THE STUDY OF TAIWANESE STUDENTS' EXPERIENCES WITH GEOMETRIC CALCULATION WITH NUMBER (GCN) AND THEIR PERFORMANCE ON GCN AND GEOMETRIC PROOF (GP)

by

Hui-Yu Hsu

A dissertation submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
(Education)
in The University of Michigan
2010

Doctoral Committee:

Professor Edward A. Silver, Chair
Professor Hyman Bass
Associate Professor Patricio G. Herbst
Associate Professor Priti Shah

DEDICATION

This dissertation is dedicated to my husband, Yu-Sheng Lin, to my daughter, Heng-Ting Lin, and to my parents, Ms. Shu-Chen Tsai, Mr. Chun-Ho Hsu, Ms. Hsiu-Hua Chang, Mr. Chen-Hsiung Lin for their support and encouragement throughout my graduate studies.

ACKNOWLEDGEMENTS

I would like to thank those persons who have helped me develop this dissertation in various ways. With their help, I would not have been able to complete this work.

I want to thank teacher Nancy (pseudonym) and her students who participated in the first two studies and the students who took part in the third study of the dissertation. Also thanks to principals, administrators and teachers in Wesley Girls' High school, Hsin-Pu Junior High School, and Jubei Junior High School for their administrative assistance. Although the teacher and students must remain anonymous, without their participation, I would not have had data for the analysis of teaching and learning in Taiwan. Particular thanks to Yu-Ping Chang and Hui-Fen Shiu for their work and feedback on coding the curriculum materials and students' survey responses.

I thank Professor Edward Silver for his work as my advisor and for chairing my dissertation committee. My appreciation goes to his generosity of providing invaluable advices on my work and his scholarship and administrative attributes. I also want to thank my other dissertation committee members. Particular thanks to Professor Patricio Herbst for his insightful comments which have inspired my investigation and have encouraged me the value of the work. I also acknowledge Professor Hyman Bass and Professor Priti Shah for investing time and efforts to shape this dissertation. Among the other Michigan faculties and students who have sharpened my thinking of mathematics education through discussions are Professor Vilma Mesa, Professor Magdalene Lampert, Professor Jay Lemki, Chialing Chen, Wendy Aaron, Gloriana Gonzalez, Sean Delaney, Ching-I Chang, and Takeshi Miyakawa. I am also grateful to my friends in Ann Arbor. They are important for me not only because of their priceless friendship at any special moments but also because of their support in taking care of my daughter at any time when needed.

I have been fortunate to take writing courses with Dr. Christine Feak and to have her help in shaping the writing through the studies. She has always encouraged me so that I can have confidence to keep going on. I owe her a great debt of gratitude. I am also grateful to Professor Fou-Lai Lin, Professor Yi-Hao Cheng, and Professor Chia-Yi Wu Yu in Taiwan for their advices specific to research on education in Taiwan. Their critical comments have deepened my thinking in conducting this dissertation.

Finally, I owe much to my family for their love and practical assistance through these years. I want to give special thanks to my mom, who did not know any English but came to Ann Arbor for taking care of my daughter Heng-Ting in my first year. Her great support made the transition for Heng-Ting and me much easier. My appreciation always goes to my husband Yu-Sheng and my lovely daughter Heng-Ting who deeply believed in me. They have always been the most important source of encouragement for me.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

LIST OF ABBREVIATIONS

GC	Geometric Calculation
GCN	Geometric Calculation with Number
GCA	Geometric Calculation with Algebra
GP	Geometric Proof

ABSTRACT

In Taiwan, students have considerable experience with tasks requiring geometric calculations with number (GCN) prior to their study of geometric proof (GP). This dissertation examines closely the opportunities provided to Taiwanese students with GCN tasks and the performance they exhibit on GCN and GP tasks. Three sequential studies were conducted, corresponding roughly to key aspects of the Mathematical Tasks Framework (MTF); namely, GCN tasks as found in curriculum materials and other instructional resources, GCN tasks as enacted by students and their teacher, and student performance on GCN tasks and related GP tasks.

Study One found that the GCN tasks used by one Taiwanese mathematics teacher were drawn from a wide variety of sources – not only the textbook series but also other sources (e.g., tests) – and the tasks varied with respect to cognitive complexity, with the tasks additionally included by the teacher being generally more demanding than those found in the textbook series. The high demand GCN tasks appeared to afford opportunities for Taiwanese students to master the types of knowledge, the reasoning and problem-solving skills that are essential not only for proficiency with GCN tasks but also for creating GP. Study Two examined how the Taiwanese mathematics teacher set up and enacted GCN tasks with her students. Of particular interest were the ways that the teacher sustained the cognitive demand levels by making the diagram configurations more complex and using gestural moves to scaffold students' visualization of the diagram

configurations so that they could sustain their work on the tasks. Through scaffolded experiences with GCN tasks containing complex diagrams, the teacher appeared to nurture students' competence in constructing and reasoning about geometric relationships in ways that are likely to support their later work with GP. Study Three presents the results of an analysis of Taiwanese students' performance on matched pairs of GCN and GP tasks, both of which use the same diagrams and require the same geometric properties to obtain solutions. The findings of this analysis strongly support the hypothesis that students' prior experience of working on GCN tasks can support their developing competence in constructing GP.

Taken together the three studies offer a glimpse at classroom instruction in Taiwan involving GCN tasks and sketch a plausible pathway through which Taiwanese students might gain competence with GP tasks through their experiences with GCN tasks. The three studies also suggest why it might be the case that Taiwanese students would both develop competence in constructing GP before formal instruction to do so in schools and develop high levels of proficiency with geometric proving and reasoning. In addition, the use of a sequence of three studies that examine different aspects of students' experiences with mathematical tasks appears to have utility as a model for other research that seeks to understand cross-national differences in mathematics performance.

CHAPTER ONE

INTRODUCTION

Rationale

For the last two decades, cross-national comparisons have been a common approach to understanding how mathematics is taught and learned in different countries. Of the many countries included in these comparisons, Taiwan has consistently scored better than others in mathematics. For example, on the TIMSS (Trends in International Mathematics and Science Study) Taiwanese 8^{th} grade students ranked first in 2008 and fourth in 2003 (Mullis, Martin, & Foy, 2008; Mullis, Martin, Gonzalez, & Chrostowski, 2004). Taiwanese students also performed well on problems requiring the construction of geometric proof (GP). In their study on geometric proving and reasoning, Heinze, Cheng, and Yang (2004) compared Germany and Taiwanese students and found that Taiwanese students performed significantly better than Germany students.

Heinze et al. also noticed a special phenomenon about Taiwanese students who demonstrated the potential to do GP before having formally learned the GP content in schools. To explore this phenomenon, I propose that Taiwanese students' considerable experience with solving geometric calculations (GC) is one of the key factors contributing to their outstanding ability to do GP. In particular, I focus on examining the geometric calculations with number (GCN)[1], a type of geometric calculations (GC) that is frequently used in Taiwanese classroom. Here a GCN is generally described as numerical calculation done in relation to mental or physical geometric diagrams on the basis of geometric principles or formulae (e.g., calculating an angle measure in a triangle given that measures of the other two angles are 30˚ and 100˚, respectively) (Aleven, Koedinger, Sinclair, & Snyder, 1998; Ayres & Sweller, 1990; Lulu Healy & Celia Hoyles, 1998; Küchemann & Hoyles, 2002; Lawson & Chinnapan, 2000).

I particularly argue that working with GCN task diagrams is critical to gaining geometric intuition and being able to solve GP tasks. This is because GCN task diagrams used in Taiwan are often diverse and complex, thus, providing students different *opportunities to learn*[2], an affordance which is worthy of investigation. To explain how GCN task diagrams can be complex, the TIMSS video study provides the following example (Stigler & Hiebert, 1999). In their investigation, Stilger and Hiebert showed that GCN tasks implemented in mathematical lessons that may influence the differences in performance of students in the U.S., Germany, and Japan. Representing the highest performance among the three countries, Japanese students were required to solve the GCN tasks that were different from those given to the U.S. and German students. A group of GCN tasks with complex diagrams (see Figure 1.1) were given to the Japanese

---

[1] GCN is one type of geometric calculation (GC). Other types include geometric calculation in algebra (GCA) and geometric calculations in coordinate system (Lang & Ruane, 1981). As GCN and GCA tasks are similar and are frequently used in Taiwanese classrooms, I detail the similarities and differences between the two types of GC tasks in Appendix 1.1.

[2] In this study, *opportunity to learn* refers to a factor that contributes to students' learning outcomes (Tornroos, 2005). In particular, the study focuses on exploring the learning opportunities afforded by the mathematical tasks situated in the curricular or instructional materials that teacher and students may enact in classroom. The enactment of the tasks may expand or degrade the cognitive demand of the tasks (Stein, Grover, & Henningsen, 1996) and in turn influence students' learning outcomes.

students to solve, whereas in Germany and the U.S., however, geometric tasks usually were assigned to students one by one.

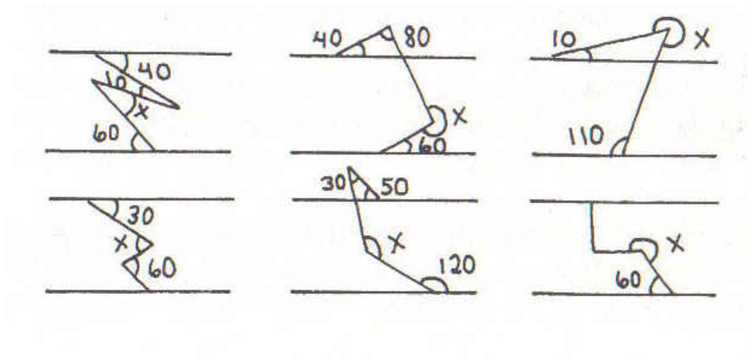


Figure 1.1 A group of GCN tasks assigned in Japanese classroom (Stigler & Hiebert, 1999, p.39)

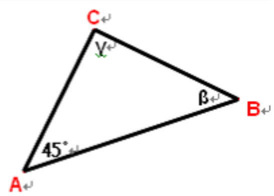
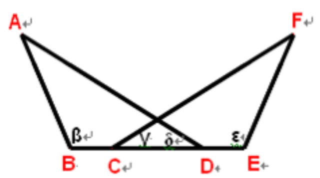
Figure 1.1 shows the GCN tasks given in the Japanese lessons. As can be seen, these GCN diagrams are diverse and complex. Although each of the GCN task diagrams contains a pair of parallel lines, the differences in the number of segments and transversals as well as their orientations and positions in the GCN diagrams make them complex and different from each other. As the complexity of the GCN task diagrams increases, so does the cognitive demand of these tasks, because most of them cannot be solved by a single geometric property or a well-known procedure (Stein, Grover, & Henningsen, 1996). To solve GCN tasks accompanying by complex diagrams, students need to visualize the sub-constructs of the diagrams and decide which corresponding geometric properties can be used to generate a solution (Zykova, 1975a). In addition, such GCN tasks also require students to draw auxiliary lines to create new sub-constructs and geometric properties to obtain solutions, which in turn also increase the cognitive demand (Hsu, 2007). As a result, the GCN tasks provide opportunities to learn beyond practicing numerical calculations with the application of a single geometric property. The latter characteristic of GCN tasks has been recognized in the literature (Schumann & Green, 2000). What learning opportunities beyond the practice of calculations and geometric properties that GCN tasks used in East Asian will be investigated in this dissertation. In particular, the dissertation focuses on exploring the ways that GCN tasks were set up and enacted by the classroom students and how the enactment of the GCN tasks can facilitate student competence in constructing GP tasks.

Another work that will be carried out in this dissertation is to examine how the role of diagram influences the relationship between GCN and GP tasks. Researchers have provided the implications regarding why geometric diagram can be the key to transferring students' experiences of solving GCN tasks to their competence in proving GP tasks. A diagram is the location where the problem solving happens (Larkin & Simon, 1987) and the schemes by which students remember the steps in solving a problem, the given statements, and the diagram labels (Lovett & Anderson, 1994). Diagrams can be parsed into chunks to cue the geometric knowledge for solutions, which mirrors how experts solve geometric tasks (Koedinger & Anderson, 1990). Moreover, diagrams can also function as artifacts in scaffolding students in learning proofs. Cheng and Lin (2006; 2007) reported that junior high school students' performance on constructing proofs were improved greatly by asking students themselves to read the given information and then color these properties on the diagrams. The colored parts of the diagram help students visualize the useful geometric properties to generate an acceptable proof. Furthermore, Fujita, Jones, and Yamamoto (2004) also indicated that creating and manipulating geometric diagrams mentally or physically can nurture students' intuition in geometric problem solving. Thus, diagrams can bring GCN much closer to GP and can be the key to influence students' performance on those two types of tasks. Another relevant focus of the geometric diagram is the requirement of geometric properties. No matter if they are solving GCN or doing GP, students need to visualize the geometric diagrams and identify the needed geometric properties in order to set up calculating sentences or form logical proving statements, but the work has been recognized as one of the main difficulties in learning geometry (Duval, 1995; Fischbein & Nachlieli, 1998; Zykova, 1975a; 1975b). As a result, diagrams can be a crucial key to establishing the relationship between GCN and GP, and creating a closer relationship between the two types of tasks, which is worthy of further investigation.

Actually, treating geometric diagrams as the crucial key to the relationship between GCN and GP has not been carefully investigated yet. Even though researchers have articulated the connection between calculations and proofs in general (Tall, 2002, 2006a, 2006b, 2007), they do not particularly deal with the relations between GCN and GP, nor the role that a geometric diagram can play in influencing the relations. Nor does the famous theory proposed by van Hiele (Fuys, Geddes, & Tischler, 1988). Furthermore, the linkage between GCN and GP also cannot be articulated in empirical studies by surveying students' performance on GCN and GP (Healy & Hoyles, 1998; Lin, Cheng, & et al., 2003), because diverse factors can confound the comparisons (e.g., the geometric properties required to obtain a solution, the number of inferring steps to solutions). Moreover, some other studies have directly treated GCN as low-level cognitive demand tasks only for the application of properties and rules and have used this perspective to compare students' performance on the two types of tasks to assess the relations (Heinze, Cheng, Ufer, Lin, & Reiss, 2008; Heinze, Cheng, & Yang, 2004; Heinze, Ufer, Cheng, &

Lin, 2008). For example, using the two tasks in Table 1.1, Heinze, Ufer, Cheng, and Lin (2008) characterized the GCN tasks (in above part of the table) as the one requiring the application of basic knowledge, and the other (in below part of the table) as GP tasks.

Table 1.1 GCN (above) and GP tasks (below) used to survey Germany students' performance (Heinze, Ufer et al., 2008)

Diagram	Givens
	The triangle is isosceles with |AC| = |BC|. Calculate the missing angles.
	C and D are points on the line BE. We have |BD| = |EC|, γ = δ and β = ε. Prove that |AB| = |EF| Give reasons for all steps of your proof

As shown in Table 1.1, the GCN task focuses on calculating unknown angle measures, whereas the GP task requires proving that two segments are congruent (AB=EF). These two tasks are different from each other in terms of the geometric properties necessary to obtain the solutions, demands on diagram visualization, and the number of proving or calculating steps. Solving the GCN task requires the triangle angle sum property and the properties related to an isosceles triangle to find the measures, whereas solving the GP task necessitates the use of the Angle-Side-Angle triangle congruence postulate to prove the conclusion. The visualization demands also differ between the two tasks. Proving the GP task requires one to recognize the sub-constructs (two overlapping triangles) in the given diagram, the work of which can be more demanding than that needed to identify the GCN task diagram as an isosceles triangle. In addition, the reasoning steps required to generate the solution in each of the two tasks are also different. For the GCN task, each of the unknowns can be inferred by applying a geometric property (e.g., measure of ∠ABC can be obtained by using isosceles triangle property). For the GP task multiple reasoning steps are needed: (1) steps of finding the needed conditional statements to conclude that two overlapping triangles are congruent; and (2) step of applying the result of congruent triangles to infer that segment AB = EF.

Specifically, considering GCN as tasks of lower-level cognitive-demand ignores the complexity and particularity a task can be (Stein, Smith, Henningsen, & Silver, 2000), and may underestimate the link between both types of tasks. For instance, using the tasks in Table 1.1 as examples again, we see that solving a one-step GCN may be much easier than constructing a multiple-step GP not because of their differences in task format, but because of the cognitive demand as determined by the number of reasoning steps needed to generate a solution. Thus, using only simple applications of basic knowledge to characterize GCN tasks as low demanding tasks, we may fail to see the relationship between GCN and GP. As a result, this study presents an argument that the link between GCN and GP can be stronger than the simple application of geometric properties. Because of the abstract nature of geometric diagrams, especially when both types of tasks share the same diagram configurations and the geometric properties necessary to obtain solutions, both types of tasks may possess the same cognitive demand level. In this regard, students' experiences with GCN tasks are very likely to contribute to students' learning of creating GP tasks later on.

To investigate the ways and the extent to which GCN tasks used in Taiwanese classrooms that can contribute to students' competence in constructing GP tasks as well as to conceptualize the relationship between GCN and GP, I conduct three sequential and independent studies roughly corresponding to the key aspects of the Mathematical Tasks Framework (MTF) (see Figure 1.2); namely, tasks as they appear in curricula/instructional materials, tasks as set up by teachers, tasks as enacted by classroom teacher and students, and the consequence of task enactment through the three stages on students' learning outcomes (Silver & Stein, 1996; Stein et al., 1996; Stein & Smith, 1998; Stein et al., 2000).

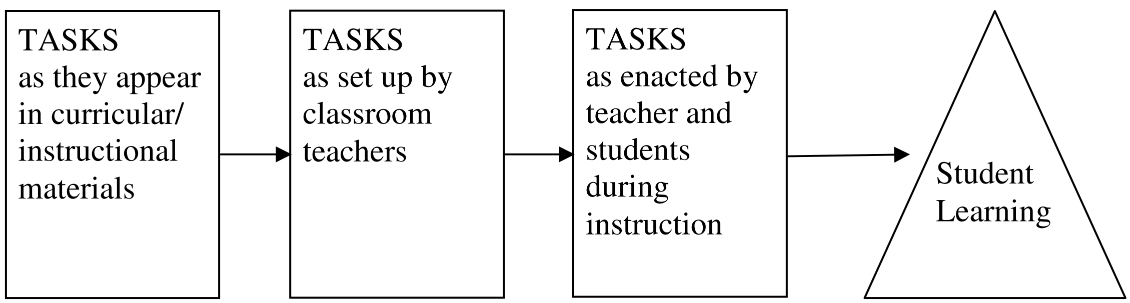


Figure 1.2 The Mathematical Tasks Framework (Stein, Smith, Henningsen, & Silver, 2000, p. 4; Silver, 2009, p. 829)

In line with the framework, Study One of the dissertation centers on the first stage, tasks as found in curricular/instructional materials, exploring what learning opportunities that GCN tasks situated in the curricular/instructional materials used by a Taiwanese mathematics teacher, Nancy[3], can provide for students to enact. Study Two, which focuses on the second and the third stages of the framework as tasks set up by teacher and tasks enacted by the teacher and students, further traces how the Taiwanese mathematics teacher sets up GCN tasks and enacts these tasks with students. In particular, as sustaining the level of cognitive demand of tasks can facilitate learning occurring (Hiebert & Wearne, 1993; Stein & Lane, 1996; Stigler & Hiebert, 2004; Tarr et al., 2008),

[3] Nancy is pseudonym.

the investigation is of interest in how Nancy sustains or increases the level of cognitive demand and facilitates student learning. Study Three emphasizes on the learning outcomes as the consequence of the enactment of GCN tasks through the three stages, examining the extent to which students' experiences with solving GCN tasks can contribute to their competence in constructing GP.

To this end, an overarching research question with three corresponding research questions (RQs) is proposed as follows.

Overarching Research Question

What are GCN tasks used and enacted in a Taiwanese mathematics teacher and to which extent can Taiwanese students' experiences with GCN tasks contribute to their competence in constructing GP?

RQ1. What opportunities are provided by the GCN tasks used by a Taiwanese mathematics teacher, Nancy, for her students to learn to handle complex geometric diagrams and solve these complex GCN tasks?

RQ2. In what ways does Nancy sustain the levels of cognitive demand and facilitate students' learning by setting up and enacting the GCN tasks with classroom students?

RQ3. To what extent is Taiwanese students' performance on GCN similar to that on GP when controlling the diagram configurations and requirements of geometric properties necessary for a solution?

Significance of the Dissertation

In this dissertation I conduct three sequential studies to investigate how GCN tasks are used and enacted by a Taiwanese mathematics teacher and her students, and how students' experiences with GCN tasks can contribute to their competence in constructing GP. Taken together the three studies that comprise the dissertation can provide insights into why it might be the case that Taiwanese students can develop potential competence in creating GP before formal instruction to do so in schools and can outperform students in other countries on geometric proving and reasoning, as has been shown in the specific case of Germany (Heinze, Cheng, & Yang, 2004). The findings taken together from the three studies can also provide implications for other research that might seek to understand cross-national differences in mathematics performance. In addition, the investigation of the key aspects in the MTF framework can also provide insight into why the framework might be the core frame for an investigation of factors accounting for the differences in students' performance in cross-national comparisons.

Overview of the Dissertation

In this chapter I provide a rationale and for the need of conducting the three sequential studies in this dissertation, as well as the research questions and significance of this study. In Chapter Two I present Study One which involves the analysis of GCN tasks situated in curricular/instructional materials, not only the official textbooks but also auxiliary materials that the mathematics teacher, Nancy, includes in a Taiwanese classroom. I discuss the diverse opportunities that the GCN tasks used by Nancy can provide for students to enact. In Chapter Three I further demonstrate Study Two as how the GCN tasks are set up and enacted by Nancy and her students. In particular, the analysis focuses on exploring the ways Nancy sustains the cognitive demand of the GCN tasks and how she uses different instructional strategies to facilitate students' learning. In Chapter Four I detail Study Three which investigates the extent to which students' experiences of solving GCN tasks can contribute to their performance on constructing GP when both the diagram configurations and the geometric properties needed to obtain solutions are controlled. In the last chapter I provide general discussions regarding how the dissertation contributes to the understanding of the role of GCN tasks to the learning of GP, how the dissertation together with the three studies can provide alternative perspectives with respect to cross-national comparisons, and an example to support the MTF framework in exploring the factors accounting for differences in students' performance in cross-national comparisons.

REFERENCES

Aleven, V., Koedinger, K., Sinclair, H., & Snyder, J. (1998). *Combatting shallow learning in a tutor for geometry problem solving*. Paper presented at the intelligent tutoring systems: Fourth International Conference, ITS '98, Berlin.

Anderson, J., Greeno, J., Kline, P., & Neves, D. (1981). Acquisition of Problem-Solving Skill. In J. R. Anderson (Ed.), *Cognitive Skills and Their Acquisition* (pp. 191-230). New Jersey: Lawrence Erlbaum Associates.

Ayres, P., & Sweller, J. (1990). Locus of difficulty in multistage mathematics problems. *The American Journal of Psychology, 103*(2), 167-193.

Cheng, Y.-H., & Lin, F.-L. (2006). Using reading and coloring to enhance incomplete prover's performance in geometry proof. In J. Novotna, H. Moraova, M. Kratka, & N. Stehlikova (Eds.), *Proceedings of the 30th Conference of the International Group for the Psychology of Mathematics Education* (vol. 2, pp. 289-296). Prague, Czech Republic.

Cheng, Y.-H., & Lin, F.-L. (2007). The effectiveness and limitation of reading and coloring strategy in learning geometry proof. In J. H. Woo, H. C. Lew, K. S. Park, & D. Y. Seo (Eds.), *Proceedings of the 31st Conference of the International Group for the Psychology of Mathematics Education* (vol. 2, pp. 113-120). Seoul, Korea.

Duval, R. (1995). Geometrical pictures: Kinds of representation and specific processings. In R. Sutherland & J. Mason (Eds.), *Exploiting mental imagery with computers in mathematics education* (pp. 142-157). Berlin: Springer.

Fischbein, E., & Nachlieli, T. (1998). Concepts and figures in geometrical reasoning. *International Journal of Science Education, 20*(10), 1193-1211.

