INSTRUCTIONAL SITUATIONS AND STUDENTS' OPPORTUNITIES TO REASON IN THE HIGH SCHOOL GEOMETRY CLASS

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Abstract

We outline a theory of instructional exchanges and characterize a handful of instructional situations in high school geometry that frame some of these exchanges. In each of those instructional situations we inspect the possible role of reasoning and proof, drawing from data collected in intact classrooms as well as in instructional interventions.

Introduction

Reasoning and proof are essential in mathematics. From the testimony of mathematics researchers as well as from scholars who study mathematical activity, it is known that reasoning and proof are vital because they fulfill several important roles. The role of reasoning and proof in validating or justifying what is believed to be true has for long fascinated philosophers and writers; reasoning and proof have often set mathematics apart from other disciplines that rely on experience for the confirmation of their truths. In his paper[2] "The history and nature of mathematical proof," mathematician Steven Krantz (2007, February) says that *proof* is "our device for establishing the absolute and irrevocable truth of statements in our subject." Likewise, renowned algebraist Hyman Bass (2009) noted that, "Each field has its own norms for certifying truth. Mathematics has evolved a unique method – deductive proof."

But confirmation of truth is not the only role played by reasoning and proof in mathematics. Imre Lakatos's (1976) study of the history of Euler's theorem helped make the case that reasoning and proof play important roles in mathematical discovery, in figuring out what is true. Gila Hanna (1982) has argued that proofs often help explain why assertions are true and thus help understand the mathematical ideas involved in the assertions proved. Michael De Villiers (1990) added the functions of systematization and communication: Reasoning and proofs help connect mathematical truths in theories of related concepts and propositions, they also help communicate those theories to others, particularly staging the permanent debate of how much needs to be shown to argue that a proof exists. Along those lines Ludwig Wittgenstein (1983) had argued that the proof of a theorem is what confers a theorem with meaning (see also Steiner, 1978). Enlarging the list of roles, in a more recent paper, Yehuda Rav (1999) has emphasized the role of proofs as containers of mathematical knowledge: a proof is not just instrumental to establishing the statement that they prove, justifying it, explaining it, communicating it,

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[2] Available in Wikipedia's entry on "mathematical proof," retrieved March 2009.

discovering it, or connecting it to other statements; actually, the statements themselves are instrumental for grounding the techniques and ideas implemented in a proof so that those techniques find their way into the record. In Rav's terms (see also Hanna & Barbeau, 2008) proofs are containers of mathematical knowledge. Furthermore, Hanna & Jahnke (1996; see also Jahnke, 2009) have emphasized that proof also plays a role in empirical sciences like physics or economics, where proofs are used to derive predictions from models that can then be confronted with experience to lend credence to such models.

Inasmuch as classroom mathematics should involve students with important mathematical content and processes and give to students the chance to "learn mathematics with understanding," mathematical reasoning and proof should become part of their mathematical experience (Hanna and Yackel, 2003; NCTM, 2000, pp. 15, 20; Thompson, 1996). The various roles of reasoning and proof noted above serve to put some nuance around why students should learn about mathematical reasoning and proof. One would expect that students would learn that mathematics aspires at establishing general statements about the relationship between concepts and that results in mathematics aspire at being apodictic, they establish the truth of a necessary relationship between two possibilities. In particular, students would learn that statements are true if they can be proved. One would also aspire that they learn particular proofs that showcase the power of some mathematical methods and ideas, one would aspire that students would experience connecting mathematical statements through reasoning, that they would be able to explain why statements are true through reasoning, that they would discover new truths by using reasoning, and communicate what they know through reasoning. We'll describe these aspirations by saying that an important goal of schooling should be to develop students' relationship to theoretical mathematics. How do classrooms actually support the creation of opportunities for students to experience and learn the various roles of mathematical reasoning? What room they make for such experience and learning? How could they improve the way they do it? The purpose of this paper is to explore those questions.

Mathematical Reasoning And Proof In School Classrooms

Policy documents and some descriptions of classroom work come across as if usual classroom work does not give reasoning and proof the place that they deserve. Classrooms are often described as focused on drill and practice, where students accept truths on the authority of the teacher. A transformation of classrooms is often called forth, one in which classrooms need to turn into places where students use their reasoning skills to work together as they study mathematics and develop habits of mathematical reasoning. This transformation is often portrayed as if it required a revolution of sorts: a wholesale change in the beliefs of a teacher, a wholesale change in the brand of pedagogy used, etc. Certainly, articulate and progressive visions like the one expressed in NCTM's (2000) *Principles and Standards for School Mathematics* (PSSM) can help transform classrooms. In particular, the emphasis of the *Principles and Standards* on fostering *reasoning and proof*—as useful tools to develop mathematical knowledge and as habits of mind to be learned (NCTM, 2000, p. 56)– can be a powerful influence on school mathematics. But students' opportunities for reasoning exist in a context, specifically in the context of classroom work and under the supervision of a teacher who is subject to demands and conditions from several sources. The goal of this report is not only to survey what opportunities for reasoning are present in the high school geometry class, but also to show how these opportunities for reasoning can be utilized in creating conditions for the emergence of theoretical mathematics.

We frame this work using the notion of didactical contract (Brousseau, 1997; Herbst, 2002b). We posit that there is a didactical contract that makes high school geometry possible as a response to conditions and demands. In turn this contract is instantiated in the form of a number of situations that include opportunities for students to reason. Our goal is to survey those opportunities and to show how the contract can be negotiated in each of them so as to both improve the students' opportunities for reason while at the same time meeting the demands that sustain the contract in the first place. A didactical contract is established in response to (societal, institutional, or professional) demands on teaching. Among those demands that condition and constrain the work of teaching, one is, notably, the *disciplinary* obligation that the discipline of mathematics can impose on schools in exchange for giving schools the license to claim that they teach mathematics: the discipline has to be appropriately represented in school activities and products (Schwab, 1978). But other demands are just as powerful. There is the *individual* obligation by any and all students who attend a school institution that no matter who they are they must be accepted as they are and enabled to learn a discipline. Further, there is the *interpersonal* (or interactional) obligation that groups of people who have to share common physical, representational, and relational spaces pose: The demand on the teacher to enable, sustain, and shepherd interaction in shared physical spaces such as the classroom, shared representational spaces like classroom discourse, and shared relational space such as the classroom "culture." Finally, there is the *institutional* obligation, a set of demands placed on teaching by the general forms of instruction in schools, which demand that students learn mathematics in courses that meet for periods of a length specified a priori (e.g., one hour a day for a yearlong course), with schedules and planning and evaluation devices that regulate the coordination of content deployment and learning time across subject matters. These four obligations--disciplinary, individual, interpersonal, and institutional--place demands on the work of teaching, on the nature of the opportunity to learn, and on the activities of learning. We propose those four conditions as four main sources that articulate how the didactical contract that binds teacher, student, and mathematics is elaborated in a particular class for a particular school (see Herbst & Balacheff, 2009). While visions for change can signal goals and direction for progress, such progress only happens against the background of existing practice. Or as Leon Wieseltier (2009, p. 48) aptly puts it (writing about foreign policy), "We cannot respond properly to what we do not perceive properly. A world misdescribed will be a world mishandled." Thus, understanding how existing instruction makes room for reasoning is important as a baseline against which to design and gauge progress.

This report takes the question of how to improve students' opportunities to encounter theoretical mathematics as requiring an incremental approach that takes into consideration existing practice. We assume that existing classroom practices have some reason of being that attest to their worth as historical solutions to conditions and demands such as those noted above (disciplinary, individual, interpersonal, and institutional). Thus changing them is a systemic endeavor that requires first and foremost that we understand what those practices are and how their existence might be instrumental to preserve order; then thinking how that system can be transformed to incorporate different, more enlightened kind of work at the same time that the system continues to serve its purposes.

We ground our work in the context of American high school geometry instruction. We make the hypothesis that the reasoning practices that one may observe in mathematics classrooms exist because they serve some purposes related to maintaining stability in the working relationships among teacher, student, and subject-matter. We look at how geometry classroom work, as it exists, makes room for reasoning and proof. We contribute to the development of a theory that describes instructional practice as the management of exchanges (see Herbst, 2006). And we use that theory in locating levers for incremental improvement. We map out the various situations in which reasoning and proof are found and for each of them we conjecture what could be done to improve their presence.

We ask how geometry instruction makes room for students to reason mathematically and we take this activity broadly to mean work in which students stand chance to develop a relationship with theoretical mathematics. What are the instructional situations in which students have the chance to reason? How are these opportunities created and sustained in geometry classes? To examine these questions we produce a description of geometry instruction based on a corpus of intact lessons collected from three different curricular options in a comprehensive high school. Rather than assuming that these opportunities can be deduced from the materials and problems provided in the textbook, we look in actual classroom instruction as these opportunities for reasoning are nested in classroom work. Our goal is to identify existing pockets of practice where specific efforts to make mathematical work more theoretical could be made. The argument we present feeds from observations in real classrooms but is fundamentally a theoretical one: this means that we are not warranting an empirical proposition about frequency or importance of instructional situations, rather we are using available records of practice to describe what practice consists of and eventually to speculate on how practice could be changed. We are saying these situations exist, this is what their bare bones look like, these are the functions they serve, and later this is how they could be manipulated to serve students' engagement in theoretical mathematics. We present an argument that looks at classroom work from the perspective of the teacher as manager of classroom knowledge.

Theoretical Framework

The framework for this paper requires elaborating on two separate realms. We provide a very brief review of the curriculum of the geometry course, connecting to other writing we have done. And point to the analysis of the specific curricula that were taught in the classes we observed. We then review the literature on mathematical reasoning and proof from the perspective of what students might be able to do when they are studying Euclidean geometry. Finally we present our own theoretical approach, the theory of instructional exchanges, which we have developed to make sense of classroom instruction as a system of regulated transactions in a symbolic economy. As a whole this report on students' opportunities for reasoning in geometry can be seen as an exploration of the potential of that theoretical proposition.

The didactical contract for geometry is articulated in response to four obligations.

A disciplinary obligation: The demand to represent geometry and mathematical reasoning in ways that are compatible to what these things are in the discipline. As we review the literature in search of indications of this demand, we ponder the question of what is geometry and how it makes room for mathematical reasoning.

An individual obligation: The demand to attend to who the geometry student is and how he or she can engage in reasoning. As we review the literature in search of indications of this demand, we ponder the question of how the geometry student has been described and what about their geometry cognition the literature provides.

An institutional obligation: The demand to teach a geometry course, located in time and space next to other courses, subject to similar schedules and devices. As we review the literature in search of indications of this demand, we ponder the question of how the geometry course location in time and charge make room for the study of reasoning.

An interpersonal or interactional obligation: The demand to maintain a safe and productive collective space, shared by many individuals with same rights and obligations. As we review the literature in search of indications of this demand, we ponder the question of what affordances the classroom discourse and interaction has and how it makes room for the practices of reasoning.

The charge on high school geometry

The high school geometry class is an intellectual space whose reasoning practices are interesting to analyze by virtue of at least two considerations. The study of geometry has historically been charged with the institutional responsibility to teach students mathematical reasoning and proof (Herbst, 2002a). Hence, the reasoning practices that could be observed in geometry classes might be seen as responding at least in part to the imperative of introducing students to mathematical practices of the discipline of mathematics (Moise, 1975; Usiskin, 1980). González and Herbst (2006) have noted how a mathematical argument was used during the 20th century to justify the study of geometry on account that the geometry course provided students an opportunity to cultivate the dispositions of mathematicians (Henderson, 1947). The value of reasoning and proof had been underscored earlier in the century, when progressives had argued that when properly taught, the skills used in developing theoretical mathematics could help develop the analytic skills for civil life. This perspective continued to be embraced by others in the 20th century who argued that geometry provides the opportunity for teaching students logical reasoning (Christofferson, 1938; Fawcett, 1938, 1970; Meserve, 1962, 1972; Schlauch, 1930; Upton, 1930). Students could conceivably apply the same skills of logical reasoning taught in the geometry course to their daily life. So, two main arguments have supported the teaching of reasoning and proof in geometry: a *mathematical argument* that intends to use geometry to teach students the values and dispositions of mathematicians and a *logical argument* which strives to transform students into informed citizens in a democratic society by means of studying the logical underpinnings of formal mathematical thinking.

The importance for all students to learn theoretical mathematics has been questioned at times during the 20^{th} century; for example, Usiskin (1980) saw as key issues in recommending the study of geometry that it is a domain of mathematics that connects uniquely which students' experiences in the real world and that it provides a graphical language for representing mathematical ideas from other areas. But Usiskin also acknowledged geometry's value in fostering theoretical mathematics by noting that it provides an accessible example of a mathematical system of axioms, definitions, and theorems. The course has continued to be the only place where students are expected to learn that in K-12.

What Is Geometric Reasoning?

In order to understand the work of reasoning and proving in geometry classrooms, we consider four strands of literature that we take as concurrent. First, we consider literature that describes reasoning as mathematical work. This literature includes accounts of what mathematicians do (as they reason mathematically) and studies on how students do and learn proofs, in geometry and in other areas. Secondly, we review literature that describes reasoning as cognitive work. This includes studies that probe into the development of the capacity to reason. Thirdly, we review literature that describes reasoning as collective work. We consider reasoning as a social process, including argumentation, and in particular reasoning in classrooms. Finally, we consider literature that describes reasoning to build geometric knowledge. From this perspective, geometric thinking involves reasoning about properties of figures. The perspectives about reasoning in these four strands help us articulate the "work of reasoning in geometry." We argue that the work of reasoning in geometry might exchange for claims on the charge of the geometry course. In particular, the geometry course is the vehicle to attain the various claims on the contract argued for by the various functions of proof.

The emphasis on reasoning has been central in the NCTM *Standards* (NCTM 1989, 2000). As Hanna and Yackel (2003) indicate, "reasoning" is often used making the assumption that everybody knows what it means, without any elaboration. However, such a colloquial use of the word has, since the publication of the 1989 *Standards*, been instrumental in bridging discussions of children's autonomy and sense making (NCTM, 1989, p. 29), the development of students' logical and verbal capacities (p. 81), and students' learning and use of inductive and deductive reasoning (p. 143). Two strands of scholarship are included in that discussion. One of those strands is focused on the kinds of reasoning used by mathematical practitioners (Hadamard, 1945; Lakatos, 1976; Poincaré, 2001; Pölya, 1945, 1954). Another strand of literature has investigated the reasoning involved in students' solving of problems and the learning of that reasoning (English, 1997; Hadar, 1977; Simon, 1996; Thompson, 1996). There have been many studies that investigate students' reasoning in connection with planning and writing deductive proofs (Greeno, 1980; Harel & Sowder, 1998; Koedinger & Anderson, 1990; Moore, 1994). These studies have yielded information on the nature of mathematical reasoning.

Reasoning As Mathematical Work: Working Like A Mathematician

Accounts by mathematicians describing their work give some insight about the reasoning skills needed to engage in such work. Mathematicians' perspectives on reasoning are useful for us in trying to understand how students' work in classrooms is similar (or different) to the work of mathematicians, especially in geometry. Here, we review the perspectives of Henri Poincaré, Jacques Hadamard, George Pölya, and Imre Lakatos.

In an essay entitled "On the nature of mathematical reasoning," Poincaré (2001) talked about the relationship between intuition and reasoning. He argued that, in mathematics, intuition is useful for finding immediate solutions to a problem. Yet, reasoning involves applying a systematic way of thinking about a problem, enabling the mathematician to apply a solution to general cases. He contrasted proofs by induction (or proofs by recurrence) with arithmetic and said,

> A chess-player can combine for four or five moves ahead; but, however extraordinary a player he may be, he cannot prepare for more than a finite number of moves. If he applies his faculties to Arithmetic, he cannot conceive its general truths by direct intuition alone; to prove even the smallest theorem he must use reasoning by recurrence, for that is the only instrument which enables us to pass from the finite to the infinite. This instrument is always useful, for it enables us to leap over as many stages as we wish; it frees us from the necessity of long, tedious, and monotonous verifications which would rapidly become impracticable. (Poincaré, 2001, p. 16)

So, according to Poincaré, a mathematical instrument—in this case proof by induction—provides the opportunity for someone to make generalizations that go beyond the use of intuition in particular cases. Mathematical reasoning has to do with using special strategies or techniques to do what one cannot do just by relying on intuition.

In other writings, Poincaré devoted particular attention to geometry. He argued that the simplicity of Euclidean geometry makes it amenable to be understood better than other geometries. Nonetheless, any geometry, as a body of knowledge, is a theoretical endeavor. The axioms of geometry anchor the development of new knowledge. He said, "*The geometrical axioms are therefore neither synthetic à priori intuitions nor experimental facts.* They are conventions" (Poincaré, 2001, p. 45). Thus, while there are possible connections between geometric objects and real world objects, physical experiments cannot prove or disprove geometric principles, since these are deductions from the axioms. Whereas experiments deal with physical phenomena, geometry deals with mental abstractions (Poincaré, 2001, p. 49). These abstractions, however, are not undisputable a prioris (as the history of geometry showed with the emergence of non-euclidean geometries) but voluntarily accepted statements, that is, conventions. One could infer that, from Poincaré's perspective, the process of making a mathematical definition based upon some axioms entails mathematical reasoning, because the mathematician uses those definitions in a systematic way to make general claims (Poincaré, 2001, p. 465-468). He said that Hilbert's attempt to organize geometry based upon a minimum set of axioms, while intending to show how logic can organize a mathematical body of knowledge, is still insufficient for a mathematician because it is in the interplay of *selecting* axioms and using the axioms where one could find a mathematician's work. Euclidean geometry shows a case where the one could observe how the selection of axioms could change the conclusions one could reach.

The notion that geometry involves special reasoning skills and particular resources to support mathematical reasoning is also evident in Hadamard's work. In *The* *mathematician's mind,* Jacques Hadamard (1945) described the kind of work that is involved in doing mathematics as a sequence of four stages: (1) *preparation*, (2) *incubation*, (3) *illumination*, and (4) *verification*. By "preparation" he meant a conscious effort to think about the problem and to use the resources one knows to solve the problem, whereas by "incubation" he meant an unconscious work of putting together pieces that would lead to a solution. The stage of illumination is where the conscious and the unconscious work on the problem coalesce to provide a solution. Finally, the stage of verification involves writing the proof and providing a legacy of the mathematical work done for others to study and to use.

Hadamard stated that diagrams are a fundamental resource for thinking about problems in geometry. He said, "When I undertake some geometrical research, I have generally a mental view of the diagram itself, though generally an inadequate or incomplete one, in spite of which it affords the necessary synthesis—a tendency which it would appear, results from a training which goes back to my very earliest childhood" (Hadamard, 1945, p. 80). This avowed use of diagrams is an example of how geometric reasoning involves the interplay with visual representations. Hadamard mentioned that Hilbert, even in his attempt to reduce geometry to logic, depended upon the use of diagrams. So diagrams, according to Hadamard, are one of the important resources bridging the conscious and the unconscious work of the mathematician. Hadamard said,

> Both mental streams, images and reasonings, constantly guide each other though keeping perfectly distinct and even, to a certain extent, independent; and we have found this to be due to a cooperation between proper consciousness and fringe-consciousness. (Hadamard, 1945, p. 97-98)

Therefore, one could expect that mathematical reasoning would involve the active use of diagrams not just for communicating ideas, but as a means towards working on a problem. In a parenthetical note, Hadamard said that his use of diagrams is different according to the work he is engaged with:

> In my own case, the role of geometrical images when thinking of analytical questions is very different from the way they intervene in geometrical research. (Hadamard, 1945, p. 114-115).

This anecdotal evidence suggests that one could expect reasoning with diagrams to vary according to the focus of the work and to the particular questions that one would pursue.

The use of diagrams is one of the strategies that Pölya (1945) suggested in his book, *How to solve it*. This book has had a big influence in school mathematics directly and indirectly, since textbooks have reproduced the problem-solving method that he explicates. This problem-solving method involves four steps: (1) understanding the problem, (2) making a plan to solve the problem, (3) solving the problem according to the plan, and (4) reviewing the solution. In this work, mathematical reasoning involves the use of a catalogue of strategies that suit the needs of the problem at hand. Moreover, mathematical reasoning involves developing the habits of mind that would enable the problem-solver to categorize a problem as one that would call for a known strategy.

