces of knowledge—such as triangle congruence, rhombi and their properties—connections that may tie the construction procedure to the edifice of declarative geometric knowledge. The work of the students in the story, noticing what the givens are and translating steps of the construction to steps of the proof seems instrumental to making those connections. Likewise the observations that students make of how the construction with its auxiliary lines resembles a rhombus, serve to further cement those connections.
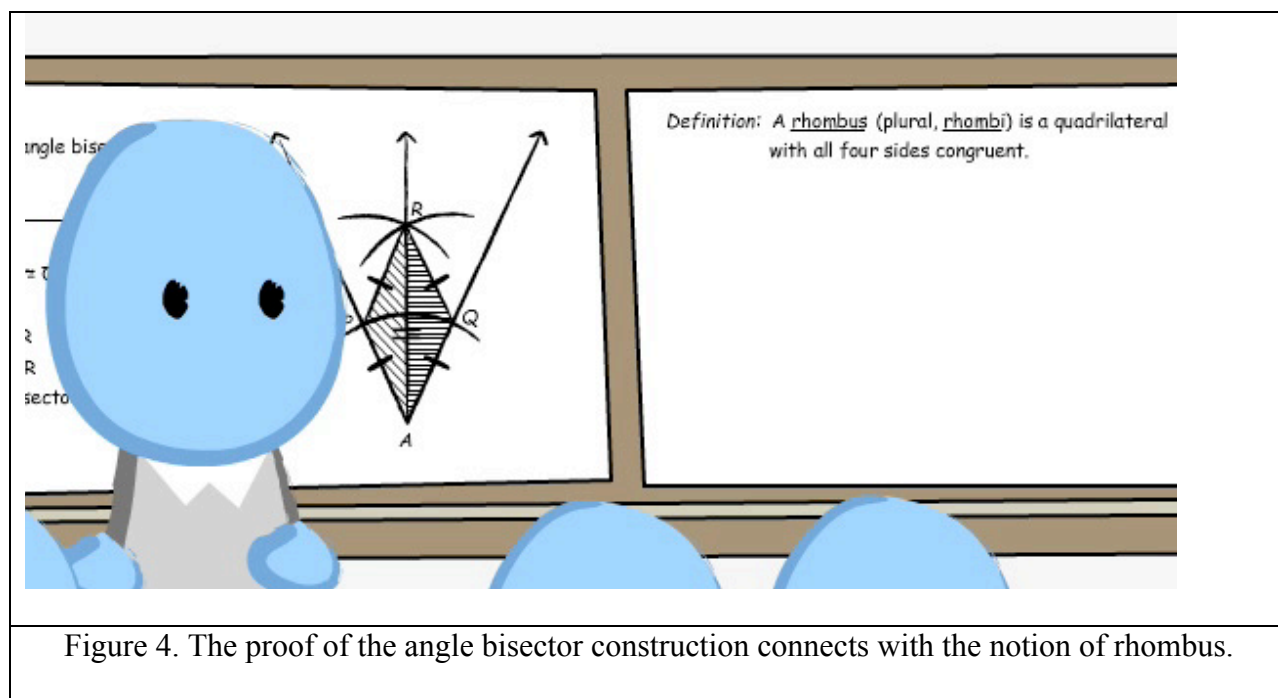

Figure 4. The proof of the angle bisector construction connects with the notion of rhombus.

Hanna has cautioned that not every proof explains; in fact it is plausible that even proofs that are explanatory in the mathematical sense of showing connections with other concepts, may fail to create in students the sense that they understand what is connected with that statement. On the other hand it is also plausible that the work that eventually accomplishes students' understanding of those connections might be questionable as a legitimate representation of what mathematicians would give as explanation.

The work of proving can thus have the exchange value of explanation—it can achieve a clarification of the conceptual grounds on which something may be true or false and prompt a statement to change status from being obvious to being a claim in need of support. The example of the "Constructions ... " story illustrates that the work of proving can have the exchange value of showing that specific connections between procedure (construction) and concept (rhombus, triangle congruency) exist. But the management required of the teacher includes more than making room for work that can count as a mathematical explanation of connections between concepts. It also requires that the teacher can ensure that students understand those connections. For a teacher to be able to exchange the work done for the claim that the statement has been explained there thus seems to be a need to attend both to what the explanation itself has as far as connections between concepts and to what the explanation does to students' activity as far as enabling them to (feel like they can) do things that they might not have thought of doing before. The various contingencies associated with whether the work done amounts to explanation, understanding, or both of them span a possible tension in teaching: if the work done to explain mathematically does not help students understand, is its value sufficient? Likewise if the work done to help students understand does not amount to a mathematical explanation, is it worth enough?