Fujita, T., Jones, K., & Yamamoto, S. (2004). *Geometrical intuition and the learning and teaching of geometry*. Paper presented at the ICME 10, Denmark.

Fuys, D., Geddes, D., & Tischler, R. (1988). *The van Hiele model of thinking in geometry among adolescents* (Vol. 3). Reston, VA: NCTM.

Healy, L., & Hoyles, C. (1998). *Justifying and proving in school mathematics*. London: Institute of Education.

Healy, L., & Hoyles, C. (1998). *Justifying and Proving in School Mathematics. Technical Report on the Nationwide Survey*. London.

Heinze, A., Cheng, Y.-H., Ufer, S., Lin, F.-L., & Reiss, K. (2008). Strategies to foster students' competencies in constructing multi-steps geometric proofs: Teaching experiments in Taiwan and Germany. *ZDM-The International Journal on Mathematics Education, 40*(3), 443-453.

Heinze, A., Cheng, Y.-H., & Yang, K.-L. (2004). Students' performance in reasoning and proof in Taiwan and Germany: Results, paradoxes and open questions. *ZDM-The International Journal on Mathematics Education, 36*(5), 162-171.

Heinze, A., Ufer, S., Cheng, Y.-H., & Lin, F.-L. (2008). *Geometric proof competence and its individual predictors: A Taiwanese-German study*, paper presented at 11th ICME. Monterrey, Mexico.

Hiebert, J., & Wearne, D. (1993). Instructional tasks, classroom discourse, and students' learning in second-grade arithmetic. *American Educational Research Journal, 30*(2), 393-425.

Hsu, H.-Y. (2007). Geometric calculations are more than calculations. In J. H. Woo, H. C. Lew, K. S. Park, & D. Y. Seo (Eds.), *Proceedings of the 31st Conference of the International Group for the Psychology of Mathematics Education* (vol. 3, pp. 57-64). Seoul, Korea.

Katz, V. J. (1993). *A history of mathematics: An introduction*. New York: HarperCollins College.

Kieran, C. (1981). Concepts associated with the equality symbol. *Educational Studies in Mathematics, 12*, 317-326.

Koedinger, K., & Anderson, J. (1990). Abstract planning and perceptual chunks: Elements of expertise in geometry. *Cognitive Science, 14*, 511-550.

Küchemann, D., & Hoyles, C. (2002). The quality of students' reasons for the steps in a geometric calculation. *Proceedings of the British Society for Research into Learning Mathematics, 22*(2), 43-48.

Lang, B., & Ruane, P. (1981). Geometry in English secondary schools. *Educational Studies in Mathematics, 12*(1), 123-132.

Larkin, J., & Simon, H. (1987). Why a diagram is (sometimes) worth ten thousand words. *Cognitive Science, 11*, 65-99.

Lawson, M. J., & Chinnapan, M. (2000). Knowledge connectedness in geometry problem solving. *Journal for Research in Mathematics Education, 31*(1), 26-43.

Lin, F.-L., Cheng, Y.-H., & et al. (2003). The competence of geometric argument in Taiwan adolescents, *International Conference on Science & Mathematics Learning*. Taipei, Taiwan.

Lovett, M. C., & Anderson, J. R. (1994). Effects of solving related proofs on memory and transfer in geometry problem solving. *Journal of Experimental Psychology: Learning, Memory, and Cognition, 20*(2), 366-378.

Mullis, I. V. S., Martin, M. O., & Foy, P. (2008). *TIMSS 2007 international mathematics report*. Chestnut Hill, MA: TIMSS & PIRLS International Study Center, Boston College.

Mullis, I. V. S., Martin, M. O., Gonzalez, E. J., & Chrostowski, S. J. (2004). *TIMSS 2003 international mathematics report: Findings from the IEA's Trends in International Mathematics and Sciences Study at the fourth and eighth grades*. Chestnut Hill, MA: TIMSS & PIRLS International Study Center.

Schliemann, A., Carraher, D., & Brizuela, B. (2007). *Bringing out the algebraic character of arithmetic-from children's ideas to classroom practice*. Lodon: Lawrence Erlbaum Associates.

Schumann, H., & Green, D. (2000). New protocols for solving geometric calculation problems incorporating dynamic geometry and computer algebra software. *International Journal of Mathematical Education in Science and Technology, 31*(3), 319-339.

Sfard, A., & Linchevski, L. (1994). The gains adn the pitfalls of reification-The case of algebra. *Educational Studies in Mathematics, 26*, 191-228.

Silver, E., & Stein, M. (1996). The Quasar project: The "revolution of the possible" in mathematics instructional reform in urban middle schools. *Urban Education, 30*(4), 476-521.

Silver, E. (2009). Cross-national comparisons of mathematics curriculum materials: What might we learn? *ZDM-The International Journal on Mathematics Education, 41*, 827-832.

Stein, M., Grover, B., & Henningsen, M. (1996). Building student capacity for mathematical thinking and reasoning: An analysis of mathematical tasks used in reform classrooms. *American Educational Research Journal, 33*, 455-488.

Stein, M. K., & Lane, S. (1996). Instructional tasks and the development of student capacity to think and reason: An analysis of the relationship between teaching and learning in a reform mathematics project. *Educational Research and Evaluation, 2*, 50-80.

Stein, M., & Smith, M. (1998). Mathematical tasks as a framework for reflection: From research to practice. *Mathematics Teaching in the Middle School, 3*, 268-275.

Stein, M., Smith, M., Henningsen, M., & Silver, E. (2000). *Implementing standards-based mathematics instruction: A casebook for professional development*. New York, NY: Teachers College Press.

Stigler, J., & Hiebert, J. (1999). *The teaching gap: Best ideas from the world's teachers for improving education in the classroom*. New York: Free Press.

Stigler, J., & Hiebert, J. (2004). Improving mathematics teaching. *Educational Leadership, 61*(5), 12-16.

Tall, D. (2002). *Differing modes of proof and belief in mathematics.* Paper presented at the International Conference on Mathematics: Understanding proving and proving to understand, Taipei, Taiwan.

Tall, D. (2006a). *A life-times' journey from definition and deduction to ambiguity and insight.* Paper presented at the Retirement as Process and Concept: A festschrift for Eddie Gray and David Tall, Prague.

Tall, D. (2006b). A theory of mathematical growth through embodiment, symbolism and proof, *International Colloquium on Mathematical learning from early childhood to adulthood*. Nivelles, Belgium: Centre de Recherche surl'Enseignement des Mathematiques.

Tall, D. (2007). Embodiment, symbolism, argumentation and proof, *The conference on Reading, Writing and Argumentation*

Tarr, J., Reys, R., Reys, B., Chavez, O., Shih, J., & Osterlind, S. (2008). The impact of middle-grades mathematics curricula and the classroom learning environment on student achievement. *Journal for Research in Mathematics Education, 39*, 247-280.

Tornroos, J. (2005). Mathematics textbooks, opportunity to learn and student achievement. *Studies in Educational Evaluation, 31*, 315-327.

Zykova, V. I. (1975a). Operating with concepts when solving geometry problems. In J. Kilpatrick & I. Wirszup (Eds.), *The learning of mathematical concepts* (Vol. 1, pp. 93-148). Chicago: University of Chicago.

Zykova, V. I. (1975b). The Psychology of Sixth-Grade Pupils' Mastery of Geometric Concepts. In J. K. I. Wirszup (Ed.), *The Learning of Mathematics Concepts* (Vol. I, pp. 149-188). Chicago: University of Chicago.

Appendix 1.1

Clarification of Geometric Calculation with Number (GCN) and Geometric Calculation with Algebra (GCA)

As discussed in the literature, a GC can be generally described as calculations done within mental or physical geometric diagrams on the basis of geometric principles or formulae. In line with this general definition, the most frequently used types of GC tasks in geometric lessons are geometric calculation with number (GCN) and geometric calculation with algebra (GCA). Such a distinction is revealed because the special characteristics of each type of GC can influence its relation to GP. Specifically, even though both types of GC tasks all involve a variety of geometric diagram shapes and require geometric properties needed to generate solutions, the extra algebra work necessary in solving GCA tasks can increase the cognitive demand of the tasks and confound the conceptualization of the relations between GC and GP, especially when algebra work is a difficult learning topic for students (Kieran, 1981; Schliemann, Carraher, & Brizuela, 2007; Sfard & Linchevski, 1994).

In order to further illustrate the special characteristics of GCN and GCA as well as the influence of extra algebra work with respect to conceptualizing the relationship between GC and GP, two comparable examples representing each type of tasks are analyzed. These two examples are similar because they use the same geometric diagram, the same written information about diagram details, and the same unknown that students have to figure out. The only difference is that the GCN task describes the length measures with numerical information, whereas the GCA task indicates givens and unknown using algebraic expressions. The comparison of the two tasks focuses on the problem-solving process which is demonstrated using the concept of *plan tree* developed based on the ACT-R theory (Anderson, Greeno, Kline, & Neves, 1981).

Anderson et al. (1981) define a *plan tree* as "an outline for actions" (p. 193) that is generated based on the logically separate stages. Using geometry proofs as subject content to exemplify the thinking mechanisms, Anderson et al. state that stages in the plans are a set of geometric rules, allowing students to get from the givens of the task through intermediate levels of statements then to the to-be-proved statements. The *plan tree* for proof is a knowledge structure in the head generated by unpacking various links of relevant knowledge and re-organizing the links and knowledge into a logic sequence. Hence, reasoning solution paths is usually a mixed forward and backward process. In this process, on the one hand, students must search forward from the givens to find sets of solution paths that can yield the to-be-proven statements. On the other hand, students also have to infer backward steps from the to-be-proven statements that may be related to the givens. The process of reasoning forward and backward gradually maps out a solution path containing relevant geometric knowledge that can link with both the givens and the conclusion statements.

Geometric Calculation with Number (GCN)

GCN is a geometric task in which numerical information and diagrammatic information are provided in the givens. A diagram shape usually accompanies a GCN task. The goal of such task is to use geometric properties embedded in the diagrams or in the givens to infer the unknowns by setting up relationships among the relevant measures. The inferring process commonly involves numerical calculations.

Table 1.2 The compared GCN task

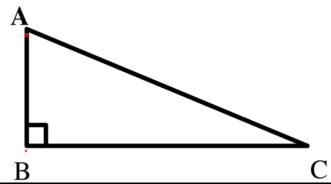
The given diagram	The written givens
	As shown on the left side, ∠ABC is a right angle. Given AC=13 and AB=5, find the length of BC[4].

Table 1.2 gives an example of GCN task. The diagram and the givens of the task specify the measures of segments in the triangle and the theoretical properties (∠ACB is a right angle) that can be used to set up the calculating step among the relevant measures. The required geometric property for this task is the Pythagorean Theorem which can be inferred based on the givens indicating that ∠ABC is a right angle.

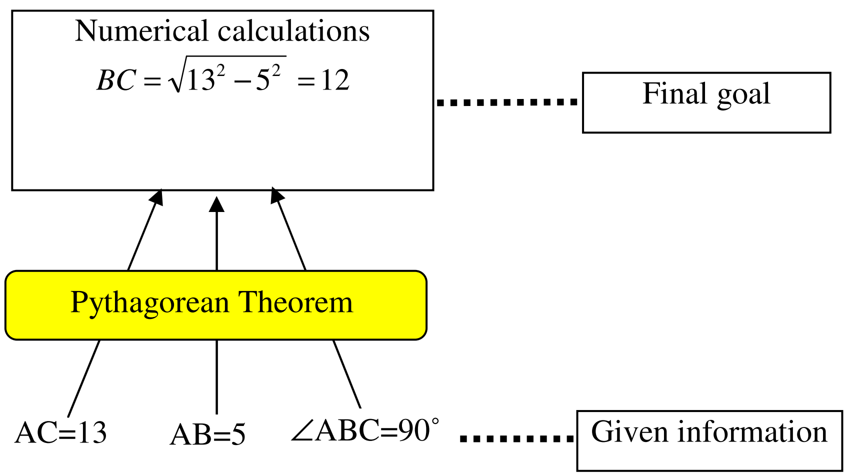


Figure 1.3 The *plan tree* for the GCN task in Table 1.2

[4] Actually, from a mathematical perspective, the unknown BC in the task can be interpreted as an algebraic expression representing the length of a segment. In this sense, GCN in a way also involves algebraic work and may cause confusion in distinguishing between GCN and GCA. As the goal of classifying the two types of GC tasks is due to the reason that algebraic work will impose extra cognitive demand on students and become a confounding variable for conceptualizing the relationship between GC and GP, the categorization of GCN and GCA is proposed from a problem-solving perspective. That means, if a GC task can be solved without using any algebraic skills to calculate measures, the GC is classified as a GCN no matter algebraic expressions are used in the givens or not. On the other hand, if the algebraic skills are needed to calculate the unknowns, the GC task is categorized as a GCA. Further clarification of the two types of tasks is illustrated in Appendix 2.4 in Chapter Two.

The *plan tree* in Figure 1.3 illustrates the stages of solving this task, in which the arrows indicate the connections of the stages. To solve this GCN task, students need to use two measures (e.g., AC=13) and the theoretical property ∠ABC=90˚ from the givens to set up a calculating sentence on the basis of the Pythagorean Theorem. The remaining work involves the numerical calculations in order to find the answer to this task.

<u>Geometric Calculation with Algebra (GCA)</u>

The definition of GCA is similar to that of GCN, in that it involves calculating some unknowns by establishing relationships among relevant measures based on geometric properties and the givens of the task. The major difference between the two types of tasks is the algebraic skills that are necessary to obtain solutions to a GCA task.

Table 1.3 The compared GCA task

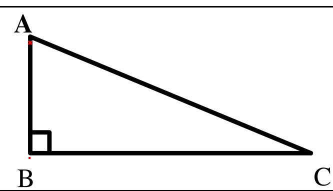
The given diagram	The written givens
	As shown on the left side, ∠ACB is a right angle. Given AB=5, BC=X+5, and AC=2X-1, find the length of BC.

Table 1.3 displays the compared GCA task. The diagram and the givens are the same as described in the GCN task shown in Table 1.2. The only difference is the use of algebraic representations describing the measures of the segments in ΔABC. Again, the Pythagorean Theorem is the geometric property needed to set up the relationship among the measures.

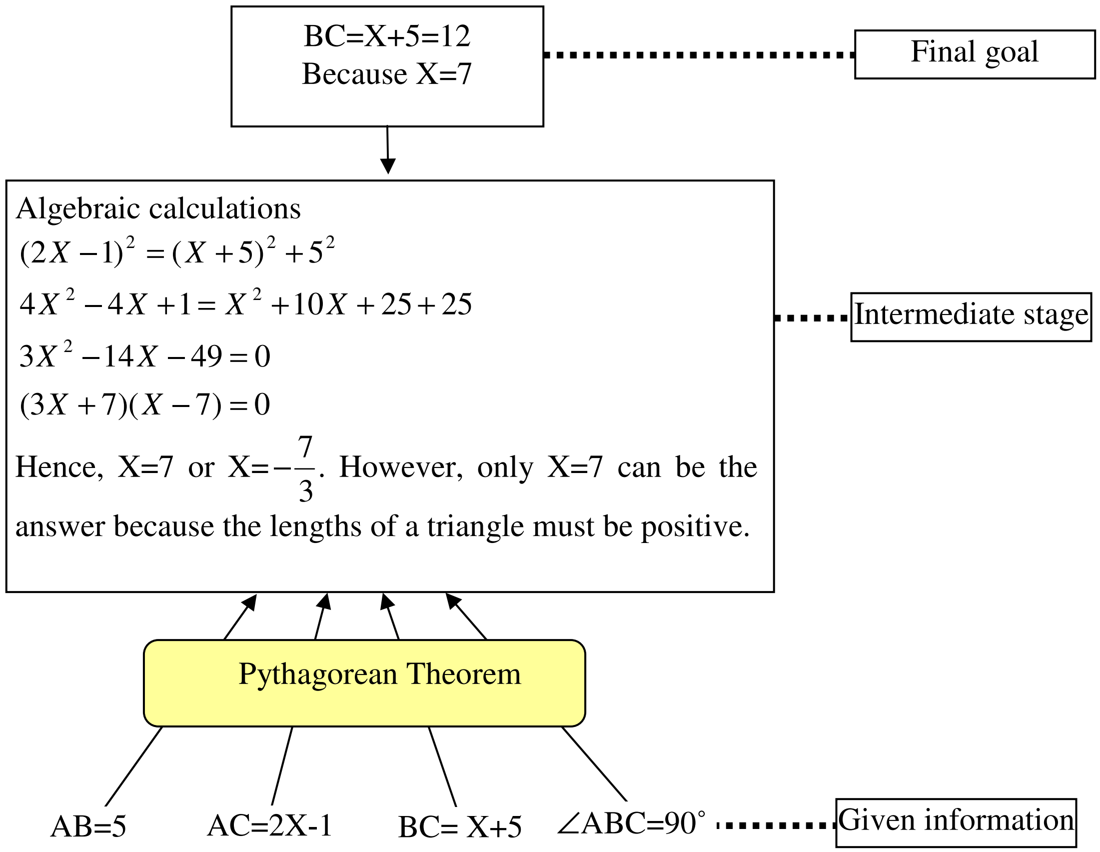


Figure 1.4 The *plan tree* for the GCA shown in Table 1.3

The *plan tree* in Figure 1.4 shows that the first step is to use three measures and the theoretical property ∠ABC=90˚ from the givens to set up the algebraic equation on the basis of the Pythagorean Theorem. The next step is to solve the equation using algebraic skills, which in turn create the intermediate stage in the *plan tree* because students have to find the value of X before calculating length of the segment BC. In addition, the *plan tree* in Figure 1.4 also illustrates that the algebraic calculations are demanding as students have to factorize the polynomials in the equation.

Despite similarities of the GCN and the GCA tasks in terms of the given diagrams, the written givens, and the geometric property needed to obtain the solutions, the plan trees shown in Figure 1.3 and Figure 1.4 highlight differences in the problem-solving processes for the two types of tasks. The first difference centers on the algebraic work. The GCA task requires extra work of students since they not only need to make inferences using the geometric property to set up the relation for the calculations, but also need to master algebraic skills to find the value of X, the intermediate stage shown in Figure 1.4. Thus, as suggested by comparing the problem-solving processes for the two tasks, the cognitive demand of the GCA task should be higher than for the paired GCN task[5].

The second difference between the GCN task and the GCA task is the uncertainty regarding the use of algebraic expressions to represent the segment measures on the diagram, which may impede students' ability to infer solutions (Koedinger & Anderson, 1990). In this regard, Koedinger and Anderson indicated that inferring from the diagram is simple a step involving recognition of the diagram configuration, but this may require several steps if algebra variables are involved. Algebraic variables impose a demand on students because they not only need to make inferences from the diagrams but also to consider how the algebra variables may regulate the inferences. Taking the GCA task in Figure 1.4 as an example, students need to evaluate the values of X (X=7 and $X=-\frac{7}{3}$) because they will influence in reasoning the measure of segment BC.

The third difference has to do with the use of given information in setting up the calculating sentences. Setting up the calculating sentence in the GCN task requires two measures from the givens, whereas that in the GCA task requires three variables. The

---

[5] The comparison on the two GCN and GCA tasks rests on the assumption that middle school students can manage the numerical calculations so that such work does not impose extra cognitive demand on students.

reason for using three measures in the GCA task instead of two measures is that the algebraic expression itself is a variable, thus decreasing a degree of freedom for setting up an equation relation similar to that of the GCN task.

To sum up, GCN and GCA can be quite different in terms of the need to apply algebraic skills to obtain solutions, the complexity of making inferences considering both the diagram and the algebraic expressions, and the number of measures from the givens needed to set up calculating sentences. While the assumption of this dissertation is that both diagram and geometric properties required to obtain the solutions are keys to students' performance on GC and GP, the algebraic work may impose the extra cognitive demands, thus, becoming a confounding variables in examining the proposed assumption. In other words, it is possible that students could not successfully solve a GCA task not because they can not visualize the diagram and retrieve geometric properties needed to obtain solutions but because they have difficulties in calculating the algebraic equations. Students' difficulties in algebra will influence the comparisons of their performance on GC and GP tasks, and cause limitations in interpreting the results from the comparisons. Hence, while exploring the relationship between GC and GP aligned with the proposed assumption, this dissertation narrows the investigation only on GCN tasks.

CHAPTER TWO

LEARNING OPPORTUNITIES AFFORDED BY GEOMETRIC CALCULATIONS WITH NUMBER (GCN) USED BY A TAIWANESE MATHEMATICS TEACHER

Introduction

For the last two decades, cross-national comparisons have been a common approach to understanding how mathematics is taught and learned in different countries. Of the many countries included in these comparisons, Taiwan has consistently scored better than others in mathematics. For example, on the TIMSS (Trends in International Mathematics and Science Study) Taiwanese 8th grade students ranked first in 2008 and fourth in 2003 (Mullis, Martin, & Foy, 2008; Mullis, Martin, Gonzalez, & Chrostowski, 2004). Taiwanese students also performed well on problems requiring the construction of GP. In their study on geometric proving and reasoning, Heinze, Cheng, and Yang (2004) compared Germany and Taiwanese students and found that Taiwanese students performed significantly better than Germany students.

Heinze et al. also noticed a special phenomenon about Taiwanese students who demonstrated the potential to do GP before having formally learned the GP content in schools. To explore this phenomenon, I propose that Taiwanese students' considerable experience in solving geometric calculations (GC) is one of the key factors contributing to their outstanding ability to do GP. In particular, I focus on examining the geometric calculations with number (GCN)[6], a type of geometric calculations (GC) that is frequently used in Taiwanese classroom. Here a GCN is generally described as numerical calculation done in relation to mental or physical geometric diagrams on the basis of geometric principles or formulae (e.g., calculating an angle measure in a triangle given that measures of the other two angles are 30˚ and 100˚, respectively) (Aleven, Koedinger, Sinclair, & Synder, 1998; Ayres & Sweller, 1990; Chinnapan, 2000; Healy & Hoyles, 1998; Küchemann & Hoyles, 2002).

I specifically argue that working with GCN task diagrams[7] is critical to gaining geometric intuition and being able to solve GP tasks. This is because GCN task diagrams used in Taiwan are often diverse and complex, thus, providing students different *opportunities to learn*[8]*,* an affordance which is worthy of investigation. To explain how GCN task diagrams can be complex, the TIMSS video study provides the following example (Stigler & Hiebert, 1999). In their investigation, Stigler and Hiebert showed that GCN tasks implemented in mathematical lessons that may influence the differences in performance of students in the U.S., Germany, and Japan. Representing the highest performance among the three countries, Japanese students were required to solve the

---

[6] GCN is one kind of geometric calculation (GC). Other kinds include geometric calculation in algebra (GCA) and geometric calculations in coordinate system (Lang & Ruane, 1981). GCA is similar GCN but the major difference between the two types of tasks is that GCA necessities the application of algebraic skills to obtain the solution whereas GCN excludes the use of algebraic skills.

[7] Not all GCN tasks provide a diagram. For these GCN tasks in which a diagram is not given, the tasks either specify the geometric shapes or describe diagram construction so that students can create diagram mentally or physically to generate a solution.

[8] In this study, *opportunity to learn* refers to a factor that contributes to students' learning outcomes (Tornroos, 2005). In particular, the study focuses on exploring the learning opportunities afforded by the mathematical tasks situated in the curricular or instructional materials that teacher and students may enact in classroom. The enactment of the tasks may expand or degrade the cognitive demand of the tasks (Stein, Grover, & Henningsen, 1996) and in turn influence students' learning outcome.

GCN tasks that were different from those given to the U.S. and German students. A group of GCN tasks with complex diagrams (see Figure 2.1) were given to the Japanese students to solve, whereas in Germany and the U.S., however, geometric tasks were usually assigned to students one by one.