With regards to diagrams, Pölya said that these are useful not just for solving geometric problems, but to solve other problems as well. He provided different suggestions to work with diagrams (1945, p. 93-97). For example, a carefully drawn diagram is helpful for someone who is starting to work on a problem, however a sketch may be more useful if one wants to reason with the diagram; a diagram that is actually drawn on paper is better than a mental image in that one could see something different about the solution of the problem at different times; a diagram with different sorts of symbols could be helpful to emphasize (or to hide) different properties of the figure. These strategies suggest that mathematical reasoning involves using different kinds of object—such as diagrams—and following some heuristics to solve a problem.

In Pölya's work, there is also a discussion about the importance of proofs as an opportunity for students to learn mathematical reasoning. He said,

> If the student failed to get acquainted with this or that particular geometric fact, he did not miss so much; he may have little use for such facts in later life. But if he failed to get acquainted with geometric proofs, he missed the best and simplest examples of true evidence and he missed the best opportunity to acquire the idea of strict reasoning. Without this idea, he lacks a true standard with which to compare alleged evidence of all sorts aimed at him in modern life. (Pölya, 1945, p. 189)

So, according to Pölya, geometry gives the best opportunity for students to prepare to reason in the real world, by exposing them to proofs. But, he contrasted proofs in geometry with other proofs that he calls, "the cookbook system," because there is a prescribed set of steps to take to do those proofs. As an example, Pölya talked about proofs in calculus, "epsilon-proofs" (1945, p. 191). This contrast is of interest here because Pölya places high value in the geometry course in teaching students an appreciation for the mathematical work involved in doing proofs. In his advocacy not only for education on deductive reasoning but also the education of plausible reasoning and habits of mind, Pölya (1954) emphasized the heuristic nature of mathematical work.

Lakatos's (1976) reconstruction of the development of the proof for Euler's conjecture demonstrates how mathematical reasoning is heuristic in nature. To reason in geometry requires not only to derive necessary consequences from accepted statements, but also to shape those accepted statements (e.g., what the definition of polyhedron is). Another conclusion from Lakatos's work in *Proofs and Refutations* is that to portray mathematical reasoning as solely deductive hides the actual process in which mathematicians create new mathematical knowledge. That is, the underlying questions that provoke the formulation of a definition the debates around a definition to include or exclude certain cases, the challenge of finding a counterexample that would test the usefulness of a definition, and the underlying motivations for proving a theorem usually remain implicit in a mathematical text. Yet, the work of the mathematician includes both the proofs and the refutations involved in shaping a theorem and finding its proof.

Lakatos's perspective is relevant in this review because it highlights that mathematical reasoning is not documented enough in written mathematical texts, since the underlying questions that motivated the creation of new mathematical knowledge are usually implicit. It also shows how misleading it is to characterize mathematical proof as an application of logic, or an exercise in deductive thinking. Lakatos's emphasis on the heuristic nature and substantial aim of mathematical reasoning illustrates that mathematical reasoning in geometry classes could be concerned with more than finding the deductive chains that connect a given premise and an anticipated conclusion. It could also be concerned with finding the premises or finding the conclusion.

In connection with this, Weiss (2009, Ch. 2) examined the ways in which mathematicians narrate accounts of their work, with the goal of articulating the elements of the "mathematical sensibility": the dispositions, or habits of mind, that mathematicians bring to bear in determining what is mathematically worthwhile. This analysis aims at understanding not how mathematicians solve problems, but rather how they pose them, and how they value work that has been done. Weiss identifies five *generative moves* that function to produce new questions from existing ones, and ten *categories of appreciation* that function as lenses with which mathematicians appraise mathematical work. These 15 dispositions are organized in dialectical pairs, which provide motivation to drive mathematics forward (Larvor, 1999).

The five generative moves operate on a conditional statement of the form "if P, then Q" by successive modifications to the hypothesis (P) and conclusion (Q). If the statement "if P, then Q" has already been determined to be true, one generative move is to *weaken the hypothesis*, i.e. to see if the same conclusion can be proved with fewer assumptions. The dialectical partner of this move is to keep the hypothesis unchanged but to *strengthen the conclusion*. In the event that the statement "if P, then Q" has been determined to be false, these same moves function in the opposite directions, as *strengthen the hypothesis* and *weaken the conclusion*. Another pair of generative moves is *generalize* and *specialize*. In these, the hypothesis and conclusion of a conditional statement are strengthened or weakened in tandem with each other: thus, one assumes less (or more), but also proves less (or more). Finally, the fifth generative move is to *consider the converse*, i.e. to interchange P and Q. This move is in a sense its own dialectical partner: It is shown how employing this same move twice in succession can do more than simply return the mathematician to the original question he began with, but rather can advance the state of his knowledge.

These five generative moves describe particular means of navigating around in a mathematical space; as such they are closely related to Pölya's heuristics, some of which they closely resemble. The five moves are complemented by a set of ten categories of appreciation that provide means for attaching value to mathematical work. One such category is *Utility*: that is, mathematical work can be appreciated for its applicability in an extra-mathematical context. On the other hand, and seemingly paradoxically, work can also be appreciated precisely for its lack of such utility — in other words for its *Abstraction.* Utility and Abstraction form a dialectical pair of categories of mathematical value. To be clear, Utility and Abstraction are not here used to refer to intrinsic qualities of mathematics itself; rather they label distinct ways that mathematics may be appreciated by the practitioner.

A second pair of categories of value is *Surprise* and *Confirmation*. These two dispositions may be activated when there is a significant temporal separation between the existence of a conjecture or belief, and the production of a proof or disproof. Results that show long-standing beliefs to be false, as in the case of Weierstrass's discovery (in 1871) of a function that is continuous but nowhere-differentiable, or Kolmogorov's construction (in 1922) of an integrable function whose Fourier series diverges everywhere, are landmarks precisely because they were never anticipated. On the other hand mathematical work may be regarded as significant precisely for the opposite reason, as when Wiles proved (in 1995) Fermat's Last Theorem, or when Hales proved (in 1998) the Kepler conjecture: In both of these cases the significance of the work lies in large part in having finally established the truth of what was long believed to be true.

A third pair of categories of mathematical appreciation is *Theory-Building* and *Problem-Solving*. Theory-Building refers to an appreciation of mathematics that attends specifically to the inter-relationship among the elements of a theory. Thus a reorganization of an existing theory with a different configuration of postulates, definitions, theorems, etc. can be regarded as a significant achievement, even if all of the elements of the theory are pre-existing. But while some areas of mathematics are built up around theorems of deep generality, others are structured around common methods of solution. Gowers (2000) notes that problem-solving heuristics "play the organizing role in combinatorics that deep theorems of great generality play in more theoretical subjects" (p. 8). Thus an individual piece of mathematical work may be valued for making novel use of an existing problem-solving method, or for contributing a new method to the problem-solvers' toolkit.

A fourth pair of categories of mathematical appreciation is *Simplicity* and *Complexity.* As others have noted (Le Lionnais, 1986; Sinclair, 2002), although the notion of a mathematical aesthetic is common among mathematicians, it is not clear that the word means the same thing to all users. Indeed some mathematical work is viewed as valuable for having yielded great simplifications, while other work is viewed as valuable precisely for having enormous complexity.

Finally, a fifth pair of categories of mathematical appreciation is *Formalism* and *Platonism*. The Formalist sensibility is brought to play when work is appreciated for its syntactic, rather than semantic, characteristics: category theory, which seeks to describe the common underlying structure behind disparate areas of mathematics, is perhaps an expression of the Formalist sensibility, as is David Hilbert's famous declaration that "one must be able to say at all times — instead of points, lines, and planes — tables, chairs, and beer mugs." On the other hand mathematical work can also be appreciated for shedding light on "things" that are described as if they are real ("This work helps lay bare the properties of Artinian rings…"). Viewed this way, the "objects" of mathematics are viewed not as mere placeholders, but as objects with an existence of their own.[3]

Taken together, these fifteen dispositions (the five generative moves and the ten categories of mathematical appreciation) comprise a first approximation to a map of the practical rationality of mathematicians (or the mathematical sensibility). They provide a language for describing not how mathematicians solve problems (as in Pölya), but rather how they pose, select from, and attach value to mathematical work.

In this section, we have reviewed mathematicians' account of their work with the purpose of distilling from their account a sense of what it takes to engage in mathematical reasoning. An important point in this review is that the goal of mathematical reasoning is not only to validate conclusions found by intuition or experiment but also to find those substantive conclusions that are worth claiming. We review now a set of studies with college students that describe the kind of reasoning they do. As a counterpart to the accounts by mathematicians, these studies give a glimpse of expectations for students to reason after high school.

In a study with undergraduate students in mathematics courses, Moore (1994) found three major sources of difficulties with proofs: Students had problems with mathematical language and notation, students had a lack of understanding of the concepts

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[3] This brings to mind the observation by Davis and Hersh (1981, p. 321) that mathematicians are Platonists on weekdays and formalists on weekends.

involved in the proof, and students did not know how to start a proof. These results are relevant in that students' difficulties with proofs prevented them to engage in the reasoning expected in their courses. Moore expanded prior work by Tall and Vinner (1981) who had proposed the constructs of *concept definition* and *concept image* to talk about the relationship between an abstract mathematical idea and the mental image of that idea. Moore proposed that the construct of *concept usage* to name actions that individuals do with concepts. With this construct, one can describe students' reasoning in relation to observable operations that students perform as they work on a proof. For example, one could study the way in which students use examples or how they apply definitions in a proof.

Harel and Sowder (1998, 2007) studied college students' proof schemes, by which they meant the cognitive structures that enable students to become convinced or to persuade others of the truth of an assertion. They documented the existence of three kinds of proof schemes (external conviction, empirical, and analytic), and for each of those kinds they identified several subvarieties. Within the category of "external conviction" proof schemes, they identify schemes as authoritarian, ritual, and non-referential symbolic. The category of "empirical" proof schemes includes within it inductive and perceptual schemes. The "deductive" category includes transformational and axiomatic proof schemes. Each of these schemes and subschemes can be understood as a set of control structures for ascertaining and persuading an individual or a community of the truth of a proposition.

Unlike other theories of proof, which begin with a theory-based notion of what a proof is or should be, Harel and Sowder's proof schemes are emic constructs; that is they provide a means for describing what counts as proof from the perspective of the student. A main finding is that in spite of courses that attempt to help students' transition to proof, college students still exhibit the full range of proof schemes, including the authoritarian, ritual, and symbolic schemes of the "external conviction" kind. Thus, there seems to be a gap between what convinces students of the truth of an assertion and the ways of knowing used by practicing mathematicians. Findings like these recommend looking into what research on cognition has to say about students' capacity for mathematical reasoning and proof.

Reasoning as cognitive work: The development of the capacity to reason

Research in psychology has reported that children are capable of making inferences using deductive reasoning at a young age (Hawkins, Pea, Glick & Scribner, 1984; Richards & Sanderson, 1999) in contrast with Piaget's theory which postulates that children demonstrate deductive reasoning skills much later, at around 12 years of age, when children are in the stage of *formal operations* (Piaget, 1947/1950, p. 148). Researchers have tried to answer the question of how children develop their ability to apply deductive reasoning. One argument explaining children's ability to solve problems using logical reasoning is related to their working memory capacity. As children increase their working memory capacity, their ability to solve problems using logical reasoning improves (Bara, Bucciarelli, & Johnson-Laird, 1995). Another argument is related to children's familiarity with the semantic content of logical propositions. When children are familiar with the way in which the meanings of the antecedent and the consequence of logical propositions are related, they are more able to make inferences deductively than when they lack of familiarity with those meanings (Ward & Overton, 1990). Johnson-Laird's mental-model theory elaborates on the relationship between deductive reasoning and semantic content. Johnson-Laird (2001) argues that deductive reasoning is based on how one can use mental models to represent possible relations between a set of premises. According to his theory, when reasoners study a set of premises, they build different mental models, each representing a possibility. Johnson-Laird (2006) says, "The principle of truth, however, specifies that mental models represent the possibilities consistent with the premises, and within each possibility they represent just those simple propositions in the premises that are true" (p. 117). So, the content knowledge that an individual possesses could contradict a premise or a conclusion deduced, thus affecting logical reasoning. One could expect that, following Johnson-Laird's argument, children's use of deductive reasoning is evidence that they can create and use a mental model to do some reasoning tasks. Then, as children increase their content knowledge, they are able to use their knowledge to make further interpretations of their conclusions and to find counterexamples that contradict inferences reached by using logical reasoning.

Researchers have identified factors that affect children's performance in deductive reasoning tasks. One of these factors is the language used in the problems. For example, in a longitudinal study where 6th and 8th graders were observed over a period of three years, Müller, Overton, and Reene (2001) found that children performed better in reasoning tasks when applying propositions that were phrased as "if-then" statements with the antecedent phrase specifying an action and the consequent specifying a condition than when the antecedent phrase specified a condition and the consequent specified an action. Their results suggest that children use single inferential schemes. Then, moving towards adolescence, children are able to integrate single inferential schemes into a systematic reasoning network, thus improving their reasoning skills. Another factor that affects children's performance in reasoning tasks is their ability to handle the language of reasoning problems. For example, in a study with children between 3 and 4 years old, Smith, Apperly, and White (2003) found that 4-year-olds can master reasoning tasks when they can handle metarepresentations—propositions that include a representation of an event within another representation of an event—and misrepresentations—propositions that falsify reality. These studies are an example of how the development of language skills affects children's use of deductive reasoning.

The teaching and learning of deductive reasoning skills to elementary school children has been a focus of attention in mathematics education research. One suggestion to foster students' development of deductive reasoning skills is to encourage them to use representations. For example, Morris (2009) argues in favor of using physical representations to model mathematical processes, to push students to apply deductive reasoning. This approach, based upon the work of Davydov and colleagues in Russia, would enable students to deduce generalities by means of using diagrams to illustrate mathematical ideas. Another example of how to encourage students to use deductive reasoning skills is by engaging them in class discussion. For example, Fosnot and Jacob (2009) show cases from a second grade classroom and from a fifth grade classroom where students used deductive reasoning in relation to the concept of equivalence. In the last section of this review we expand on how classroom discussions could provide the opportunity for individual students to reason in collaboration with others. What the studies in mathematics education add to the tradition in psychology about student reasoning is other means that students possess to attain and to apply deductive reasoning skills such as the use of representations and the use of classroom discussions.

Researchers have also found evidence for children's use of other reasoning skills besides deductive reasoning skills such as analogical reasoning. Analogical reasoning involves matching pieces of information from one domain with another domain, and then transferring conclusions reached in one domain, the base, to another domain, the target (Gentner, 1989). Children, including 4-year-olds are capable of using analogical reasoning (Alexander, White, & Daugherty, 1997; Pierce & Gholson, 1994). However, children who do not possess the conceptual knowledge involved in the analogical reasoning tasks assigned have difficulties to perform those tasks (Alexander, White, & Daugherty, 1997, p. 136). In addition, children's ability to identify similar components depends on the kind of problem and the kind of similarity that the problem calls upon. For example, Pierce and Gholson (1994) performed a set of experiments where children were asked to solve *isomorphic* and *nonisomorphic* problems. By isomorphic problems they meant "problems with identical goal structures, constraints, and problem spaces" (Pierce & Gholson, 1994, p. 725). They found that when the problem involved isomorphic transfer, kindergarten children tended to use *surface similarity*—such as features of the objects—whereas older children based their results on *relational similarity*—such as identifying causal relationships common in the base and the target. But, when the problem required nonisomorphic transfer, most children relied on surface similarity. These studies show that children can perform analogical reasoning, but features of the problem such as the concepts that these problems tap into and the kinds of reasoning that is required to solve the problem affect children performance.

From the point of view of mathematics learning, analogical reasoning is one skill that students could apply when solving mathematical problems. For example, in English's (1997) study about reasoning with word problems, sixth graders received a set of four problems that included two pairs of isomorphic problems. Students had to match problems that could be solved similarly, describe the base problems and their solutions in their own words, solve the base problem, describe the target problems and their solutions in their own words, and solve the target problems. At the end, students had the chance to revise the pairing of problems with similar solutions using their work on the problem. Students who had difficulties matching the problems correctly relied on superficial characteristics of the problem instead of using the similar features of their mathematical structure. In discussing the results of this study, English suggests that instructional practices geared towards teaching problem solving skills should help students to have a better understanding of the mathematical concepts underlying problems which solutions have a similar structure and to establish relationship between base and target problems.

In sum, research in psychology shows that children use reasoning to solve problems starting at a young age. Research in mathematics education has been useful to identify some of the resources that students could use to develop their reasoning skills such as representations and classroom discussions. The way in which classroom discourse provides the means for students to develop reasoning skills is the focus of our next section.

Reasoning as collective work: Reasoning in classrooms

A strand of scholarship on reasoning has emerged from the influence of discursive, cultural, and social approaches to the study of mathematics classrooms (see Seeger, Voigt, & Wachescio, 1998; Steinbring, Bartolini-Bussi, & Sierpinska, 1998). In this strand, reasoning is considered as the structures that emerge from experiences as one participates in, and interacts with, the social environment. The work of Lave and Wenger (1991) on situated learning, has been instrumental in thinking about classrooms as places where learning happens through interactions with others—teachers and students—and in relation to the work students do. From this perspective, learning results when there are shifts in participation toward an active engagement within a community of practice. The metaphor of learning as participation contrasts with the metaphor of learning as acquisition in that evidence for learning is supposed to be observable as changes in a discourse instead of the result of changes in inferred mental schemes (Sfard, 1998, 2008).

Whereas this literature has not been completely clear about what constitutes mathematical reasoning in classrooms, it has provided ways to think about the roles of reasoning in the construction of shared knowledge. Thus, reasoning is not so much something that is explicitly learned by individuals, it rather is a set of strategies used by a group to consider individual assertions within the shared way of knowing that a classroom community is developing. Important contributions, empirical and theoretical, come from sociocultural perspectives (Forman & Larreamendy-Joerns, 1998; Lerman, 1996), from epistemological perspectives (Balacheff, 1991, 1999; Mariotti, et al., 1997; Steinbring, 1998), and from emergent approaches that blend constructivism with interactionism (Cobb & Bauersfeld, 1995; Yackel & Cobb, 1996). The work of Krummheuer (1995, 1998) on reasoning and argumentation (based on the ideas of Toulmin, 1958; Toulmin et al., 1984) has been particularly helpful in describing the formats of collective argumentation that enable children to shape their habits of mathematical reasoning. Finally the work of Ball and Bass (2000, 2003) has brought attention to the work of a teacher managing a class' collective reasoning, and to the roles that mathematical language and a base of public, prior knowledge play in the reasoning of justification.

In alignment with the metaphor of learning as participation, some studies suggest that children develop their reasoning skills through classroom discussions, as they engage in the process of argumentation (Ball & Bass, 2003; Diezmann, Watters, & English, 2002; Krummheuer, 1995; Maher, 2005; Maher & Martino, 1996; Reid, 2002a, 2002b; Zack, 1999; Zack & Graves, 2001). Teachers have a major responsibility of creating a classroom culture where students have opportunities to justify their ideas in public. In those classrooms, students would create arguments in trying to convince others—the teacher and other students—and would gradually notice the need of a proof. Thus, students' participation in classroom discussions where teachers enable students to argue about mathematical ideas is a vehicle for them to develop their reasoning skills.

There are particular actions that a teacher could do to encourage students to reason in public. Prior research has shown that the kinds of questions that a teacher asks to students could lead them to justify their answers. For example, in a longitudinal study with students from 3rd to 5th grade, Martino and Maher (1999) found that students would not make justifications on their own. However, because the questions that the teacher asked, students started giving justifications and generalizations. The researchers concluded that teachers who follow students' thinking and that also ask questions at the right moment enable students to justify their answers and move towards generalizations. In another study, Maher (2005) found that when students need to convince each other about their ideas, they start producing proofs. In a longitudinal study over five years with one student, Maher and Martino (1996) found that inviting students to use multiple representations, especially when their original ideas do not make sense, pushes students to evaluate their thinking and to move towards doing proofs. So one could infer that a teacher who encourages students to make sense of their mathematical ideas, who asks for explanations, and who requires students to use multiple representations promotes students' opportunities to reason in a classroom. In addition, students' opportunities to reason may also stem from students' interactions in small groups. Walter and Maher (2002) identified patterns of interactions that developed when students asked questions to each other, leading them to engage in reasoning about mathematical ideas. So, students' interactions with the teacher and with their peers could foster their opportunities to reason in a mathematics class.