The work of proving may count as discovery of a reasonable statement

A third set of stakes of the work of proving is associated with the contingencies of on the one hand representing a rational discovery of a mathematical statement and on the other hand, of attesting to students' perception that the statement is plausible or reasonable. The management work of the teacher, effecting a transaction of the work done for the claim that the statement is

reasonable, includes not only to attest that the statement can be produced by reasoning but also that students can reason their way through to the statement.

The work of Imre Lakatos (1976) has illustrated the notion that proofs and refutations are part and parcel of the generation of mathematical knowledge: Proof is not merely a process done after the formulation of statements but actually a process that enables the production, the shaping of plausible statements. With his study of the history of Euler's theorem for polyhedra, Lakatos illustrates how the work of proving a naïve conjecture can lead to formulating a more precise conjecture and even defining precisely the concepts (e.g., polyhedron) alluded to in the naïve conjecture. De Villiers (1990) also alludes to this function of proof by noting that mathematicians working on advanced areas of mathematics such as non Euclidean geometries would be hard pressed if they had to depend only on intuition or experimentation to conjecture statements and only used proof to verify them. Hanna & Jahnke's (1996) discussion of the exploration function of proof is also captured by this discovery function—in particular as proof plays the role of exploring the consequences of a definition or assumption.

In the story "Constructions, Theorems, and Corollaries" described above, the task of proving that the ray constructed is indeed an angle bisector leads the class to consider two adjacent isosceles triangles, that they prove congruent (see Figure 4). The initial interest in that triangle congruence is to show that the two angles at the vertex A, ($\angle$PAR and $\angle$QAR) are congruent by being corresponding parts of a triangle congruency. But when the class's attention is turned to recognizing that the quadrilateral APRQ is a rhombus, they notice that the congruency also entails that AR is the angle bisector of $\angle$PRQ. The work of doing a proof produces reasonable statements that might not have been anticipated before.

The prior example shows how one important stake to be claimed by way of having done a proof is that a statement, which emerges from the proof, is, by virtue of the proof, reasonable. Likewise this function of proof also gives resources for the teacher to appraise what students do when they reason deductively. A teacher can call on this discovery function of proof to appraise the work of a student (or of the students with the help of the teacher) arriving deductively at a statement.
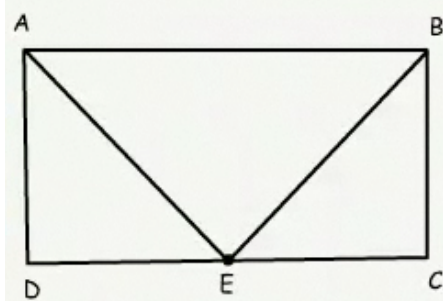

Figure 5. What could be proved about this figure?

The animated story "A Proof about Rectangles" illustrates how a student could engage in the work of proving to find out what is possible to claim. We argue that a teacher could use the "discovery" function of proof to appraise such student work. At the very beginning of the story the teacher has asked the class to think about a rectangle as in Figure 5 where it is known that E is the midpoint of segment DC and that ∠AEB is a right angle. The teacher asks what can be proved about the sides AB and BC. Rho contributes his thinking: "Well the corner triangles are isosceles so BC is half of DC which is the same of … so, I know… we could show it's half of the other…." Rho seems to see that right triangles △ADE and △BCE are not only congruent to each other but also isosceles which (conceivably added to the facts that AB = DC = DE + EC, and that DE ≅ EC) suggest to him that AB is twice as long as BC. The teacher answers, "I know what you mean, and that is what we will be proving." The work Rho has done has value for the

teacher who knows that Rho finds the statement that AB = 2 BC a plausible thing to conclude from the givens.