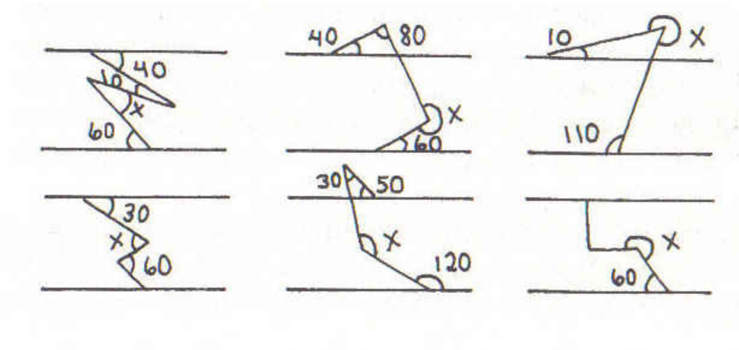


Figure 2.1 A group of GCN tasks assigned in Japanese classroom (Stigler & Hiebert, 1999, p.39)

Figure 2.1 shows the GCN tasks given in the Japanese lessons. As can be seen, these GCN diagrams are diverse and complex. Although each of the GCN task diagrams contains a pair of parallel lines, the differences in the number of segments and transversals as well as their orientations and position in the GCN diagrams make them complex and different from each other. As the complexity of the GCN task diagrams increases, so does the cognitive demand of these tasks, because most of them cannot be solved by a single geometric property or a well-known procedure (Stein, Grover, & Henningsen, 1996). To solve GCN tasks accompanying by complex diagrams, students need to visualize the sub-constructs of the diagrams and decide which relevant geometric properties can be used to generate the solutions (Zykova, 1975). In addition, such GCN tasks also require students to draw auxiliary lines to create new geometric properties to obtain solutions, which may involve using a transformational proof scheme (Hsu, 2007).

As a result, the GCN tasks provide opportunities to learn beyond practicing numerical calculations with the application of a single geometric property. The latter characteristic of GCN tasks has been recognized in the literature (Schumann & Green, 2000).

To explore what additional learning opportunities are afforded by GCN tasks with complex diagrams enacted in Taiwanese classrooms, I examine the curriculum materials used in an 8th grade mathematics class. Particularly, the materials include both textbooks and auxiliary curricular/instructional materials that Taiwanese mathematics teachers may use in their instruction. Analysis of both textbooks and auxiliary curricular/instructional materials is important because each contains GCN tasks for students to enact and thus influences the learning outcomes (Stein & Smith, 1998). Accordingly, I selected a Taiwanese mathematics teacher, Nancy[9], and examined GCN tasks situated in the curricular/instructional materials that she used and implemented in her classroom.

Taiwan has a very strong examination culture, which is another reason to analyze both textbooks and auxiliary curricular/instructional materials such as tests. Because of this examination culture, Taiwanese instruction at the middle school level can be described as examination-driven teaching style with instruction generally aimed at helping students get good grades on the high school entrance examination. In this regard, practice on tests plays an important role in helping students obtain good scores because the tests usually consist of diverse and demanding tasks. Given their contribution, the auxiliary curricular/instructional materials (e.g., tests) should be included in the analysis of the GCN tasks used in Nancy's class.

---

[9] Nancy is a pseudonym.

To understand the nature and contribution of GCN tasks, especially the contribution to the learning of GP, this study investigates two research questions.

1. *What opportunities are provided by the GCN tasks used by a Taiwanese mathematics teacher, Nancy, for her students to learn to handle complex geometric diagrams and solve these complex GCN tasks?*
2. *To what extent and in what ways do the learning opportunities afforded by the GCN tasks used in Nancy's class differ according to curricular/instructional sources?*

Literature Review

Diagram is a type of representation involving "a data structure in which information is indexed by two-dimensional location" (Larkin & Simon, 1987, p. 68). A diagram in geometry is more than a representation conveying spatial information by its appearance of data structure. In a way, a geometric diagram can be viewed as an abstract object that possesses, mentally or externally, conceptual (theoretical) and figural (spatio-graphical) properties simultaneously (Fischbein, 1993; Laborde, 2005).

The abstract nature of a geometric diagram can be more deeply understood in terms of the notion of *apprehension*, which is used to describe the cognitive work of "several ways of looking at a drawing or a visual stimulus array" (Duval, 1995, p. 143). Duval further used the term, *discursive apprehension*, to refer to the reasoning of geometric properties that are embedded in a diagram as a result of the givens. Duval introduced the more specific concept of *operative apprehension* to describe the various mental or physical operations on the diagram to obtain an intuitive sense of solutions. These operations typically include such activities as dividing the whole diagram into parts of shapes, regrouping parts to create another diagram, enlarging or shrinking the given diagrams, changing the position of the diagrams, and inserting auxiliary lines. *Operative apprehension* is similar to the concept of *transformational observation* (Harel & Sowder, 1998), which involves "operations on the objects and anticipations of operations' results" (p. 258) as well as the *generative* mode of interaction (Herbst, 2004). The *generative* mode of interaction is used to describe how diagrams can be manipulated to create new referents (e.g., drawing something on the diagrams) and new signs (e.g., new labels) to make reasonable conjectures. Because of the abstract nature of geometric diagram, these manipulations involve transformation observations and are often necessary to solve GCN tasks. For instance, transformation observations are particularly relevant when auxiliary lines need to be drawn on diagrams to create new angles and segments to obtain solutions (Hsu, 2007).

To solve geometric tasks students need to identify relevant diagrammatic information. This involves tapping into both figural and conceptual properties of diagrams, which Fischbein and Nachlieli (1998) argued should emerge together when students learn geometric reasoning. On the one hand, figural properties are important for students to determine the diagrammatic information to obtain intuitive senses of a solution. On the other hand, students also need to rely on conceptual properties to validate each reasoning step related to the diagram. Regarding the cognitive work related to the transitions between the two diagrammatic properties, Duval (1998) used the term "*anchorage change*" to elaborate this transition. *Anchorage change* refers to the transitions from visual (figural) to discursive (conceptual) perspective and vice versa. To elaborate anchorage change associated with the visual and discursive perspectives, the following geometric task is useful. "In a parallelogram ABCD, if the measures of ∠BAD are 30˚, find the measures of ∠ABC”. Solving the task requires students to observe the visual (figural) perspective, gestalt organization of the diagram, either as a two-dimension object (e.g., the appearance of the parallelogram as a roof) or as one-dimension object (e.g., the sides constituted the shape). Given these observations, students need to change their *anchorage* from the gestalt organization of the diagram, visual perspective, to the specific geometric properties embedded in the gestalt shape, discursive perspective, (e.g., opposite sides of a parallelogram are parallel). Alternatively, instead of working from a gestalt observation, students may start from the theoretical properties related to a parallelogram and then visualize how these theoretical properties are embedded in the diagram sub-constructs. Thus, the anchorage changes back and forth between the visual perspective on gestalt organization of the diagram and discursive perspective on the embedded theoretical properties provide students geometric intuition, and in turn enable them to envisage a path to generate a solution plan in the diagram.

Given the movement between the visual perspective and the discursive perspective, properties of the diagram, generating a geometric solution can proceed smoothly on the basis of diagram information, but not as a result of written information (Larkin & Simon, 1987). This is because a diagram is organized in terms of locations that can lead to information needed for the next inferring step, which is often in an adjacent location in the diagram. However, inferences for the next step in an adjacent location cannot easily be made when reasoning on the basis of written statements. If reasoning rests on written statements, students first have to apply the sentential information to the diagram in order to understand what these written sentences mean and then integrate both sentential information and diagrammatic information to determine the next reasoning step in the diagram. This reasoning process usually results in a heavy cognitive load because of the split-attention effect (Mousavi, Low, & Sweller, 1995). This effect arises when students must divide their attention among different information (both the diagrammatic information and sentence information in working memory) when inferring the solutions. The resulting heavy cognitive load can prevent students from obtaining solutions, which could otherwise be achieved if students simply make references to the key location on the diagram.

Geometric diagrams are potentially more helpful than written statement in solving geometric tasks because diagrams can serve as the underlying structure for remembering solutions (Lovett & Anderson, 1994). In their study exploring student memory of solutions to geometric tasks, Lovett and Anderson (1994) controlled two factors: similarity of the diagram used in geometric tasks and the similarity of reasoning structure in written sentences from the givens to the proving statements. They then examined how these two factors influence student recall of geometric solutions. Lovett and Anderson found that if two geometric tasks use the same diagram, students' experience in solving the first task can facilitate their ability to obtain a solution of a subsequent task. The reason is that students can remember solution steps in the diagram and use the diagram to cue relevant reasoning steps to generate solutions in the subsequent task. However, if two geometric tasks have a similar reasoning structure based on written sentences beginning with the givens and leading to proving statements, students' experience with solving the first task can not successfully transfer to their work on the second task. Lovett and Anderson's findings, to some extent, suggest that students' experiences with solving

GCN tasks can transfer to their work on GP tasks later if both types of tasks are based on similar diagram configurations.

Further evidence supporting the conclusion that diagram similarity is central to students' problem solving can be found in Koedinger and Anderson. Koedinger and Anderson (1990) noticed that geometry experts usually parse geometric diagrams into perceptual chunks and use these perceptual chunks to cue relevant geometric knowledge. According to their investigation, the diagram configurations are used as operation schemes in the abstract planning stage as well as in remembering a complex combination of geometric properties (e.g., SAS congruence triangle postulate). Based on the operation schemes, geometry experts can focus on key steps and ignore less important ones when generating solutions.

Diagrams are even more central to geometry tasks because they can also function as artifacts that can scaffold students as they learn proofs. In this regard, Cheng and Lin (2006; 2007) reported that junior high school students' construction of proofs was greatly improved after an instructional intervention that had students read the given information and then color the property information on the diagrams. They concluded that the colored parts of the diagram facilitate students' ability to visualize useful geometric properties and then use these properties to generate proof solutions.

Finally, diagrams are also the key to developing geometry intuition. According to Fujita, Jones, and Yamamoto (2004), the development of student intuition can be achieved by creating and manipulating geometric diagrams mentally or physically. Such manipulations of geometric diagrams can direct students to visualize relevant geometric properties and relative images and relate them to geometric concepts and theorems.

Manipulations of geometric diagrams also help students decide where and how to start solving a problem. As a result, Fujita et al. concluded that using a series of well-designed tasks with diverse diagrams can nurture students' geometry intuition.

Methods

The methods contain five sections. The first section offers the operative definitions of terms relevant to the framework development and the coding procedure. The second section describes the framework with its (sub) categories specific to geometric diagrams for capturing diverse learning opportunities afforded by GCN tasks. The following two sections further detail the data and the procedure in analyzing the data. The final section demonstrates the process of establishing the reliability of the (sub) categories in the framework.

Operative Definitions of the Terms Used

A Task

Tasks identified in this study include both worked examples in the instructional blocks used to scaffold students understanding mathematics and facilitating skill acquisition (Renkl, 2002) and problems in the exercise blocks planned to be solved by students. Considering that the lengths and requirements of tasks situated in curricular/instructional materials may vary, thus causing the inconsistency of coding, this study defines tasks as problems asking for an answer (Charalambous, Delaney, Hsu, & Mesa, 2010; Zhu & Fan, 2006). For instance, if a GCN problem listed in curricular materials requires of inferring three different unknown measures, then the problem is coded as three tasks.

A Geometric Property

Geometric properties refer to those geometric statements or definitions formally introduced in the textbooks. In particular, this study defines a geometric property as that which supports a problem-solving step to visualize a specific location or a relation in a diagram configuration (e.g., two congruent segments in an isosceles triangle) so that a solution can be obtained. Thus, a geometric statement described in the textbook may contain more than one geometric property. Further clarification of a geometric property in regard to a geometric statement listed in textbooks is provided in Appendix 2.1.

Solution Steps

A reasoning step is defined as a calculating step set up by applying a geometric property. Recognizing that a GCN task usually can be solved by multiple solutions, which may require different reasoning steps, the coding will cause inconsistency. To solve this problem, this study defines the solution identified in a GCN task as the one that requires the minimum number of reasoning steps to obtain the answer. Also important here is that each reasoning step in the solution identified should be supported by a geometric property that students have learned.

Reference Diagram

A reference diagram is defined as the geometric diagram accompanying a geometric property that is formally introduced in textbooks. With the idea of treating a diagram as an external representation (Laborde, 2005), this definition is important because it can prevent inconsistencies in coding as it does not consider differences in the mental images of a diagram concept processed by individuals. For example, Figure 2.2 provides different images of a parallelogram that one may possess.

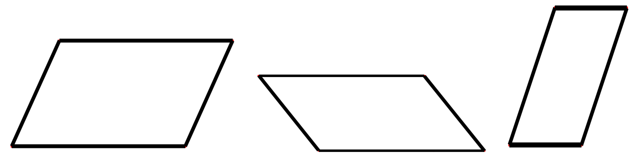


Figure 2.2 Different mental images of a parallelogram

The term "*reference*" is borrowed from the notion of *reference example*s (Michener, 1978), conveying the special characteristics that the diagram accompanying a geometric property has. Michener elaborates *reference examples* as

> "…examples that one refers to over and over again. They are basic, widely applicable and provide a common point of contact through which many results and concepts are linked together" (p. 366)

According to Michener, an example of a reference diagram can be an isosceles triangle in Figure 2.3 that not only conveys the definition of a shape with two congruent segments, but is also linked to other geometric properties (e.g., the sum of interior angles of a triangle is 180°).

Moreover, a reference diagram also possesses salient features, which can help distinguish other relevant geometric properties. For example, the reference diagram for an isosceles triangle (see the diagram on left side in Figure 2.3) shows that the lengths of two congruent legs stand symmetrically on the two sides with the base side on the bottom parallel to the horizontal axis. Thus, the congruence of the segments in the isosceles triangle is visible and easy to be perceived. The reference diagram for an isosceles triangle also reveals the difference in lengths between the congruent segments and the bottom segment so that there can be no confusion with the reference diagram for an equilateral triangle (the diagram on the right side in Figure 2.3).

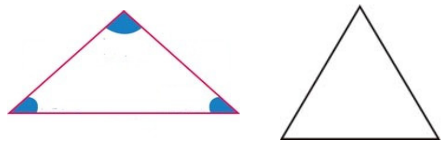


Figure 2.3 Reference diagrams for the isosceles triangle definition (on the left side) and for the equilateral triangle definition (on the right side) (Nan-I, vol. 4, p. 39)

Framework Development

This study developed an innovative framework because existing frameworks employed in textbook analysis studies associated with geometry are often too general to depict the specific characteristics of diagrams and to capture how diagrams influence opportunities for students to learn geometry (Fujita, Jones, & Kunimune, 2009; Haggarty & Pepin, 2002; Schmidt, McKnight, Valverde, Houang, & Wiley, 1996; Valverde, Bianchi, Wolfe, Schmidt, & Houang, 2002). The framework developed contains *diagram complexity* and *problem-solving complexity* with (sub) categories. The *diagram complexity* aims to describe the characteristics of a complex GCN task diagram that in turn may prevent students from generating a solution because of the visual obstacles. The *problem-solving complexity* focuses on exploring the kinds of cognitive work associated with geometric diagrams that students have to do when solving a GCN task. Details of developing the framework with corresponding (sub) categories are described as follows.

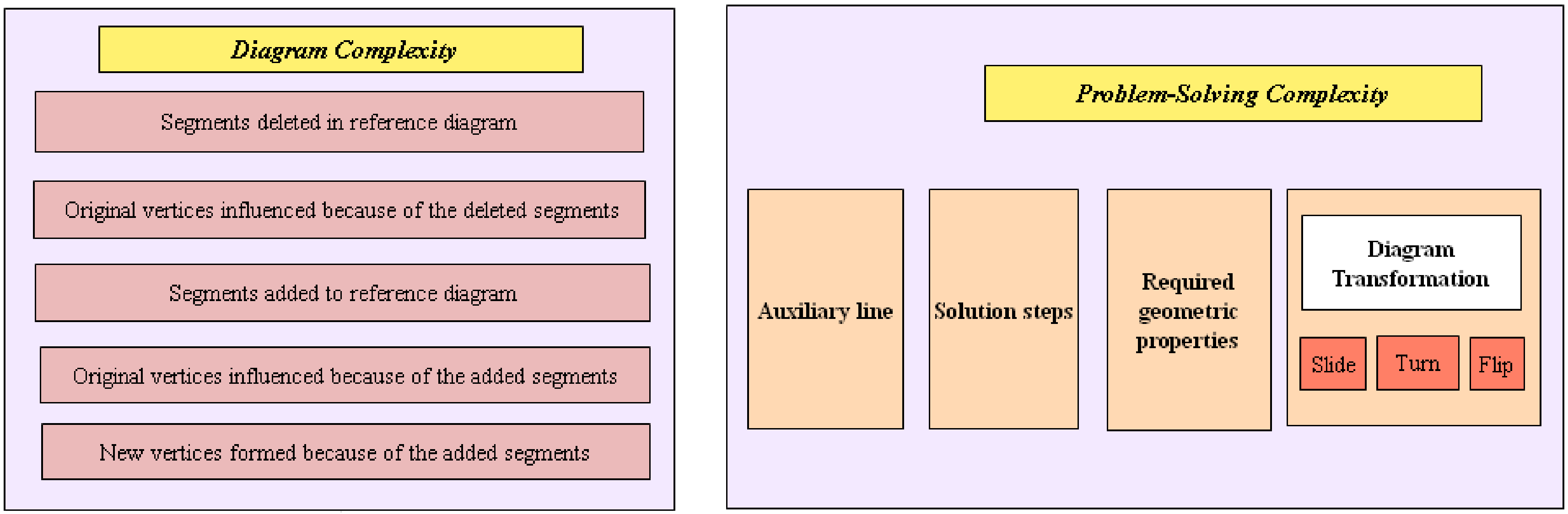


Figure 2.4 The developed framework with (sub) categories

Diagram complexity

The initial idea of developing *diagram complexity* with corresponding categories was to use segments and vertices, essential elements in constituting a diagram, to describe how a diagram given in GCN task can be complicated. In particular, this study illustrates the *diagram complexity* representing a GCN task as changes of the vertices and segments by comparing the GCN task diagram with a reference diagram[10]. Analyzing GCN task diagrams through a comparison to a reference diagram can provide information regarding possible visual obstacles that students may encounter when identifying the reference diagram along with its corresponding geometric property in the given GCN task diagram.

Accordingly, by applying the grounded-theory approach (Strauss & Corbin, 1998), this study developed five categories[11] to describe the complexity of a diagram given in a GCN task (see Figure 2.4) including: (1) the number of deleted segments in reference diagram; (2) the number of original vertices influenced because of the deleted segments; (3) the number of segments added to reference diagram; (4) the number of original vertices influenced because of the added segments; and (5) the number of new vertices created because of the segments added.

To clarify the analysis used to categorize GCN task diagrams with respect to the five categories, a comparison of a GCN task diagram with a reference diagram embedded in the diagram is illustrated here.

---

[10] The reference diagram used for the comparison with a GCN task is the one that represents a particular geometric property necessary to obtain the selected solution to the GCN task. Detail of the criteria for determining the reference diagram for a GCN task is described in the coding procedure section.
[11] The framework does not consider changes to diagram positions and those to angle or segment measures as another category in *diagram complexity* because these changes does not alter the numbers of vertices or segments in the original diagram.

Table 2.1 GCN task diagram and a reference diagram embedded in the diagram

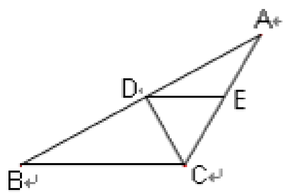
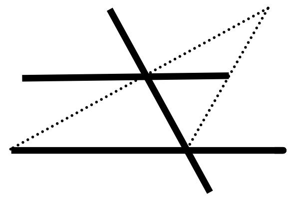
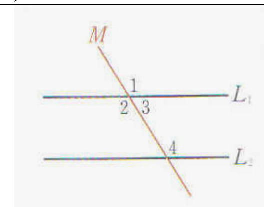
GCN task diagram	Reference diagram embedded in GCN task diagram	Reference diagram for the alternate interior angles property (Nan-I, vol. 4, p. 156)
		

Table 2.1 provides a GCN task diagram (on the left side) and a reference diagram representing the alternate interior angles property (on the right side), as well as how the reference diagram is embedded in the GCN task diagram (the middle diagram). An initial look at the GCN task diagram and the compared reference diagram gives the impression that the two diagrams do not look the same. Nor are the shapes of reference diagram and that of the corresponding sub-construct embedded in the GCN task diagram. Some segments in the reference diagram are absent and some have been added to the GCN task diagram. The following is the analysis that details the differences in the two diagrams in terms of the five categories.

Table 2.2 Changes of reference diagram by deleting segments

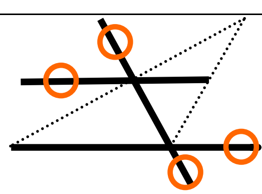
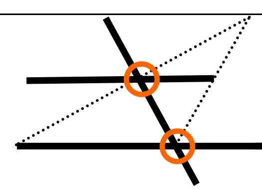
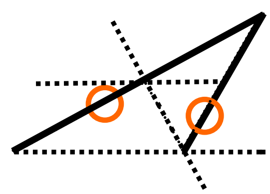
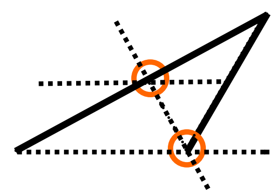
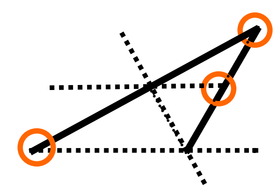
Category One: Number of deleted segments in the reference diagram	Category Two: Number of original vertices influenced because of the deleted segments
	

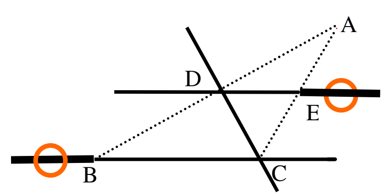
Table 2.2 shows the changes to the reference diagram made by segment deletion. Category One counts the number of deleted segments, showing that a total of 4 segments (labeled with circles) is deleted from the reference diagram[12] (diagram on the left side of Table 2.2). Category Two, which analyzes the number of original vertices in the reference diagram influenced because of the deleted segments, indicates two original vertices in the reference diagram (labeled with circles) are influenced (diagram on the right side of Table 2.2).

Table 2.3 The addition of segments and its influence on vertices

Category Three: Number of new segments added to the reference diagram	Category Four: Number of original vertices influenced because of the added segments	Category Five: Number of new vertices formed because of the added segments
		

The remaining three categories used to describe *diagram complexity* regarding how a reference diagram is made more complex by segment addition, which in turn influences the original vertices or create new vertices on the reference diagram. Category Three takes into account the number of segments that are added to a reference diagram.

[12] The below diagram shows another two segments (labeled with circles) which are also absent in the GCN task diagram. However, these two segments are not be coded as deleted segments because they are segment extension and do not influence the original vertices in the reference diagram (point D and point C).



As shown on the left side of Table 2.3, two segments (the thicker black lines) are added. Categories Four and Category Five aim to describe the changes to the vertices caused by the segment addition. Category Four identifies the number of original vertices influenced by the segment addition, which in this example are 2 vertices in the reference diagram. Category Five captures how many new vertices are created because of the segment addition. The GCN task diagram is coded as having three new vertices (designated by the circles) (see the diagram on the right hand column in Table 2.3).

Because of these changes in the segments and vertices, the GCN task diagram looks different from the compared reference diagram. These Changes to the segments and vertices also yield a variety of sub-constructs of geometric shapes (see Appendix 2.2), most of which are absent from the original reference diagram. The variety of geometric shapes in the GCN task diagram provides task designers an opportunity to create diverse GCN tasks, allowing students to practice identifying different geometric shapes and different combinations of geometric properties embedded in the given diagram in order to obtain solutions.

Overall, the above analysis shows that the combination of the five categories can comprehensively capture the ways in which a diagram is made more complex in a GCN task, using the reference diagram representing one of geometric properties embedded in the diagram as a starting point.

The following section further discusses the cognitive complexity underlying the GCN task diagram in Table 2.1 and the related increase in cognitive demand on students when working on the diagram. As Category One reports that four segments are deleted in the reference diagram when GCN task diagram (Table 2.2), the segment deletion occludes the reference diagram embedded in the diagram of GCN task because the reference diagram does not resemble its original shape. If students recognize the diagram only based on its appearance of shape, which Duval (1995) terms perceptual apprehension, they will not be able to identify or they will misinterpret which part of the configurations of the GCN task diagram represents the reference diagram. Nor can they figure out the geometric properties that correspond to the reference diagram in the GCN task diagram.