Besides the kinds of question that a teacher asks, the choice of a problem for students to work on is important, to make sure that students have opportunities to reason with that problem (Koyama, 1996). In their work, Deborah Ball and Hyman Bass attempt to address how teachers nurture the work of the classroom in the development and use of mathematical reasoning in elementary school mathematics. Ball and Bass (2000) conceive of the teacher as someone who must juggle emergent mathematical ideas, practices, and dispositions in the context of tasks envisioned for the learning of culturally valued mathematical ideas, practices, and dispositions (see also Boaler, 2000; Boaler & Greeno, 2000; Voigt, 1995; Yackel and Cobb, 1996). Within their perspective, it is part of the work of a teacher to listen to students' emergent mathematical ideas and to use these ideas in moving them to new understandings. They claim that mathematical reasoning is not auxiliary but fundamental to understand mathematics proficiently. They say, "Reasoning, as we use it, comprises a set of practices and norms that are collective, not merely individual or idiosyncratic, and rooted in the discipline" (Ball & Bass, 2000, p. 29). They argue that promoting a reasoning of justification is essential in a kind of teaching that aspires to keep the integrity of the discipline, attend to students' thinking, and create an intellectual community. Ball and Bass further suggest a teacher can provide tasks that provoke students to develop their own ideas while, at the same time, allow students to do mathematical work with others. As a result of working with others, students can learn to listen attentively and critically to different points of view. Moreover, students can learn to evaluate and to justify their own mathematical claims in public. Within this perspective, the teacher has the responsibility of listening carefully to students' reasoning and encouraging them to make their mathematical ideas public.

However, listening to students' reasoning and making use of their ideas in class is not easy. Lampert (1985, 1990, 1993, 2001) has proposed that the teacher is a dilemma manager who learns how to cope with problems of teaching. Lampert says, "The work of managing dilemmas…requires admitting some essential limitations on our control over human problems. It suggests that some conflicts cannot be resolved and that the challenge is to find ways to keep them from erupting into more disruptive confrontations" (1985, p. 193). According to Lampert (2001), some of the difficulties teachers face are dealing with individual students and with the whole class at the same time, complying with the demand of covering the curriculum, and helping students to make connections with what they know from the past. In particular, Lampert (1993) said that geometry teachers reported that they follow the sequence of topics in the geometry curriculum with the intention of showing students how to develop an axiomatic system. Therefore, according to teachers, the use of software tools that would alter that order that is established in the curriculum could threaten how they usually organize knowledge in the geometry class. In addition, students would rely upon other source of knowledge—their individual prior knowledge from earlier mathematics classes or their explorations with the software—instead of using deductive reasoning. This study illustrates some of the problems of teaching that geometry teachers have to cope with when introducing a novel tool.

Students' engagement in classroom discussions could be an opportunity for them to learn to reason. However, it has been argued that children may incur into reasoning patterns that are different to the reasoning accepted by mathematicians. For example, in a set of lessons in a fifth grade class taught by Vicki Zack, David Reid (2000b) found that students' questioned the initial step of a proof by induction. For them, the proof by induction did not help them to explain why something worked. In addition, students took empirical evidence as a basis for explanations, especially in cases where they had to reformulate a solution. Reid (2000a) also found that students, under the teacher's guidance, were able to observe patterns, formulate conjectures, and make generalizations when working on a problem. However, students did not use deductive reasoning to make their arguments. In another study of Zack's fifth grade classroom, Zack and Graves (2001) found that students were able to use analogical reasoning in their dialogue with the teacher and with other students. This move towards using analogical reasoning is also related to a teacher's choice of related problems.

In sum, classrooms are places where students could learn to reason in their interaction with other students and with the teacher. Reasoning, from the perspective of situated activity involves students' change in participation in their classroom. This involves students' use of their reasoning skills in public as they debate their mathematical ideas with others. In relation to instruction, a teacher's work of listening to students' ideas, asking them for justifications, and pushing students to make generalization could provoke a change in students' mathematical work, thus engaging students in mathematical reasoning.

Reasoning to build knowledge: Geometry

We focus now our attention on studies regarding reasoning in geometry. First, we review studies of the cognition involved in doing proofs (Anderson, 1982; Greeno, 1976; Koedinger &Anderson, 1990). Then, we review work related to the van Hiele model on levels for geometry thinking (Battista, 2007; Clements & Battista, 1992, 2001; Fuys, Geddes, & Tischler, 1988; Usiskin, 1982). In addition, we look at studies that inspect geometric reasoning in particular with reference to diagrams (Duval, 1995; Herbst, 2004; Laborde, 2005; Larkin & Simon, 1987; Parzysz, 1988). Finally, we include studies about how dynamic geometry software influence students' opportunities to reason in geometry (Arzarello, Olivero, Paola, & Robutti, 2007; González & Herbst, 2005, 2007; Healy & Hoyles, 2001; Jones, 2000; Laborde, 1998, 2000, 2001, 2005; Mariotti, 2000, 2001;

Yerushalmy & Chazan, 1990). Our main point is that geometry—because of the use of proofs and because of the availability of diagrams—provides special opportunities for students to reason.

Cognitive science approaches to the study of geometric reasoning. Cognitive studies have used geometric problems to study how individuals reason, particularly when doing geometric proofs. For example, John Anderson (1976) developed a simulation model based on his ACT-R (adaptive control of thought-rational) cognitive architecture. With this model, Anderson intended to study learning by examining how the simulation would manage a network of propositions. Anderson and colleagues used the simulation to model "the learning processes in geometry" (Anderson, 1983, p. 192), with the case of studying the skills needed for doing two-column proofs. The model divided the process of doing a proof into two episodes: making a plan and executing the plan. In his work, Anderson compared the behavior of the simulation when solving a proof problem and that of a typical student. He found that their behavior were similar. However the model was more amenable to trace back solutions that did not work than the student. Anderson described two stages of learning in the production of proofs. One stage consists of the acquisition of knowledge, or "knowledge compilation" (Anderson, 1983, p. 202). In this stage, knowledge that was present as a set of declarative sentences becomes adapted into production rules, or knowledge that could be use to perform a process. The other stage involves the application of knowledge to solve achieve a goal, and thus, solve the problem. He found four processes that learners can apply to solve proof problems: analogy, generalization, discrimination, and composition (Anderson, 1982, 1983). According to Anderson, the identification of these four processes is important because it shows specific cognitive actions that individuals perform in learning a skill such as doing proofs. For the sake of this review, an important outcome of Anderson's work is a description of geometric knowledge: Anderson shows that items of declarative knowledge (for example, knowledge of the statement that two triangles are congruent if they have two sides and the included angle congruent) are cognitively different from production rules (e.g., to show that two triangles are congruent, search for corresponding pairs of sides and included angles). Their difference is established in the examination of performance doing proof problems and one representation can devolve into the other with practice (Anderson calls this process *compilation*).

Further elaborations of Anderson's work involve an examination of strategies for doing proofs problems in geometry (Greeno, 1976) and the study of how experts use diagrams when doing geometric proofs (Koedinger & Anderson, 1990). Greeno (1976) used high school geometry proofs as an example of a well-structured problem—a problem with a specific goal that could be solve by using a set of operators. With the aid of a computer program, Perdix, he found that a strategy related to the solution of ill-structured problems applied to well-structured problems as well: the use of pattern recognition to achieve subgoals, thus moving towards a solution of the problem. One could infer that when doing proofs in geometry, students' reasoning skills involve identifying subgoals where they could apply the theorems they know to parts of the proof. Students would then need to link the various subgoals achieved in an argument to make the whole proof.

Koedinger and Anderson (1990) extended the prior work of Anderson's ACT-R theory, by designing a computer model that followed strategies that have been observed in experts when they solve geometry proof problems. Experts concentrate on solving key steps of the problem (instead of following through details of a solution). In addition, experts make use of the diagram as they work on a problem to look for possible solutions and to record partial conclusions. The Diagram Configuration model, DC, simulated problem solving in geometry with three processes: (1) *diagram parsing*, (2*) statement encoding*, and (3) *schema search* (Koedinger & Anderson, 1990, p. 520). *Diagram parsing* involves studying the diagram of the problem to recognize geometric configurations that are associated with theorems in geometry (e.g. segment-based configurations, angle-based configurations, and triangle-based configurations). *Statement encoding* is the process where there is identification of the givens and of the statement to prove. *Schema search* involves matching the first two processes, so as to find a possible solution to the problem using the configuration of the diagram and the givens. The simulation selects among a number of possible schemas to solve the problem. In their experiment, the researchers have the computer simulation solve the problem, and then compare those solutions to those of experts. From this study there are two main results regarding experts' reasoning in geometry. First, there is the idea that experts are more amenable to work forward towards a solution, as opposed to working backwards. In particular, an expert (a high school geometry teacher) used purely forward search in most of the answers (5 out of 8). This means that the expert solved the problem without considering the goal. A second result is related to the process of using *perceptual chunks*. Experts, instead of thinking of individual steps towards solving a problem, consider a network of steps or "chunks." Here the diagram is instrumental because they perceive pieces of the diagram together as they use the diagram to solve a problem. According to Koedinger and Anderson, these results have implications for instruction in identifying a method for teaching students how to work on geometry proofs.

Van Hiele's Levels of geometric thinking and the research they inspired. The work of the van Hieles provides another perspective about reasoning when learning geometry. Pierre Marie van Hiele and Dina van Hiele-Geldof, developed a theory about levels of geometric thinking (Fuys, Geddes, & Tischler, 1988). According to the van Hiele model, there are five different levels for learning geometry. Students cannot achieve one level without overcoming the previous one. A summary of these five levels follows. At level 0, students identify geometric figures by recognizing geometric configurations and by naming geometric objects. At level 1, students relate to geometric figures empirically and discover some of their properties. At level 2, students make informal arguments about properties of geometric figures. At level 3, students apply deductive reasoning to do proofs and relate to theorems. At level 4, students understand how different axioms yield a different body of knowledge of geometry.

The van Hieles also gave detailed descriptions of instructional activities that they conducted when developing their theory. They posited that these instructional activities helped students to move from one level to the next. Students are supposed to move from a level to a new level by going through five different phases: information, guided orientation, explicitation, free orientation, and integration. (Fuys, Gedes, & Tischler, 1988, p. 7). These phases are relevant in the design of instructional activities that would enable students to move from one level to the next. For example, a lesson could start by providing the opportunity to get some informal ideas about a concept, such as the names of special quadrilaterals (information). In the next phase, students start classifying similarities and differences between properties of specials quadrilaterals (guided orientation). Then, the teacher asks students to make a table for classifying special quadrilaterals according to information about their angles or about their sides (explicitation). With the aid of the table, students are asked to classify some figures given by the teacher (free orientation). Finally, students create a flowchart with a classification of special quadrilaterals (integration). This example shows that the phases could help students to move from level 1 to level 2.

The categories of geometric thinking in the van Hiele model are analogue to other psychological models that describe reasoning in terms of levels or stages. Other researchers have elaborated further on van Hiele's model. For example, Usiskin (1982) led a study with around 2700 students in 13 American high schools. In the final report of this study, Usiskin stated the objectives of the study as follows, "The fundamental purpose of this project is to test the ability of the van Hiele theory to describe and predict the performance of students in secondary school geometry" (Usiskin, 1982, p. 8). Usiskin and colleagues had difficulties identifying descriptions of specific actions of students' behaviors for the last two levels, especially for the highest level. They decided to modify the van Hiele model and omit level 4.[4] Even though they constructed a test that included level 4, Usiskin and colleagues concluded that level 4 either did not exist or was not testable. In terms of the other levels, Usiskin and colleagues found that they were good descriptors and predictors of students' performance in their geometry class. They also found that students who had difficulties with proofs tended to be at the lower levels of the van Hiele model. They suggested that students should study geometric concepts prior to the geometry course so that in the geometry course they can move to higher levels.

There have been other developments regarding the van Hiele model (see Battista, 2007; Clements & Battista, 1992). A relevant contribution for this review is Clements and Battista's (2001) work. While they have agreed with the notion that the van Hiele levels represent the kind of reasoning that students do in geometry, they found that, at times, there are discontinuities in students' shifts from one level to the next. Also, a student's level can vary depending upon the content of the problem. Therefore, Clements and Battista posited that students' development through the levels is incremental. But, instead of moving linearly from one level to the next, students' geometric reasoning in different types of knowledge—syncretic knowledge, descriptive verbal knowledge, and abstract symbolic knowledge—increases simultaneously. According to this view, the characterization of a student's reasoning skills as belonging to a particular level means that there is evidence showing that that student uses the reasoning skills in that level, more so than another. They said, "We hypothesize that, even though all types of thinking do grow in tandem to a degree, a critical mass of ideas from each level must be constructed before thinking characteristic of the subsequent level becomes ascendant in the students' orientation toward geometric problems" (Clements & Battista, 2001, p. 137). An implication for instruction that we can infer from this work is that the van Hiele

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[4]In Usiskin's (1982) reformulation of van Hiele's levels, level 1 corresponds to van Hiele's level 0. Therefore, Usiskin talked about difficulties with level 5, which corresponds to level 5 in van Hiele's theory.

model could be useful for conceiving of instructional activities that would enable students to move towards higher levels as students learn different geometric content. That is, because the same student may show a different level of geometric reasoning according to the geometric content that that student is learning, then instructional activities could consider moving student towards higher levels in specific content areas.

The van Hiele model has also been used in the study of teachers' geometric reasoning and in how they apply this reasoning when making instructional decisions. A study by Afonso, Camacho and Socas (1999) focused on in-service teachers' geometric reasoning, using the van Hiele model. They studied possible relationships between teachers' performance in geometric reasoning tests that placed them into a van Hiele level and their instructional moves. For example, they looked at teachers' use of instructional materials, the organization of activities in the class, the tasks chosen, and their use of vocabulary. This work extends the theory of van Hiele beyond the study of geometric reasoning from the perspective of students, to consider the interplay between teacher knowledge and teachers' instructional decisions in helping students develop geometric reasoning.

In conclusion, the van Hiele model identifies levels of geometric reasoning and phases for moving from one level to the next. Further research has refined the idea of levels, showing that students' performance in different tasks may show evidence that they reason at different levels. The work of the van Hieles included the creation of instructional activities to foster students' shifts towards higher levels of geometric reasoning but researchers have paid more attention to the levels that to the instructional activities in the original theory.

Reasoning with geometric diagrams. A third group of studies pertaining to reasoning in geometry involves studies focusing on reasoning with diagrams. The question of how individuals use diagrams to process information is one that has been of interest in psychology. For example, with the aid of a computer simulation, Larkin and Simon (1987) studied what kind of processing is needed in order to use a diagram to solve a problem in geometry. They found that one of the most important features of a diagram is that it eases the process of recognizing information that is useful towards solving a problem. The process of recognition requires matching elements (or configurations) in the diagram and propositions (theorems, postulates, and definitions) about geometric relationships. They said,

> The major difference in a diagrammatic representation, we believe, is difference in recognition processes. We have seen that formally producing perceptual elements does most of the work of solving the geometry problem. But we have a mechanism—the eye and the diagram—that produces exactly these "perceptual" results with little effort. We believe the right assumption is that diagrams and the human visual system provide, at essentially zero cost, all of the inferences we have called "perceptual." (Larkin & Simon, 1987, p. 92)

The process of recognition is one of the three processes that Larkin and Simon had identified when working on a problem: search, recognition, and inference. They argued that, even though the work of solving a geometric problem requires these three processes, diagrams are most helpful when performing the process of recognition, enabling the problem solver to connect information in the diagram and knowledge that would lead them towards solving the problem.

In addition, Larkin and Simon said that the labels in the diagram are useful when the information in the diagram should be written in a proof. This is an important point when considering the relationship between working with a diagram in geometry and using elements of the diagram in a proof. So one could conceive that the process of working with the diagram and finding a solution to the problem is separate to that of writing the proof. However, in terms of the computer simulation program, the labels in the diagram were very important in the process of recognition to match parts of the diagram with known propositions. Therefore, one could expect that the labels in a diagram help the problem solver to pay attention to parts of the diagram that otherwise one could ignore. Overall, the work of Larkin and Simon underscore how diagrams provide information that is cognitively different in nature from the information provided by declarative statements. While this work underscores the informational singularity of any diagrams, epistemological considerations on the teaching and learning of geometry have added distinctions between the cognitive processes associated to the diagram itself and the knowledge of the objects represented by those diagrams—the geometric figures.

An early attempt to distinguish between diagrams and figures is present in the work of Bernard Parzysz (1988) who distinguished between *figure* and *drawing*. He says, "the *figure* is the geometrical object which is described by the text defining it" (p. 80). *Drawings*, on the other hand, are two-dimensional representations of figures. Further addressing the complex distinction between *diagrams* (drawings) and *figures*, Raymond Duval (1995) identified four different processes (grasps or apprehensions of the figure) that individuals could deploy when working with diagrams: (1) *perceptual*, (2) *sequential*, (3) *discursive*, and (4) *operative apprehension*. Perceptual grasp involves paying attention to characteristics of parts of a diagram or the diagram as a whole. By applying perceptual grasp an individual looks at characteristics of the diagram (such as the size, the shape, and the orientation of the diagram), divides the diagram into smaller sub-figures, and recognizes a figure by its name. Sequential grasp involves organizing the construction of a diagram into a series of steps, especially when using construction tools. The choice of construction tools limits the kind of procedure involved for making a diagram. For example, if one were to construct the angle bisector of a given angle with a compass and a straightedge, the procedure would involve a sequence of steps in a particular order. In contrast, if one were to draw freehand the angle bisector of an angle, the procedure would not require that many steps and a sketch of the angle bisector would suffice. Therefore, sequential grasp yields different "figures" depending on what kind of tools are being used to produce them. Discursive grasp is the process of producing or using information about the diagram. This information could be encoded in the diagram by means of imposing symbols, but also could be expressed in descriptive statements about the diagram. Finally, operative grasp is the process of modifying a diagram, for example, by sub-dividing the diagram into its components or by adding other elements to the diagram such as auxiliary lines.

Duval has argued that the work with geometric figures involves perceptual grasp plus another one of the other three processes. According to Duval, mathematical work with diagrams uses the four kinds of grasps. These grasp processes are not isolated, but connected, especially when considering the kinds of resources that students have available to solve problems. For example, if students were to have available dynamic geometric software to solve a problem, it is likely that they would have to use sequential grasp to consider the order of steps to make a diagram. Knowledge of the geometric object—the figure—involves coordinating the different apprehensions of a diagram, work that may rely notably on proving, for example when showing that the diagram produced by executing as series of steps (sequential grasp) indeed has the properties described in informational text about the diagram (discursive grasp).

Further elaborating on the distinction between diagram and figure, Colette Laborde (2005) has proposed a distinction between theoretical and spatio-graphical properties of a diagram. Laborde stated that as an empirical object, a diagram possesses some spatio-graphical properties such as the orientation on a paper (or on a screen), the relative lengths of sides, or the thickness of the lines. One could perceive spatio-graphical properties of a diagram by looking at the diagram. A diagram, on the other hand, also possesses some features that make reference to relationships about geometrical objects—theoretical properties of the figure being represented. Students' work with diagrams involves navigating between these two kinds of properties to solve a problem in geometry. Students could use the spatio-graphical properties of a concrete diagram and also could draw upon their knowledge of theoretical properties of figures. According to Laborde, the study of geometry requires students to confront understanding of the two different kinds of properties: the spatiographical properties of actual diagrams and the theoretical properties of the abstract figures represented by those diagrams. Part of our argument will be that navigating the distinction between these two kinds of properties provides unique opportunities for students to reason.

Herbst (2004) has identified four modes of interaction with diagrams base upon observations of students' work in geometry classes. These four modes of interaction are: *empirical*, *representational*, *descriptive*, and *generative.* The empirical mode of interaction includes using measurements of a diagram to make inferences about geometric properties of the geometric figure. The representational model involves using (accurate) diagrams to illustrate the geometric properties of figures. The descriptive mode includes abducting properties of figures from perception of the diagram (where visual perception may be corrected by additional symbols such as hash marks or arcs that would override what the diagram appears to be). The generative mode involves making hypotheses about geometric figures from observing how the diagram responds to actions on it (including measurements) and modifications such as adding auxiliary lines or decomposing some of its parts. Herbst has argued that the generative mode of interaction can help students make reasoned conjectures—conjectures that come from reasoning rather than from perception.

These perspectives about students' work with diagrams show that reasoning in geometry is special because of the use of diagrams to represent geometric figures. In particular, the use of diagrams pushes students to combine their visual perception about geometric objects and their knowledge about geometric relationships. Thus, the work with diagrams is not separate from the work of reasoning on geometry. On the contrary, reasoning in geometry happens as students work with diagrams to solve a problem. The novel tools of dynamic geometry software have made more evident students' work with diagrams as these tools enable students to use diagrams differently then when using paper-and-pencil diagrams.