This discovery function of proof stresses how the work of proving can be the source of mathematical propositions. In terms of the theory of instructional exchanges, this encourages us to look at the work the teacher does or the students do to come up with or introduce a new piece of mathematical knowledge, a definition or a theorem and to place special value to this work when it represents mathematical knowledge as reasonable (Ball & Bass, 2003). The various contingencies associated with whether the work done amounts to representing a rational discovery, coming up with a plausible statement, or both of them span a possible tension in teaching: if the work done to represent a rational discovery does not contribute to students' capacity to propose the statement, is the rational derivation sufficiently valuable? Likewise if the work done to enable students to come up with a statement employs not deductive reasoning but only intuition or empirical work, is the rationality of mathematical practice sufficiently well represented?

*The work of proving may count as negotiation or demonstration of standards for communication*

A fourth exchange value on the work of proving is the claim that a mathematical argument has been communicated. This requires the teacher to manage an exchange that involves not only whether the argument has been communicated but also whether students know how to communicate it. In his article "The role and function of proof in mathematics," Michael de Villiers (1990) points to the importance of proof as a locus of mathematical debate. Building on positions taken by Davis & Hersh (1986) as well as by René Thom (1971), De Villiers argues cogently that the practice of constructing a proof stages the ongoing debate of what counts as a

mathematical argument within the mathematical community. This position about the function of proof as communication has some resonance also in Hyman Bass's (2009) observation that mathematicians don't quite do proofs (in the formal sense) but rather concern themselves with making other mathematical "practitioners convinced of the existence of proofs" (p. 3). Most of the time the work of showing that a proof exists, for example in communicating a result through a journal publication, includes some negotiation of how much about the argument is needed in order to communicate the result as one for which a proof exists. The recent story of Fermat's last theorem, in particular the fact that the result initially claimed by Andrew Wiles in a lecture in 1993 was eventually accepted only after firmed up in a paper by Richard Taylor and Andrew Wiles (1995), which filled a gap in the original proof (Faltings, 1995), attests to how the work of showing that a proof exists functions as a stage for the mathematical community to negotiate what would count as enough of such demonstration.

Any proof done in class can therefore have as an exchange value not just that it lays claim on the truth or on the explanation of the statement proved but also that it instantiates the nature of what counts as proof. Along those lines, the work of proving can be accounted for as demonstration of how much students have internalized and can reuse or transform existing standards for proving.

The animated story "The Square," in which students discuss Alpha's claim that in a square angle bisectors meet at a point because they are the diagonals, eventually showcases how a classroom could stage a debate on the nature of mathematical argument as the class deals with showing that a proof exists for the claim that diagonals of a square bisect opposite angles. To argue that the diagonal of a square bisects opposite angles Lambda adduces that a diagonal splits the square into two congruent isosceles triangles. Lambda adds that the same thing is the case

with the other diagonal. Thus Lambda's argument unpacks Alpha's argument (by showing why

one could say that a diagonal is an angle bisector) and glosses over some details (e.g., it avoids

showing in detail that both diagonals are angle bisectors and it overlooks the formal distinction

that angle bisectors are rays while diagonals are segments; see Figure 6). Lambda's argument

illustrates the communication function of mathematical proof. The story also illustrates, how

such attempts at communicating an argument might not be perceived as enough demonstration.

In fact in the story, immediately after Lambda provides his argument, the teacher asks for a

proof, what motivates Lambda to say, "I just did that." When pit against the background of what

proofs usually look like in geometry classes (for example as shown in the latter part of the story

"A proof about rectangles") one can understand why Lambda's argument might seem to some as

not abiding by enough of the standards for communication of a proof.