Another challenge for students is that, as discussed earlier, the segment deletion also influences the original vertices of reference diagram; thus, these constituted angles in the original reference diagram are also changed. Detail is illustrated as follows.

Table 2.4 Reference diagram and its changes in GCN task diagram after segment deletion

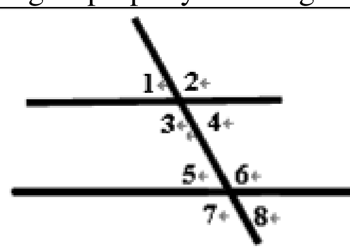
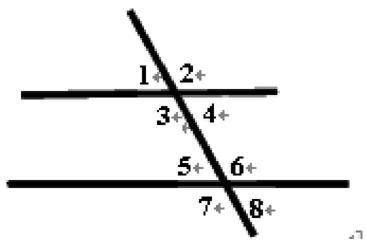
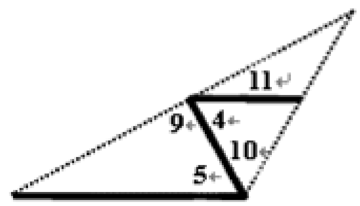
Reference Diagram for the alternate interior angles property with angle labels	The reference diagram embedded in part of the GCN task diagram configurations
	

Table 2.4 shows cognitive complexity of the GCN task diagram when compared to the reference diagram after segment deletion. The reference diagram is constituted by a pair of parallel lines and a transversal with eight angles, two of which (∠3 and ∠6, ∠4 and ∠5) represent the alternate interior angles property. The right side of Table 2.4 shows what happens to the reference diagram after segment deletion. As can be seen, only two of the original eight angles are retained in the GCN task diagram, namely ∠4 and ∠5.

The change in angles increases the cognitive demand of inferring the alternate interior angles property because students have to recognize where embedded in the given diagram are the parallel lines and the transversal as well as angles corresponding to this property.

Table 2.5 The new angles created because of the two added segments

Reference Diagram for the alternate interior angles property with angle labels	Two segments added to the reference diagram
	

In addition, when the segments are added to the reference diagram, as shown on the right of Table 2.5, the angle formations are even more complex. Specifically, the three angles ($\angle$ 9, $\angle$10, and $\angle$11), which do not exist in the reference diagram, are created because of the two added segments. The perspective of this study is that these created new angles may cause difficulties for students in recognizing the alternate interior angles property for two reasons. First, $\angle$9 and $\angle$10 are situated in diagram locations similar to $\angle$3 and $\angle$6 in the reference diagram corresponding to the alternate interior angles. Hence, if students use the perceptual images of the related locations of these angles to identify the alternate interior angles property, they may conclude that $\angle$9 and $\angle$10 are congruent, which is not correct. Second, the addition of segments and the formation of $\angle$9 and $\angle$10 also occlude the inference of the congruence of $\angle$4 and $\angle$5 corresponding to the alternate interior angles property. The segment deletion and addition make the GCN task diagram look quite different from the original reference diagram.

Consequently, students have to physically or mentally decompose and recompose the given diagram (Gal & Linchevski, 2010) in order to identify the sub-construct with the alternate interior angles in the GCN task diagram.

In sum, this section compares a GCN task diagram with a reference diagram to elaborate the coding with respect to the five categories and how complicating a diagram may cause visual obstacles in recognizing the sub-constructs of the diagrams with corresponding geometric properties.

Problem-Solving Complexity

Problem-solving complexity developed in the framework consists of four categories: auxiliary lines required or not, number of solution steps, number of geometric properties required, and number of diagram transformations (slides, flips, and turns) required in a solution. The four categories are included because they represent the kinds of cognitive work essential for solving a GCN task.

Auxiliary lines

The first category included in *problem-solving complexity* captures cognitive demand of drawing auxiliary lines necessarily to obtain the solution with minimum number of reasoning steps in a GCN task. For some GCN tasks, geometric properties embedded in a GCN task diagram are not sufficient to generate the solution, thus creating cognitive work on adding lines to create new sub-constructs and new geometric properties helpful for solution generation. Pólya (1945) indicates that to construct auxiliary lines students need to recall prior knowledge and previous problem-solving experiences, which help them decide the locations of the lines in diagrams that can contribute to the solution generation. Yerushalmy and Chazan (1990) further specify that constructing the new lines facilitates students to "generate new insights into the diagram, both questions and conjectures" (p. 213). However, the work of drawing auxiliary lines on a diagram usually is highly demanding because it forces students to anticipate the creation of sub-constructs associated with corresponding geometric properties that can be used to form a solution plan. This anticipation further requires students to view the diagram dynamically and apply the *transformational observation* (Harel & Sowder, 1998) to visualize a solution generated with the help of such auxiliary lines (Hsu, 2007).

Solution steps

The second category contained in *problem-solving complexity* involves analyzing the solution steps, i.e. the minimum number of reasoning steps that are necessary to obtain the solution in a GCN task. Here, a reasoning step is a problem-solving action set up by applying a geometric property.

Previous studies have addressed number of solution steps that can influence the cognitive demand of a task because generating a multiple-step solution requires students not only to retrieve geometric properties for each reasoning step based on either the diagram or the givens, but also to chain the reasoning steps into a logic sequence (Ayres & Sweller, 1990; Cheng & Lin, 2008; Heinze, Reiss, & Rudolph, 2005; Ufer & Heinze, 2008). Specially, inferring the intermediate reasoning steps for a multiple-step GCN task is usually more challenging for students than obtaining the answer to the task (Ayres & Sweller, 1990). This is because the intermediate steps require students to reason forward and backward between the givens and the conclusion statements in order to map out a solution path that satisfies both the givens and the conclusion statements, the cognitive work termed as "hypothetical bridge" (Cheng & Lin, 2008).

However, textbook analysis studies usually categorize solutions steps for tasks in a binary matter as single-step verses multiple-step (Li, 2000; Son & Senk, 2010; Stigler, Fuson, Ham, & Kim, 1986; Zhu & Fan, 2006). This binary classification scheme provides very limited information regarding cognitive demand of tasks situated in curricular or instructional materials because the classification could not elaborate the differences in cognitive demand for different number of solution steps. In contrast to these studies, the present framework can systematically determine the number of solution steps and use the coding result to describe the cognitive demand that a GCN task requires.

Required geometric properties

In addition to determining solution steps for a GCN task, *problem-solving complexity* also analyzes the minimum number of geometric properties required in a solution. Three considerations underline the decision to include this category in the framework. First, the number of geometric properties necessary to obtain a solution for a GCN task may not be same as that of solution steps because different reasoning steps in a solution may require the use of the same geometric property. Therefore, analyzing required geometric properties can provide richer information regarding the opportunities that a GCN task can provide students to learn. Second, the coding of the required geometric properties for GCN tasks also influences the subsequent identification of reference diagrams because these are necessarily the diagrams representing the geometric properties needed to obtain the solution with minimum reasoning steps. Third, analyzing the geometric properties are also significantly related to the examination of diagram transformations, which compare the transformation actions involved in mapping reference diagrams representing these geometric properties necessary to obtain the least-step solution onto the given GCN task diagram.

Diagram Transformation

The fourth category included in analyzing the complexity of *problem-solving process* is minimum number of diagram transformations (e.g., rotating) because it reveals the cognitive work created when students are required to map reference diagrams onto a GCN task diagram in order to retrieve geometric properties that are necessary to generate the fewest-step solution to the task. During this mapping process, students may need to mentally or physically transform the reference diagrams (e.g., rotating) to check if they resemble a part of the diagram configurations in a GCN task. This manipulation causes cognitive difficulties for students because the orientation and position of a GCN task diagram may influence the identification of corresponding reference diagrams (Fischbein & Nachlieli, 1998). In this process, three types of manipulative activities are relevant for the extant study: slides (translations), turns (rotations), and flips (reflections) (Kidder, 1976; Piaget & Inhelder, 1967; Williford, 1972). These actions leave a diagram unchanged except for its position and orientation.

Data

This study focuses on a geometry chapter with the content of properties related to parallel lines and quadrilaterals as the data boundary to collect both textbook materials and auxiliary instructional materials from the mathematics teacher, Nancy, when she taught an 8th grade class with about 40 students in a private girls' high school in Taiwan. The reason to select Nancy is that she represents a typical expert Taiwanese teacher at the middle school level because she has several attributes that are part of the cultural script

(Stigler & Hiebert, 1999) of Taiwan instruction, including the profound understanding of subject content knowledge for mathematics (Ma, 1999); good *mathematics knowledge for teaching* (MKT) (Ball, Hill, & Bass, 2005); and, the most important, examination-oriented teaching style. The high-stakes entrance examination plays a crucial role in influencing Nancy's instruction because she aims to help her students obtain high scores in the entrance examination and believes that students' success on the examination is a very important criterion for evaluating the quality of teaching.

The reason for selecting this geometry chapter is because of its particular role in facilitating students' cognitive movement from empirical explorations into formal GP. In line with the National Curriculum Standard (Ministry of Education, 2003), this geometry chapter not only aims to help students become familiar with geometric content (e.g., definition, properties), but also to gradually introduce students to the concept of GP through different instructional activities (e.g., hands-on activities). Accordingly, mathematics textbooks for this grade level contain GP that show students what a GP looks like and how to make general inferences based on geometric properties in a proof construction. However, students at this stage are not expected to construct formal proofs themselves; rather, they only complete already partly-constructed proof tasks (e.g., writing down one of the proving sentences) or to figure out the geometric properties that support a proving sentence. To achieve the curricular goal, a high proportion of GCN tasks are used as alternative materials that help students acquire the geometric knowledge and solution skills that can be used to solve GP later.

Totally four types of curricular/instructional materials were collected when Nancy taught this geometry chapter in an 8th grade class, including the textbook series, the supplemental materials, the tests, and the tasks created by Nancy that she used when teaching this geometry chapter. Details of the four types of curricular/instructional materials are provided in Appendix 2.3.

Coding Procedure

Figure 2.5 provides a flowchart showing the procedure for coding geometric tasks collected from Nancy's class.

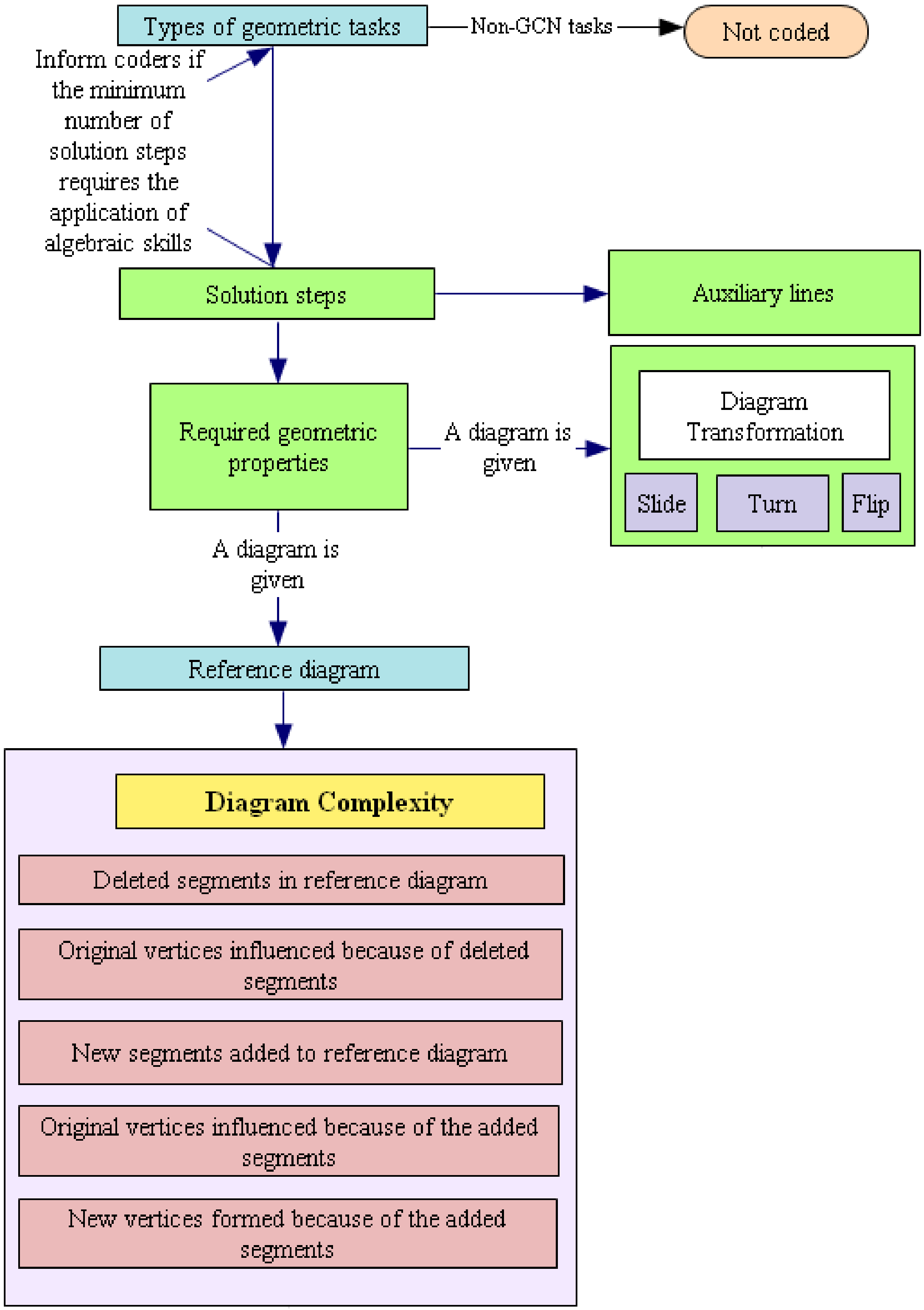


Figure 2.5 Flowchart for the coding procedure

The coding procedure started with determining whether a geometric task was a GCN task. A problem related to the determination of a GCN task was encountered when a GC task does not use any algebraic expressions in the givens but requires the application of algebraic calculations to obtain the solution with the fewest reasoning steps. To solve this problem, this study clarified a GC as a Non-GCN task if its solution requires the application of algebraic skills whether or not the givens use algebraic expressions. Further details regarding the problem and the coding decision are illustrated in Appendix 2.4.

The next coding procedure centered on the *problem-solving complexity* with its four categories (auxiliary lines required or not, minimum number of solution steps, minimum number of geometric properties required in a solution, and diagram transformations). In particular, the coding began with the determination of minimum number of solution steps, which in turn affected the next steps of analysis. Therefore, Step Two of the coding procedure examined solution steps necessary to obtain the answer requiring the fewest reasoning steps. After the number of the solution steps for a GCN task was identified, Step Three checked if the solution required the drawing of auxiliary lines. Step Four assessed the number of geometric properties used to support minimum number of reasoning steps in the identified solution. Specifically, the geometric properties were those that were formally introduced in textbooks previously. Any geometric properties that had not been learned were excluded even if these properties could be used to generate the solution to a GCN task.

Step Five of the coding procedure furthered the investigation of transformation together with its three sub-categories. Because the coding of diagram transformations necessitated the present of given diagrams in the task presentation, GCN tasks without accompanying diagrams were excluded from the analysis. The coding computed the minimum number of transformation actions that were required to map the reference diagrams representing the geometric properties identified in the solution onto the given GCN task diagram.

The final two coding steps focused on *diagram complexity* dimension. Again, as the analyses of *diagram complexity* also required the presence of given diagrams in the GCN tasks, these tasks in which diagram were not provided were excluded from the analysis. Step Six determined the reference diagram that was the basis for examining the complexity of a diagram given in a GCN task. Several steps were carried out to determine the reference diagram represented in a GCN task. First of all, the reference diagram used to analyze the complexity of a GCN task diagram must be one of the reference diagrams corresponding to the geometric properties that were necessary to obtain the minimum number of solution steps in this task. Secondly, when several reference diagrams were identified in a GCN task, reference diagram representing the geometric property that was one of the to-be-learned properties in the current teaching unit was selected as the one used to analyze the *diagram complexity* with corresponding categories. After a reference diagram was identified for a GCN diagram task, Step Seven of the coding procedure furthered to analyze the *diagram complexity* with its five coding categories based on the reference diagram identified. After coding the five categories, this study computed coding results as the minimum number of changes needed to transform a reference diagram into the diagram that accompanied the task. This minimum number was used an indicator to describe the complexity for a GCN task diagram.

Inter-Rater Reliability

Inter-rater reliability for the sets of (sub) categories in the framework was established with a coder who has master's degree in mathematics education and has teaching experience in Taiwan. To ensure the consistency of the coding results, several steps were taken. First, we selected tasks from each unit of the geometry chapter and each type of curricula/instructional material to examine if the categorizations in the framework could capture the characteristics of a GCN task in terms of its diagram and the kinds of cognitive work involving the diagram when solving the task. We deliberately selected tasks that were difficult to classify and used the coding from these tasks to validate and modify the (sub) categories.

Secondly, about 10 % of the tasks selected from each unit of the geometry chapter were coded by the author and the coder individually. In the first trial, very good inter-rater reliability[13] was achieved for the categories of auxiliary lines, and the sub-category of slide in diagram transformation category (kappa statistics (κ) =0.95 for auxiliary lines, κ=1.00 for slide) (Altman, 1991). For the coding of the five categories used to access the *diagram complexity*, the strength of inter-rater agreements was good (κ=0.73 for deleted segments in reference diagram; κ=0.70 for original vertices influenced because of deleted segments; κ=0.71 for new segments added to reference diagram; κ=0.69 for original vertices influenced because of the added segments; and κ=0.75 for new vertices formed because of the added segments). But for the rest of (sub) categories in the framework—the solution steps, required geometric properties, and rotation and flip in diagram

[13] The Cohen's κ was used to assess the inter-rater agreement. According to Altman (1991), κ=<0.20 reflects poor strength of agreement, 0.21-0.41 fair, 0.41-0.60 moderate, 0.61-0.80 good, and 0.81-1.00 indicates very good strength of agreement.

transformation category—the reliability was low. The discrepancies can be attributed to the fact that the answer to a GCN task could be obtained in different ways. Even though this study defined the solution to a GCN task as the one requiring the fewest number of reasoning steps, identifying this solution was still demanding because the fewest-step solution could be identified only when all solutions to a task could be determined. To address this difficulty, we proposed an alternative approach to ensure the coding reliability for those (sub) categories by discussing all the possible solutions to a GCN task before formally coding the task with respect to (sub) categories and calculating the inter-rater agreements. The author and the coder analyzed each task individually and then discussed together to merge our knowledge of a GCN task to determine the fewest-step solution to the task. After achieving consistency in selecting the solution for a GCN task, we re-coded the low-reliability (sub) categories separately again. Importantly, recognizing that coding of the five categories in *diagram complexity* dimension was also related to determination of the fewest-step solution to a GCN task, we also re-coded these five categories. The calculations of inter-rater reliability showed that the recoding of these (sub) categories all achieved very good strength of agreement. For the solution steps and required geometric properties, kappa statistics (κ) was 0.98 and 0.97 respectively. Regarding both the rotation and flip sub-categories, we had kappa statistics (κ) =1.00. We also achieved outstanding reliability for the coding of the five sub-categories of *diagram complexity* (κ=1 for all the five categories in *diagram complexity*).

Limitations

The major limitation of this study originates from the decision to select one Taiwanese mathematics teacher of teaching an 8^{th} grade class with a population of 40 students and collect the curricular/instructional materials from one geometry chapter. Although well-justified, selecting a Taiwanese mathematics teacher and her instruction related to a single geometry chapter limits any generalization because the study cannot capture how learning opportunities are afforded to students engaged in other geometry chapters or with other geometric teachers. As a result, the results cannot be generalized to other Taiwanese mathematics teachers or to other grade levels. Different teachers may have different viewpoints regarding what tasks from which curricular/instructional materials should be chosen for students to enact. Furthermore, considering students' naïve competencies in learning the mathematics and the pressures from students and their parents aimed at influencing teacher's choice of the curricular materials (Herbel-Eisenmann, Lubienski, & Id-Deen, 2006), it is also very possible that the same teacher may use different curricular/instructional materials for a different student population.

Another limitation of this study is the focus on middle school level. While recognizing the role of examination in influencing the curricular/instructional materials given to students to enact at this level, the use of auxiliary curricular/instructional materials at the elementary school level[14] may be very different from those found in this study. In addition, a limitation may also derive from the geometry content analyzed in this study. As each geometry topic has its special characteristics in terms of shapes, properties, and relation to calculations and proofs, it is possible that the analysis of GCN tasks associated with other geometry topics may generate results different from those in this study.

[14] For high school level, because students also need to take the university entrance examination, the use of auxiliary curricular/instructional materials may be similar to those used in middle school level. However, the use of curricular/instructional materials implemented in high school classes requires further investigation.

Findings

GCN Tasks Used in Nancy's Class

A total of 1084 geometric tasks were identified in the curricular/instructional materials as available for use by Nancy when she taught the geometry material that is the focus of this study. Among these geometric tasks, about one-half (529 tasks) were categorized as GCN tasks and the remaining half were non-GCN tasks (e.g., GCA, construction, exploration, or GP). In general, given that this dissertation focuses on GCN tasks, the analyses presented here focused on those 529 tasks. More specifically, some analyses (e.g., *diagram complexity* and diagram transformations) required the presence of given diagram in the task presentation, so these analyses found on the 423 of the 529 GCN tasks in which a diagram was provided[15]. First, I report the results regarding *diagram complexity*, after which I detail the analyses regarding *problem-solving complexity*.

Diagram Complexity

The analysis of *diagram complexity* was based on the 423 tasks that included a given diagram. *Diagram complexity* was determined by the minimum number of changes needed to transform a reference diagram into the diagram that accompanied the task. The number of changes (i.e., segment addition or segment deletion) ranged from 0-22. Table 2.6 gives the observed frequencies for the changes. A very high percentage (81.7%) of GCN task diagrams used in this class involved at least one change from a reference

[15] For these GCN tasks in which a diagram is not provided, their written information either indicates the geometric shape or elaborate the diagram constructions for the tasks. In this regard, students require to mentally or physically construct the geometric shapes themselves in order to set up the calculating steps to obtain solutions.

diagram[16]. In general, these changes to the reference diagram influenced the original vertices or created new vertices. As can be seen in Table 2.6, about 55% of the GCN tasks involved diagrams with at least 6 changes from a reference diagram.

Table 2.6 Distribution of frequency and percentage of diagram complexity

Numbers of changes	Frequency	Percent (%)
0	80	18.3
1-5	117	26.8
6-10	178	40.8
11 through the highest (22)	61	14.1
Total	436	100

A total of 356 GCN tasks had diagrams that involved at least one change from a reference diagram. For these diagrams, I identified five categories of changes from a reference diagram and tabulated the frequency of each type of change. Table 2.7 summarizes the means and standard deviations for each category of change. Several observations can be made from the data reported in the table. First, the GCN task diagrams were more likely to be obtained by adding segments to reference diagrams (mean value=2.04) than by deleting original segments from the diagrams (mean value=1.04). Second, segment additions to a reference diagram were more likely to create new vertices (mean value=2.19) rather than to influence the original vertices (mean value=1.17) of the diagram. Third, the number of original vertices in a reference diagram influenced by segment deletion was less than those affected by segment addition. In addition, the analysis shows that, on average, GCN task diagram had about 7 changes (mean value=7.38) involving segments and vertices.

[16] Though the diagrams used in about 18% of the GCN tasks did not involve changing a reference diagram by adding or deleting segments, these diagrams were sometimes different from the corresponding reference diagrams because of the changes in diagram positions or measures of angles and segments.

Table 2.7 Descriptive analysis of categories in diagram complexity dimension per diagram

Category	Mean	Std. D
Deleted segments in reference diagram	1.04	1.390
Vertices influenced because of the deleted segments	.95	1.386
New segments added to reference diagram	2.04	1.438
New vertices formed because of the added segments	2.19	1.699
Original vertices influenced because of the added segments	1.17	1.243
Sum of the five categories for a GCN task diagram	7.38	4.438

Problem-Solving Complexity

In this section, I present the analysis of complexity of *problem-solving complexity* of the GCN tasks. This analysis consisted of four components: whether auxiliary lines required or not, the minimum number of solution steps, the minimum number of geometric properties required in a solution, and whether diagram transformations (slides, flips, and turns) were required or not and if required, how many frequencies. The first three components were investigated for the set of 529 GCN tasks, but the analysis of transformations used only the 436 tasks that contained diagrams. An example of a GCN task and the analysis of this task with respect to each of the four components can be found in Appendix 2.5.