Dynamic Geometry. In a study that focused on students who used The Geometric Supposer, Yerushalmy and Chazan (1990) found that these students incorporated novel strategies when working with paper-and-pencil diagrams as a result of their interaction with the software. Some of these strategies involved making a sequence of diagrams to show different instances of a case, balancing their focus on parts of the diagram and on the whole diagram, and adding auxiliary lines on their own. Yerushalmy and Chazan concluded that students' use of diagram was more flexible when compared with students who did not have access to The Geometric Supposer. The software allowed students to overcome usual obstacles when working on geometric problems, such as the assumption that the characteristics of one diagram could be taken as representing a class of geometric figures. Thus, the tools of the software enabled students to incorporate useful strategies when working with diagrams, improving their ability to make conjectures using a diagram.

Other dynamic geometry software packages, such as Cabri (Laborde & Bellemain, 1993), Sketchpad (Jackiw, 1991), and GeoGebra (Hohenwarter, 2001), include the opportunity for users to drag parts of geometric diagrams that are displayed on the screen (see Goldenberg and Cuoco, 1998). This feature enables users to change a diagram or parts of a diagram and see those transformations in real time on a computer screen. Arzarello and colleagues (2002, 2007) have identified a hierarchy of dragging modalities. These dragging modalities are useful for characterizing how students use dragging to make and to validate their conjectures, especially in transitions from gathering empirical evidence towards proving some of the conjectures. Students' use of different dragging modalities provides evidence of how they solve problems in geometry with the mediation of technological tools. The research shows that the dragging features of dynamic geometry software allows students to reason differently than when working with static diagrams because students can get instant feedback about how changes to part of a diagram affect other parts of the diagram. By dragging, students can focus on relationships between geometric objects.

There are other features besides dragging, or in combination with dragging, that make reasoning with dynamic geometry software tools different than when using paper and pencil. For example, Healy and Hoyles (2001) reported the case of a pair of students who used the measuring and the dragging capabilities of dynamic geometry to establish the initial conditions of a problem, while if they had had to work with compass and a straightedge, they would have used other procedures to establish the initial conditions of the problem. In our work we have found concurrent evidence that students who combine the measuring and dragging capabilities of dynamic geometry interact differently with diagrams than when using static diagrams (González & Herbst, 2005, 2007). While such interaction includes elements of the empirical and the generative mode of interaction (Herbst, 2004), the dynamic links provided by the software between each version of the diagram and its measurements, enable the dragging operation to produce collections of measurement information about a diagram associated to controlled variation of its spatiographical properties. We have called this mode of interaction with diagrams the *functional* mode of interaction with diagrams, where students reason looking for patterns that relate controlled inputs that result from dragging with the outputs that the software provides from measurements.

The literature on uses of dynamic geometry in instruction has made evident that students' opportunities to reason with the aid of technological tools do not happen solely by exposing students to these tools. A teacher's actions requesting students to provide justifications for their findings and providing opportunities to share their work in public are some elements that push students to move from making discoveries with technological tools to proving. For example, a study by Jones (2000) showed that students using dynamic geometry software in a unit on quadrilaterals shifted from giving informal explanations about their discoveries to more sophisticated mathematical justifications. The teacher's questions provoked students to make references to earlier work in the unit, thus providing the basis for deductive reasoning. At times, other artifacts aid a teacher's work of making public students' discoveries when working with dynamic geometry. For example, Mariotti (2001) reported that students' use of a notebook for students to keep track of their discoveries and for the teacher to ask students for specific references to their work on a problem was useful to make students' individual work in the computer lab public. These examples show that teachers can broaden students' opportunities to reason in classroom with the use of dynamic geometry software by incorporating instructional practices that would enable students to interpret the feedback from the screen.

Mathematics education research on proof in geometry. The nature of the activities that teachers and students undertake in geometry provides yet another set of reasons for us to care about understanding the geometry class. The description of geometry (attributed to Pölya) as "the science of correct reasoning on incorrect figures" suggests a set of issues for which the geometry class affords a fundamental opportunity. Being concerned with understanding basic things like shape and space, the study of geometry may find uses for students' intuition, perception, and physical experience (Hershkovitz, van Dormolen, & Parzysz, 1996; Lehrer et al., 1999; Parzysz, 1988). Yet as geometry seeks to systematize those through the development and use of rigorous thinking and proving, it is also to be expected that geometry instruction might impose challenges on students' intuitions, sometimes even conflict with them. Thus, interesting manifestations of geometric reasoning, combining form and substance, may take place in classrooms. Alan Schoenfeld (1986, 1987) provides an important example: He documents how, in checking the correctness of a construction, students did not think about using a theorem they had just proved. They would rather limit their work to accurately following a construction procedure. He used that observation to point out a problematic separation between the deductive reasoning of justification and the purely empirical reasoning involved in constructions. Chazan (1993) also documents how students who would use diagrams as anchors to develop proofs that an observer would recognize to be general would also think that the validity of the proposition proved would be limited to the particulars of the figure in the given diagram.

Observations of these kinds of phenomena have often been taken as informing characterizations of individual conceptions. Schoenfeld's (1987) contribution, however, suggests that these phenomena might actually attest to the influence of larger structures finding their way into the classroom. He suggested that a notion of "good teaching" attuned to good performance in a high stakes exam might actually be commending, even promoting, the problematic separation of reasoning practices that he had observed. We suggest it is highly relevant to the development of a knowledge base for instructional improvement that we look for more examples like Schoenfeld's, indeed for a theory grounded on such examples. We need to know how the actions that emerge in mathematics classrooms and the structures that frame the classroom combine to influence how teachers foster reasoning practices in the context of mathematics instruction.

In sum, reasoning in geometry has been characterized as special from the point of view of cognitive studies and of mathematics education. The work of doing proofs, the use of diagrams, and the availability of technological tools to construct and to manipulate diagrams are some of the elements that make geometry special for mathematical reasoning. These elements can also make it challenging for a teacher when considering how to sustain students' opportunities to reason in a geometry class.

The geometry of instruction

The fourth constraint on the work of teaching geometry comes from the expectation that geometry be represented and learned in the context of a specific course, which students take at particular times in their lives and subject to particular kinds of schedules and assessments. Our work focuses on the high school geometry course, which is ordinarily taken in 9th or 10th grade. Students take this course in daily periods of about 50 minutes all days of the week for a year, or in blocks of about 90 minutes, three times a week for a year, or in blocks of about 90 minutes every day for a semester, or in periods of 75 minutes every day for two out of three trimesters per year. Scheduling the deployment and learning of geometry responds to constraints that are irrespective of the subject being studied.

One way to describe the objects of contract is by itemizing each of the topics covered in each of the chapters. The contents of the textbook used in the classes we studied (Boyd et al., 1998) are a good representation of the mainstream geometry content at the end of the 20th century. The book starts with an introduction to geometry through the coordinate plane and measurement, followed by an introduction to proof by way of logic (converse, contrapositive, universal and existential quantifiers, etc.) and algebra. The synthetic geometry content starts to develop in chapter 3, which introduces perpendicular and parallel lines, and properties of angles formed by transversals to parallel lines. Chapters 4 and 5 introduce triangle congruence postulates, prove theorems about triangles and their special segments, and provide multiple opportunities for students to prove triangles are congruent. Chapter 6 explores special quadrilaterals and their properties. Chapter 7 presents similarity and similar triangles and proves the theorem about the segment connecting two midpoints in a triangle. Chapter 8 introduces trigonometric and Pythagorean relationships in right triangles. Chapter 9 develops the study of circles, angles, arcs, chords, and tangents. Chapter 10 deals with area of polygonal shapes and of the circle, while chapter 11 develops volume and surface area formulas for basic solids. Chapter 12 returns to coordinate geometry to revisit basic synthetic properties using algebra and vectors. And Chapter 13 looks at locus and transformations using coordinates. There are definitions, postulates, and theorems through the book though most especially in chapters 3-9, those chapters are also heavily populated by proof exercises.

But the textbook is not the only source of authority regarding what teachers are obliged to teach and students obliged to learn. Policymakers at the national, state, and local level also play a role in establishing objects of the contract. For example, an examination of the high school content expectations (HSCE) for the state of Michigan would add to the above list many additional expectations of what students are expected to learn, including: how to carry out and justify compass-and-straightedge constructions; pattern recognition and conjecturing based on inductive reasoning; the significance of irrational numbers and how they emerge from geometry; solving triangles using trigonometry; and formal proofs of various incidence properties. The following table contains as cross listing of important objects of knowledge in the geometry curriculum and the expectations set on high schools in the state of Michigan. It illustrates how accountability for the teaching of geometry ideas is infused to the teachers.

Object of knowledge	High School Content Expectations
Geometric concepts (in any dimension) and their definitions	G1.4.1. Solve multistep problems and construct proofs involving angle measure, side length, diagonal length, perimeter, and area of squares, rectangles parallelograms, kites, and trapezoids.
Geometric relationships among objects and their definitions	G1.1.1. Solve multistep problems and construct proofs involving vertical angles, linear pairs of angles, supplementary angles, complementary angles, and right angles.
The notion of definition	G1.1.6. Recognize Euclidean geometry as an axiom system. Know the key axioms and understand the meaning of and distinguish between undefined terms, axioms, definitions, and theorems.
Theorems about and properties of geometric concepts	G1.2.1. Prove that the angle sum of a triangle is 180 and that an exterior angle of a triangle is the sum of the two remote interior angles.
Theorems about and properties of geometric concepts	G1.4.4. Prove theorems about the interior and exterior angle sums of a quadrilateral.
Theorems about and properties of geometric relationships	G1.1.1. Solve multistep problems and construct proofs involving vertical angles, linear pairs of angles, supplementary angles, complementary angles, and right angles.
Theorems about and properties of geometric relationships	G1.1.2. Solve multistep problems and construct proofs involving corresponding angles, alternate interior angles, alternate exterior angles, and same-side (consecutive) interior angles.
Measures of geometric objects or relationships	G1.6.1. Solve multistep problems involving circumference and area of circles.
Measures of geometric objects or relationships	G2.1.2. Know and demonstrate the relationships between the area formula of various quadrilaterals.
Ways of calculating the measure of an object or a relationship	G1.3.2. Know and use the Law of Sines and the Law of Cosines and use them to solve problems. Find the area of a triangle with sides a and b and included angle Θ using the formula Area=(1/2)a*b*sin(Θ)
Transformations involving geometric objects	G3.1.1. Define reflection, rotation, translation, and glide reflection and find the image of a figure under a given isometry.
Transformations involving geometric objects	G3.1.2. Given two figures that are images of each other under an isometry, find the isometry and describe it completely.
Theorems about and	not specified

properties of transformations	
The notions of theorem and postulate	L3.1.3. Define and explain the roles of axioms (postulates), definitions, theorems, counterexamples, and proofs in the logical structure of mathematics. Identify and give examples of each.
Proofs of key propositions (specific theorems)	G1.2.3. Know a proof of the Pythagorean Theorem, and use the Pythagorean Theorem and its converse to solve multi-step problems.
Techniques of proof (kinds of proof and logical background)	L3.2.1. Know and use the terms of basic logic.
Techniques of proof (kinds of proof and logical background)	L3.2.3. Use the quantifiers "there exists" and "all" in mathematical and everyday settings and know how to logically negate statements involving them.
Tools for proofs (specific theorems applied)	G2.3.1. Prove that triangles are congruent using the SSS, SAS, ASA, and AAS criteria and that right triangles are congruent using the hypotenuse-leg criterion.
The notions of proof and counterexample	L3.1.3. Define and explain the roles of axioms (postulates), definitions, theorems, counterexamples, and proofs in the logical structure of mathematics. Identify and give examples of each.
The notions of proof and counterexample	L3.3.2. Construct proofs by contradiction. Use counterexamples, when appropriate, to disprove a statement.
Notations and conventions	not specified
Representational Fluency	L1.2.3. Use vectors to represent quantities that have magnitude and direction, interpret direction and magnitude of a vector numerically, and calculate the sum and difference of two vectors.
Fluency applying geometric theorems and formulas for measure to real world and applied scenarios	G1.6.1. Solve multistep problems involving circumference and area of circles.
Construction procedures	G1.1.3. Perform and justify constructions, including midpoint of a line segment and bisector of an angle, using straightedge and compass.
Construction procedures	G1.1.4. Given a line and a point, construct a line through the point that is parallel to the original line using straightedge and compass. Given a line and a point, construct a line through the point that is perpendicular to the original line. Justify the steps of the constructions.
Visual Dexterity	G2.2.1. Identify or sketch a possible three-dimensional figure, given two-dimensional views. Create a two-dimensional representation of a three-dimensional figure.
Drawing capacity	G2.2.2. Identify or sketch cross sections of three-dimensional figures. Identify or sketch solids formed by revolving two-dimensional figures around lines.

The previous paragraphs and the table summarize what one could find if one were to describe the objects of contract as a list of topics to be covered. Yet it is important to note that as far as how each of those topics is an object of contract, one would have to do more than listing all the topics. One would also have to describe in what way would students need to come to own those topics. We produce below a synthetic response to the question of what geometry knowledge, skills, and dispositions students are expected to lay claim on in the geometry class.

Usiskin (1980) provides a framework for answering that question that takes into account the curriculum debates of the 20th century. First, geometry is a domain of mathematics that connects uniquely which students' experiences in the real world. In that sense, one would expect the geometry course to enable students to lay claim on mathematical ideas that would help them master the physical world. In particular one can think of the following knowledge, skills, and dispositions:

- knowing and being able to use the formulas for the perimeter and area of basic shapes, the surface area and volume of basic solids
- knowing how to measure directly unknown dimensions (angles, sides) of shapes and solids
- knowing how to use properties of figures to find indirectly the measures of unknown dimensions
- knowing how to reduce a problem of measuring a complex shape or solid to a combination of simpler shapes and solids
- being able and amenable to model a real world object as a combination of (or a transformation of) basic geometric shapes and solids
- being able to predict the dimensions of a real shape or solid based on an analysis of its geometric model

Geometry also provides a graphical language for representing mathematical ideas from other areas including calculus and linear algebra. In that sense one would expect that students would

- Be able to name shapes and solids they encounter as pictorial representations
- Be able to visualize shapes and solids given their properties
- Being able to produce shapes to represent mathematical ideas

Finally, Usiskin (1980) notes that geometry provides an accessible example of a mathematical system of axioms, definitions, and theorems. From this perspective, students should be able to

- Know what giving a definition means
- Know when an object is a particular case of another and when a statement follows from another one.
- Know how to show that the truth of a statement derives from the truth of other statements
- Know how to show the falsity of a statement by providing a counterexample to it
- Know one or more examples of a mathematical system of postulates, definitions, theorems and their proofs

- Know how to use various methods of proof, including both direct and indirect argumentation

A perhaps more succinct list of objects of study involved in the geometry class is the following (also shown on the Table above).

Geometric objects (in one, two, and three dimensions) and their definitions
Geometric relationships among objects and their definitions
The notion of definition

Theorems about and properties of geometric objects
Measures of geometric objects and ways of calculating those

Transformations involving geometric objects
Theorems about and properties of transformations
The notions of theorem and postulate

Proofs of key propositions (specific theorems)
Techniques of proof (kinds of proof and logical background)
Tools for proofs (specific theorems applied)
The notions of proof and counterexample

Notations and conventions
Representational Fluency
Fluency applying geometric theorems and formulas for measure to real world and applied scenarios
Construction procedures

The sections just reviewed account from four different perspectives on the way the territory of this article has ordinarily been mapped. The question of reasoning in geometry has been seen as one that concerns the subject of study (geometry), the manner in which practitioners in this subject operate (mathematicians), the individual development and actions of children and their social interaction and argument. While attentive to the work of reasoning in geometry from disciplinary, individual, interpersonal, and institutional perspectives, the prior review attests also to a scarcity of research on geometry instruction that focuses on the work of the teacher. In what follows we immerse the consideration of the work of reasoning in geometry into the work of teaching, especially in the context of the management of exchanges that the teacher needs to do. This paper contributes to the exploration of reasoning in high school geometry instruction by undertaking an investigation of how geometric reasoning and proof are enabled by the instructional practices developed and used in geometry classrooms.

A Theory of Classroom Exchanges with Particular Reference to Geometry

The review on the work on reasoning done above underscores that reasoning and proof are the key tools of mathematical discovery. Perhaps more than any other academic discipline, mathematics is concerned with objects and relationships whose existence is intellectual (even if they may at times be represented physically). Thus propositions about them are arrived at and confirmed through intellectual work. In place of the material agency of objects to which other disciplines submit, mathematical entities are subject to what Pickering (1995) calls *disciplinary agency*, the constraints determined by the relationships that the community assumes mathematical objects entertain with each other. Deductive reasoning is the way in which such disciplinary agency gets enacted, deriving necessary consequences from postulated relationships. If indeed the didactical contract in a mathematics class is accountable to the discipline of mathematics, it is natural to expect that work in mathematics classes would, among other things, lay claim on that reasoning as a tool of the trade: That it lay claim on the methods of mathematical discovery and confirmation. The geometry class has traditionally been presented as a place where students engage in theoretical mathematics, not only because they encounter an example of a mathematical system but also because they have the opportunity to learn to reason mathematically and engage in mathematical reasoning.

In the strands of literature surveyed above, we documented what the work of reasoning in geometry can look like. To ask the question of what opportunities to reason mathematically are created in the geometry class, however, is fundamentally a question about viable activity inside an institution. If students are to have opportunities to engage in mathematical reasoning, these will happen as part of the activities that do in classrooms. It is critical to understand what is customarily done and on what regulations that depends.

Geometry classrooms stage an encounter between students, geometry content, and a teacher. This encounter is framed by an unspoken contract, the didactical contract, according to which those participants enter into relationships vis-à-vis the communication of that content and for some limited time (see Brousseau, 1984, 1990; Brousseau & Otte, 1991). This relationship can be seen as a transaction of sorts—a certain body of knowledge, skills, and dispositions exists as part of the culture at large, the student enters into a contractual relationship to acquire that knowledge from the teacher, while the teacher enters into the relationship expected to facilitate and certify that acquisition. The teacher needs to ensure that students have had the opportunity to acquire that knowledge.[5]

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[5] A metaphor of learning as acquisition of information and skill is pervasive in that rendering of the contract. In presenting the contract in that way we are not necessarily claiming that such acquisition metaphor describes well what learning is as a phenomenon but rather claiming that the acquisition metaphor describes how educational institutions frame learning (that is, what learning is as a second order phenomenon, as a folk theoretical construct or what “learning” means as a process contracted about). Likewise, the contract does rely on an existing notion of teaching that, while not necessarily desirable or effective, describes what a teacher is expected to do, broadly speaking. This notion of teaching includes the communication of knowledge to students and the care for the wellbeing of students as individuals and as a collective. We do not contend that those notions of learning and teaching are desirable or that they are universal, we do contend that they are normative in the US in the sense that other notions of teaching and learning, other contracts, are presently defined as departures of those.

Our theoretical contribution is to describe how a practice of teaching has been organized to satisfy the achievement of that learning, around which the contract is established.

In earlier work (particularly Herbst, 2003, 2006) we have elaborated on how the hypothesis of the existence of a didactical contract helps account for the tensions and contradictions experienced in teaching. Our work builds on the work of scholars who have described teaching as the management of tensions (Cohen, 1990) and dilemmas (Lampert, 1985). Such tensions are endemic in the work of teaching. Teachers encounter these tensions all the time, and they condition and constrain the mathematics that can be known and done in classrooms (Arsac, Balacheff, & Mante, 1992; Ball, 1993 Chazan & Ball, 1999; Chazan, 2000). We propose that potential for these tensions and dilemmas is inscribed in teaching by the presence of multiple stakeholders that constrain the unfolding of the didactical contract over time. A main contribution of the present work is to show that specific instructional situations in geometry function to simplify potential tensions associated with the work of reasoning and proof.

One of the constraints on the didactical contract for the geometry class is that it utilizes a process of institutionally managed instruction for the teaching and learning of target knowledge. In particular, content and skills are taught and learned through curricular units (represented in book chapters and sections and corresponding work assigned to be done in allocated days and times). Also, learning is attested through assessments deployed also over time in the form of graded assignments and tests. This apportioning of different content for different periods of time and certification of learning of different elements of content at different periods of time was common to all the geometry classes we examined.