Figure 6. Diagonals of a square bisect angles.
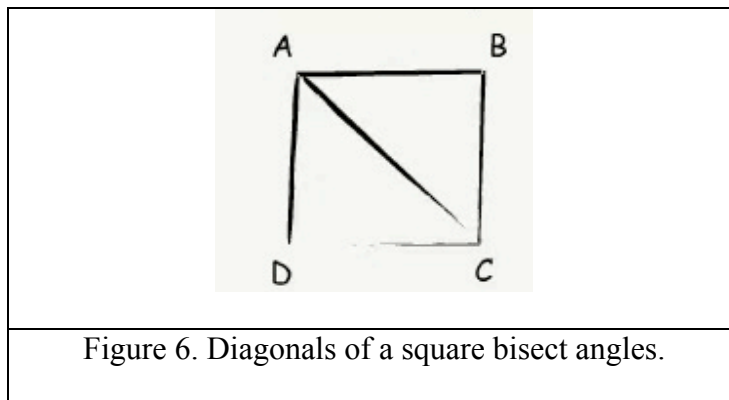
In terms of the theory of instructional exchanges these observations help put forward the notion

that work done to produce a proof can have the exchange value of communication. This

exchange value puts a premium on the extent to which the work done creates a representation of

the process of negotiating a mathematical argument. Since the work of the teacher includes

creating (with students) representations of the mathematics to be learned and since this

mathematics to be learned includes the practice of producing proofs acceptable to a community,

some of the work to be done includes transacting elements of an argument in formation so that the linguistic tokens useful to demand and comply with standards for argument can be claimed to have been publicly represented. In particular, language uses adapted to requesting further argument (e.g., "how can you say that… ") or to avoiding further argument (e.g., "it follows easily that…," "without loss of generality, let's assume… ") can be the elements of knowledge at stake to be exchanged for the work of showing that a proof exists.

On the other hand, to the extent that performance also has to show that students have acquired the target knowledge, it is understandable that moments of negotiation of what counts as an acceptable mathematical argument, however they are implemented, might be followed by periods of use and transformation of those negotiated features. In particular, it is understandable that students in the high school geometry course would be asked to use particular forms (such as the two column form; see Herbst, 2002) for writing their proofs, that they would do problems where they practice using such forms, and that future negotiations of new arguments might be done against the background of such forms (see Weiss, Herbst, & Chen, 2008).

The various contingencies associated with whether the work done amounts to negotiating the communication of a particular mathematical argument, employing learned standards for such communication, or both of them span a possible tension in teaching: if the work done to communicate an argument in a mathematically acceptable way does not contribute to students' sense that they know what a good proof is, is that work sufficient? Likewise, if the work done to enable students to practice and transform standards for communication of an argument produces a communication that is reiterative, redundant, or protracted, is the work done mathematically justifiable?

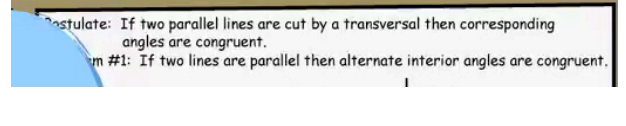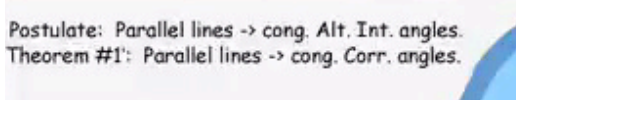*The work of proving may count as systematizing mathematical knowledge*

A fifth exchange value on the work of proving is the claim that a mathematical statement has been incorporated into a theory or mathematical system of postulates, definitions, and other theorems. This has been referred to as the systematizing function of proof by De Villiers (1990) as well as by Bell (1976) and also alluded to by Hanna & Jahnke's (1996) "incorporation" function of proof, whereby a proof may incorporate a known proposition into a different theory, thus enabling the representation of that proposition in a new light. Accordingly, the work of producing a proof of a statement may be accounted for as showing that a statement is deducible or derivable from some other statements. While research grounded on the van Hiele levels of geometric thought has shown that high school geometry students tend not to achieve understanding of this aspect of mathematical rigor (Usiskin, 1982) the expectation that students will learn that geometric knowledge is organized as a system of axioms, theorems, and definitions has continued to be present in geometry textbooks. This expectation is present, at the very least, in the fact that a large part of the plane geometry material that students learn in high school geometry is material they already know; however, they are getting to know it differently, through deductive means. One might expect that teachers would create opportunities for students to show that they understand how new statements relate to known statements, including postulates, definitions, and theorems.