Auxiliary lines

About one in four of the 529 GCN tasks required students to draw auxiliary lines in order to obtain a solution. In general, the solutions to the GCN problems used in Nancy's classroom required the drawing of at most one auxiliary line.

Solution Steps

Table 2.8 gives the distribution for the frequency and percentage of the minimum number of solution steps for the GCN tasks. The number of required solution steps ranged from 0 to 17. More than 70% of the GCN tasks required multiple solution steps with nearly half of the tasks requiring 2-4 steps. About one task in five required at least 5 solution steps.

Table 2.8 Distribution of frequency and percentage of solution steps

Number of solution steps	Frequency	Percent (%)
0	4	0.8
1	152	28.7
2-4	260	49.1
5-7	75	14.2
8 through the highest (17)	38	7.2
Total	529	100

Required geometric properties

Similar to the analysis of the minimum number of solution steps, I also analyzed the minimum number of geometric properties required for a solution of a GCN task. Table 2.9 displays the distribution for frequency and percentage of the minimum number of geometric properties for the GCN tasks. The number of required geometric properties ranged from 0 to 13. About two-thirds of the tasks required students to apply at least two geometric properties in order to obtain a solution. In addition, one can notice the similarity of frequencies in Table 2.8 and Table 2.9 which indicate that most GCN tasks situated in Nancy's class require of different geometric properties to obtain solutions.

Table 2.9 Distribution of frequency and percentage of geometric properties needed

Required geometric properties	Frequency	Percent (%)
0	4	0.8
1	172	32.5
2-4	285	53.9
5-7	47	8.9
8 through the highest (13)	21	4.0
Total	529	100

Diagram Transformations

For the 436 GCN tasks that had accompanying diagrams, I examined whether a solution required a transformation of diagram using a rigid transformation (i.e., slide, flip, or rotation). Table 2.10 gives the distribution for the frequency and percentage of required transformations for the GCN tasks. The number of required transformation actions ranged from 0 to 10. More than two-thirds of these tasks required students to perform at least one transformation action. Nearly four in ten of these GCN tasks required at least 2 transformation actions.

Table 2.10 Distribution of frequency and percentage of transformation actions required

Number of required transformation actions	Frequency	Percent
0	132	30.3
1	142	32.6
2	78	17.9
3 through the highest (10)	84	19.2
Total	436	100.0

An examination of the specific transformation actions required for the GCN tasks used by the Taiwanese teacher, Nancy, revealed that (a) none of GCN tasks required the action of sliding, because the diagrams in the GCN tasks were static and did not indicate any dynamic changes in location; and (b) the GCN task diagrams that required transformations were almost twice as likely to require a turn (mean value=0.91) than a flip (mean value=0.52).

GCN Tasks Situated in Different Curricular/Instructional Materials

Here I further discuss the GCN tasks situated in different curricular/instructional materials used by the Taiwanese teacher, Nancy. In her classroom, four types of curricular/instructional sources were used, including the textbook series, the supplemental materials, the tests, and the tasks created by her. Of the 529 GCN tasks analyzed, more than three-fourths (414 tasks) were situated in the auxiliary curricular/instructional materials that Nancy added for her students to practice. Almost one-half (256 GCN tasks) were found in the tests and roughly 29% (151 tasks) were in the supplemental materials. Very few tasks (1%) were created by Nancy.

Table 2.11 Descriptive analysis of different types of curricular/instructional materials

	Diagram Complexity			Problem-Solving Complexity										
				Auxiliary lines		Solution Steps			Required Geometric properties			Transformation		
	N	Mean	Std. D	N	Needed	N	Mean	Std. D	N	Mean	Std. D	N	Mean	Std. D
Textbook series	115	3.94	3.726	115	8 (7.0%)*	115	1.92	1.874	115	1.69	1.314	115	0.63	.755
Supplemental materials	125	5.73	4.500	151	26 (17.2%)	151	2.85	2.523	151	2.33	1.719	125	1.14	1.150
Tests	189	7.47	4.291	256	81 (31.6%)	256	4.07	3.027	256	3.19	2.047	189	2.08	1.754
Tasks created by Nancy	7	6.86	4.180	7	4 (57.1%)	7	3.14	1.215	7	2.43	1.134	7	1.18	.378
Total	436	6.03	4.438	529	119 (22.5%)	529	3.24	2.786	529	2.61	1.903	436	1.42	1.495

* Number of tasks and percentages in (parenthesis)

Table 2.11 presents a summary of the number of GCN tasks used by Nancy, the mean values, and standard deviations for different curricular/instructional materials with respect to *diagram complexity* and the four components of *problem-solving complexity* (auxiliary lines, minimum number of solution steps, minimum number of geometric properties required in a solution, and diagram transformations). Several observations can be made from the data reported in the above table. First of all, the GCN tasks in the textbook series had the lowest mean values or percentage in all of the categories analyzed. Second, GCN tasks included on the tests had the highest mean values for most of categories analyzed, except for the category of auxiliary lines. For the category of auxiliary lines, the percentage of GCN tasks on the tests ranked the second highest among the four types of curricular/instructional materials. Third, the mean values or percentage for the supplemental materials in the categories analyzed are all higher than those for the textbook series and lower than those for the tests and the tasks created by Nancy.

The statistical analyses provide further information regarding cognitive complexity of GCN tasks used in the types of curricular/instructional materials. As presented in Table 2.12 and Table 2.13, ANOVA analysis for GCN tasks with respect to *diagram complexity* and the three components of *problem-solving complexity* (minimum number of solution steps, minimum number of geometric properties required in a solution, and diagram transformations) as well as Chi-square test for auxiliary line category in *problem-solving complexity* all show the significant differences in the mean values or percentages among the three types of curricular/instructional materials[17].

Table 2.12 ANOVA analysis for *diagram complexity* and the three components of *problem-solving complexity*

ANOVA test and P values						
		Sum of Squares	df	Mean Square	F	Sig.
Diagram Complexity	Between Groups	904.318	2	452.159	25.557	.000*
	Within Groups	7554.568	427	17.692		
	Total	8458.886	429			
Solution Steps	Between Groups	398.788	2	199.394	28.031	.000*
	Within Groups	3691.825	519	7.113		
	Total	4090.613	521			
Required Geometric Properties	Between Groups	195.102	2	97.551	29.622	.000*
	Within Groups	1709.174	519	3.293		
	Total	1904.276	521			
Transformation	Between Groups	74.711	2	37.355	18.422	.000*
	Within Groups	1052.425	519	2.028		
	Total	1127.136	521			

*: A significant difference at the .05 level

Table 2.13 Chi-Square Test for auxiliary lines in *problem-solving complexity*

Chi-Square Test and p value			
	Value	df	Asymp. Sig. (2-sided)
Pearson Chi-Square	31.012a	2	.000*
N of Valid Cases	522		

a: 0 cells (.0%) have expected count less than 5. The minimum expected count is 25.34
*: A significant difference at the .05 level

[17] For the tasks created by Nancy, because of the small sample size (N=7) which can cause the difficulty in statistically comparing the means and percentages, this type of curricular/instructional material was excluded from the following analyses.

Post hoc tests for the *diagram complexity* and the three components of *problem-solving complexity* (see Table 2.14) as well as posteriori comparisons for auxiliary lines (see Table 2.15) further show that any two of the three curricular/instructional materials all achieved significant differences, except for the comparison of the supplemental materials and the textbook series in the diagram transformations. The mean values or percentages for the tests are all significantly higher than the textbook series and the supplemental materials in *diagram complexity* and components of *problem-solving complexity*. The mean values or percentages for the supplemental materials are also significantly higher than the textbook series in the coding categories, except for the diagram transformations. The statistical analysis confirms that GCN tasks used in the tests were the most cognitively complex, whereas those situated in the textbook series were the least demanding.

Table 2.14 Post hoc analyses for *diagram complexity*, solution steps, required geometric properties and diagram transformations

Dependent Variable	(I) Type of curricular/instructional materials	(J) Type of curricular/instructional materials	Mean Difference (I-J)	Std. Error	Sig.
Diagram Complexity	Tests	Textbook series	3.524	.497	.000*
	Tests	Supplemental materials	1.735	.484	.002*
	Supplemental materials	Textbook series	1.789	.543	.005*
Solution Steps	Tests	Textbook series	2.149	.299	.000*
	Tests	Supplemental materials	1.216	.274	.000*
	Supplemental materials	Textbook series	.933	.330	.019*
Required Geometric Properties	Tests	Textbook series	1.501	.204	.000*
	Tests	Supplemental materials	.856	.186	.000*
	Supplemental materials	Textbook series	.644	.225	.017*
Transformation	Tests	Textbook series	.913	.160	.000*
	Tests	Supplemental materials	.566	.146	.001*
	Supplemental materials	Textbook series	.347	.176	.144

* A significant difference at the .05 level

Table 2.15 Posteriori comparisons for auxiliary lines

(I) Type of curricular/instructional materials	(J) Type of curricular/instructional materials	Percentage Difference(I-J)	Simultaneous Confidence Interval	
			Lower Bound	Upper Bound
Tests	Textbook series	-24.6% (*)	-.338	-.154
Tests	Supplemental materials	-14.4% (*)	-.197	-.006
Supplemental materials	Textbook series	-10.2% (*)	-.247	-.040

* A significant difference at the .05 level

Summary

These analyses of the GCN tasks used in Nancy's classroom indicate that her students had frequent opportunities to encounter tasks that embodied diagram complexity and problem-solving complexity. The GCN tasks in her classroom frequently required students to use two or more geometric properties to solve a problem, and they often required multiple solution steps. A portion of GCN tasks also required the drawing of an auxiliary line or a transformation of a diagram by flipping or rotating. In addition, the vast majority of tasks were accompanied by diagrams that were complex variants of reference diagrams obtained by adding or deleting segments in the reference diagrams.

Furthermore, the analyses also show that a high portion of the GCN tasks involving diagram complexity and problem-solving complexity came from the auxiliary curricular/instructional materials, especially the tests, that Nancy added for her students to enact. These GCN tasks situated in the auxiliary curricular/instructional materials had more complex diagrams and higher problem-solving requirements than those in the textbook series. Among the three types of auxiliary curricular/instructional materials, the GCN tasks in the supplemental materials had the lowest complexity in terms of the diagram and problem-solving requirements.

Discussion

Developed Framework

This study demonstrates that the innovative framework consisting of (sub) categories can capture specific relevant characteristics of GCN tasks with respect to geometric diagrams and the cognitive work involving diagrams. No previous study has undertaken this work (Fujita, Jones, & Kunimune, 2009; Haggarty & Pepin, 2002;

Valverde, Bianchi, Wolfe, Schmidt, & Houang, 2002). In particular, since the construction of GP tasks involves geometric diagrams and the kinds of cognitive work analyzed, the proposed framework can also be applied to examinations of GP.

Another contribution of the framework is that it allows investigators to systematically and scientifically analyze the geometric tasks without the consideration of students' prior knowledge, which may influence the coding of mathematical tasks. Numerous studies have proposed frameworks for classifying cognitive levels of mathematical tasks (Doyle, 1983; Li, 2000; Stein, Smith, Henningsen, & Silver, 2000; Valverde, Bianchi, Wolfe, Schmidt, & Houang, 2002), and have used these levels to categorize the cognitive demand of mathematical tasks situated in curricular materials. However, applying these frameworks based on levels to analyze mathematical tasks often are limited in their ability to determine the cognitive demand of mathematical tasks because of students' prior knowledge and experiences. As Stein et al. (2000) pointed out factors relevant to

> "...deciding the level of challenge provided by a task is the students (their age, grade level, prior knowledge and experiences) and the norms and expectations for work in their classroom" (p. 17-18)

The proposed levels of cognitive demand of a task can be debated when considerations are made with regard to students' age, level, prior knowledge and experiences. These considerations are important, but they may cause problems in coding mathematical tasks in textbooks. For example, as Stein et al. (2000) suggest, tasks involving a well-established procedure be categorized as procedures without connections. However, in a textbook analysis, it is difficult to determine whether an established procedure is targeted at a particular group of students as well as aimed at distinguishing between a procedure without connection and procedure with connection. The difficulty arises due to the challenges in examining the extent to which tasks drew connections to the meaning of the underlying procedures or not (Charalambous, Delaney, Hsu, & Mesa, 2010). The framework proposed in this study can avoid this coding dilemma because the basis for analyses is the geometric properties and diagrams in the textbooks which are not related to students' prior knowledge or previous learning experiences.

GCN Tasks in Curricular/Instructional Materials

Tasks play a very important role in mathematics learning because they can structure the way students think about mathematics, direct students' attention to a particular perspective of mathematics, and specify ways to process the information (Doyle, 1988; Henningsen & Stein, 1997; Silver, 2009). In line with this viewpoint, the analyses in this study show that the tasks used in Nancy's class not only helped students to become familiar with certain geometric properties and numerical calculations, which were possibly treated only as a minor instructional goal, but aimed to offer opportunities to learn geometry specifically in the following areas: (1) visualizing the sub-constructs in complex diagram configurations; (2) determining if auxiliary lines are necessary to solve the tasks; (3) generating solutions that require multiple reasoning steps; (4) retrieving sets of geometric properties needed to obtain solutions; and (5) transforming corresponding reference diagrams to map them onto the GCN task diagrams. Because of these learning opportunities the GCN tasks used in Nancy's class are likely to play a particularly important role in sharpening students' geometry thinking and reasoning, especially in relation to geometry proofs.

Of particular interest are the GCN tasks that Nancy added to her instruction to supplement the tasks found in the textbook series. Though we do not know all of the reasons for this teacher's decision to supplement the textbook tasks as she did, there are some plausible motivations that are worth noting. One reason is very likely related to the examination culture in Taiwan (Lin & Tsao, 1999; Lin & Li, 2009). Nancy stated that the goal of instruction is to help students obtain high grades on the high school entrance examination. Given that she perceived the Taiwanese high school examinations as requiring diverse problem-solving knowledge and skills beyond that likely to be acquired only through the practice of the textbooks problems, it is reasonable to conclude that Nancy decided to include auxiliary instructional materials containing highly demanding GCN tasks as opportunities for her students to learn.

A closely related question is how the students in this classroom managed to learn to solve the high demand GCN tasks. Though this study did not directly investigate this issue, the findings of this study and a companion investigation (see Chapter Three of this dissertation) point to three possible explanations. The first relates to Nancy's instructional skill in sustaining students' intellectual work and in motivating the students to take on intellectually demanding work, which will be discussed in Chapter Three. The second explanation pertains to the use of the supplemental materials. Although the basis for Nancy's decision to include the supplemental materials as a major source of instructional material was not directly investigated here, the findings requiring the cognitive demand of the GCN tasks in the supplemental materials were that the cognitive demand fell between the low-demand tasks that were abundant in the textbook and the very high-demand tasks created by the teacher and those found on the tests. Thus, it seems reasonable that the teacher would include the supplemental materials tasks because these would scaffold students' learning of how to solve the tasks she created and the highly demanding tasks on the tests. The analysis of cognitive demand suggested that the supplemental materials contained both low-level—providing students opportunities to practice using geometric properties by solving simple and single-step tasks—and tasks of higher cognitive demand that could help them become accustomed to the very challenging tasks included on the classroom tests, as well as those they might be likely to encounter on the high school entrance examination.

A third explanation may be related to the philosophy of "practice make perfect" (Fwu & Wang, 2006), an ancient Chinese proverb, that sums up the rationale for asking students to work on highly demanding tasks. This belief emphasizes that the more students practice on high demanding tasks, the greater the possibility they will understand the mathematics and solve challenging tasks themselves in the future. In line with this belief is also the notion that students may not succeed in their initial attempts to solve GCN tasks. Rather, through repeated practice on those demanding tasks, Nancy expected her students to develop intuitions for finding solutions and understanding the mathematics. This expectation is widely held among other Taiwan mathematics teachers as well.

We cannot conclude from the findings of this study that solving highly demanding tasks during classroom lessons will necessarily ensure that all students in Nancy's class learn mathematics well. Nevertheless, the findings of this study does suggest possible ways that students' mathematics proficiency might be leveraged through their experiences in solving many cognitively demanding tasks, especially given evidence that

Chinese students can learn mathematics effectively by repeatedly working on mathematical tasks (Zhu & Simon, 1987). Practice on well-arranged mathematical tasks, according to Zhu and Simon, helps Chinese students internalize mathematics knowledge and skills with understanding even though classroom teachers do not lecture or teach these students. On the other hand, the findings of other research suggest that teaching mathematics using many challenging tasks may not be effective for low-ability students (e.g., Gal, Lin, & Ying, 2009). In fact, Gal et al. (2009) reported that low-achieving students in Taiwan often could not participate in classroom discussions. Thus, those students may have little chance to learn mathematics through an approach that requires students to solve frequently high cognitive demand tasks.

While cross-national comparisons have consistently reported the outstanding ability of East Asian students as demonstrated by their superb achievements in mathematics, the curriculum, especially the curriculum embodied in textbook materials, is treated as having a significant influence on learning and teaching (Charalambous, Delaney, Hsu, & Mesa, 2010; Li, Chen, & An, 2009; Son & Senk, 2010; Stevenson & Bartsch, 1992; Valverde, Bianchi, Wolfe, Schmidt, & Houang, 2002; Zhu & Fan, 2006). However, in line with the Mathematical Tasks Framework (MTF), the argument proposed is that relying on textbook analysis as predictor of students' learning outcomes may fail to identify the actual reasons for the differences in students' performances in cross-national comparisons. A reliance on textbook analyses alone is risky because it ignores how teachers adopt the textbook and include auxiliary materials for their instructional activities, especially in Asian countries where instruction is often strongly influenced by an examination culture.

REFERENCES

Aleven, V., Koedinger, K. R., Sinclair, H. C., & Synder, J. (1998). *Combatting shallow learning in a tutor for geometry problem solving*. Paper presented at the Intelligent tutoring systems: Fourth International Conference, ITS '98, Berlin.

Altman, D. (1991). *Practical statistics for medical research.* Lodon: Chapman and Hall.

Ayres, P., & Sweller, J. (1990). Locus of difficulty in multistage mathematics problems. *The American Journal of Psychology*, 103(2), 167-193.

Ball, D., Hill, H., & Bass, H. (2005). Knowing mathematics for teaching: Who knows mathematics well enough to teach third grade, and how can we decide? *American Educator*, 14-17 & 20-22 &43-46

Charalambous, C. Y., Delaney, S., Hsu, H.-Y., & Mesa, V. (2010). A comparative analysis of the addition and subtraction of fractions in textbooks from three countries. *International Journal for Mathematical Thinking and Learning*, 12(2), 117-151.

Chen, M. (2008). *8th-Grade Mathematics* (Vol. 4). Tainan, Taiwan: Nan-I.

Cheng, Y.-H., & Lin, F.-L. (2007). The effectiveness and limitation of reading and coloring strategy in learning geometry proof. In J. H. Woo, H. C. Lew, K. S. Park, & D. Y. Seo (Eds.), *Proceedings of the 31st Conference of the International Group for the Psychology of Mathematics Education* (vol. 2, pp. 113-120). Seoul, Korea.

Cheng, Y.-H., & Lin, F.-L. (2008). A study on left behind students for enhancing their competence of geometry argumentation. In O. Figueras, J. L. Lortina, S. Alatorre, T. Rojano, & A. Sepulveda (Eds.), *Proceedings of the Joint Meeting of 32nd Conference of the International Group for the Psychology of Mathematics Education (PME) and PME-NA XXX* (vol. 2, pp. 305-312). Morelia, Mexico.

Lawson, M., & Chinnapan, M. (2000). Knowledge connectedness in geometry problem solving. *Journal for Research in Mathematics Education*, 31 (1), 26-43.

Doyle, W. (1983). Academic work. *Review of Educational Research*, 53(2), 159-199.

Doyle, W. (1988). Work in mathematics classes: The context of students' thinking during instruction. *Educational Psychologist*, 23(2), 167-180.

Duval, R. (1995). Geometrical pictures: Kinds of representation and specific processings. In R. Sutherland & J. Mason (Eds.), *Exploiting mental imagery with computers in mathematics education* (pp. 142-157). Berlin: Springer.

Duval, R. (1998). Geometry from cognitive point of view. In C. Mammana & V. Villani (Eds.), *Perspectives on the teaching of geometry for 21st century* (pp. 37-52). Dordrecht: Kluwer.

Fischbein, E. (1993). The theory of figural concepts. *Educational Studies in Mathematics,* 24(2), 139-162.

Fischbein, E., & Nachlieli, T. (1998). Concepts and figures in geometrical reasoning.

Instructional Journal of Science Education, 20(10), 1193-1211.

Fujita, T., Jones, K., & Kunimune, S. (2009). The design of textbooks and their influence on students' understanding of 'proof' in lower secondary school. In F.-L. Lin, F.-J. Hsieh, G. Hanna, & M. de Villiers (Eds.), *Proceedings of the ICMI Study 19 Conference: Proof and Proving in Mathematics Education* (vol. 2, pp. 172-177). Taipei, Taiwan.

Fujita, T., Jones, K., & Yamamoto, S. (2004). *Geometrical intuition and the learning and teaching of geometry*. Paper presented at the ICMI 10, Denmark.

Fwu, B.-J., & Wang, H.-H. (2006). Practice makes perfect on the blackboard: A cultural analysis of mathematics instructional patterns in Taiwan. *ZDM-The International Journal on Mathematics Education*, 38(5), 368-375.

Gal, H., Lin, F.-L., & Ying, J.-M. (2009). Listen to the silence: The left-behind phenomenon through the lens of problematic learning situations. *International Journal of Science and Mathematics Education*, 7, 405-429.

Gal, H., & Linchevski, L. (2010). To see or not to see: Analyzing difficulties in geometry from the perspective of visual perception. *Educational Studies in Mathematics*, 74, 163-183.

Haggarty, L., & Pepin, B. (2002). An investigation of mathematics textbooks and their use in English, French and German classrooms: Who gets an opportunity to learn what? *British Educational Research Journal*, 28(4), 567-590.

Harel, G., & Sowder, L. (1998). Students' proof schemes: Results from exploratory studies. *CBMS Issues in Mathematics Education*, 7, 234-283.

Healy, L., & Hoyles, C. (1998). *Justifying and proving in school mathematics*. London: Institute of Education.

Heinze, A., Cheng, Y.-H., & Yang, K.-L. (2004). Students' performance in reasoning and proof in Taiwan and German: Results, paradoxes and open questions. *ZDM-The International Journal on Mathematics Education,* 36(5), 162-171.

Heinze, A., Reiss, K., & Rudolph, F. (2005). Mathematics achievement and interest in mathematics from a differential perspective. *ZDM-The International Journal on Mathematics Education*, 37(3), 212-220.

Henningsen, M., & Stein, M. K. (1997). Mathematical tasks and student cognition: Classroom-based factors that support and inhibit high-level mathematical thinking and reasoning. *Journal for Research in Mathematics Education*, 28(5), 524-549.

Herbel-Eisenmann, B. A., Lubienski, S. T., & Id-Deen, L. (2006). Reconsidering the study of mathematics instructional practices: The importance of curricular context in understanding local and global teacher change. *Journal of Mathematics Teacher Education*, 9, 313-345.

Herbst, P. (2004). Interactions with diagrams and the making of reasoned conjectures in geometry. *ZDM-The International Journal on Mathematics Education*, 36(5), 129-139.

Hsu, H.-Y. (2007). Geometric calculations are more than calculations. In J. H. Woo, H. C. Lew, K. S. Park, & D. Y. Seo (Eds.), *Proceedings of the 31st Conference of the International Group for the Psychology of Mathematics Education* (vol. 3, pp. 57-64). Seoul, Korea.

Kidder, F. R. (1976). Elementary and middle school children's comprehension of Euclidean transformations. *Journal for Research in Mathematics Education*, 7(1), 40-52.