Over time, students and teachers not only engage in instructional work of that kind but they also lay different kinds of claims on the various objects of the contract. The teacher lays claim on having represented mathematical concepts and skills for students to learn them; and on behalf of students, the teacher lays claim on their attainment of learning. We thus propose that the fulfillment of the didactical contract can be understood as the execution of a series of transactions between on the one hand the work that students do (including the reasoning they do) and on the other hand the claims made (by the teacher) on the students' knowledge of what is at stake in the contract. Specifically, over the time allocated to a particular contract (say over a year or a semester) a number of pieces of knowledge need to be acquired by students, often in a sequence set by the curriculum. Each of those curricular requirements we call an object of contract. Each of these objects needs to be represented in the class and students are expected to have it taught and to learn it in the class—these objects of study are laid over as discrete points along the timeline of execution of the contract. The procurement of each of those contracted objects is done in and through classroom work—the continuous, moment-to-moment interaction between teacher and students apropos of various classroom tasks. Students do and say many things in response to assignments and questions, often those assignments and questions are posed on behalf of an object of knowledge that they are expected to learn. The work that they actually do attests to the extent to which they have had the chance to learn it. While at least in theory the student is responsible for doing the lion's share of the work, the teacher is the party responsible to create and sustain opportunities to do it and to effect this transaction—namely the teacher has to create work opportunities for students that enable them to lay claim on objects of knowledge and the teacher has to attest that their involvement in such work amounts to laying such claim.

We are interested in understanding these transactions in regard to the role they adjudicate to mathematical reasoning. It is thus important to go beyond the observation that students do work that has some exchange value in terms of the knowledge to be acquired. It is important to find what kinds of work are done and for what objects of knowledge they are traded. We propose that the exchanges between work done and knowledge claimed are not uniform over the course of a year of study in geometry—rather, over the course of the year of study, different kinds of exchanges are operated. Empirically one may note for example that on some days it is encouraged of students to say what they think is true about a diagram from merely looking at it, while on other occasions students are expected to justify their statements with reasons and chastised if they are "going by looks." Our theoretical proposition to account for this diversity in instructional exchanges revolves around the notion of an *instructional situation*. Specifically, we argue that different instructional situations preside over exchanges in which different kinds of work are allocated value as knowledge.

We propose that the fulfillment of the didactical contract by way of effecting transactions or exchanges between work done and objects of knowledge laid claim on is done through the recurrent deployment over time of a finite number of instructional situations. That claim has two sides. On the one hand we say that more than just one kind of work is valued in geometry, in particular, different kinds of reasoning by students are given value at different times. Yet we are also saying that this valuing of diverse work by students is not open to anything that students might happen to do, and in particular that it is not necessarily negotiated apropos of each specific topic at each moment in the year of study.

Instructional situations are frames for the exchange of work and knowledge claims—they enable participants, in particular teachers, to know what kind of work is expected (appropriate or desirable) for students to do and what kinds of objects of knowledge can be claimed on account of such work. In principle, a heuristic to think about the different instructional situations that exist in geometry is to implement an epistemological reading of the curriculum—anticipating that various kinds of objects of study might call for unique kinds of instructional situations. Instructional situations are mathematically specific frames for effecting those transactions, specific to the elements of an epistemology of school geometry. Since the course of studies in geometry revolves around the study and learning of geometric concepts, theorems, formulas, proofs, and construction procedures, specific situations exist that usher students into the learning of those every time one of them is at stake.

To the extent that different instructional situations summon students to do different kinds of work, we contend that different instructional situations make room for different kinds of reasoning. The problem of describing what are the opportunities to reason that students have in geometry classes can be reduced to one of describing the different instructional situations in which students have the chance to participate. The most important contribution of this report lies in fleshing out for the geometry class an initial inventory of instructional situations that make room for students' reasoning work, exemplifying the tasks that are customary in those situations and describing the kind of reasoning that students have the chance to engage in. We conclude our description with suggestions for how research and development in geometry instruction can make use of the theoretical proposition that these instructional situations are the ones recurrently used in geometry.

The notion of *instructional situations* as frames for exchanges has been explored in our earlier work studying specific activities in geometry classes. We have elaborated in more detail on two instructional situations that we called "doing proofs" and "installing a theorem" (see Herbst & Brach, 2006; Herbst et al., 2009; Herbst & Miyakawa, 2008; Herbst & Nachlieli, 2007; Herbst, Nachlieli, and Chazan, in press). In those contributions we have sketched also a way to study particular instructional situations that relies on two techniques. One of them consists of modeling an instructional situation as a system of norms that regulate the work done, its exchange value, the division of labor, and the organization of time. Any of those models consist basically of a list of statements that describe normal (unmarked) action. In Herbst & Miyakawa (2008) we show how representations of teaching (in the form of animations) can be used to construct prototypes of those models. In Herbst, Nachlieli, and Chazan (in press; see also Herbst & Nachlieli, 2007) we show how those models can be tested empirically by confronting practitioners with those animated prototypes. The other technique consists of examining how practitioners warrant deviations from the norms hypothesized by those models. It relies on analyzing the ways in which practitioners describe and appraise courses of action alternative to those displayed in a representation of teaching where the class is purportedly involved in an instructional situation. The aim of this paper is to cast a much wider net on the landscape of instructional situations that one can find in geometry classrooms. Rather than developing a detailed model of each we describe each of a handful of situations and illustrate them with records of practice observed in four geometry classes offered in a comprehensive high school.

In an earlier section we showed examples from the geometry curriculum of the knowledge and skills students are expected to acquire as these were laid out in the textbooks and materials used by the teachers in the school we observed and in the state standards and benchmarks that those classes had to meet. We now look for examples in the records of those classes in which there is evidence that such objects of knowledge were at stake. We use those records to provide broad strokes descriptions of the work to be done in order to lay claim on those objects of knowledge. Each of those descriptions is done in the context of proposing one instructional situation. Finally we demonstrate how canonical tasks in each of those instructional situations could be extended to give students opportunities to reason mathematically.

Reprise of the Theory

To summarize what we have done so far is the following. In the literature review we have outlined references that describe in broad strokes what can be considered mathematical reasoning in geometry. This work includes a number of activities and learnings that observers might notice in children working alone or together, as they work with geometry content. It includes reasoning that feeds from empirical and perceptual interaction with diagrams as well as reasoning that feeds from stipulated assumptions and uses only prior knowledge. All this reasoning work is clearly just a part of what could be considered "mathematical" work in geometry classes. The section thus outlines one of the elements a teacher needs to deal with in managing instructional exchanges. In the last section we have outlined the other element by describing in three different ways (curriculum outline, state goals and benchmarks, and an epistemological reading of those) what the didactical contract for the geometry class includes, as far as the kinds of objects of knowledge around which a contract is established.

Our main contention is that a teacher of the high school geometry course needs to manage exchanges between objects of the contract and chunks of mathematical work (which, we submit, includes the reasoning work outlined in the review). A second contention builds on the notion of negotiation of the contract. The existence of a didactical contract which specifies in broad strokes what needs to be learned, what it means to learn, and what it means to teach does not quite specify what teacher and students will (have to) do when they encounter specific work and exchange it for specific objects of contract. Lacking other devices, every new object of knowledge and learning, every new chance to learn something new, might require teacher and students to negotiate how the terms of the contract apply to it—including what it means to know it, what the student has to do to learn it, and what the teacher has to do to teach it. Such a position, according to which every new object of knowledge requires a new negotiation of the contract is perhaps desirable in terms of ideal adaptation to the needs of learners and knowledge. Along those lines Hanna and Jahnke (1996) say that for each proof a teacher should decide (possibly with her students) how much detail is needed. In particular, decisions to implement innovations carry with them the need to do these negotiations as we have shown in our earlier work Herbst (2003, 2006). But engaging in those negotiations is arduous and often distressing. We hypothesize that to avoid the need for such negotiation of the contract for each task, or at least to simplify those negotiations, instructional situations have developed historically in the geometry teaching profession and are implemented in a class that participants use to frame their exchanges of work for objects of knowledge. An instructional situation is a frame that fast tracks the exchange between certain kinds of interactive work and certain kinds of objects of knowledge at stake. Rather than being adaptations of a single didactical contract under continuous negotiations[6] these instructional situations operate like a menu of available ways of organizing activity, and are summoned through the participants' use of key words that account for what all are supposed to do (e.g., construct, prove, solve).

Instructional Situations in High School Geometry

We propose that to handle the responsibility to attain the geometry contract, a number of instructional situations have developed in geometry instruction. These situations are defaults for classroom activity, which means that classroom tasks default to these for the rules by which people go. In the following we describe those situations summarily below and then we provide thicker accounts of the work that they make

[6] An alternative way of conceptualizing how the exchange could be made more efficient would be by saying that frames get adapted in chunks of time, so that, for example the contract is negotiated to operate in a certain way early in the year, facilitating some kinds of exchanges then. Later in the year the contract is negotiated to facilitate other exchanges and so on. This approach may be compelling when dealing with mathematics in earlier grades. In high school geometry, partly because of the epistemological diversity of the knowledge at stake, it makes more sense to hypothesize that several different frames are available at any one time.

possible as we illustrate them with examples from the classes observed. The list is not exhaustive. We note some elements missing at the end.

Installing a concept (defining a new idea). At stake in this situation are at least three things: students' appropriation of the official meaning of a word in terms of the attributes of objects that respond to that word, students' visualization of a prototypical shape and of a range of shapes that respond to that word, and students' disposition to recur to a definition when using a term. This situation makes room for the teacher's work eliciting prior uses of the word in students' experiences, spelling out the definition of the word, illustrating the meaning of the word by drawing or constructing different shapes that respond to the word, asking students to consider a given object in regard to whether the word applies to it, and relating or prompting students to relate the term being defined with other objects or terms. As we will show below, students could have opportunities to reason inductively when a definition is generated from the consideration of commonalities in different shapes, and deductively when examples or non examples are put to their consideration.

Installing a theorem, property, or formula about known ideas. At stake in this situation are students' knowledge and belief that a declarative statement is true, students' knowledge of the statement's connection to other statements, and students' authority and disposition to use the statement in future work. The situation makes room for the spelling out of the statement, the engagement in verification activities leading to students' conviction, the proof of the statement, the drawing and recognition of diagrams that illustrate the theorem, the instantiation of the theorem in the case of generic objects, and the development of verbal means to remember the theorem in future work. Students could have opportunities to reason deductively when translating or verifying the translation of the statement of the theorem to the case of a particular geometric object, and when proving or following the proof of a theorem.

Calculating the unknown dimensions of a shape or solid. At stake in this situation is students' knowledge and ability to use a theorem or a formula. The situation makes room for the recognition of figures and their properties, for the proposition of arithmetic or algebraic relationships between geometric quantities using known theorems and known dimensions. The situation also makes room for the recognition of properties in a figure based on the information provided by a diagram. Students could have opportunities to reason deductively by inferring that unknown dimensions have particular numerical values or algebraic expressions on account of givens and known theorems.

Doing a proof that a conclusion statement follows from given statements. At stake in this situation are students' knowledge and ability to use known theorems, definitions, and postulates. It is also at stake students' ability to make deductive inferences and to read given information from a diagram. The situation makes room for the proof of propositions that might or might not be used later, for the drawing of diagrams that respond to given conditions, for the recall of ideas that might be associated to the argument to make. Students have the opportunity to reason deductively both in recognizing that a given statement is a case of a more general definition, theorem, or postulate, and in using deduction rules (like modus ponens or modus tollens) to derive a new statement.

Constructing a shape or solid. We use the word "construct" here in a loose sense. At stake in this situation is knowledge of the prototypical shape of a figure, knowledge of the definition or properties of a figure, knowledge of how properties of a figure aid in constructing a diagram of it, knowledge of a construction procedure, knowledge of a diagrammatic register (including conventions to support the diagram in conveying properties; see Weiss & Herbst, 2007) and the disposition to translate statements to this diagrammatic register. The situation makes room for the work of drawing diagrams with no tools, with all sorts of tools, or with a limited tool set,[7] for the translation of conceptual properties of a figure (e.g., that two sides are congruent) to particular controls in the production of diagrams (e.g., that two strokes look just as long, that both measure 4 cm. with the ruler, or that the same opening of the compass covers each of their span). The situation also makes room for the construction of models through folding and gluing paper, connecting sticks or piling blocks. And the situation makes room for the work of enlarging, reducing, or otherwise transforming a diagram into another diagram according to a rule. One particular case of this situation often consists of a strict usage of the word "construct," according to which a procedure is followed to create a diagram using only straightedge and compass. The geometry class is the place where the looser "constructing" situation may get differentiated into a stricter "construct" situation and another, looser, "make (draw, sketch)" situation. Students have the chance to reason deductively in inferring action rules for constructing a specific diagram from the definitions and properties of an abstract figure.

Exploring a shape or solid. At stake in this situation is knowledge of the definition, names, and properties of geometric figures, knowledge of symbols to express the properties of a particular geometric object, skill manipulating instruments for measuring and other instruments (like mirrors) to check on properties, practice identifying various objects after geometric terms, and experiential learning of geometry. The situation makes room for visual inspection, measurement, and manipulation of geometric models, for the translation of properties of the object modeled from the register of the model (e.g., diagrammatic or concrete) to a generic or to a conceptual register. One particular case of this situation consist of "conjecturing" based on perception—when students are expected to produce generic or conceptual statements after observing or measuring a diagram (see also Aaron, 2010). A situation of exploring may also include the construction of several models for the purpose of examining their common properties. Students have the chance to reason inductively and abductively in inferring general properties from the exploration of particular models and deductively in verifying that a given model matches the definition of a known figure.

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[7] These variations are summoned with different choices of word (construct vs. draw or sketch) which summon different norms. One could refine the analysis to establish really different situations.

The Role of these Situations

A teacher of geometry is responsible to teach all objects of the contract. In order to do that the teacher needs to engage students in some work that provides evidence that the teaching and the learning of such objects of contract has happened. Our theoretical framework says that there are basically two options for a teacher. One is that given the ideas that need to be learned and the specific reasoning tasks where those ideas are at stake, the teacher engages in negotiating a specific didactical contract within which students can take responsibility for that mathematical reasoning task. This would require, for example negotiating how to devolve students' responsibility for some aspects of the work in a task or series of tasks and for the teacher to accept responsibility for other aspects of this work. The other possibility is to rely on customary instructional situations as defaults within which to engage students in tasks that are quite customary in terms of their reasoning demands for students (and the work demands for the teacher). This second option does not eliminate all need for negotiation but they reduce the need for those negotiations dramatically as well as the chances of breaches in the students' activity.

Thus we propose that instructional situations have the following properties. As far as the work that they enable students to do, we contend that different instructional situations in geometry enable (by default) different kinds of reasoning work. We illustrate this claim by examining cases of each of the basic situations listed above and examine the reasoning that it allows. In doing so we account for our research question—how the geometry class makes room for students engagement in reasoning (including proof). As far as the exchange of such work for claims laid on the objects of contract, we contend that for a given item of knowledge, one or more situations can be used to organize work with which students can lay claim on that item of knowledge.

A Grounded Look At Selected Instructional Situations

Sources of Empirical Material

We collected video records of instruction in four high school geometry classes at a comprehensive high school (Midwestern High School). The records were collected to represent instruction in geometry across all three offerings of the class. We videotaped weekly the accelerated geometry classes taught by Cecilia Marton and Megan Keating, the regular geometry classes taught by Lucille Vance, and the informal geometry classes taught by Emma Bello. All of these classes were purportedly college preparatory though the informal geometry class was described to us as a class involving no proofs. We have reported on work based on those records in a number of other publications and reports. In particular, Hamlin (2006) is a dissertation that contains a case study of Emma Bello's informal geometry class.

Situations of Exploration

A case of making conjectures at the beginning of the year. In late September, three weeks after the beginning of classes, Megan Keating introduced a "little activity" by passing out and asking students to silently read a piece of paper describing what they would do. The sheet read:

We are now at the very beginning stage of learning to do formal and informal proofs in geometry. Before you can begin trying to prove something, you need to have some hypothesis or conjecture. Today, you are going to work individually for about ten minutes, looking at some diagrams. You can do whatever you wish with those diagrams including draw on them, measure things, draw similar figures to look for pattern etc. You may want to start over with a new and similar picture. All of these things are part of the exploration process. You are a discoverer.... You are boldly going where no man has gone before (ok, maybe some people have looked at some of the same things). After everyone has looked and hypothesized alone, we are going to break into groups and compare some of our ideas. Argue with others if you think their idea doesn't work. Ask why people think some ideas do work. Then we are going to come together as a group and share some things. I will make a master list of the hypothesis that get made today and we will see how many we can prove before the end of the year.

After students read the sheet Megan continued

We have thirty-eight minutes or so. So I figure what we will do is we'll have 10 minutes when I'll let you look at this diagrams … I'll pass out rulers and protractors if you want to use something like that… and I want you to try and come up with stuff and then we're going to break up into little groups and compare what you got with the other people in your little group and then we'll compare with the whole class.

Now, today is just the beginning of this… what I was thinking is I think it might be interesting to make a list of things… if you randomly noticed something in your homework that you think "I wonder if that's true all the time"… that will go on our list, so that'll be some hypothesis that you'll have and we'll see if we can prove it by the end of the year. We'll see how many of these things people come up with that we can prove… some of them I'll just tell you right out, "yeah, we're gonna prove that" but some, people may come up with some stuff that's not in your book and then we're gonna have to fit that right some time…

I randomly sat at my computer and thought okay what what should I draw here so I just drew like six figures … these in no means are the most important things I could have thought of or anything… (…)… see if you can find anything interesting about it… it doesn't need to be earth shattering, okay, though we might get some earth shattering stuff, but don't be worrying about writing down something you think that is something that's obvious. That Geometry is very intuitive and a lot of this is gonna be obvious [keeps distributing sheets]

Nina: There is no way to mess up right?

Megan: No way to mess up! No….. I'm gona give you protractors in just a second.

…

Megan: you are observing and you are writing down things you think are true, even obvious stuff … [passes rulers and protractors] you might have twenty things in any one picture .., you don't need to use these but some people… you know feel better with mathematical implements

The excerpt illustrates a number of features of a situation of exploration. The first one is that the object of exploration is given through a representation. Students are asked to explore figures given concretely, in this case through diagrams. Considering that this lesson occurs at the beginning of the year, it is safe to say that the activity is done in order to claim the territory of future learning—as a way to say, 'these are some of the things we will be learning about' rather than to claim that these things have been learned. In particular the figures chosen, albeit not "the most important things I could have thought of" they are figures that the class would be learning about (they include rectangles, parallelograms, and isosceles triangles, for example). The teacher also legitimates some tools for students to use when exploring—protractors, rulers, drawing on stuff, and visually inspecting figures. Megan also anticipates what is at stake for students—"no way to mess up!" and "it does not have to be earth shattering" but "you might have twenty things in any one picture." The proceeds from the following day's lesson show what sort of statements would be expected as conjectures or hypotheses—previously described as something that students might "wonder if that's true all the time." As the lesson moves on we see a student making marks on a diagrams of a quadrilateral to indicate that its four angles are right, and that opposite sides are congruent. She also writes these statements on the side. Another student investigates the diagram of a triangle that looks isosceles by drawing a line perpendicular to the base through the vertex common to the congruent sides, then he marks the two angles thus determined as congruent, using equal arcs. Likewise we see the student investigating the parallelogram by drawing its diagonals in, and marking with equal arcs the angles determined by one diagonal at a vertex (which is a false statement); we also see students making measurements on a diagram to inspect whether dimensions of a diagram are in a numerical relationship.