The story "Postulates and theorems on parallel lines" illustrates a kind of work that could be accounted for or justified by recourse to this systematization function of proof. In this story, the teacher first gives as a postulate the statement that a transversal that intersects two parallel lines creates congruent corresponding angles. Then she proves as a theorem the statement about the congruence of alternate-interior angles determined by a transversal line that intersects parallel

lines. The proof given for this theorem shows this statement as deducible from the one offered as postulate earlier. Later on in the same lesson, however, the teacher says they could just as well have taken as a postulate that alternate interior angles are congruent and use it to prove the theorem that parallel lines imply corresponding angles are congruent. In one of the variants of this story, the teacher asks students to disregard the earlier postulate, proposes as postulate what they had last proved as theorem, and sets about to prove as theorem what had earlier been offered as a postulate. In this way the story stresses that a proof establishes a logical relationship of deducibility between postulate and theorem, adding to the knowledge of those statements knowledge about the organization of a mathematical theory.

Other variants of this story show that students may not necessarily understand what the point could be of changing postulates and proving as a theorem something that was before a postulate: In version C of the story, a student reacts to the proposition that parallel lines imply congruent alternate interior angles by saying "I don't get it, that's a theorem;" someone else reacts to the proposition that they would prove as a theorem that parallel lines imply congruent corresponding angles by saying "that's a postulate." With expressions like these the story illustrates the difficulties that students might have understanding the nature of a mathematical system or differentiating the notion of truth from that of deducibility. In this way the story illustrates how the systematization function of proof can be useful for a teacher to justify why spend time proving that by taking different statements as postulates one might be able to prove different theorems. The story also illustrates that students may have difficulty accepting that the same statement may be a postulate in one system and a theorem in another (see Figure 7a and 7b).

Postulate: If two parallel lines are cut by a transversal then corresponding angles are congruent. m #1: If two lines are parallel then alternate interior angles are congruent.	Postulate: Parallel lines -> cong. Alt. Int. angles. Theorem #1': Parallel lines -> cong. Corr. angles.
Figure 7a. One possible systematization.	Figure 7b. Another possible systematization.

This function of proof, to show the dependency of a proposition on others, also offers the teacher a resource to manage the work of allocating value to arguments that students produce even when they might not have done enough to establish the truth of all the propositions that sustain the one whose truth they claim. The systematization function of proof can help the teacher manage allocating value when the work done shows that the student knows how the provability of a statement depends on the truth of other statements. An example of this was provided above when, in the context of "The Midpoint Quadrilateral," we discussed Kappa's proof of the claim that the dual of an isosceles trapezoid is a rhombus.

From the perspective of the theory of instructional exchanges this systematization function of proof is important to show that inasmuch as classroom work has to create a representation of mathematics, the work of doing of a proof serves to represent the architecture of mathematical theories. The doing of a particular proof in class may have as its exchange value the representation of the deducibility of a given proposition from other propositions. As a resource for the teacher this function of proof identifies a source of values that can be used to appraise the work of showing the dependence of a theorem on other theorems or the equivalence of two propositions. It also provides resources for a teacher to appraise work students do when they conceive of the architecture of a proof but are unable to provide all the details. But it also permits to anticipate dilemmas a teacher may need to manage. The various contingencies associated with whether the work done amounts to representing the deducibility of a statement

from other statements, eliciting students' understanding of what a mathematical system is, or both of them span a possible tension in teaching: If the work done to show that one or more propositions entail another one makes students think they can just assume propositions to be true, is the work done justifiable? (see Nachlieli & Herbst, 2009). Likewise if the work done to enable students to learn what a mathematical system is includes making them take as postulates statements that are not postulates in mathematics, is the architecture of mathematical theories appropriately represented?

The work of proving may count as containing or showing a mathematical technique

A sixth exchange value for the work of proving is the claim that a mathematical technique (procedure, method) has been represented and/or learnt. Rav (1999) brings up a function of proof in mathematics that has not been addressed by any of those listed above: Proofs are bearers of mathematical knowledge. He argues that the entire mathematical know-how is embedded in the collection of proofs. This is one of reasons why mathematicians are keen to attend to the proofs rather than only to the theorem statements when they read scholarly articles. In their commentary on Rav's (1999) contribution, Hanna & Barbeau (2008) further show how relevant this function of proof is in mathematics education by exemplifying how school-level proofs are bearers of mathematical knowledge. By participating in the proof of a statement, techniques become part of the body of mathematical knowledge. In arguing that the work of proving could be justified on account of creating a context where to anchor particular mathematical techniques or practices we call attention to the teacher's possibility to use the work of proving to (1) represent a particular mathematical technique and (2) observe students' capacity to use a technique.