Koedinger, K., & Anderson, J. (1990). Abstract planning and perceptual chunks: Elements of expertise in geometry. *Cognitive Science*, 14, 511-550.

Kuchemann, D., & Hoyles, C. (2002). The quality of students' reasons for the steps in a geometric calculations. *Proceedings of the British Society for Research into learning Mathematics,* 22(2), 43-48.

Laborde, C. (2005). The hidden role of diagrams in students' construction of meaning in geometry. In J. Kilpatrick, C. Hoyles, O. Skovsmose & P. Valero (Eds.), *Meaning in mathematics education* (vol. 37, pp. 157-179). New York: Springer.

Lang, B., & Ruane, P. (1981). Geometry in English secondary schools. *Educational Studies in Mathematics*, 12(1), 123-132.

Larkin, J., & Simon, H. (1987). Why a diagram is (sometimes) worth ten thousand words. *Cognitive Science*, 11, 65-99.

Li, Y. (2000). A comparison of problems that follow selected content presentation in American and Chinese mathematics textbooks. *Journal for Research in Mathematics Education*, 31(2), 234-241.

Li, Y., & Chen, X., & An, S. (2009). Conceptualizing and organizing content for teaching and learning in selected Chinese, Japanese and US mathematics textbooks: The case of fraction division. *ZDM-The International Journal on Mathematics Education*, 41, 809-826.

Lin, F.-L., & Tsao, L.-C. (1999). Exam math re-examined. In C. Hoyles, C. Morgan & G. Woodhouse (Eds.), *Rethinking mathematics curriculum* (pp. 228-239). London: Falmer Press.

Lin, P.-J., & Li, Y. (2009). Searching for good mathematics instruction at primary school level valued in Taiwan. *ZDM-The International Journal on Mathematics Education*, 41, 363-378.

Lovett, M., & Anderson, J. (1994). Effects of solving related proofs on memory and transfer in geometry problem solving. *Journal of Experimental Psychology: Learning, Memory, and Cognition*, 20(2), 366-378.

Ma, L. (1999). *Knowing and teaching elementary mathematics*. New Jersey: Lawrence Erlbaum.

Michener, E. (1978). Understanding understanding mathematics. *Cognitive Science*, 2, 361-383.

Ministry of Education. (2003). *Grades 1-9 curriculum guidelines*. Taipei, Taiwan: Author.

Mousavi, S., Low, R., & Sweller, J. (1995). Reducing cognitive load by mixing auditory and visual presentation modes. *Journal for Educational Psychology*, 87(2), 319-334.

Mullis, I., Martin, M. & Foy, P. (2008). *TIMSS 2007 international mathematics report*. Chestnut Hill: TIMSS & PIRLS International Study Center.

Mullis, I., Martin, M., Gonzalez, E., & Chrostowski, S. (2004). *TIMSS 2003 international mathematics report*. Chestnut Hill: TIMSS & PIRLS International Study Center.

Piaget, J., & Inhelder, B. (1967). *The child's conception of space*. New York: the Norton Library.

Polya, G. (1945). *How to solve it*. New Jersey: Princeton University Press.

Renkl, A. (2002). Worked-out examples: Instructional explanations support learning by self-explanations. *Learning and Instruction*, 12, 529-556.

Schmidt, W., McKnight, C., Valverde, G., Houang, R., & Wiley, D. (1996). *Many visions, many aims: A cross-national investigation of curricular intensions in school mathematics (vol. 1)*. Dordrecht: Kluwer.

Schumann, H., & Green, D. (2000). New protocols for solving geometric calculation problems incorporating dynamic geometry and computer algebra software. *International Journal of Mathematical Education in Science and Technology*, 31(3), 319-339.

Silver, E. (2009). Cross-national comparisons of mathematics curriculum materials: What might we learn? *ZDM-The International Journal on Mathematics Education,* 41, 827-832.

Son, J.-W., & Senk, S. (2010). How reform curricula in the USA and Korea present multiplication and division of fractions. *Educational Studies in Mathematics*, 72(4), 117-142.

Stein, M., Grover, B., & Henningsen, M. (1996). Building student capacity for mathematical thinking and reasoning: An analysis of mathematical tasks used in reform classrooms. *American Educational Research Journal, 33*, 455-488.

Stein, M., & Smith, M. (1998). Mathematical tasks as a framework for reflection: From research to practice. *Mathematics Teaching in the Middle School, 3*, 268-275.

Stein, M., Smith, M., Henningsen, M., & Silver, E. (2000). *Implementing standards-based mathematics instruction: A casebook for professional development*. New York, NY: Teachers College Press.

Stevenson, H., & Bartsch, K. (1992). *An analysis of Japanese and American textbooks in mathematics*. Ann Arbor: Center for Japanese Studies, University of Michigan.

Stigler, J., Fuson, K., Ham, M., & Kim, M. (1986). An analysis of addition and subtraction word problems in American and Soviet elementary mathematics textbook. *Cognition and Instruction, 3*(3), 153-171.

Strauss, A., & Corbin, J. (1998). *Basics of qualitative research: Techniques and procedures for developing grounded theory*. Thousand Oaks, California: SAGE

Publications.

Tornroos, J. (2005). Mathematics textbooks, opportunity to learn and student achievement. *Studies in Educational Evaluation, 31*, 315-327.

Ufer, S., & Heinze, A. (2008). *Development of geometric proof competence from grade 7 to 9: A longitudinal study*. Paper presented at the ICMI 11. Mexico.

Valverde, G., Bianchi, L., Wolfe, R., Schmidt, W., & Houang, R. (2002). *According to the Book: Using TIMSS to investigate the translation of policy into practice through the world of textbooks*. Dordrecht: Kluwer.

Williford, H. J. (1972). A study of transformational geometry instruction in the primary grades. *Journal for Research in Mathematics Education, 3*(4), 260-271.

Yerushalmy, M., & Chazan, D. (1990). Overcoming visual obstacles with the aid of the Supposer. *Educational Studies in Mathematics, 21*, 199-219.

Zhu, X., & Simon, H. A. (1987). Learning mathematics from examples and by doing. *Cognition and Instruction, 4*(3), 137-166.

Zhu, Y., & Fan, L. (2006). Focus on the representation of problem types in intended curriculum: A comparison of selected mathematics textbooks from mainland China and The United States. *International Journal of Science and Mathematics Education, 4*, 609-626.

Zykova, V. I. (1975). Operating with concepts when solving geometry problems. In J. Kilpatrick & I. Wirszup (Eds.), *The learning of mathematical concepts* (Vol. 1, pp. 93-148). Chicago: University of Chicago.

Appendix 2.1

Number of Geometric Properties in a Geometric Statement

Geometric Statement: "Diagonals of a rhombus are perpendicular and bisect each other" (Nan-I, vol. 4, p. 181).

Below provides one example of GCN task to show the number of geometric properties for the geometric statement listed above.

Table 2.16 GCN task used to analyze the number of geometric property related to the geometric statement listed above.

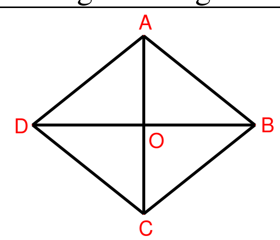
The given diagram	The written givens
	As the diagram on the left side shows, given the diagonals $\overline{AC}$ and $\overline{BD}$ of the rhombus are 6 centimeters and 8 centimeters respectively. Find the length of $\overline{AD}$.

Table 2.17 Solution for the GCN task in Table 2.16

Steps	Calculating sentences	Geometric reasons
Step 1	AO=OC=3 cm	Diagonals of a rhombus bisect each other
Step 2	DO=OB=4 cm	Diagonals of a rhombus bisect each other
Step 3	$AC \perp DB$ so that $\angle AOD=90^\circ$	Diagonals of a rhombus are perpendicular to each other
Step 4	AD= $\sqrt{3^2+4^2}$ =5 cm	The Pythagoras Theorem

To obtain the solution, each calculating step requires part of the geometric statement related to a rhombus listed above. For example, inferring the measure of AO requires the part of the geometric statement indicating that the diagonals of a rhombus bisect each other. The inference process directs students' attention to the segment AC and leads them to recognize that this segment is bisected by segment DB so that AO=OC.

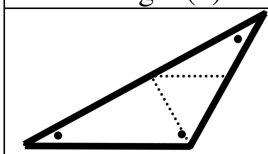
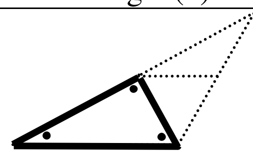
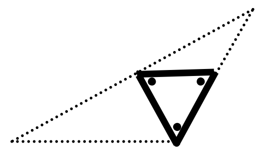
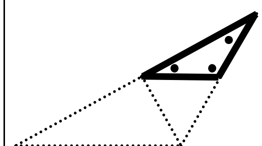
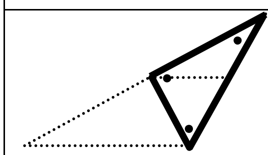
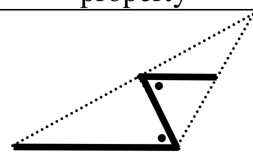
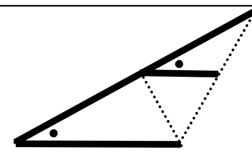
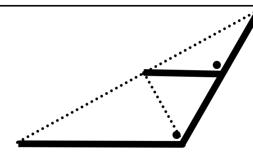
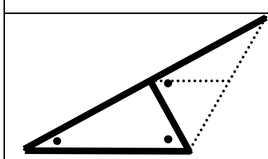
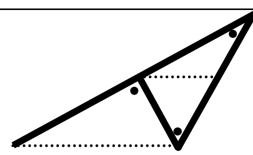
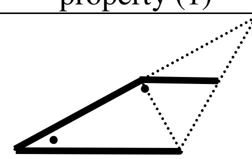
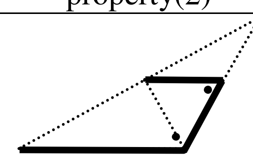
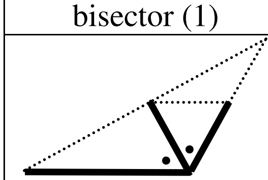
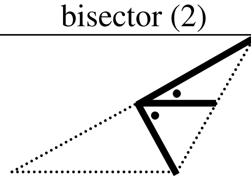
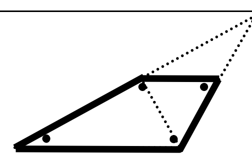
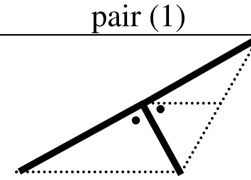
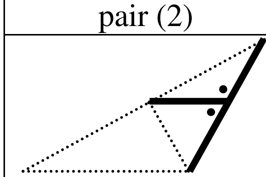
This process is a problem-action involving a specific location of the diagram (segment AC). Similarly, students have to apply the same geometric property to infer another segment DO=OB, which is treated as another reasoning step because inferring segment DO=OB involves visualizing segment DB, which is a different location in the diagram. In addition, since the givens of the GCN task indicate only that AC and DB are diagonals, the task requires students to infer that these two diagonals are perpendicular. As a result, the geometric statement related to a rhombus involves three geometric properties that can be applied to set up different calculating sentences.

Appendix 2.2

List of Sub-Constructs Embedded in the GCN Task Diagram

Table 2.18 demonstrates the sub-constructs labeled by thick bold lines with corresponding geometric properties embedded in the GCN task diagram in Table 2.1.

Table 2.18 Sub-constructs with corresponding geometric properties embedded

Triangle (1)	Triangle (2)	Triangle (3)	Triangle (4)
			
Triangle (5)	The alternate interior angles property	The corresponding angles property (1)	The corresponding angles property (2)
			
The exterior angle property (1)	The exterior angle property (2)	The consecutive interior angles property (1)	The consecutive interior angles property(2)
			
Property of angle bisector (1)	Property of angle bisector (2)	Quadrilateral	Property of linear pair (1)
			
Property of linear pair (2)			
			

Appendix 2.3

Four Types of Curricular/Instructional Materials Collected from Nancy's Class

Four different types of curricular/instructional materials were collected during Nancy's teaching of this geometry chapter. The first type is the textbook series[18] which includes the student textbook and the student workbook (Chen, 2008). The student textbook contains both instructional blocks that comprise of diverse mathematical activities (e.g., paper folding activity, diagram construction, proving, and measuring) and exercise blocks. The student workbook encloses only exercises for students to practice. A total of 55 pages in the student textbook and 14 pages in the student workbook associated with this geometry chapter were analyzed in this study.

The second type of instructional material is the supplemental materials. Supplemental materials used in Nancy's classroom contain not only the sequence of mathematical topics but also the content and exercises for students to practice. Although every student in Nancy's class had the textbook series, she rarely used the mathematical tasks in the textbook series for her instruction. Alternatively, she taught from the supplemental materials which were written by her school mathematics teachers as the major instructional materials. She taught some of tasks in the supplemental materials and asked her students to practice others. Sometimes, she also assigned a portion of the exercise tasks in the supplemental materials as student homework. The supplemental materials were available to all mathematics teachers and all 8th grade students in the

[18] The teacher guide book was excluded because it is to be used by class teachers and could not be accessed by class students.

school, but the mathematics teachers had the right to decide whether or not they wanted to use it in their instruction.

The supplemental materials written for this geometry chapter contain 18 dense pages, each of which covers either many geometric properties and definitions or many geometric tasks. For example, one of pages in supplemental materials associated with the second unit of this chapter consists of 31 tasks. To understand more of the supplemental materials, several characteristics of the supplemental materials need to be highlighted. First of all, the supplemental materials designed for this chapter are divided into the same number of units as those listed in the textbooks. Each unit in the supplemental materials presents several topics, focusing on the main geometry content (e.g., the properties related to rectangles) that should be learned in this unit. Secondly, only few of the geometric tasks used in the supplemental materials are exactly the same as those designed in the textbook series. Most of the geometric tasks in supplemental materials differ from the textbook series in terms of geometric diagrams and the description of the task settings.

Thirdly, at the beginning of each geometric topic, the supplemental materials summarize the definitions and geometric properties relevant to the topic that students have learned previously or will learn in the unit. After the summary sections, the supplemental materials include diverse geometric tasks for students to practice and sequence those tasks from low to high cognitive demand. For example, in the first unit of the chapter under study here, after a one-page summary of the geometric content, the supplemental materials present a very high percentage of one-step tasks (74%) in the first two pages. These tasks can be solved using a single geometry property. After these one-step tasks, the supplemental materials include tasks that require multiple reasoning steps to obtain solutions. Finally, the supplemental materials contain worked examples, but those examples are not fully worked out and do not provide solutions for students.

In addition to using the supplemental materials as a major source of instructional material, Nancy also used geometric tasks that she created herself. She posed those geometric tasks when she thought they were highly relevant to the class content. Twenty-five such tasks were created by Nancy during the instruction of the geometry chapter.

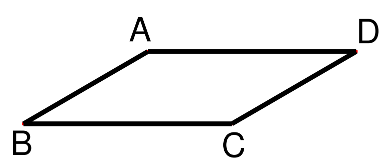
Another main type of curricular/instructional material collected from Nancy's class was the tests that were used for formative or summative purposes. Some of the tests were assigned as student homework and some were used on a weekly basis to evaluate students' learning of newly introduced geometry content. The remaining tests were employed as summative evaluations to check if students understood the mathematical content in the chapter and were able to connect the learned content with relevant mathematical topics. Usually, Nancy discussed the test tasks in classes to address the difficulties students had in solving them. A total of eighteen tests were administrated to students during the instruction of the chapter under study.

Appendix 2.4

Decision of Determining a GCN task

In this study it was observed that a group of GC tasks had givens used no algebraic expressions (e.g., 2X+1) to describe angles or segments of geometric diagrams, but did require algebraic skills to generate a solution. In dealing with situation, this study used a problem-solving perspective to categorize a GCN task. A GC that can be solved without using any algebraic skills, the task was coded as a GCN. If algebraic skills are necessary to find the unknowns, the GC is a Non-GCN task. The following is an example that illustrates this coding decision.

Table 2.19 Task Description (selected from the tests used by Nancy)

The given diagram	The written givens
	In a parallelogram ABCD. Given that two times ∠B plus three times ∠D equal ∠A, find the measure of ∠C.

As shown in Table 2.19, the written givens for this task do not use any algebraic expressions (e.g., 2X+1) except for the notations of the four angles (e.g., ∠A) of the parallelogram. A GC task is categorized based only on the description of the givens, the task is considered as a GCN. However, further examination of the solution for this task shows that algebraic skills are essential to obtain the answer.

Table 2.20 A solution for the GC task in Table 2.19

Steps	Calculating sentences	Geometric reasons
Step One	$\angle B=\angle D$ So that $\angle A=5\angle B$	Opposite angles of a parallelogram are congruent
Step Two	AD//BC[19]	Opposite sides of a parallelogram are parallel
Step Three	$\angle A+\angle B=180^\circ$	The consecutive interior angles property
	Equation 1: $\angle A=5\angle B$ *Equation 2:* $\angle A+\angle B=180^\circ$ *Answer:* $\angle A=150^\circ$; $\angle B=30^\circ$	
Step Four	$\angle C=\angle A=150^\circ$	Opposite angles of a parallelogram are congruent

As Table 2.20 shows, solving the task requires four reasoning steps, which are set up based on the corresponding geometric properties. In particular, before obtaining the measures of $\angle A$ and $\angle B$, students have to solve two equations, $\angle A=5\angle B$ and $\angle A+\angle B=180^\circ$ between the third step and the fourth step. Solving these two equations requires the application of algebraic skills, and, thus, this task should be coded as a non-GCN task.

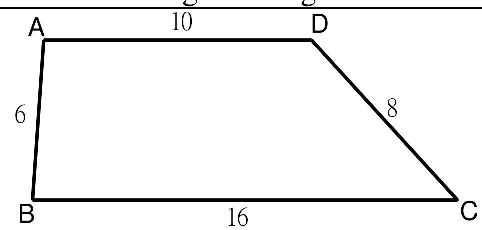
This example shows that using the information on the givens alone is not a reliable way to determine whether or not a GC task is a GCN. This is because solving a geometric task may require of algebraic work, even though the givens do not use algebraic expressions (e.g., 2X+1). Equally, it is also possible the givens in a GC task use algebraic expressions, but the solution does not necessitate the application of algebraic skills to find the solution.

[19] Here, the geometric properties used to support calculating sentences were determined on the basis of those geometric properties listed and introduced in the Nan-I textbooks analyzed. Because Nan-I textbook series do not include the geometric statement that the consecutive interior angles of a parallelogram are supplementary, it suggests students to reason this property themselves when solving or proving geometric tasks.

Appendix 2.5

An Example of GCN Task Requiring the Drawing of Auxiliary Line

Table 2.21 Description of the GCN task selected from the tests

The given diagram	The written givens
	As shown in the diagram, in a quadrilateral ABCD given that AD//BC and the length measures for AD=10, BC=16, AB=6, CD=8, and measure of ∠DCB=48°, find the measure of ∠BAD=_________.

The drawing of auxiliary line and its solution

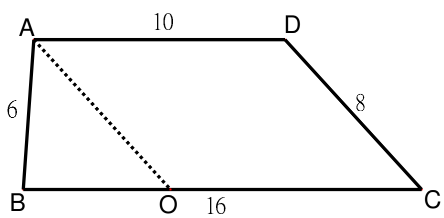


Figure 2.6 The drawing of auxiliary line AO on the given diagram

The GCN task in Table 2.21 cannot be solved unless an auxiliary line is added to the given diagram. In order to obtain solutions that involve the fewest reasoning steps, it is possible to construct the auxiliary line by creating an AO segment such that OC=10 and BO=6. The segment construction in turn creates new sub-constructs and new geometric properties that can be used to generate the solution, as shown in Table 2.22.

Table 2.22 The solution for the GCN task in Table 2.21

Steps	Calculating sentences	Geometric properties
Step One	Because AD//OC and AD=OC=10, AOCD is a parallelogram	If one pair of the opposite sides is parallel and congruent, the quadrilateral is a parallelogram.
Step Two	∠DCO=∠DAO=48˚	Opposite angles of a parallelogram are congruent
Step Three	AO//CD	The opposite sides of a parallelogram are parallel
Step Four	∠DCO=∠AOB=48˚	The corresponding angles property
Step Five	BA=BO=6 ∠AOB=∠BAO=48˚ So that ∠BAD=48˚+48˚=96˚	Properties of isosceles triangle

The drawing of auxiliary line AO creates AD=OC and leads to the inference that the quadrilateral AOCD is a parallelogram because of the property stating "if one pair of the opposite sides is parallel and congruent, the quadrilateral is a parallelogram". After recognizing that AOCD is a parallelogram, the next step is to deduce ∠DOC=∠DAO=48˚ on the basis of the property of "opposite angles of a parallelogram being congruent". The third step requires the inference that AO//CD based on the property that "the opposite sides of a parallelogram are parallel". In the fourth step, ∠AOB=48˚ can be obtained using the corresponding angles property. The final step is to apply the properties of isosceles triangles to figure out ∠BAO=48˚ and use this result to calculate ∠BAD by adding up the measures of ∠BAO and ∠DAO, which is 96˚.

Analysis of cognitive demand involving the drawing of auxiliary lines

This section tries to unpack the cognitive demand involving the generation of solutions for this task, which can be discussed from two perspectives. The first perspective involves recognizing the necessity to draw auxiliary lines because the geometric properties embedded in the diagram and properties in the given are not sufficient to generate a solution. Accordingly, these embedded geometric properties that can be inferred from the givens and the diagram include the identification of ABCD as a trapezoid and its relevant geometric properties (e.g., area formula for a trapezoid, property of trapezoid median) on the basis of the given AD//BC. The given parallel lines AD//BC also can be used to deduce that $\angle DAB+\angle ABC=180^\circ$ and $\angle ADC+\angle DCB=180^\circ$ using the consecutive interior angles property. Another geometric property that can be retrieved is the angle sum of a quadrilateral. In addition, the measures of the four sides of the trapezoid ABCD in the written givens also inform students that the trapezoid is not special (e.g., isosceles trapezoid); thus geometric properties specific to special trapezoids cannot be applied to obtain a solution in this task (e.g., base angles of an isosceles triangle are congruent).

In the application of those embedded geometric properties to generate a solution, the cognitive demand comes from recognizing the reasoning gap between the unknown and the givens that cannot be bridged by these identified geometric properties. As the given measure of $\angle DCB=48^\circ$ is located in the opposite angle to the unknown $\angle BAD$ in the trapezoid, the search is initiated for any geometric property that can be directly applied to bridge between the given $\angle DCB$ and the unknown $\angle BAD$ to obtain a solution. However, no geometric property is available to support this bridging action because the trapezoid is not special. As a result, the task creates the demand to infer the sides or angles adjacent to the given $\angle DCB$ or the unknown $\angle BAD$. One strategy is to start the reasoning from $\angle DCB=48^\circ$ and obtain the measure for the adjacent angle of $\angle ADC$, which is 132° according to the consecutive interior angles property. After $\angle ADC$ is inferred, the next step is to try to bridge between ∠ADC=132˚ and the unknown ∠BAD by searching for any geometric property that can support this inference. However, this strategy does not work because no existing geometric property in the diagram can be used for this bridging action. An alternative strategy is to reason backward from the unknown to the given. Here, the first step is to deduce ∠DAB+∠ABC=180˚ based on the consecutive interior angles property. The second step is to establish the relationship between ∠ABC and the given ∠DCB=48˚. Nevertheless, similar to the first strategy, the second strategy does not work because of the same difficulty encountered by the first strategy. As a result, the task requires students to recognize the reasoning gap between the givens and the unknown and see the need to construct auxiliary lines to create new sub-constructs and geometric properties that can help them generate a solution.

The second perspective regarding the cognitive demand involved with solving this GCN task is to determine where to draw needed auxiliary line on the diagram, which is the most difficult part of this task. This task requires reasoning what geometric properties can possibly be created by adding auxiliary line and how the created geometric properties can contribute to the generation of a solution. In particular, to solve this GCN task students have to observe the relationship among the measures of the trapezoid sides, AD+AB=BC (10+6=16), which leads to the problem-solving strategy of dividing the segment BC=16 into BO=6 and OC=10 (see Figure 2.6). The divided segments BO and OC in turn reveal the possible places where the auxiliary line can be drawn on the given diagram. As shown in Figure 2.6, an auxiliary line is constructed by connecting vertex A to the point O on segment BC so that a solution requiring the fewest number of reasoning steps can be generated.