The class continued the individual work into group work, but close to the end of the period, Ms. Keating stopped the group work to indicate that the activity would continue over homework, students would have some "time to look at these tonight and [they'd] do this again tomorrow." But before continuing she indicated that one thing that would be good would be to "take one of the problems and talk about it as a group and then maybe that would give [them] some insight into looking at some of [those] other problems." Thus she picked problem 1, saying "I'm just gonna draw this freehand so it may not end up being as accurate as yours [draws a quadrilateral that looks like a rectangle with a longer side parallel to the floor]. Okay, who has some hypothesis that they thought about this figure." A student, Robert, contributed that "the parallel sides are of equal length;" then Megan asked, "who thinks they knew what this was? Let's start with that, what shape is this?" Students say that it is a rectangle and she refers back to the property of equal side lengths and says "this may be something we try and prove about rectangles… that parallel sides are of equal length…that opposite sides are of equal length. I tell you right now the definition of rectangle does not say anything about the sides so this would be a theorem that we can maybe prove." In like manner Norman proposed that "all the angles add up to 360 degrees," which Megan acknowledged would be true, "maybe we'll look at that," and prompted that that would not just be true for rectangles but more broadly for all quadrilaterals and asserted "maybe we'll look at it in a broader thing later." She prompted, "who drew something else in their picture?" and Yuri proposed that the "diagonals are equal and their intersection point is the midpoint" which

Megan said “that’s really two” so she would write them separate. In recording the property of diagonals bisecting each other, Megan originally made two sets of equal hash marks (as in Figure 1), and then revised those to be four equal hash marks (as in Figure 2), saying “if they are equal they’ll all be the same.” Sharing of conjectures about the rectangle continues including trivial properties like Corey’s “it has four lines” which Megan accepts saying “we are just brainstorming here,” until Lydia says that “opposite angles formed by the diagonals are equal.” Pointing at two of those angles, Megan asks Linda “these two would be equal? Why are they?” and then she asserts “right now we can prove this with the properties that we already have.” Then she asks whether someone can say why those angles are equal, “why do we already know that.” Someone says those are vertical angles, and Megan takes that as the reason for that claim. Later someone claims that the two diagonals make two pairs of equal, isosceles triangles and Megan asks “which one of these hypotheses [the ones already written on the board] would I already have to believe to say that those were isosceles?” Upon Rhea’s identification that diagonals are equal and meet at their midpoint, Megan indicates, “I’d probably have to prove some of that first.” The discussion continues during that day and into the next day thus allowing us a glimpse on the characteristics of the work included in the situation that we call “exploration,” to which we turn next.

A situation of exploration. Students could look at, measure, mark, or draw in a given diagram. They could state any property for the diagram using a combination of informal language and mathematical notation and as many properties as they wanted for each diagram. These properties were then accepted as is or revised by the teacher. The teacher revised properties by (1) separating composite properties into simple ones, that assert only one thing; (2) probing for and eventually indicating whether the property actually applies to a larger class of figures than what the diagram is supposed to showcase; (3) asking for reasons for some properties that could be derived from other ones or from existing knowledge; (4) marking those that are trivial. The teacher indicated whether some properties could already be proved, would be proved later, or maybe would not be proved later. Conceivably this would be a time where the teacher could take on some false conjectures, put them under consideration, reject them, or propose them for future discernment. Students were given the chance to speculate on the possibility to prove some of those conjectures.

The preceding discussion illustrates that a situation of exploration gives context to the work of making claims about figures. The stakes of an exploration are of two different kinds. There are engagement stakes, according to which it is important to involve students in actively doing something self directed in geometry. There are also content stakes that include new general statements about a generic instance of an abstract concept—a situation of exploration might frame work oriented to state a new definition, a theorem, a postulate, or a formula or to know how those apply in a generic case. The work to be done includes the students’ free choosing among a range of material operations to apply on concrete (physical or pictorial) embodiments of the concept depending the tools available to them, their reading of the particular results of those operations, and the translation of those results into general statements made in the conceptual register. The reasoning that students could thus have the opportunity to engage in can be described as abductive, proceeding from particular to general.[8]

In terms of the tasks that could fit in a situation of exploration, one could describe those as being established around the resources available (the representations and the tools) and the operations implied by the authorized physical relationship between the agent and those resources. For example a situation of exploration would frame the task of finding something interesting about a figure (a generic triangle, *ABC*) given by a diagram drawn in a sheet of paper, copies of which are distributed to all students along with some geometric instruments (such as a ruler and a protractor). Some operations may be stated explicitly as allowances (e.g., if the teacher says "you can measure the angles, you can draw auxiliary lines, you can fold," etc.). The goal or product of the task may be formulated more or less vaguely in terms of substance (e.g., find something interesting about that triangle) but students will be responsible to state it in abstract terms (e.g., "sides *AB* and *CD* are congruent," rather than "the left side measures 15, the bottom side measures 8, and the right side measures 15"). Students may choose what to look for but in general they will choose which operations to execute and in which order, among all of those afforded by the tools available. Students' accountability for finding a particular, expected property is low, meaning that individuals will not be found at fault if they contribute general statements with low cash value or if they fail to notice the facts that translate into one target general statement. One particular case of "exploring" goes under the label of "conjecturing" in some classes and involves students making statements that reflect general properties they perceive to be true about figures from visual inspection of diagrams.

Students use those tools and reason abductively or perhaps inductively (e.g., if they have access to many examples) to produce general statements out of their work with instruments on a diagram (or other embodiment, for example a dynamic diagram), and to state their findings using a hybrid register. We contend that the situation of exploration (and one of its variants, the situation of conjecturing) can be useful for the work that precedes the installation of a definition or a theorem about a particular kind of objects. In those cases they would normally be followed by a situation of installation. xx

Situations of installing (concepts, theorems and formulas)

We define situations of installing as those that mediate the allocation of responsibility for the class to know and use an official element of knowledge. We are particularly interested in the installation of concepts, theorems, and formulas.

Installing a concept. To account for the installation of a concept we use excerpts from three episodes, including Lucille Vance's installation of the concept of *central angle*, Cecilia Marton's installation of the concept of *tangent to a circle*, and Megan Keating's installation of the concept of *distance from a point to a line*. All of these episodes show the high involvement of the teacher in installing a concept. They show

[8] Abduction is the process by which given a particular statement, one finds a more general statement that entails (deductively) the given statement. The characterization of this kind of reasoning is attributed to C. S. Peirce.

how the teacher identifies and puts forth the concepts to be installed while marking those actions as important for students to attend to.

In the first example, we see how Lucille Vance erases the board and writes on the board “Arcs and angles;” then she asks the class to stop individual conversation and to take notes. She identifies the new section to be on “arcs and angles” and relates it to what they had studied before, angles and circles. She does so by representing a circle by way of a diagram and reviewing with the class the notation for that circle. She also speaks of angles that could be created. Moving to the installation of the first concept, she draws an angle whose vertex is on the center of the circle and labels the intersections of the legs of the angle and the circumference (see Figure 1). Then Lucille traces with her finger the segment of circumference from one of those intersections to the other. She names that as “the arc” and characterizes it as a portion of a circumference of the circle, but indicates her desire to focus on the “angle right here” marking it with a small arc (see Figure 2). Lucille prompts for students’ prior knowledge “tell me something that you know about any angle,” handling a diversity of contributions but focusing on those that are relevant such as the notion of vertex of an angle. Then she focuses the specific case on the new concept to be installed: “the vertex of this angle is at the center of this circle.” She then writes the definition “If the vertex of an angle is at the center of a circle, we call it a central angle.”

Figure 1. An arc

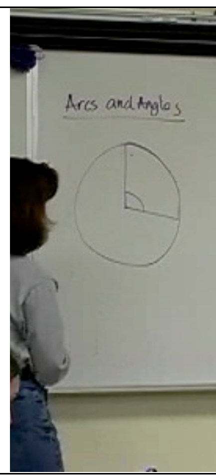


Figure 2. An angle

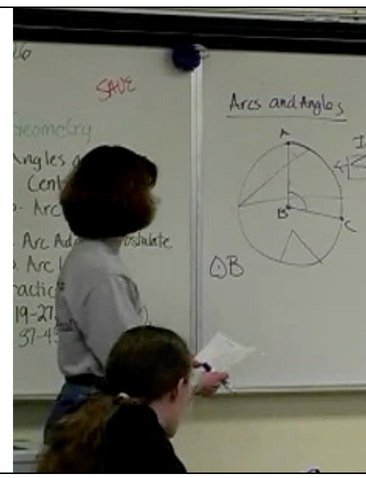


Figure 3. Examples and nonexamples of central angle

Lucille then asks students to consider two examples, shown in Figure 3, “are these central angles?,” obtaining a negative answer and reinforcing that the “key thing is that their vertex needs to be at the center.”

In the episode from Cecilia Marton’s class, the concept being installed is that of line *tangent to a circle*. The case provides a glimpse onto another move that might be part of the work of installing a concept—relating the concept to other concepts that use similar signs (in this case, the same word). Cecilia marks the occasion as important by noting that they will acquire an item of vocabulary, yet she notes that, “the word tangent is not unfamiliar to [students].” She alludes to students’ knowledge of the trigonometric ratio among the legs as something that the students know about, indicates that they “are going to learn a different meaning of the word tangent” and confesses that while “the trigonometric tangent is related in a way to the tangent [they] are gonna [be] talking about" she "cannot possibly begin to explain to them why." She then defines a tangent as "a line that intersects the circle in exactly one point." Cecilia names this point the "point of tangency." While the lack of a deeper connection with the trigonometric tangent may be something to regret, the episode still illustrates how these connections are part and parcel of the work of installing a concept and the signs used to evoke it.

In the episode from Megan Keating's class, the concept being installed is that of *distance from a point to a line*, and it illustrates two possible components of the work of installing a concept. On the one hand, Megan asks students for their opinion as to what they think should be the distance from a point to a line, eliciting in that process the notion that the distance should be defined as the shortest path and that this one would be located in the perpendicular segment from the point to the line. On the other hand, the episode illustrates how the work of installing a concept might involve proving that the concept is well defined. Megan proposes, "maybe there is another line that is shorter [than the perpendicular segment from the point to the line]" and suggests that one would show that such assumption leads to a contradiction.

Taken together, these episodes point to what the teacher may expect (and be expected) to do in occasions when what is at stake is students' responsibility for proper usage of a concept. Clearly the students may have awareness and knowledge of the signs used to denote the concept (including words, symbols, and icons) and still may not be knowledgeable about the concept—this is shown in all cases in different ways: The teachers attempt to connect the new concept with prior knowledge or other usages of the same words. In addition to signs (such as words and conventions), geometric concepts have diagrammatic representations that are summoned when a concept is installed. One could expect that the literacy associated to representing the new concept in the diagrammatic register would need to be addressed here as well: How to use a diagram to represent a concept otherwise defined in the conceptual register and how to read in a diagrammatic representation the presence of a concept. In all three episodes, the teachers took charge of drawing diagrams that represented examples of the concepts being introduced. Additionally, they took care of presenting nonexamples—angles whose vertex was not on the center in Lucille's case, segments that joined point and line but were not perpendicular to the line in Megan's case. Relatedly, the spelling out of definitions of new concepts invites naming new objects that may initially be visible through the diagrammatic representation, as is the case with the "arc" in Lucille's episode, the "point of tangency" in Cecilia's episode, and potentially the "foot of the perpendicular" in Megan's episode. While the concept being installed may not require that those ancillary objects have names, we'd argue it is normative in school to provide names for those and that the responsibility to do so rests on the teacher. While most of the work of installing a concept is initiated and evaluated by the teacher, students may be involved providing connections to known ideas and proposals for the definitions sought. Students are also involved in relating signs to referents and those with representations, thus involved albeit passively in the translation of propositions across registers and in the raising of questions about the objects involved in the whole process. An important moment in this process that creates for students the opportunity for reasoning is in the abduction from geometric object to geometric concept: while students are unlikely to play a role in creating and legitimating conventions, they can capture essential properties that define a concept while investigating perceptually (and perhaps also empirically) a well suited set of examples and non examples. Clearly, the teacher can do most of this work by herself, as illustrated by Cecilia, or share some of this work with students, as shown by Megan. In the latter case, while students may play a key role in bringing the elements of the definition in, the teacher plays a key role prompting such work with appropriate representations and resources (such as wait time). At stake in this exchange is not only knowledge of what the signs (technical words and conventions) mean, but also knowledge of the meanings targeted (knowing that certain things, such as tangency of a line to a circle, are interesting to track on) and of the representations (especially the prototypical ones) associated to those important concepts.

Installing a theorem, property, or formula about known ideas. At stake in this situation are students' knowledge and belief that a declarative statement is true, students' knowledge of the statement's connection to other statements, and students' authority and disposition to use the statement in future work. The situation makes room for the spelling out of the statement, the engagement in verification activities leading to students' conviction, the proof of the statement, the drawing and recognition of diagrams that depict the theorem, the instantiation of the theorem for the case of generic or diagrammatic objects, and the development of verbal means to remember the theorem in future work. Students could have opportunities to reason deductively when translating or verifying the translation of the statement of the theorem to the case of a particular geometric object, and when proving or following the proof of a theorem.

We define the situation of installing a theorem as the frame that enables the exchange between work done around the production of a true statement and the rights for the student to use that statement as truth thereafter and for the teacher to hold students accountable for using that statement as truth thereafter. The work to produce the statement may make use of other frames such as "doing proofs" or "exploration," yet ad hoc work to install a theorem is normatively done. In particular we propose as normative that (1) the statement be represented in a diagrammatic register, (2) the statement be represented in a conceptual register, (3) relevant concepts be activated, (4) the statement be sanctioned as theorem or property, and (5) the proof of the statement be mentioned as possible, sketched, assigned, or provided (see Herbst, Nachlieli, and Chazan, in press). To illustrate the situation of installing a theorem we narrate a few cases.

After defining the concept of tangent line, Cecilia Marton introduced as an item of vocabulary the notion of "tangent segment" by considering a line tangent to a given circle (of center *C*, which she had drawn) through a given point (draws *P* exterior to the circle) and marks *X* the point of tangency for that line: A tangent segment is the segment whose endpoints the given point and the point of tangency. Then she notes that she could draw another tangent through *P* to the circle—meeting the circle at another point of tangency, which she identifies as *Y*. She then reminds the class about a tip she had given earlier—whenever they have a circle they should draw its radii—and draws radii to the two points of tangency; students requests that she marks the right angles between radii and tangents, thus achieving Figure 4. Cecilia lets the class consider the figure and hears the conjecture that "it's a kite." She then asks what they would have to prove, student voice various ideas, and Cecilia keeps listening until someone says, "adjacent sides are congruent." Students notice immediately that one pair of sides (the radii) are congruent. While many have no idea about how to prove the other sides (the “tangent segments”) are congruent, Cabe offers the idea of drawing an auxiliary line through *P* and *C* and suggest that the two triangles could be congruent. “How do I prove it?” asks Cecilia and several students offer “hypotenuse-leg;” Cecilia adds marks to Figure 5. Then Cecilia states, “if you have two tangents from the same point, they are going to have to be congruent.” She takes the opportunity to reiterate what that would entail for a new figure, on Figure 6 and then indicates, “those would have to be congruent. Not only that but we’d also proved something else in there. We also proved that if you do draw a line through the center those angles would have to be congruent, wouldn’t they? For the same reason… That’s probably another big thing in the homework.” As she anticipates the homework, she points to places in the textbook and says, “everything we talked about here are reflected as theorems.” We note in this case several features of the installation of theorem that we submit are normative. The bringing up of key words such as “tangent segment” is instrumental for students to understand what the textbook means when it says, “tangent segments are congruent.” Cecilia also identifies the two statements made as theorems and expresses the statement of those theorems both in the conceptual and the diagrammatic register. An interesting feature of her explanation is that the proof is used to warrant the making of each statement.

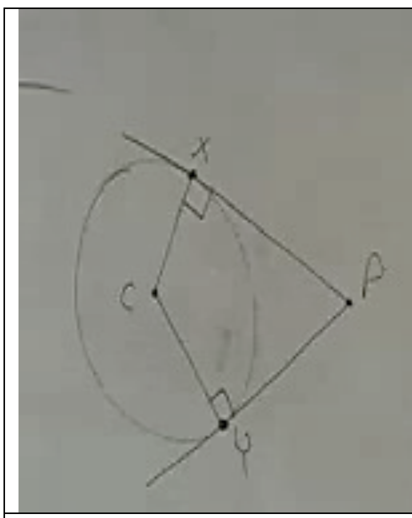


Figure 4. Tangent segments and radii

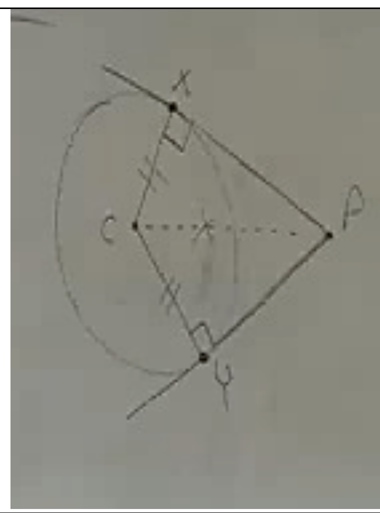


Figure 5. Tangent segments and an auxiliary line

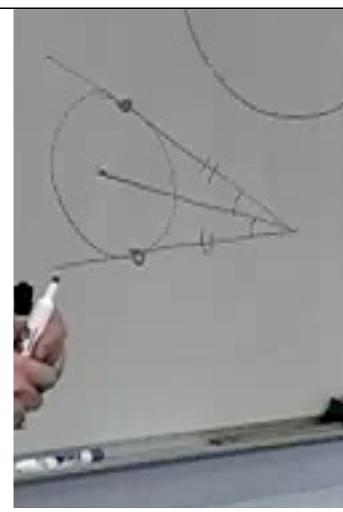
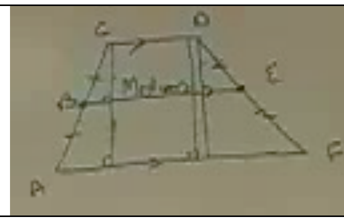
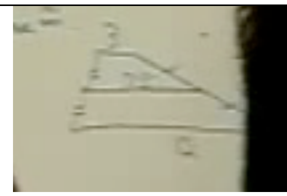


Figure 6. Tangent segments are congruent.

The second case of installing a theorem contrasts with the first one in regard to the role the proof plays in the installation. It happens in Megan Keating’s geometry class, during a lesson where she had installed the concept of trapezoid and its median. Megan prompts, “does anybody have any theories about the median?” Jade suggests the median might be parallel to the bases. Megan accepts that as correct and asks the class to talk about why that would be correct, while showing the diagram on Figure 7. Students offer various ideas, including that the height of the median is the same. After rehearsing various arguments with figure 8, Megan indicates that “Jade is right, that’s the first part of the theorem… the second part of the theorem I don’t think you’re gonna guess. Okay, I originally was gonna have you measure stuff to figure this out, so I’m gonna show you a picture with measurements. So we are gonna say they are all parallel. There’s some other property they have too. Turns out that [draws a trapezoid] say I know this [one base] is 4, this [the other base] is 10, then this [the median] will be 7, every time. If I draw this one and I say that this [one base] is 3, this [the other base] is 12, this [the median] is 7.5 (see Figure 9). If you drew your trapezoid and you measured, you would see this pattern after a while, that the median has a relationship to the two bases, what is the relationship?" Students say "it's half" and Megan assents, "it is half, it is the average." Then she writes the theorem as "Theorem 6-16: The median of a trapezoid has length one half the sum of the bases and is parallel to the bases." We see in this case that the theorem is not only sanctioned as theorem but also given a conceptual formulation and a name (theorem 6-16). In both cases narrated we see that students are having the chance to conjecture the theorem, while the sources of the conjecture are different: the proof in one case, and perception of the diagram in the other.

		
Figure 7. The median of the trapezoid	*Figure 8*. Distances between parallel sides	*Figure 9*. Measures of median and parallel sides

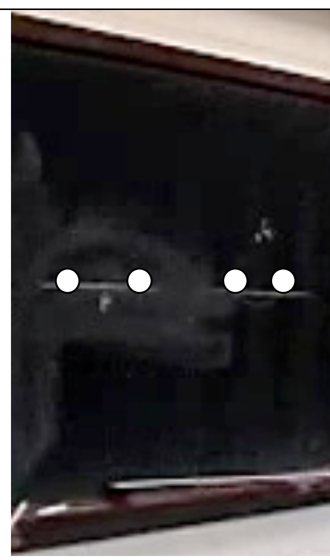
Situations of Construction

A situation of construction gives context to the work of making diagrams that respond to particular descriptions. The stakes of the work are twofold. On the one hand at stake is a claim that students are able to use various geometric tools (including in particular straightedge and compass, but also protractor, ruler, and possibly the construction menus of a software tool) to turn verbal descriptions into diagrams. On the other hand, at stake are particular properties and procedures of construction that are included in what students need to know about geometric figures (e.g., how to make the perpendicular bisector of a segment, what a kite looks like).

The work to do is somewhat the inverse of exploration. Students are given verbal descriptions of figures in the conceptual register, along with more or less directive requests, and tools to be used. Students are expected to produce diagrammatic versions of the objects mentioned in the description. The work is framed by the statement of the goal of a task to make a diagram with certain properties and by the available resources (especially the construction tools) to do it. Depending on the nature of the former (how detailed and ordered the directions are) and the available tools, students may be more or less involved in making choices (of what to do next and what tool to use). The reasoning students are to deploy involves controlling deductively the fit between a particular diagrammatic object and the conceptual description that it is supposed to represent. This deduction can simply be the deduction from general definition to specific example, or more generally it can imply using known properties of the diagram constructed to infer others.