The story "Intersection of Medians" provides a good example of how a proof could be used to represent a technique. In this story the teacher shows that the intersection of medians in a triangle is a point such that it determines, with the three vertices of the triangle, three triangles of equal area. The proof that the teacher gives shows the statement as a consequence of applying the notion that one median splits a triangle into two triangles of equal area and that when one subtracts equal areas from equal areas one gets equal areas (see also Herbst, 2005, 2006). While the statement proved is not obvious it may not be highly consequential either. But the proof is important in its capacity to show how a seemingly obvious statement, the additive rule for areas, underpins a powerful method: To compare the areas of two figures, see if they can be represented as juxtapositions of simpler figures that can be matched. In the story, the teacher makes very little explicit about why he thinks this is "a cool proof" (see Figure 8) but the fact that it bears this important technique could be a justification for why having students learn it.

Figure 8. The centroid splits a triangle into three equal areas.
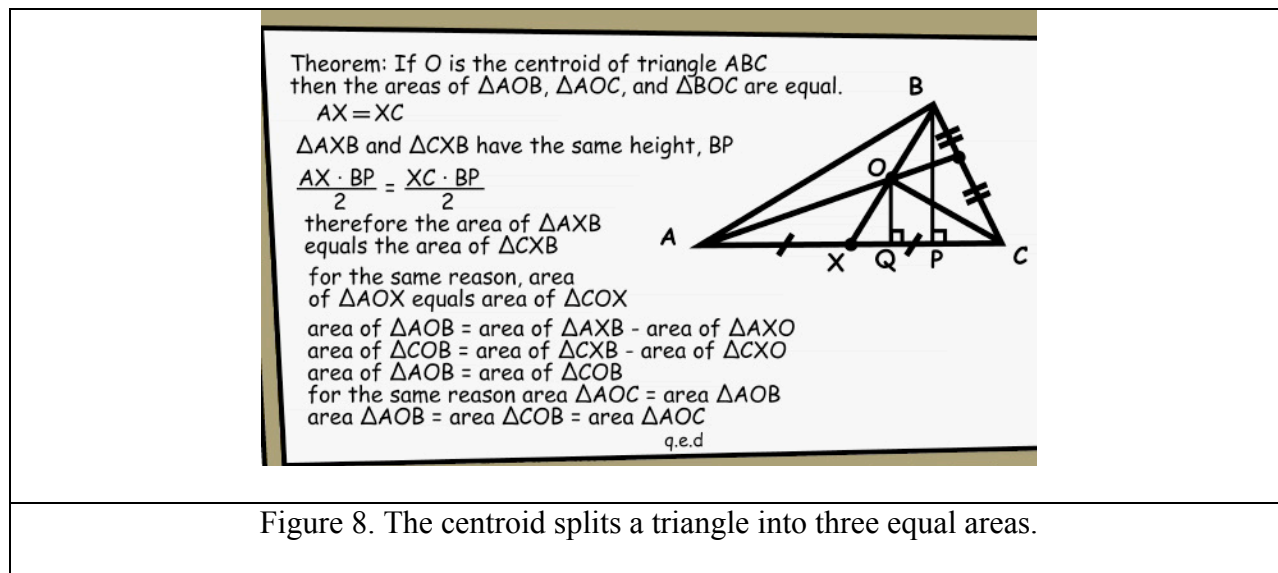
While Rav's (1999) article has been a landmark in the scholarly literature on proof in mathematics education, one could hardly argue the novelty of its claim to teachers of geometry.