Analysis of minimum number of solution steps, minimum number of geometric properties, and number of diagram transformation required

As shown in Table 2.22, the minimum number of solution steps required for this GCN task is five and the number of required geometric properties is also five. Regarding the transformations, Table 2.23 provides the reference diagrams corresponding to the geometric properties necessary to obtain the solution listed Table 2.21. The identification of individual reference diagrams forms the basis for analyzing what transformation actions are required to map the reference diagrams onto the GCN task diagram, which is described in Table 2.24.

Table 2.23 Reference diagrams corresponding to the geometric properties required in the solution

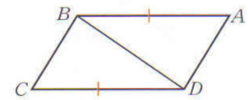
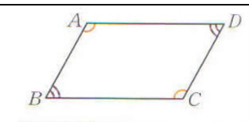
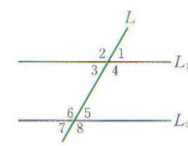
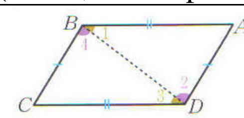
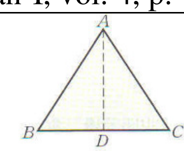
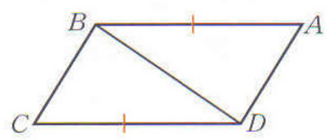
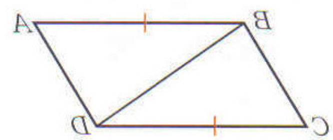
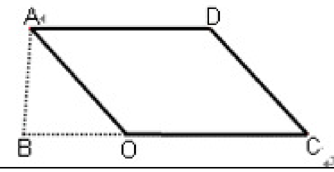
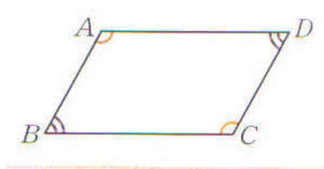
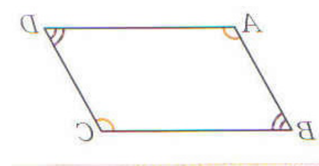
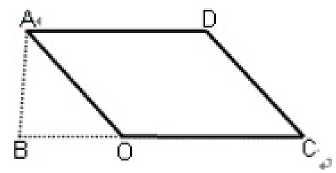
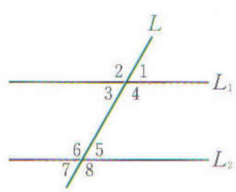
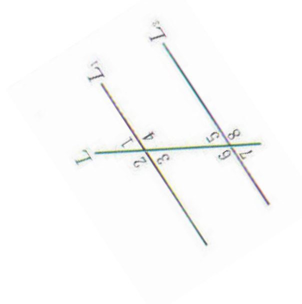
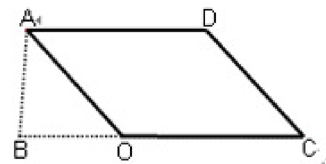
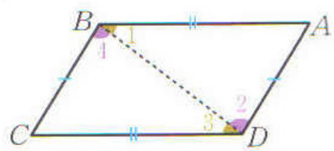
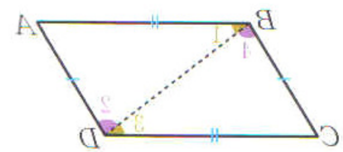
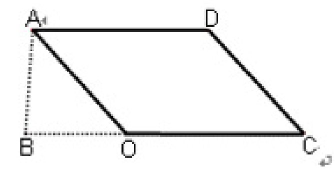
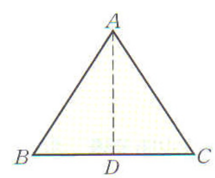
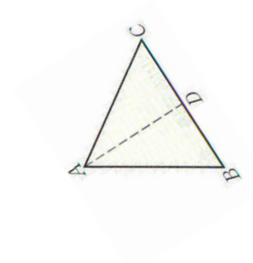
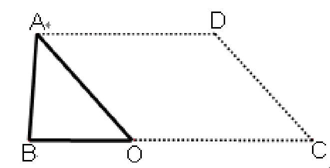
Steps	Calculating sentences	Geometric properties	Corresponding reference diagrams
Step One	AOCD is a parallelogram	If one pair of the opposite sides is parallel and congruent, the quadrilateral is a parallelogram.	 (Nan-I, vol. 4, p. 166)
Step Two	∠DCO=∠DAO=48°	Opposite angles of a parallelogram are congruent	 (Nan-I, vol. 4, p. 178)
Step Three	∠DCO=∠AOB=48°	The corresponding angles property	 (Nan-I, vol. 4. p. 154)
Step Four	AD=OC=10 So that BO=6	Opposite sides of a parallelogram are congruent.	 (Nan-I, vol. 4, p. 174)
Step Five	BA=BO=6 ∠AOB=∠BAO=48° So that ∠BAD=48°+48°=96°	If two sides of a triangle are congruent, then their corresponding angles are congruent.	 (Nan-I, vol. 4, p. 133)

Table 2.24 Transformation actions required to map each reference diagram onto part of given GCN task diagram configuration

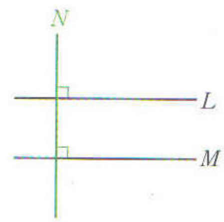
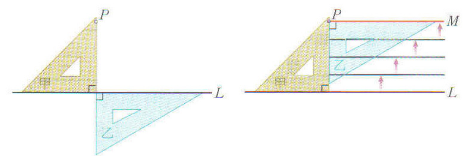
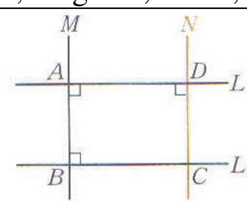
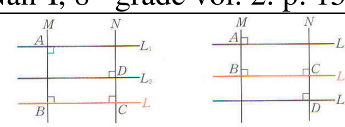
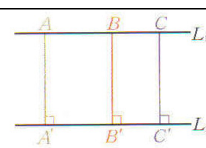
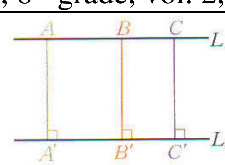
Solution steps	Transformation Actions	Reference diagrams appearing in the textbooks	Reference diagrams after transformations	Sub-constructs in the given GCN task diagram
Step One	Flip (Reflection)			
Step Two	Flip (Reflection)			
Step Three	Turn (Rotation)			

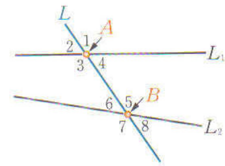
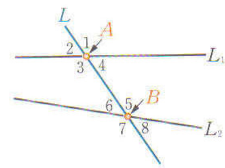
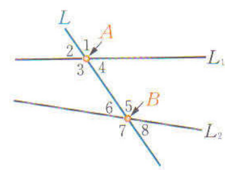
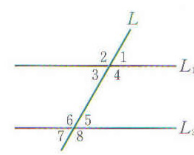
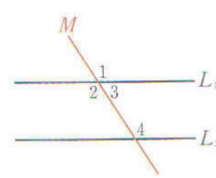
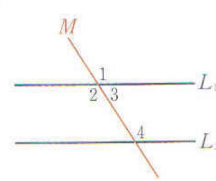
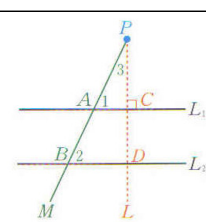
Solution steps	Transformation Actions	Reference diagrams appearing in the textbooks	Reference diagrams after transformations	Sub-constructs in the given GCN task diagram
Step Four	Flip (Reflection)			
Step Five	Turn (Rotation)			

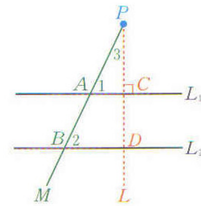
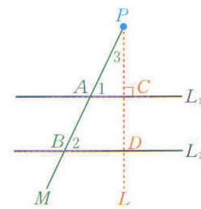
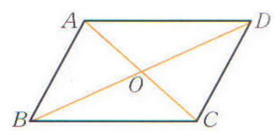
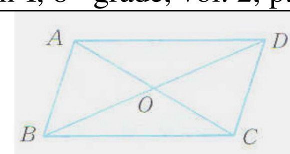
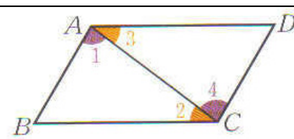
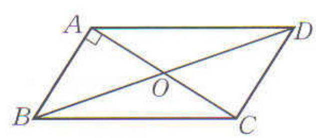
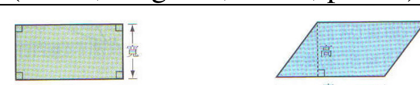
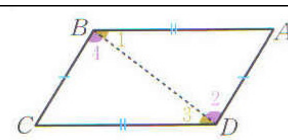
As shown in Table 2.24, a total five transformation actions were required in order to map the corresponding reference diagrams onto the sub-constructs of the GCN task diagram. For step one, step two, and step four, flipping actions (reflection) were needed. For step three and step five, turning actions (rotation) are needed to move the reference diagrams to the same orientations as they appear in the sub-constructs of the GCN task diagram.

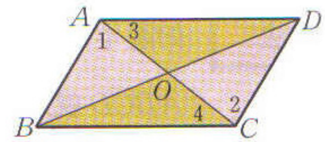
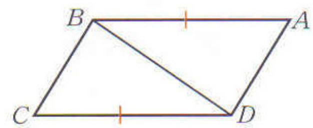
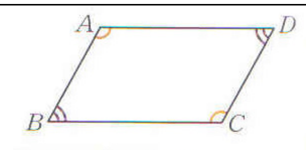
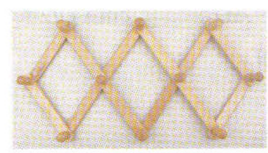
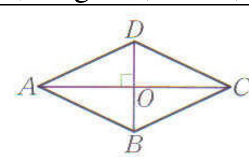
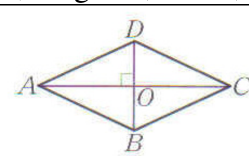
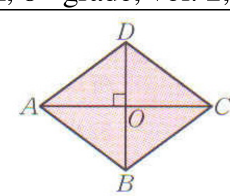
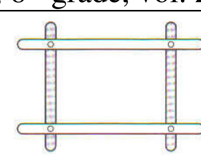
Appendix 2.6

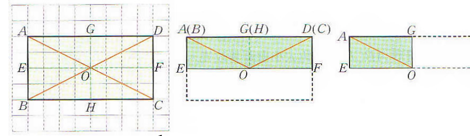
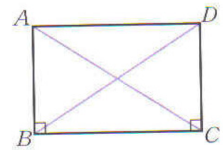
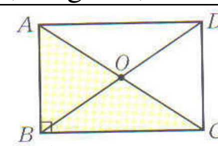
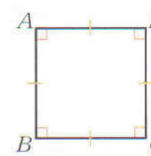
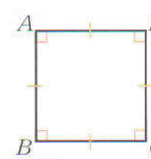
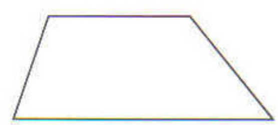
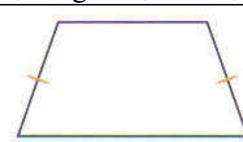
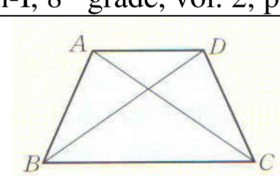
Geometric Statements with Corresponding Reference Diagrams Listed in the Analyzed Textbook Chapter

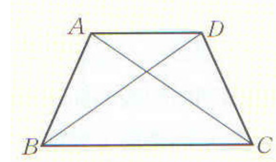
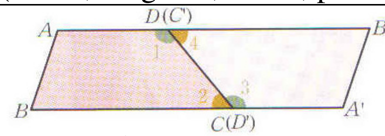
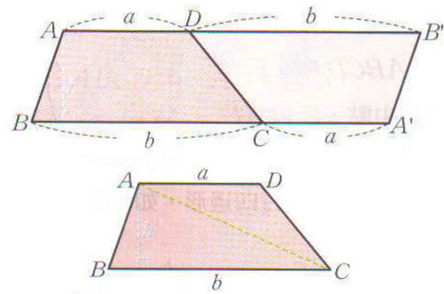
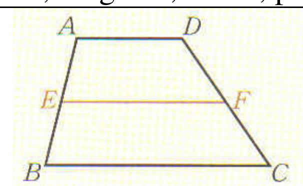
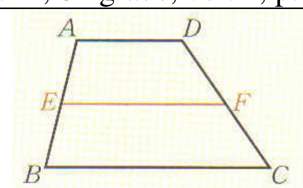
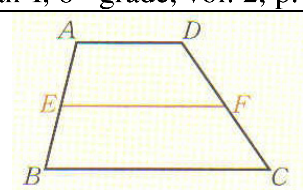
Translation	Geometric statements in Chinese	Corresponding Reference Diagrams
Definition of parallel lines: If two lines are all perpendicular to a third line, then the two lines are parallel lines.	平行線定義：若兩條直線都垂直於同一直線，則此兩直線叫做平行線。	 (Nan-I, 8th grade, vol. 2, p. 148)
There exists only one line that is parallel to line L through a given point outside line L.	通過直線外一點，有一條且只有一條直線與 L 平行。	 (Nan-I, 8th grade, vol. 2, p. 149)
If a line is perpendicular to one of two parallel lines, then this line is also perpendicular to the other line.	如果有一直線垂直於兩條平行線中的一條直線，那麼此直線也垂直於另一條平行線。	 (Nan-I, 8th grade vol. 2. p. 150)
If two lines are parallel to a third line respectively, then these two lines are parallel to each other	如果有兩條直線都分別與第三條直線平行，那麼這兩條直線也互相平行。	 (Nan-I, 8th grade, vol. 2, p. 150)
The distance between two parallel lines is constant.	兩平行線之間的距離處處相等。	 (Nan-I, 8th grade, vol. 2, p. 151)
Two parallel lines never intersect.	平行的兩條直線永不相交	 (Nan-I, 8th grade, vol. 2, p. 151)

Corresponding angles (non parallel lines)	同位角（非平行線）	 (Nan-I, 8th grade, vol. 2, p. 153)
Alternate interior angles (non parallel lines)	內錯角（非平行線）	 (Nan-I, 8th grade, vol. 2, p. 153)
Consecutive interior angles on the same side of the transversal (non parallel lines)	同側內角（非平行線）	 (Nan-I, 8th grade, vol. 2, p. 153)
Corresponding angles property: If two lines are cut by a transversal, then the corresponding angles are congruent.	同位角性質（平行線）	 (Nan-I, 8th grade, vol.2, p. 154)
Alternative interior angles property: If two parallel lines are cut by a transversal, then the alternate interior angles are congruent.	內錯角性質（平行線）	 (Nan-I, 8th grade, vol. 2, p. 156)
Consecutive interior angles property: If two parallel lines are cut by a transversal, then the interior angles on the same side of the transversal are supplementary.	同側內角互補性質（平行線）	 (Nan-I, 8th grade, vol. 2, p. 156)
If two lines are cut by a transversal and the corresponding angles are congruent, then the two lines are parallel.	兩條直線被一直線所截，如果同位角相等，那麼這兩條直線必平行。	 (Nan-I, 8th grade, vol. 2, p. 159)

If two lines are cut by a transversal and the alternative interior angles are congruent, then the two lines are parallel.	兩條直線被一直線所截，如果內錯角相等，兩條直線平行。	 (Nan-I, 8th grade, vol. 2, p. 159)
If two lines are cut by a transversal and the consecutive interior angles on the same side of the transversal are supplementary, then the two lines are parallel.	兩條直線被一直線所截，如果同側內角互補，兩條直線平行。	 (Nan-I, 8th grade, vol. 2, p. 159)
A parallelogram is a quadrilateral with both pairs of opposite sides in parallel.	兩組對邊分別平行的四邊形叫做平行四邊形	 (Nan-I, 8th grade, vol. 2, p. 166)
The opposite sides of a parallelogram are congruent; the opposite angles are congruent; and the two diagonals bisect each other.	平行四邊形的對邊相等、對角相等、兩條對角線互相平分	 (Nan-I, 8th grade, vol. 2, p. 168)
A parallelogram is divided into two congruent triangles by its diagonal.	平行四邊形的對角線把此平行四邊形分成兩個全等的三角形	 (Nan-I, 8th grade, vol. 2, p. 169)
The diagonals of a parallelogram bisect each other.	平行四邊形的對角線互相平分	 (Nan-I, 8th grade, vol. 2, p. 171)
The area of a parallelogram is equal to the base times the altitude.	平行四邊形的面積公式=底 X 高	 (Nan-I, 8th grade, vol. 2, p. 172)
If both pairs of opposite sides of a quadrilateral are congruent, then the quadrilateral is a parallelogram	平行四邊形的判斷一兩組對邊相等	 (Nan-I, 8th grade, vol. 2, p. 174)

If the diagonals of a quadrilateral bisect each other, then the quadrilateral is a parallelogram.	兩條對角線互相平分的四邊形是平行四邊形	 (Nan-I, 8^{th} grade, vol. 2, p. 174)
If one pair of opposite sides of a quadrilateral is both parallel and congruent, then the quadrilateral is a parallelogram.	平行四邊形有一組對邊平行且相等的四邊形	 (Nan-I, 8^{th} grade, vol. 2,p. 175)
If both pairs of opposite angles of a quadrilateral are congruent, then the quadrilateral is a parallelogram.	兩組對角分別相等的四邊形是平行四邊形	 (Nan-I, 8^{th} grade, vol. 2, p. 176)
Rhombus is a quadrilateral whose four sides all have the same length.	四邊都相等的四邊形叫做菱形。	 (Nan-I, 8^{th} grade, vol. 2, p. 180)
The diagonals of a rhombus are perpendicular and bisect each other.	菱形的對角線互相垂直、平分	 (Nan-I, 8^{th} grade, vol. 2, p. 181)
Rhombus is symmetric across each of its diagonals.	菱形是線對稱圖形，他的對秤軸就是兩條對角線	 (Nan-I, 8^{th} grade, vol. 2, p. 181)
The area of a rhombus is half the product of the diagonals.	菱形的面積等於兩條對角線乘積的一半	 (Nan-I, 8^{th} grade, vol. 2, p. 183)
A rectangle (oblong) is a quadrilateral where all interior angles are right angles.	四個角是直角的四邊形叫做長方形（矩形）	 (Nan-I, 8^{th} grade, vol. 2, p. 185)

A rectangle (oblong) is symmetric across lines that connect midpoints of opposite sides.	矩形是線對稱圖形，兩組對邊中點的連線就是兩條對稱軸	 (Nan-I, 8th grade, vol. 2, p. 187)
The diagonals of a rectangle (oblong) are congruent and bisect each other.	矩形的對角線互相平分且等長	 (Nan-I, 8th grade, vol. 2, p. 187)
The midpoint of the hypotenuse of a right triangle is equidistant from three polygon vertices. In other words, OA=OB=OC.	直角三角形斜邊上的中點 O 到三個頂點的距離相等，即 OA＝OB＝OC	 (Nan-I, 8th grade, vol. 2, p. 187)
A square is a polygon with four equal sides and right angles.	四邊等長、四個角都是直角的四邊形就是正方形	 (Nan-I, 8th grade, vol. 2, p. 189)
The diagonals of a square are congruent, perpendicular, and bisect each other.	正方形的對角線相互平分，垂直且等長	 (Nan-I, 8th grade, vol. 2, p. 189)
A quadrilateral with one pair of parallel sides is referred to as trapezoid.	一組對邊平行而另一組對邊不平行的四邊形叫做梯形	 (Nan-I, 8th grade, vol. 2, p. 191)
A trapezoid in which non-parallel sides are equal is called as an Isosceles Trapezoid.	兩腰等長的梯形叫做等腰梯形	 (Nan-I, 8th grade, vol. 2, p. 191)
The base angles of an isosceles trapezoid are congruent.	等腰梯形兩底角相等	 (Nan-I, 8th grade, vol. 2, p. 192)

The diagonals of an isosceles trapezoid are equidistant.	等腰梯形的兩條對角線等長	 (Nan-I, 8th grade, vol. 2, p. 192)
Two congruent isosceles trapezoids constitute a parallelogram.	兩個全等的等腰梯形可以組成一個平行四邊形	 (Nan-I, 8th grade, vol. 2, p. 192)
The area of a trapezoid is half the product of the altitude and sum of its two bases.	梯形的面積= 1/2 x (上底+下底) x 高	 (Nan-I, 8th grade, vol. 2, p. 193)
The length of the trapezoid median is the average length of the bases.	梯形的中線長等於兩底和的一半	 (Nan-I, 8th grade, vol. 2, p. 194)
The median of a trapezoid is parallel to the bases.	梯形的中線平行於兩底	 (Nan-I, 8th grade, vol. 2, p. 195)
The area of a trapezoid is the product of its median and altitude.	梯形的面積=中線長乘以高	 (Nan-I, 8th grade, vol. 2,p. 195)

CHAPTER THREE

SUNTAINING THE COGNITIVE DEMAND OF MATHEMATICAL TASKS: AN EXAMINATION OF ONE TAIWANESE TEACHER'S SEQUENCING OF GEOMETRIC CALCULATION WITH NUMBER (GCN) TASKS

Introduction

Investigations of mathematics classroom teaching across countries have provided rich information regarding the effectiveness of schooling and ways of teaching that can contribute to student learning (Hiebert et al., 2003; Stigler, Gallimore, & Hiebert, 2000; Stigler & Hiebert, 1999). Although it cannot be concluded from these studies that there is one best approach, numerous studies have explored models of exemplary mathematics teaching, especially those in East Asian countries, which are believed to be an important factor contributing to the outstanding performance of students in those countries.

One the matter of excellence in mathematics teaching, the 2009 issue of the ZDM—International Journal on Mathematics Education (Volume 41, Issue 3) highlighted characteristics of exemplary mathematics instruction in East Asian countries, including the clear flow of lessons, the emphasis on the key mathematical concepts, the use of multiple representations and solution strategies, and the emphasis on explanations and reasoning (Huang & Li, 2009; Kaur, 2009; Lin & Li, 2009). The publication also specifics another two characteristics of East Asian mathematics teachers, namely they carefully select and sequence mathematical tasks and maintain the cognitive demand of these tasks during instruction (Mok, 2009; Pang, 2009; Yang, 2009). To help clarify why Taiwanese students have consistently scored at the top in cross-national comparisons, the present study focuses on the latter two instruction characteristics and investigates what opportunities the selection of mathematical tasks set up by a Taiwanese mathematics teacher, Nancy[20], can provide students to learn and how she sustains the levels of cognitive demand when enacting the mathematical tasks with classroom students.

In fact, emphasizing the importance of selecting mathematical tasks and making connection among these tasks to facilitate students' learning in classroom is not new to mathematics education (Bell, 1993; Krainer, 1993; Lampert, 2001). According to Bell (1993), mathematical instructional tasks have significant features that can contribute to student learning. These include the connection among mathematics topics, the flexibility in adjusting the degree of challenge in a mathematical task, and extending a single task into multiple tasks by changing the elements (e.g., type of number), structure, and context of a single task. While providing general criteria of selecting and making connection among mathematical tasks, to explore what learning opportunities the tasks can provide the examination of a single type of task has been overlooked and not fully explored in previous studies. Specifically, as Herbst (2006) used the term *instructional situation* to identify various systems of norms students have to know and to do when working on different mathematical tasks, he addressed that different mathematical tasks can constitute different intellectual context for students to think about the mathematics at stake in a problem. In this regard, examining the intellectual context of a type of

---

[20] Nancy is a pseudonym.

mathematical tasks can provide insight into its special role in students' learning of mathematics.