Situations of construction are really a family of situations that should be better unpacked in future work, pending access to a finer corpus of lessons. The lessons in our corpus of intact lessons do not afford enough construction lessons. We anticipate that these different situations are normally evoked with expressions like “copy,” “construct,” “enlarge,” “sketch” or “draw,” noting that in each case the student is expected to produce a diagram that abides by certain properties. To do so the student uses different kind of tools—in the extreme cases of “sketch” or “draw,” markings on the diagram play the role of tools, repairing the lack of perceptual accuracy of the diagram with indications of what one should see. In the other extreme, “construct,” students are limited to straightedge and compass and specific operations with those. In the situation that we anticipate as “copy,” many tools are used included rulers and protractors. Exactitude is as much prized as in “construct” but the range of operations and tools is wider. We illustrate a case of “construction.”

In Emma Bello’s 2nd period class on 09/25/03, she led students through a review of the construction of a perpendicular to a line. The procedure encompassed the construction of the perpendicular bisector of a segment as well as the construction of a line perpendicular to another line passing through a given point—Emma referred to this construction as “the fish situation” (see Hamlin, 2006). In this episode we see the teacher enacting both the roles of teacher (giving the construction task) and student (performing the construction). Emma describes the three possibilities for the task, being “given two points ‘cause you are given a segment, [being] given one point [on a line], or [being] given no points [on a line, but one point off the line].” The work of the teacher includes more than saying what the student is given; she actually provides those givens in the diagrammatic register, as shown in Figure 10. The work of the student includes making sure that they “have two points” even though they “may not be given two points.” The student is expected to “create [the points needed] by going to the only point we have, drawing arcs left and right, or going to the only point we have, pulling down [the compass] past the line and get two points on the line.” After making the arcs in each diagram, Emma marks the intersection with thicker dots (as shown in Figure 11).

	
Figure 10. The givens	*Figure 11*. The obtained two points.

Once students have the two points they can do the construction (the “fish design) by “open[ing] up [the compass] more than halfway [between the two points]… and draw semicircles from either side.” Students’ actions include not only drawing the two semicircles so that they intersect (see Figure 12) forming something that looks like a fish, but also “draw a line that cuts through its tail and its head, and you’ve got a perpendicular that cuts the line in half” (see Figure 13). Emma repeats the construction for the other two diagrams showing how they can “draw a perpendicular through a point on the line or off the line [given],” adding later that “these constructions are all basically the same” (see Figure 14). The review illustrates how the roles of the teacher and student are divided in regard to the situation of construction. The student is supposed to execute a procedure on the given objects with the given tools. In executing this procedure, the student can use objects previously given or constructed to create new objects. Note however that while a possible task could have been to draw the perpendicular to a line (with no points given), that one was not among the ones reviewed for students. We hypothesize that since students would have to create a point to begin any procedure, such construction would be unlikely—that is, giving the original objects from which the construction will derive is part of the work reserved for the teacher. A second observation to make is that while the discourse with which the teacher describes the construction alludes to most objects conceptually (semicircle, line, point), the students’ work relates to those objects diagrammatically. Lemke’s (1996) distinction between typological and topological meanings is helpful here to describe where the difference lies—while the language used by the teacher is made of discrete particles, the distinctions that the student needs to control are continuous, and the teacher’s assistance (in specifying the construction procedure), is one of providing benchmarks (draw arcs right and left, pull compass past halfway, etc.).

		
Figure 12. Two semicircles	*Figure 13*. A perpendicular	*Figure 14*. Two perpendiculars

Situations of (Geometric) Calculation

A situation of geometric calculation (see also Küchemann and Hoyles, 2002; Hsu, 2010) gives context for students’ demonstration of knowledge of properties of figures. The work consists of using properties to set up one or more calculations and to solve that calculation to obtain the dimensions of a figure. A geometric calculation relies on a diagram albeit in very different ways than a situation of exploration: The diagram provided includes extra information in the form of other signs (numbers, hash marks, or algebraic expressions) that correspond to some of the geometric objects in the diagram. The student is supposed to use that information to ascertain what is known about the figure represented by the diagram, to bring from his or her own knowledge other properties of such figure, and to use the information given and those properties to set up calculations of one of two kinds. We characterize two kinds of situations of calculation in geometry: geometric calculations in number and geometric calculations in algebra.

Geometric Calculations in Number. In a situation of geometric calculation in number, the work to be done includes finding deductively the measure of an element of a figure, using geometric properties to set up arithmetic calculations. What is at stake is a claim on students' capacity to use a property they already know. Usually the diagram of a figure is given with some of its dimensions set numerically while others are unknown. The role of the teacher or the textbook is to provide a diagram with just enough information about a figure to enable the student to use that information along with theorems about that figure to find other information. The role of the student includes to interact descriptively with the diagram, that is, to use the diagram as container of information of two kinds: (1) information about existence and incidence of objects may be read off the diagram while (2) information about parallelism and congruence (or measurements) can only be read off when supported by additional signage (hash marks, arrows, arcs, or numbers). This use of the diagram is similar to that allowed in situations of doing proofs and very different from what is possible in situations of exploration or construction. The work of the student also includes remembering theorems studied before and using them as action rules. Hsu (2007, 2010) has elaborated more on the notion of a geometric calculation in number in the context of Taiwanese middle school mathematics.

To illustrate this we consider two episodes, one of them in Lucille Vance's class was an exercise that gave students practice on the notion of similarity. The problem from the textbook gave that segments *QN* and *RP* are parallel and asked to identify the similar triangles and find the measure of segment *TR* (see Figure 15). In the resolution of the problem with the class, Lucille showed how the students were expected to use the information given. She asked first how students would use the concept that *QN* is parallel to *RP*. Upon obtaining the response "corresponding angles" she extended the segments to show better how prior knowledge about parallel lines cut by a transversal would play out, obtaining a figure like Figure 16. She prompted students to think which angles would be congruent if she were using the line through *NR* as a transversal, then did similar thing with the line through *QP*, eventually getting students to agree that triangles *NTQ* and *RTP* are similar by the angle-angle criterion of similarity. Lucille emphasized here that the letters needed to be in the correct order so that the triangles would correspond.

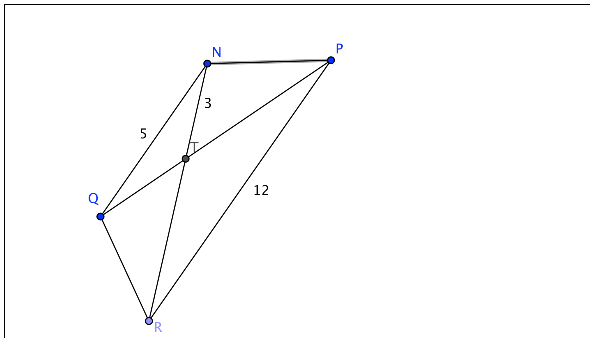


Figure 15. Triangles determined by diagonals of a trapezoid

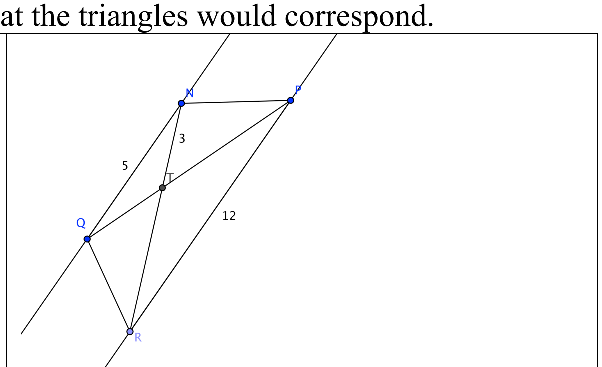


Figure 16. Parallel lines imply triangles are similar.

Going to the second part of the problem, Lucille asked what would segment *TR* correspond to in the similarity proposed, getting students to answer that it would correspond to segment *NT*. They thus posed the proportion $\frac{x}{3} = \frac{12}{5}$ which the class later solved to say that *TR* equals 7.2.

This simple example yields some observations of a situation of calculation. The objective of a situation of calculation is to find dimensions of geometric objects. Yet the means to do such work exclude empirical interactions with diagrams. Rather the work includes using the diagram descriptively: For example, students use the sense that *TN* is smaller than *RT* to make sure that in the proportion they are setting *x* corresponds with 12 and 3 with 5. While that kind of information is retrieved from the diagram, other information such as the fact that the triangles are similar is not taken from any interaction with the diagram but deduced using information given and available (previously known) theorems—in this case, the theorem that if two pairs of corresponding angles are congruent, the two triangles are similar. A second use, this time of a definition, shows how statements (definitions, postulates, and theorems) are used in calculations. The definition of similarity says that similar figures have corresponding sides proportional, which basically states that once the correspondence is defined, corresponding sides stand in the same ratio. The goal of a situation of calculation is to produce a measure specific of a single geometric object, but unlike in a situation of exploration, the method by which students can do that is deductive. They use reasoning similar to that of proof to derive the measure of an object. The definition of similarity now is used not to assert the equal ratio or even to find what the equal ratio is; rather the definition is used to hypothesize the equal ratio and use such hypothesis to calculate the value of the unknown dimension. It is important to note that while in proof exercises students normally know what they are going to prove and what is at stake is the connection between the statements (given to conclusion), in these exercises of geometric calculation, the measure to be found deductively is unknown. So while these problems are not as abstract as proof problems, their demands to students' reasoning can be quite high.

The next example, taken from one of Cecilia Marton's lessons on circles (Figure 17), shows how complicated a geometric calculation could get. By looking at the figure, Max and Cabe provide the observations that "triangle *PRT* is equilateral" and "angle *PGS* is 60 degrees." Both observations are met with the question "how do you know that?" and subsequently withdrawn by the students. Cecilia says, "I know that *PGS* is an isosceles triangle but that does not make the angle 60." Note that the triangle would eventually be shown to be equilateral, so Cecilia is only denying the entitlement to assert its equilateral nature thus far, not the fact that it is so. But then Jeff notes that *PG* as well as *SG* are radii, and that makes them congruent, with which they can assert that triangle *PGS* is equilateral. Cecilia approves of this and stresses that the important property used by Jeff is that all the radii of a circle are congruent to each other by the definition of a circle. After Jeff's observation that *PSG* is an equilateral triangle, Cecilia notes, "all of a sudden we start to fill stuff in" and she adds measures to the diagram starting from noting that the interior angles of the triangle *PSG* are 60 degrees each. She notes that $\angle PGR$ the supplementary angle to $\angle PGS$ is 120 degrees, notes that the triangle *TGS* is likewise equilateral, making angles ∠*TGS* and ∠*TSG* be 60 degrees each. Cecilia prompts for more information and Karen observes that the angle made by lines *PT* and *SG* measures 90 degrees, Karen justifies that by noting that the quadrilateral *PGTS* is a rhombus. With that piece of information, Cecilia has a warrant to say that the two angles at *P* and at *T* are 30 degrees each. As regards to the angles outside the rhombus, Tarina contributes that since *GR* is a radius, the triangle *PGR* is isosceles and so the angles ∠*GPR* and ∠*GRP* would be 30 degrees each. Cecilia notes that there are a couple of reasons that make that true: one because they need to add up to 60 to make 180 degrees along with the given ∠*PGR* which was known to be 120 degrees and two that they needed to be equal since the triangle *PGR* is isosceles (Figure 18).

In this more involved calculation we see once again that interaction with the diagram is descriptive (Herbst, 2004): Students can derive some information from the diagram, including reading the hash marks that say that some segments are congruent and reading off incidence properties to say for example that some segments are radii. But perceptual information (such as that provided by Cabe and Max originally) is not accepted unless there is a reason that allows one to derive the information required deductively. The use of theorems is slightly different here than in the prior case because the metric properties of the configuration studied (the angle sizes in particular) are independent of the measurements of specific objects (such as the radius of the circle). Using the definition and theorems about the rhombus, about the sum of angles in a triangle, about base angles of an isosceles triangle, and about supplementary angles, students can deduce all angle measures without knowing any one of them. We see however a different feature of the work to do in a situation of calculation: To obtain one numerical value (say, the measure of ∠*RPT*) one may need to obtain other numerical values (say, the measures of ∠*RPG* and ∠*GPN*) which in turn may require separate applications of different theorems (in this case, base angles of an isosceles triangle on one hand and properties of an equilateral triangle on the other).

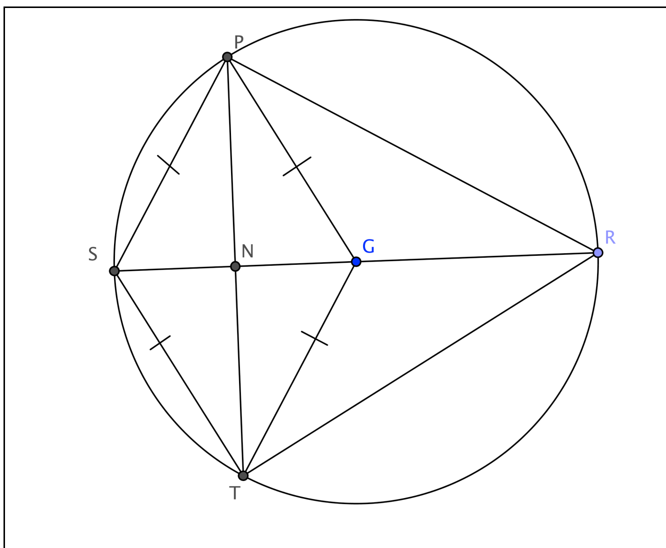


Figure 17. Original problem, no angles are given

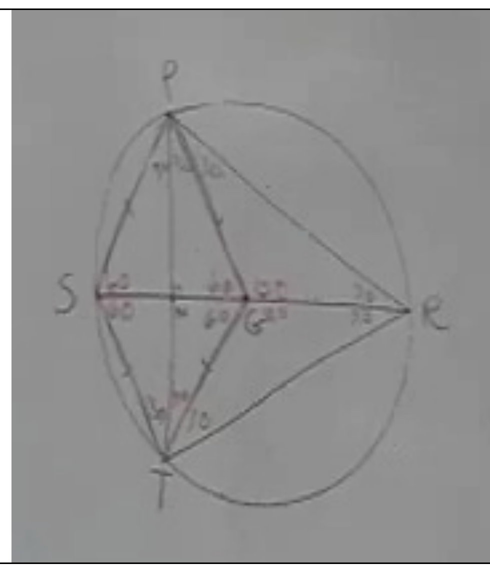


Figure 18. All the angles are found

Geometric calculation in algebra. The work to be done includes finding deductively the numeric value of an unknown (and of expressions that use that unknown) where some of those expressions using the unknown are given as representations of the measure of an element in a figure. Geometric properties known about the figure are used to set up algebraic equations that can be later solved. What is at stake is a claim on students' capacity to use a property they already know as well as a claim on maintaining knowledge of algebra skills. These exercises could vary dramatically in complexity and this complexity occurs along two dimensions. On the one hand the complexity of the algebraic manipulation could get very involved, for example requiring students to solve simultaneous linear equations or solving quadratic equations. On the other hand one or more geometric properties could be used to set up those equations. The two examples that we present below give a modest illustration of that complexity.

In Cecilia Marton's class, early in the year and after they had studied the theorem on the sum of angles in a triangle, students were asked to find the value of x in the diagram provided in Figure 19. The interaction about the problem illustrates how complex the work could get. First Cecilia asks for the "easiest way to set this up" and Erie contributes "$x + 80 = 3x – 22$" which is avowedly an application of the exterior angle theorem (that says that the measure of the exterior angle of one angle in a triangle equals the sum of the measures of the two other angles). Students' work would then involve manipulating both sides of the equality to get $x = 51$. But Cecilia asks, "what if you did not know about that theorem? Could anybody figure out a way to get around that?" A student proposes to add another variable, y to the third interior angle and "you could say $3x – 22 + y = 180$" and then "$80 + x + y = 180$." They equate both expressions "$3x – 22 + y = 80 + x + y$," Cabe notes that they could subtract y, and eventually Cecilia notes that "we are back where we were." Cecilia then remarks that the exterior angle theorem would save them some time but they could do the problem even if they did not know that theorem. What is interesting however are the operations students would have to do in that second case. On the one hand they would be applying two statements: that adjacent angles are supplementary (hence they add to 180 degrees) and that the sum of the interior angles of a triangle is 180 degrees. On the other hand students would have to posit a new variable to be able to bring to the calculation one of the geometric objects at play in the given diagram. The use of the diagram is notably similar than in the geometric calculation in number, in that students would not be entitled to measure the missing angle value but rather interact with it descriptively. The work of geometric calculation in algebra, as in that of geometric calculation in number, is deductive.

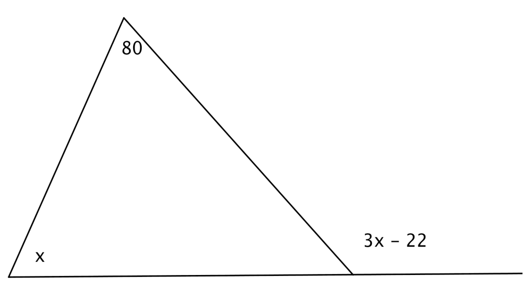


Figure 19. Find x

Situations of Doing Proofs

We have written more extensively about the situation 'doing proofs' in many other publications, especially Herbst et al. (2009) but also Herbst and Brach (2006) and Herbst (2002b, 2004). To complement the present exposé of other situations we note how a situation of doing proofs creates opportunities for students to reason. In a situation of doing proofs there are two main forms of work that are valued. One of them has to do with students capacity to "do proofs," that is to produce chains of statements and reasons that connect a given statement to a conclusion, both of which are provided in advance. At stake in this work, which pays no attention to what the proof is about, is the notion of proof itself—that students can do proofs. The second one is students' capacity to recall and apply known theorems, definitions, and postulates to justify specific statements. At stake in these proofs are specific theorems, thus proof exercises can be given after students have learned specific theorems or definitions and their performance in those proof exercises has the cash value of knowledge of those theorems. Thus proof exercises fulfill two different functions in high school geometry. While the first one (knowing how to do proofs) may be accomplished just as many times as students do a proof, the latter (remembering theorems and knowing how to apply them) gives proof exercises a renewed reason to be requested.

The situation of doing proofs gives context for students' demonstration of deductive reasoning skills and knowledge of theorems. The work involved includes writing descriptive statements about a diagram, then justifying those statements deductively through showing that those statements are particular cases of known properties about that figure, stated as reasons. Four kinds of inference are made. In one of them a statement of a property of a given figure is translated from diagram to symbols, a symbolic description of diagrammatic features that involves a kind of analogical reasoning: The statement made in pictorial language is equivalent to the symbolic statement. In the second kind of inference, an abstract statement of a geometric property is written down as reason because it entails (as particular case) the symbolic statement about the diagram written before. The relationship between symbolic statement and reason is one of particular to general, namely the reason implies the statement. The third kind of inference is observed occasionally when symbolic statements are used to make new symbolic statements, for example when an equality is used to assert another equality by virtue of a convenient arithmetic operation or when two or more lines are used to claim a new line of proof. In this case while deductive reasoning can be used to justify that the step taken is legal, some strategic reasoning is also present in asserting the new statement. The fourth kind of reasoning, similar to the first takes symbolic statements arrived at in the writing of a proof and translates those into marks on the diagram. This reasoning step also makes use of analogy.

Provoking A Negotiation Of The Contract

In this section we describe a few experimental lessons that were implemented in geometry classrooms. We do this to illustrate the following general point. A novel task (that is, a task that does not fit in any existing instructional situation) demands a negotiation of the contract that invokes and shapes the characteristics of one or more existing instructional situations. Thus, negotiating the contract for a particular, novel task calls for evolution of, rather than revolution against, existing situations (frames). We call these lessons instructional experiments, taking the word experiment in the sense of Francis Bacon's crucial experiments—they present one opportunity to visualize a predicted phenomenon—rather than in the current, psychological sense, of isolating the cause of the phenomenon. Our instructional experiments are close in kind to the collaborative design experiments that are common in mathematics education research (see Cobb, 2000). The instructional experiments are also close in kind to the breaching experiments of early ethnomethodologists (Garfinkel & Sacks, 1970; Mehan & Wood, 1975) who tested models of practical actions by creating conditions for observing empirically how practitioners restored order to a breach of a model of practical actions. In our case, we are proposing as a model of instruction the notion that a general didactical contract is articulated as the juxtaposition of a finite number of subject-specific instructional situations. We are testing the value of those situations as tools to restore order by way of showing how classrooms handle disruptive events, novel tasks that do not easily fit in those situations. From an analysis of how teachers and students organized their activity in activity types, interesting differences can be observed in the number of switches of activity structure in intact and experimental lessons as well as in the time spent on average on one activity structure: While intact lessons had between 6 and 8 activity types per lesson, experimental lessons had a median of 11.5 activity types per lesson. The intact lesson's activity types lasted between 6.5 and 8 minutes while the experimental lessons' activity types average 5 minutes in duration. While experimental lessons had on average one segment of group work per lesson, intact lessons had on average 0.22 segments of group work, or differently said, for 4 lessons that had group work, 14 would not have it. But the more interesting differences between intact lessons and experimental lessons can be described in terms of the way the tasks were handled in experimental lessons.