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Geometry instruction, at least in the United States, shows a concentration of proof problems for students in the first half of the course, when students study parallelism and congruence. Two key techniques could be shown to account for most of the proof exercises students do in this period: the technique of finding corresponding angles determined by a transversal cutting parallel lines and the technique of finding corresponding triangles that could be proved congruent. As Herbst & Brach (2006) have noted, the proofs that students do are rarely proofs of important propositions; but these proof exercises always stage applications of a technique for proving. In terms of the theory of instructional exchanges, then, this function of a proof underscores that the work of doing of a proof may be instrumental to representing a technique, which is the knowledge at stake in that work. One can understand why a teacher might then ask students to prove a particular theorem in a particular way or why a teacher might want to show a different proof of a theorem that had already been proved before: To do these things helps represent new techniques or helps demonstrate how those techniques can be put to use in doing authentic mathematical work.
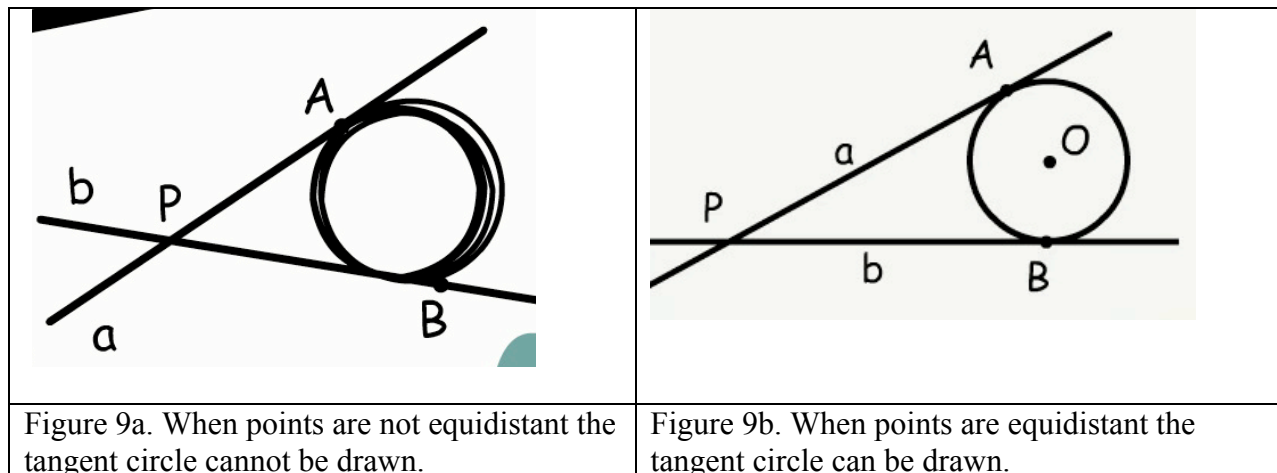
The various contingencies associated with whether the proving work done amounts to representing a mathematical technique, giving students opportunity to show they have learned a technique, or both of them, span a possible tension in teaching: If the work done to prove a proposition allocates too much attention to the proposition proved in detriment of the technique used, would this put the value of the time spent doing the proof at risk? Conversely, ad hoc proof exercises for learning or practicing a technique could keep these techniques in focus, but their lack of grounds on an authentic problem of the discipline could put their mathematical value at risk.

The work of proving may count as establishing the theoretical predictability of an empirical fact

A seventh exchange value for the work of proving is the claim that a mathematical theory can predict an empirical fact. From a reflection on the relationship between mathematical proof and empirical sciences such as physics, Hanna and Jahnke (1996) identified three functions of proof. Two of them (exploring and incorporating) have been covered by two of the functions described above (discovery and systematization, respectively). The third function of proof identified by Hanna and Jahnke (1996) is the construction of an empirical theory: Proofs are crucial elements for constructing an empirical theory. An empirical theory (e.g., mechanics) can be seen as a system of propositions, each of which asserts empirical statements of fact, connected by deductive relationships that are established by proofs. This function of proof is particularly important as it helps relate theoretical knowledge of mathematics to some empirical aspects of students' mathematical activity including drawing geometric diagrams, sketching graphs, estimating calculations, and so on.

The story "The Tangent Circle" showcases how the work of proving could help establish the theoretical predictability of a successful drawing, which had originally been achieved through trial and error, hands-on experiences with a diagram. The story shows how a class works on the problem of drawing a circle tangent to two intersecting lines at two specified points. Through interacting with the diagram, the class discovers what could be seen as a collection of empirical facts: if the points of tangency are chosen equidistant from the point of intersection they succeed in drawing a circle tangent at those points but when those points appear not equidistant, the circle turns out to be not tangent (see Figure 9). At the end of the story, the teacher indicates that only a proof could confirm that such empirical success is mathematically valid. Indeed, a proof that two right triangles (with a common hypotenuse at the segment whose

extremes are the circle's center and the intersection of the lines) are congruent if and only if their legs are congruent establishes the tangency of such circle as a consequence of Euclidean geometry. In this case, the proof here enables a connection between the empirical fact of having given equidistant points and the empirical fact of being able to draw a circle tangent at those two points.

| Figure 9a. When points are not equidistant the tangent circle cannot be drawn. | Figure 9b. When points are equidistant the tangent circle can be drawn. |

From the perspective of the theory of instructional exchanges, this function of proof is important in high school geometry as a way to organize a disorganized set of concomitant facts about diagrams that might be known from earlier courses and through empirical or perceptual means. While students may already be able to lay claim on many of the truths about geometric figures after their earlier experiences with geometry, the high school geometry class puts them again as stakes of the learning process. If they exist as prior knowledge they need to be "forgotten" and reconstructed, this time deductively (González, 2009). At the same time, diagrams continue to provide perceptual or empirical grounds for plausibility (Herbst, 2004; Polya, 1954). Thus just like in physics, proof can articulate relationships between possible facts, effectively representing possible facts as predictions based on known facts and deductive inferences from those known facts; predictions that can be confronted and perhaps confirmed by

empirical interactions with diagrams (see Herbst, 2003, for an example). Such function of proof helps the teacher represent geometry as a theoretical field of study and thus underscores how measurement and proof can coexist in the work that students do in class. This function of proof helps represent the relationships between the theoretical notion of geometric figure and its concrete representations (in diagrams or other forms) as a case of the relationships between theory and field of experience (see also Boero, Garuti, and Mariotti, 1996). The teacher can use this function of proof to account for the work of proving: Statements that are known to be true through intuitive or empirical means can still be proved in order to show that they are predictably true.

The various contingencies associated with whether the proving work done amounts to establishing the predictability of an empirical fact, enabling students to know what to expect from an experience, or both of them span a possible tension in teaching: If the work done to prove a proposition allocates too much attention to the conclusion established in detriment of the necessity established between givens and conclusion, would this put the value of the time spent doing the proof at risk? Conversely, problems that give students the opportunity to predict empirical facts given some empirical conditions might elicit their use of any prediction method thus possibly compromising the mathematical value of the work done.

Conclusion and Implications

Seven functions of proof in mathematical activity have been reviewed and discussed: verification, explanation, discovery, communication, systematization, containment of techniques, and theoretical prediction of empirical observations. The paper illustrates with scenarios of geometry instruction how a teacher could use those functions of proof to allocate value to

different kinds of proving work. Those scenarios are useful not only because they permit to illustrate the more general point that, as a diverse set of performances, the work of proving can be valued along a many-valued scale. They are useful also because they permit a common way of inquiring on the perspective of teachers.

Indeed, the functions of proof discussed above constitute a hypothesis: That various stakes of knowledge are to be claimed through the work done in class--the truth of a proposition, the connections between concepts, the reasonableness of mathematical propositions, the norms of mathematical argument, the relationship between propositions, specific techniques for handling mathematical concepts, and the relationships between mathematical theory and concrete representations. Those stakes of knowledge are important elements of mathematical practice and knowledge. But, do practitioners perceive them as important?

Our discussion of functions of proof based on the theory of instructional exchanges has made heavy use of the hypothesis that a teacher is obligated not only to mathematics (he or she has the responsibility to represent the discipline) but also to students (he or she needs to promote and certify their learning). That theoretical examination led us to derive possible dilemmas that a teacher may need to manage as s/he manages the exchanges between work done and the values to be claimed. The analysis permits to ask further empirical questions. Do teachers perceive those dilemmas? And, how do they value instructional actions such as those observed in the illustrative scenarios where teachers attempt to make room for work of proving that could be accounted for in these many ways? The analysis also provides a plausible justification for the relatively small place that proof actually plays in classrooms—the existence of those dilemmas might account for why only some of the work of proving exemplified here actually takes place in real geometry classrooms.

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