The Intersecting Lines Lesson: Angles Formed by Intersecting Lines

One of the instructional experiments that we have done in geometry classrooms is called the "Intersecting Lines Lesson" (see also Chen & Herbst, in review). The lesson is taught before introducing the parallel postulate and includes giving students a diagram with 6 different lines, some of them intersecting in the page, some of them apparently parallel, some apparently not parallel but intersecting outside the page. Students are asked how many angles they would need to measure in order to know the measures of all the angles determined by those lines.

We created this activity as a context for students to develop the preference for answering questions based on rational inferences from plausible assumptions, and to create a contrast with knowledge based on collection of empirical data. Clearly, measurement of every single angle would serve to answer the question though it would be hard work to extend every pair of lines till they meet and then measure the angles: There are as many as 60 angles to measure. However, by thinking about the quantitative relationships among those angles entailed by a few hypotheses about the geometric objects one would be able to reduce the number of actual measurements significantly. Possible hypotheses to make include that adjacent angles are supplementary, that vertical angles are congruent, that the sum of the angles in a triangle add to 180 degrees, that corresponding angles among parallels cut by a transversal are congruent, etc. All of these would eventually be theorems in Euclidean geometry but not all of them had that status at the time of the intervention. But the task is such that a student could know the measure of some angles as entailments of knowledge of other angles and some hypotheses. The task could help students lay claim on those theorems as conjectures that are useful to have because they produce useful results. The activity thus provides the context for the emergence of mathematical work that includes substantial mathematical reasoning: students make hypotheses, deduce information from those hypotheses, eventually making contextualized conjectures (e.g., only three measures are needed) that they might be motivated to prove.

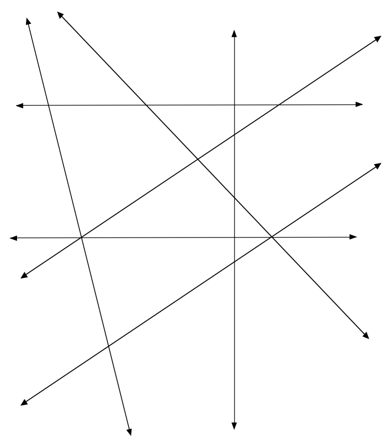


Figure 20. The intersecting lines

Our question has been what does it take for a teacher to use that diagram, and the fundamental question of what the measures of those angles are, to engage students in formulating and proving claims of the sort: *if* this and that lines are parallel then knowing this and that angle will be sufficient to also know that other angle or if we accept that angles in a triangle add to 180 degrees then we would only need to measure this and that angle to know them all. At this stage any conjecturing may be contextualized to the problem of determining specific angle measures but we submit that much of the reasoning students make to decide that measuring an angle is not needed, or to decide what the measure of an angle is, is deductive.

As far as the work we have done identifying the instructional situations in geometry classes the questions that arise are, what instructional situations from the geometry class might enable such activity to proceed, what stakes of knowledge those situations would ordinarily enable to claim, and how such framing could affect the nature of the mathematical work students could do. Note by the way that this approach to experimentation is different in nature to the approach known as didactic engineering (Artigue & Perrin-Glorian, 1991): In didactic engineering, an experimental contract is established so as to enable fidelity of implementation—in particular, the work that the teacher will need to do to make knowledge evolve in and through the activity is anticipated and communicated to the teacher. In our approach, only the task that students are envisioned to do is communicated to the teacher but she is left on her own to negotiate with students how they will undertake that work, what will be accepted by the teacher, and what stakes they will accomplish. Such lack of specification of the work of the teacher in our experimental contract permits us to observe how the teacher incorporates the activity to their usual practice, in particular how she negotiates the task with the students by possibly invoking the framing of one or more of the existing instructional situations.

There are three situations that could compete for framing the "intersecting lines" task. One of them is "doing proofs" and could be clearly useful in framing contextualized arguments like "if two lines intersect, it is enough to know the measure of one angle in order to be able to find the other three." Another is the situation of calculation which could be useful framing the work of finding actual measures of those angles: For example, if two lines don't visibly meet but they do meet a third line making (cointerior) angles of known measures with it, students could argue that the invisible angle measures 180 degrees minus the sum of the two visible angles. In particular, such calculation could help substantiate that the two lines are parallel. Finally there is the situation of exploration, which could be useful to actually get started in the work of determining angle measures and making contextualized conjectures—students could measure one angle and upon perception that such angle is congruent to another one, predict that the other angle would be equal in measure.

In Megan Keating's class the activity was introduced just as planned, she gave protractors to students and indicated they could use them but also said she thought they might be able to find most of the angles without measuring. Students did engage in the kind of reasoning anticipated, for example using the triangle sum theorem to probe the assumption that two lines might meet outside the page—something akin to a proof by contradiction—and calculating specific angle values using that and other properties. They did not however formulate general conjectures and Ms. Keating maintained the discussion in the realm of how many angles were needed to measure. Our interpretation is that Ms. Keating was able to negotiate the acceptance of responsibility for minimizing the number of measurements and then the situation of calculation absorbed this task. Theorems about angles formed by parallel and intersecting lines never became explicit though they were used to find angles. A similar set of interactions happened in Ms. Vance's class, while in Ms. Bello's class the task was absorbed by the situation of exploration—students used the protractor profusely, turning the task into one of discovering the actual number of angles and measuring as many of them as they could identify.

As a task that could lead to proving propositions that connect properties of angles, this task would have to surmount a hurdle: Students not only need to accept responsibility to minimize the number of measurements (as they did in Ms Keating's class and eventually also in Ms. Vance's class). They also need to forgo any actual measures on behalf of operating with generic angles and they would have to make assumptions explicit in order to justify symbolic calculations. We contend that as a context for a teacher to engage students in proving, this activity is problematic for a teacher, because (1) it holds students accountable for doing things that would ordinarily fall within the teacher's share of labor; (2) because it challenges the temporal structure of knowledge on which 'doing proofs' rely; and (3) because it subverts the economy of classroom exchanges.

First point: The intersecting lines lesson holds students accountable for doing things that ordinarily fall within the teacher's share of labor. In particular the identification of initial conditions such as whether any pair of the given lines is parallel or not and the activation of the triangle sum property as relevant knowledge to be used are not done in ways akin to how these actions are done when students are engaged in proving. In normal activities when students are engaged in proving, the given has a factual rather than hypothetical character and it is identified by the teacher. Students' assumptions are never seen as acts of resourcefulness to simplify a problem, but rather as transgressions to the norms that regulate how students should interact with diagrams.

Second point: The intersecting lines lesson challenges the temporal structure of knowledge on which 'doing proofs' relies. In particular the temporal division between the set up of a proof problem and the production of a proof are challenged when the diagram is given without being final and when the statement of the task continues to be negotiated over the whole lesson. For example the extent to which the actual measure of an angle is ambiguously known or not known, measured, inferable, or inferred, desired or undesired, has to be ambiguous for some time, but the decision to do a proof (and to find the moment when a proof can be done) hinges on resolving that ambiguity.

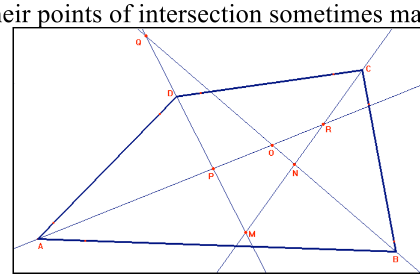
Third point: The intersecting lines lesson subverts the economy of classroom exchanges, particularly as it relates to what kind of work can be valued as proving. The presence of measures of angles in the context of eventually proving conjectures about what could be the case with the measures of those angles is a miscue for students, who are often taught that properties of figures cannot be measured but need to be proved. The extent to which it is personally valuable for them to measure in order to have an idea but publicly pretend that they have not measured (or better argue that their measurement was not needed) to make a strategic move, challenges an order in which students are usually guided (at times even taken by the hand) in finding out where they should allocate their efforts.

In defaulting to a situation of calculation, these teachers were able to implement a substantive version of the task, one in which students were involved in deductive reasoning yet one in which properties of angles never became explicit.

The Quadrilateral Bisectors' Lesson:
What One Can Say About the Intersection of Angle Bisectors in a Quadrilateral?

The quadrilateral bisectors lesson was based on the problem shown in Figure 21.

We know that the **angle bisectors** of a **triangle** meet at **a single point**. For a general **quadrilateral,** however, the angle bisectors might meet at as many as 6 points, but for some quadrilaterals you might get fewer points. Their points of intersection sometimes make interesting shapes.



What would it take to construct a quadrilateral so that the intersection of its angle bisectors make an interesting shape, perhaps intersecting at less than 6 points?

Figure 21. The angle bisectors problem

We implemented this task in several of Megan Keating's classes, in all cases using dynamic geometry software (Cabri Geometry ^{TM} in the TI Voyage 200^{TM}) as a resource to provide diagrams to students and to enable their initial work on the problem. In all cases it was observed that the existence in the class of situations of exploration was useful for Ms. Keating to launch the task as one of exploring the figure. She encouraged students to drag the vertices of the figure, observe what happened, and come up with conjectures. We had predicted that the intervention would perturb the usual contract in geometry in that neither the exploration work nor the conjectures that students could make as a result of the exploration are usually part of the course. In the lessons recorded we observed two important shifts in the task that we interpret as a response to that circumstance.

The first shift was a shift of the situation—from a situation of exploration to a situation of doing proofs. Rather than negotiating the task so that students would use deduction to enable their exploration of the shape, the teacher defaulted to the usual exploration in which all operations are allowed. As students made findings the teacher bypassed continuing the exploration. Instead the teacher indicated that her goal was to prove some of the properties students had found and brought individuals to the board to share their conjectures and work on proving those. The work of proving these conjectures did not always default to the "doing proofs" situation in that various of the results that were proved were not proven in two columns: At times work noting information by marks on a diagram (rather than by writing statements and reasons) or statements not followed by reasons were done. Yet other norms of "doing proofs," such as the norm of not altering the given diagram by taking objects away or adding new objects, were present and enforced. An important way in which the work defaulted to "doing proofs" has to do with the role of the proposition proved: While the exploration converged to the formulation of conjectures, the proofs of those conjectures did not as much emphasize the conjectures as other aspects of the proof process. This was predictable inasmuch as the propositions conjectured were not among the theorems that students needed to learn. The second shift was thus one of focus on what is at stake: The teacher shifted the stakes of the task from the goal of reaching an important conclusion to noting important elements of the process of proving. Substantive techniques for proving were the big stakes of the work: How to use a proven congruence of triangles to feed the work of proving another congruence of triangles and how to use segment addition to prove two segments are congruent.

The main accomplishment of this special lesson was to show that the didactical contract in geometry has enough resources (in the form of the situations of exploration and "doing proofs") to contain the kind of mathematical work afforded by a task like the angle bisectors problem. Rather than having to negotiate anew what kind of work students would need to do and what would be claimed as an accomplishment of that work, the teacher could utilize existing frames of activity to contain a large part of the possible mathematical work: She did this by unfolding the task into two tasks, one of coming up with conjectures and another one of doing the proofs of selected conjectures. Two consequences were observed of this unfolding. One is that the most general properties that could have been searched for (e.g., that when angle bisectors don't meet at a point, opposite angles of the resulting quadrilateral are supplementary) were never the object of conjecturing. We would explain this on account of the norms for exploration, that privileged free manipulation and perception and did not sufficiently encouraged deduction as a tool to find things out. The second is that the results that got proven were not made memorable, to the point that the use of a proven result to justify a new result was never observed (e.g., the angle bisectors of a rectangle make a square and the angle bisectors of a parallelogram make a rectangle but neither of these propositions was used in the proof of the other, rather they were proved anew). We would explain this on account of the norms for doing proofs that place low accountability on the remembering of the conclusions proved and high accountability on the proving process.

The Special Quadrilaterals Lesson: What are the Properties of Special Quadrilaterals?

Herbst, González, and Macke (2005) reported of a special lesson we designed to help students understand one key aspect of making definitions in mathematics: a definition places minimal boundaries on the object being defined. The lesson was built around a game we called "Guess my quadrilateral" (a variation of the Milton Bradley game "Guess who?") in which students need to discover a hidden shape by asking yes/no questions with the goal of minimizing the number of questions needed to make a certain "guess." In that paper we reported of how the game was utilized in Megan Keating's class to bring up students' prior knowledge of special quadrilaterals and use it as a base to provide official definitions. We showed that the lesson led to the class's development of the preference for succinct definitions as well as the discovery that alternative definitions could be given for the special quadrilaterals. After that experience, the lesson plan was brought to Ms. Emma Bello for replication in her informal geometry class. In this section we report on how the negotiation between us and Ms. Bello resulted in modifications to the lesson that attest to the presence of default instructional situations (see also Hamlin, 2006, chapter 6).

An important emergent of the negotiation with Ms. Bello were an activity preliminary to the playing of the game and four resources for the scaffolding of this activity. The resources included (1) a list of possible properties that a quadrilateral might have, (2) a diagrammatic catalogue of shapes that depicted different special quadrilaterals and gave them names, (3) a "draw a quadrilateral" worksheet that gave students several properties and for each of them asked to draw a quadrilateral that had the property, and (4) a tracking sheet which included those possible properties in rows, and columns for students to keep track as to whether a specific shape had those properties, to be used in the midst of the "Guess my quadrilateral" game. The details of how the game was played have been extensively described in Hamlin's (2006) dissertation. In here we only make connections with the notion of instructional situation, using for that an examination of the worksheets. We note that three important modifications of the lesson included (i) providing a list of possible properties as opposed to expecting students to conjecture those possible properties, (ii) providing the shape catalogue and names as opposed to expecting students to remember those, and (iii) creating an opportunity for students to draw shapes that fit properties as opposed to merely imagining what shape they might be getting at with a specific property. We can see in these modifications the use of a situation of exploration and a situation of construction in order to manage the "guess my quadrilateral" task. Specifically, at one point students would be exploring the shape catalogue along with the list of possible properties thus "conjecturing" (really, associating or seeing the fit between) properties and canonical shapes. This would allow, for example to see how shapes that have a special name also have a cluster of properties. At another point in time students would be constructing (in fact just drawing) specific shapes after specific properties, thus seeing how much about one shape they could bound by abiding to the constraint of one property. These preliminary activities were useful in constructing resources for students to play the game "Guess my quadrilateral," which in spite of being novel it required much less negotiation of task between teacher and the students: About the unknown shape, students could ask "does it have this property" and use their drawn figures as well as the clusters of properties represented in the shape catalogue to know whether the answer (yes or no) was narrowing down their search and what possible shapes it could be. Again, this illustrates how the existence of default instructional situations was instrumental to enable the taking up of a novel task, but also how this situational framing of the task modified the nature of the work done (Herbst, 2006).

The Triangle Congruence Lesson: Construction Problems or Theorems to Install?

The triangle congruence lesson was implemented in four classes, two taught by Megan Keating and two by Lucille Vance. Students were given a set of segments and angles and asked to

> Choose a number of those data, as many as you think would be needed, to construct a unique triangle. Your group may choose any combination of angles and segments; then another group will verify whether you were correct. You will successfully meet the challenge if the data that you choose permits the other group to construct one triangle and only one triangle.

We anticipated that this task would perturb the usual contract between teacher students and knowledge at stake. The knowledge at stake in the task includes the so-called congruence criteria (some of them usually introduced as postulates and others usually introduced as theorems). Yet the task required students to produce a construction and argue that the object constructed was unique. We anticipated that the mismatch between declarative statements (how theorems and postulates are usually stated) and the expectation that students use construction procedures might create tensions. Tensions were present—for teachers the need to manage long chunks of time when students were working on their own, and for students the need to handle a task with subordinate requirements was at times frustrating. One of the teachers felt moved to disclaim that the purpose of the task had not been to make them feel incompetent. Yet on the whole the lesson did converge to a rather comprehensive discussion of what elements are sufficient to make a triangle and some discussions of uniqueness.

While the kind of examination done does not enable causal claims, we suggest that these tensions were eased out through the existence of two situations: the situation of installation and the situation of construction. In the observed lessons we noted that teachers divided them naturally into two phases. One of those phases consisted of settling on a choice of elements and implementing a construction procedure: Students thus specified how they could construct a triangle using two sides and an included angle or two angles and the included side. They utilized rulers or compasses to measure (or capture) the given segments and they used protractors to measure angles and implemented in general appropriate procedures to construct the triangles. However none of the procedures reported included considerations of choice (where to add an element to a construction in process) or of uniqueness (is there another triangle that could be done with these same elements). The second of those phases consisted of the teacher's overview of what had been done and discussion of whether a unique construction was possible. Two impossible cases were brought up by students: (1) that a construction using three angles might be impossible if the three angles don't add to 180 degrees and (2) that a construction may be impossible if two sides don't add to more than the third side. Additionally, one non unique case was brought up in one class: that a triangle would not yet be unique if only two angles were specified. These discussions converged naturally to installing the congruence criteria but did not address issues of choice in construction procedures; instead talk about the diagrams was descriptive. We observe that the existence of situations of installing (theorems or postulates) was useful to make room for a collective reflection on the statements subjacent to students' procedures while the existence of situations of construction was useful to make possible that initial work. While the task did not fit canonically and completely in any one of those situations, it was made viable by separating it into phases that somewhat fit with those situations.

Discussion and Conclusions

The main contention of this report is that instructional situations specific to the high school geometry class facilitate the management of exchanges between the tasks that students do in class and the contractual claims that their teacher can lay claim on. To be clear, all institutionalized teaching can be modeled as consisting of those exchanges and an important part of the work of the teacher is that of managing those exchanges. In completely novel tasks, a teacher needs to manage the negotiation of the task (or the negotiation of how the contract applies to the task) by agreeing with students on what kind of work they will accept responsibility for doing and what they can expect from the teacher. We've shown that in an established institutionalized setting such as the geometry class, kinds of tasks recur and customary ways of working are used to envelope tasks of the same kind: We call those customary ways of working instructional situations.

The report accounts for several instructional situations observed in our archive of high school geometry lessons. We named those "doing proofs," "installing theorems," "installing concepts," geometric calculation in number, geometric calculation in algebra, construction, and exploration. We examined cases of each of those situations, characterizing the situations in terms of the opportunities for reasoning that they afford to students and the way geometric objects (and diagrams) are used. We demonstrated that while different objects of the contract are at stake and different work is required of students in each of these situations, opportunities for students to reason mathematically can be developed within each of those situations.

In the last section we inspected how those instructional situations support the work of teaching in handling novel tasks. Expanding on our earlier work (Herbst, 2006) we showed that instructional situations can be of use for a teacher to handle instructional interventions that involve novel tasks: The work required by those tasks can be subdivided into tasks that are closer to canonical tasks in various of the instructional situations which are customary in that class. We illustrated this phenomenon by examining records from four different instructional interventions.

The significance of this phenomenon is two fold. One could say that the glass is half empty or that it is half full. Seeing the glass half empty, one could use the existence of instructional situations as a principle to explain why traditional classroom instruction resists change. Tasks that challenge customary ways of working can be transformed to the point of disappearance by defaulting to ways of working that fit more canonically existing instructional situations. The observation we have heard from teachers that one should set up the class to do that challenging kind of work from early in the year seems to call not for early socialization into the process of negotiating tasks but actually for developing a new kind of instructional situation, one that could contain problem oriented work that calls for the making of deductive arguments. Seeing the glass half full, one could note that instructional situations are natural emergents of a process of adaptation: The complex work of managing exchanges between varied kinds of work and varied objects at stake is simplified, streamlined by the existence of those situations. Like all complex activities, the existence of some routines for handling more predictable aspects of work enables the allocation of more energy to emergent problems and tactical decisions. Thus the design of lessons that foster students' reasoning could use an inventory of instructional situations productively: It could help to anticipate how the work could unfold as a sequence of tasks enveloped by different instructional situations and by identifying in each of those the aspects that are specific to the task that demand more close monitoring and tactical decision making.

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