More specifically the extant this study examines the learning opportunities of geometric calculations with number (GCN)[21] tasks set up by the Taiwanese mathematics teacher and how she sustains the level of cognitive demand of the tasks. The reason that GCN tasks are well suited for this investigation is because these tasks together with their related diagrams can reveal how diverse learning opportunities are afforded by selecting a sequence of mathematical tasks. In particular, since GCN tasks usually require the same reasoning and knowledge as geometric proofs (GP) do (Hsu, 2007; 2008), this investigation can further specify what learning opportunities associated with GP that the GCN tasks can afford.

To explore the extent to which the selection and sequencing of GCN tasks can contribute to students' learning, the Mathematical Tasks Framework (MTF) (Silver & Stein, 1996; Stein, Grover, & Henningsen, 1996; Stein & Smith, 1998; Stein, Smith, Henningsen, & Silver, 2000) is applied to model and trace learning opportunities. As shown in Figure 3.1, in the MTF framework mathematical tasks lead to student learning as a result of how they appear in the pages of textbooks or other curricular mathematics, how they are set up by teacher, and how these mathematical tasks are enacted by the teacher and students in classroom. The first three stages in the MTF framework have a strong influence on students' learning outcomes (Silver & Stein, 1996; Stein et al., 1996; Stein & Smith, 1998; Stein et al., 2000). In particular, the examination focuses on the

[21] A GCN is generally described as numerical calculation done in relation to mental or physical geometric diagrams on the basis of geometric principles or formulae (e.g., calculating an angle measure in a triangle given that measures of the other two angles are 30˚ and 100˚, respectively).

second and the third stages, namely tasks as set up and enacted by classroom teacher and students.

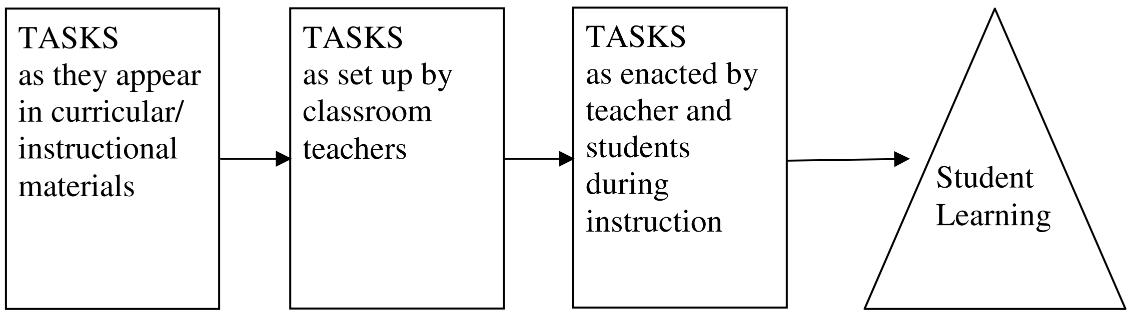


Figure 3.1 The Mathematical Tasks Framework (Stein, Smith, Henningsen, & Silver, 2000, p. 4; Silver, 2009, p. 829)

Furthermore, the MTF framework not only helps this study trace the influence of sequencing and the implementation of GCN tasks, but also examine how the cognitive demands of the tasks are sustained. While sustaining a high demand of a mathematical task through the stages can have a significant impact on students' learning (Boaler & Staples, 2008; Hiebert & Wearne, 1993; Stein & Lane, 1996; Stigler & Hiebert, 2004; Tarr et al., 2008), this study investigates how Nancy was able to facilitate student learning by maintaining the cognitive demand through her efforts of setting up a sequence of GCN tasks and enacting these tasks with her students.

Specifically, the investigation focuses on how GCN task diagrams influence the maintenance or the increase in cognitive demand when enacting the sequence of tasks in classroom. Another facet that will be examined in this study is the use of gestures. Instruction in East Asian countries can be described as teacher-centered (Leung, 2002; 2005) while classroom teachers often talk more than students. In this regard, how students can learn mathematics in this environment is of interest. From the perspective of embodied cognition (Lakoff & Núnez, 2000), non-verbal communication can be a possible way to make learning happen in the classroom in East Asian countries. This study particularly focuses on examining the use of gestures, one type of non-verbal communication, because gestures can function as a semiotic aid to facilitate students' learning of mathematics (Nemirovsky & Ferrara, 2009). To this end, the research question proposed in this study is described as follows:

In what ways does a Taiwanese mathematics teacher sustain or increase the levels of cognitive demand and facilitate students' learning by setting up a sequence of GCN tasks and how she uses gestures to facilitate the learning?

Literature Review

Mathematical Task

In recent years, considerable attention has been given to mathematical tasks in order to improve the quality of mathematics teaching and learning (Bell, 1993; Henningsen & Stein, 1997; Herbst, 2003; 2006; Krainer, 1993; NCTM, 2000; Simon & Tzur, 2004; van Boxtel, van der Linden, & Kanselaar, 2000). As defined by Doyle (1988), a task is

> (a) a goal state or end product to be achieved; (b) a problem space or set of conditions and resources available to accomplish the task, (c) the operations involved in assembling and using resources to reach the goal state or generate the product, and (d) the importance of the task in the overall work system of the class (p. 169).

In line with Doyle, a mathematical task can be considered as the context that conveys the message of what mathematics must be done while solving the task. When task elements are manipulated, tasks have the potential to influence and structure how students think about mathematics. Tasks can also direct students' attention to a particular perspective of the mathematics and broaden or limit their viewpoints of mathematics

(Henningsen & Stein, 1997; Silver, 2009). As such, a good mathematical task can scaffold students' development of a proficient understanding of mathematics.

Herbst (2006) further elaborated a mathematical task as a specific *unit of meaning* that "constitutes the intellectual context in which individuals think about the mathematical ideas at stake in a problem" (p. 315). He argued that teaching requires more than understanding what mathematics problems are and how students interact with them. Significance as well is recognizing how a teacher is able to sustain the intellectual demand of a task in the classroom. To explore this, he proposed the term *instructional situation*, which identifies what various systems of norms students have to know at a particular time and what they have to do to fulfill the implicit didactical contract (Brousseau, 1997) agreed upon between teacher and students in the classroom.

Researchers also suggest that mathematical tasks should be used to organized instruction, especially the tasks of a high-level (Krainer, 1993; Lampert, 2001; Stein & Smith, 1998; van Boxtel et al., 2000). For example, Krainer (1993) suggests using powerful tasks (i.e. high level) so that teachers can manage complex and conflicting instruction (e.g., the conflict between the well-developed mathematics system and students' prior knowledge). Here, a powerful task refers to a task that can be strongly interconnected with other tasks and can facilitate students in actively and reflecting confronting mathematics, which can further generate interesting questions (Krainer, 1993). Bell (1993) also investigated the effectiveness of mathematical tasks and suggested several principles that should be followed to improve mathematics teaching, including (1) selecting a mathematical task that contains mathematical concepts to be learned and ideas that students will want to work on; (2) determining and adjusting the degree of challenge in a mathematical task to allow students to explore the mathematics concepts in accordance with their ability; (3) extending a single task into complex and multiple tasks so that students will broaden their experiences working on certain mathematics topics and then relate those learned mathematics topics to other mathematical content; and (4) providing students opportunities to reflect and to review their self learning in relation to a broader sense of mathematics, different types of problems, and different methods of solutions within the field.

Despite the value placed on tasks of high cognitive demand, researchers have recognized that such tasks do not always guarantee that students' enactment remains at the same high level in terms of their mathematics understanding (Stein, Grover, & Henningsen, 1996; Stein, Smith, Henningsen, & Silver, 2000). One reason for this put forth by Stein, Grover, and Henningsen (1996) is that teachers can not exactly predict students' responses when enacting these tasks because students may generate diverse mathematics ideas in solving them. In this regard, these tasks require that teachers possess good instructional skills to maintain the cognitive demand during classroom instruction. If teachers do not effectively manage tasks with a high level of cognitive demand, students' enactment may remain at a level of demand that is lower than the initial cognitive demand.

A lowering of cognitive can be attributed to six factors articulated by Stein et al. (1996). The first factor occurs when the challenge of the tasks becomes a non-problem for students. This results when teachers respond to students' request that teachers reduce the demand of the task by providing more explanation for the tasks, which in turn reduces the complexities and the challenges of a task. Teachers may take over the challenge of the work by unintentionally performing the work for students or directly telling students how to do it. The second factor that reduces the level of the cognitive challenge is when inappropriate tasks are assigned to students. In this circumstance, students fail to engage in high–level cognitive activities because of a lack of motivation, interest, or prior knowledge related to the mathematics work. The third factor is related to the process of implementing the task. If teachers shift the learning focus from understanding mathematical meaning, and concepts to emphasizing the correctness or completeness of the answers, the cognitive challenge will be reduced. The fourth factor is related to the time allocated to tasks. Time constraints that prevent students from going through the mathematical ideas can lead to a lower level of cognitive demand. The fifth factor is the lack of student accountability if students do not take responsibility for learning the mathematics and monitoring the work progress themselves. For example, students will not learn mathematics if they think the classroom teacher should always give them the answers to problems and, thus, refuse to reason out the problem themselves. The final factor that can lower cognitive demand has to do with classroom management. If teachers lack sufficient skills to manage the flow of task implementation, a high-level cognitive activity may become a low-level one.

Gestures as Semiotic Aid to Facilitate Student Learning

In recent years, many researchers have investigated how gestures play a role in human communication and in influencing the teaching and learning (Chen & Herbst, 2007; Hsu, 2008; Maschietto & Bussi, 2009; Nemirovsky & Ferrara, 2009; Pozzer-Ardenghi & Roth, 2005; Radford, 2003; 2009; Roth, 2001; 2002; Roth & Lawless, 2002; Roth & Thom, 2009; Sabena, 2004; Williams, 2009). Previously, gestures were viewed as spontaneous accompaniments of speech realized through the movements of figures, hand, and arms; in other words gestures and speech are usually co-present in the social-interactive context of communication (McNeill, 1992). But, recent studies have deepened our understanding of gestures by recognizing their cognitive nature and arguing that gestures are more than accompaniments to spoken messages during communication. Language cannot be separated from imagery (McNeill, 2005), and gestures are representations of thought. In this regard, gestures have cognitive possibilities as they universally and automatically occur with speech to convey meanings during communication and have an important influence on the communication context.

The cognitive possibilities of gestures were revealed in a study by Radford (2009), which uses a broader context of the interplay of the various sensuous sources to explain the learning. Rodford (2009) argued that knowing is an experience resulting from multi-sensorial experiences of the world and then self-sensuous apprehension of to-be-learned objects. Thus, gestures are one of importance in conveying abstract thinking. Other support for the importance of gestures comes from the viewpoint that treats mathematics learning as the development of a particular type of imagination (Nemirovsky & Ferrara, 2009; Radford, 2009). In this sense, gestures have a cognitive function , this possibly facilitating students' understanding and imagination of mathematics.

Considering that teachers are responsible for scaffolding students in learning mathematics, they should take advantage of all resources available to do so. Since gestures can convey mathematical meaning to students in classroom (Arzarello & Paola, 2007), they provides teachers an alternative way to externalize their thoughts (Roth, 2002) and then potentially enhance students' understanding of the content embodied in the inscriptions (e.g., diagram, photograph) (McNeill, 1992; Pozzer-Ardenghi & Roth, 2005). Given that teachers need to interact with diagrams to illustrate geometric meaning for students, the interplay between diagrammatic properties and gestures is worthy of further investigation (Chen & Herbst, 2007).

Methods

The Methods here consist of four sections. The first section describes the rationale of selecting a Taiwanese teacher for the investigation of this study. The second section explains the data source and the selected episodes for analysis. Section Three provides information on the selected episodes in the context of the instruction. The final section elaborates the methodological approaches used to analyze the selected episodes.

Rationale for Teacher Selection

One teacher, Nancy, was selected for this study because she represents a typical expert Taiwanese mathematics teacher at the middle school level. As teaching is a cultural activity (Stigler & Hiebert, 1999), Nancy's teaching reflects several attributes specific to Taiwan instruction. First, Nancy has a profound understanding of subject content knowledge for mathematics (Ma, 1999). Given her more than twenty years of teaching experience, Nancy can clearly describe the teaching sequence for mathematical topics and elaborates how these topics relate to other mathematics content. She can do this from the perspective of both student learning and mathematics curriculum development. In this regard, she is capable of evaluating whether or not the arrangement of a topic in a textbook is appropriate and aligned with students' learning trajectories. The second reason that Nancy is representative of the Taiwanese teaching culture has to do with her teaching style in the classroom. Nancy's classroom is teacher-centered

(Leung, 2002; 2005). She usually dominates the classroom activities by deciding what work her students have to do and what mathematics should be discussed in class. Although she assigns students to demonstrate their task solutions in front of the classroom, students are constrained in elaborating their solutions because the solutions should be relevant to the tasks assigned. It seems that the underlying didactical contract (Brousseau, 1997) between Nancy and her students is to work on the mathematical tasks assigned, while avoiding moving beyond the scope of the tasks. The classroom norm is that Nancy grants students some freedom to solve a task by different solution paths, while being very clear about the mathematics content to be learned. As a result, Nancy typically expects only those answers that can be used to solve the tasks being discussed or can be used as materials for subsequent instructional activity. The third typical characteristic that Nancy has is consistent with the examination culture in Taiwan (Lin & Tsao, 1999). Nancy believes that students' success on the high school entrance examination is a very important criterion for evaluating the quality of teaching. Thus, this high-stakes entrance examination plays a central role in her instruction. Specifically, in relation to geometry, Nancy comments that this is the most difficult topic for middle school students, especially the construction of GP. However, she also believes that geometric tasks can provide students an opportunity to acquire mathematics intuition because of the particularity and complexity of individual geometric task. Considering that each geometric task is special and complex, students can not simply apply a well-developed procedure to derive the solution. This lack of a ready-made procedure forces students to visualize the underlying structure of the geometry, especially the structure of geometric diagrams, and to develop the intuition of geometry. This intuition, Nancy believes, is the key to the success of mathematics learning and to getting good grades on the high school entrance examination.

Data Source and Episode Selection

The data used in this study consists of video records of 30 lessons[22] collected from Nancy's instruction in 8th grade mathematics class with about 40 students. The geometry content taught in those lessons was properties related to parallel lines and quadrilaterals.

Among these 30 video records, two episodes were selected for a close analysis. The selection of the first episode was of interest because it indicates how Nancy sustained or raised the cognitive demand of the work when she and her students solved a sequence of GCN tasks in class. The investigation of this episode focused on the role of diagrams in sustaining or raising the cognitive demand of the tasks and possibly in turn changing instructional mathematical tasks (Doyle, 1983; Stein, Smith, Henningsen, & Silver, 2000).

The second episode was selected because it revealed how Nancy used gestures to scaffold her students in learning geometry. Visualizing the diagram configurations with sub-constructs are essential to retrieving useful geometric properties and there are one of keys to solving GCN tasks. To aid in visualization, teachers may use gestures. Thus, the analysis here focuses on the interaction between the diagram and gestures that can convey geometric meaning to students without lowering the level of cognitive demand of the GCN tasks Nancy gave her students to solve.

---

[22] About 50% of tasks used by Nancy during the 30 lessons were GCN tasks.

Sustaining the level of cognitive demands by selecting and sequencing GCN tasks as well as the use of diverse gestures to facilitate learning is very common in Nancy's classroom. Selecting the two episodes as convenient examples can allow author to explicitly and inclusively demonstrate the ways to facilitate students' learning through the selection of GCN tasks and the use of gestures.

Background of the Two Selected Episodes

Episode One

Episode One involves Nancy's setting up of a sequence of GCN tasks. When setting up the sequence of GCN tasks, Nancy altered the diagram configurations to make them more complex. The geometric content associated with the sequence of GCN tasks was the properties related to parallel lines. Before working on the sequence of GCN tasks, Nancy introduced students to the definitions and properties of parallel lines (e.g., the alternate interior angles theorem). Together she and the students worked on several GC tasks which required students to apply the learned properties.

In the selected episode, Nancy first set up a task (the task on the left side of Figure 3.2) and led the classroom discussion to solve the task using the relevant geometric properties for each reasoning step in the solution. After obtaining the solution, Nancy set up another two tasks (the one in the middle of Figure 3. 2 and the one on right) which were similar to the first task in terms of the shape of the diagrams (two horizontal parallel lines and several segments which constitute angles in between the parallel lines), the given numerical measures of the angles, and the unknowns that the students had to figure out. The major difference in the three tasks was the number of segments in between the two parallel lines and the angles formed by both the segments and the parallel lines.

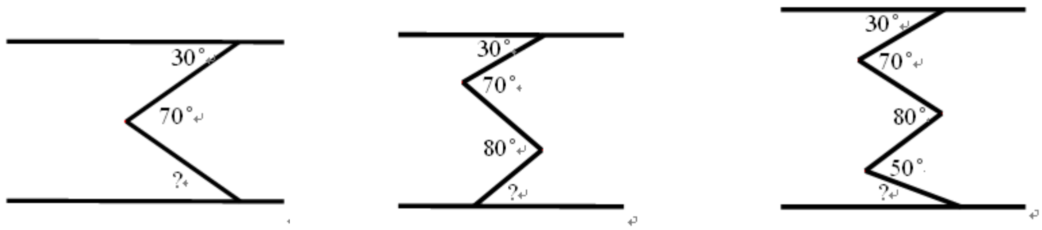


Figure 3.2 A sequence of GCN tasks set up by Nancy

Episode Two

Episode Two aims to explore how Nancy used gestures as semiotic aid to facilitate student learning. This episode involves a GCN task from a test sheet that Nancy used for formative purpose to evaluate students' learning of the properties related to quadrilaterals. The GCN task was selected because the task diagram is complex and involves a transformation action that offers an opportunity to observe the interplay between Nancy's use of gestures and the diagram. The following is the description of the GCN task.

Table 3.1 The GCN task used to illustrate the analysis of gestures

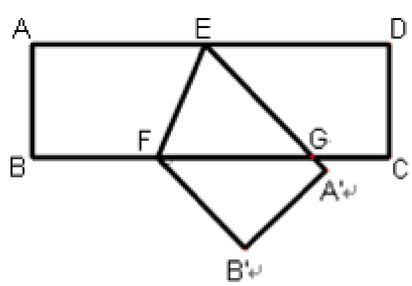
The given diagram	The written given
	As shown on the left side diagram, a rectangle ABCD was folded along the segment EF so that the point A and point B are moved to the new positions A' and B'. Given that measure of $\angle EGB=45^\circ$, measure of $\angle GFB'=45^\circ$, and AB=8 cm, find the area of ΔEFG=________.

Table 3.1 shows that the GCN diagram is constituted by different sub-constructs (e.g., a rectangle) and involves a folding action. The task requires students to understand how the folding action is accomplished and influences the given diagram as well as the

geometric properties that are created as a result of folding. In addition, because the geometric properties embedded in the given diagram are not sufficient to obtain a solution, the task also requires students to draw auxiliary lines that create new sub-constructs and relevant geometric properties and to anticipate how these added lines can contribute to the generation of a solution (Hsu,2007).

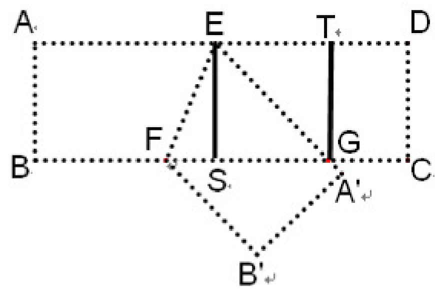


Figure 3.3 One of the methods to draw auxiliary lines on the diagram

Figure 3.3 demonstrates an approach to drawing auxiliary lines on the diagram to create sub-constructs and new geometric properties that can be used to generate a solution. As can be seen in Figure 3.3, the first line is constructed through point E so that it is perpendicular to segment BC; the second line is drawn through point G so that it is perpendicular to segment AD. By constructing the two auxiliary lines, one can obtain a solution, which is detailed in Appendix 3.1. For this problem, managing and sustaining the cognitive demand of the task without telling students where to draw the auxiliary lines requires good instructional skills, including the use of gestures as a semiotic aid. The gestures Nancy used facilitated students were the efforts to visualize the sub-constructs in the GCN diagram with corresponding geometric properties embedded as well as helped them recognize the need to draw auxiliary lines to generate a solution.

Mode of Data Analysis

This case study documents the instruction of a Taiwanese mathematics teacher with a view toward systematically illuminating the reality of the instruction phenomenon in detail (Merriam, 1998). Discourse analysis tools were used to analyze the video records (Erickson, 2004; Johnstone, 2002; Lemke, 1990). Similar to most other research in education, this study focuses on the interaction among teacher, students, and subject matter content as manifested in speech, gestures, written symbolization, and non-verbal actions (Erickson, 2006). In addition, multimodal analysis suggested by Thibault (2000) is used to frame the transcription of the video clips in order to reveal the co-deployed and co-contextualized textual meaning as created by the distinct semiotic resources systems.

The examination of the teacher's gestures involved in solving a GCN task in Episode Two employs a comprehensive classification of gestures developed by Pozzer-Ardenghi and Roth (2005) developed from observations the use of gestures by biology teachers when interacting with visual representations (e.g., photographs, diagrams).

Pozzer-Ardenghi and Roth identified eight types of gestures, which are described as follows.

- *Representing* refers to the gestures used to denote the objects or phenomena that are not directly available in the visual representations but related to some features of it (e.g., to present the movement of something real but can not be shown in the static diagram).

- *Emphasizing* describes gestures that emphasize an entity directly available in the visual representation.

- *Highlighting* gestures are those used to draw attention to an approximate area in the visual representation where an object can be identified. Usually, these gestures are circular or elliptical.
- *Pointing* refers to the gestures that clearly point to specific objects in the visual representation or entire representation from some distance.
- *Outlining* gestures are used to trace a shape in the visual representation. Hence, the shape of this gesture is determined by the shape of the reference object.
- *Adding* gestures are similar to outlining, but the objects outlined are not available in the visual representation, even though it could have been there.
- *Extending* describes gestures that add something beyond the boundary of the visual representation.
- *Positioning* refers to the gestures that strongly relate to the body orientation and then constitute a form of extension into three-dimensional space.

This comprehensive classification reveals what geometric meaning was conveyed through the interplay between the gestures and the diagrams during the process of solving the GCN task.

Findings

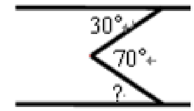
Episode One

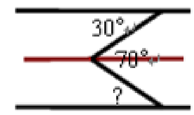
Episode One focuses on exploring how Nancy set up a sequence of GCN tasks and enacted these tasks with her students. The sequence of classroom discussion with respect to the three tasks is demonstrated in the following transcripts[23], which are framed into three sections: the subjects (Nancy or the students), the verbal and non-verbal (e.g., physical actions related to the mathematics) communications, and the diagram alterations or locations to which the physical actions (e.g., gesturing) refer.

The key to solving the first GCN task is to recognize that auxiliary lines need to be drawn on the given diagram so that new sub-constructs and corresponding geometric properties can be created and used to generate solutions. Nancy left the essential work deciding where to draw the auxiliary lines to the students and facilitated the class discussion regarding solutions by asking what angle measures students were inferring and what geometric properties were used to support their inferences.

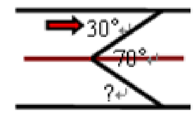
[23] Episodes were translated into English by the author and then validated by a Taiwanese whose expertise is in English.

[24] T: How to solve this task?
SG: The answer is 40.
T: Why 40?
SG: To draw a parallel line [gesture a horizontal line virtually][25]

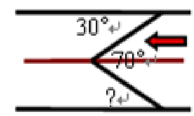


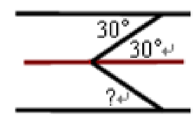


T: To draw a parallel line (1.0). Some one just said to draw a parallel line. Ok, we draw it here [draws a parallel line]. Then?
SG: The measures for the acute angle are 30 [point at the 30˚ angle virtually].

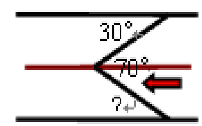


T: So, the measures for this angle are 30 [points at the 30˚ angle]. What will be the next?
SG: That (0.5) the measures of the alternate interior angle are 30 [point at the angle corresponding to the alternate interior angle].

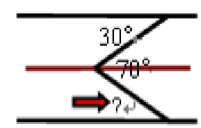




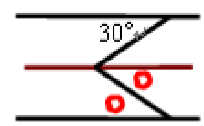
T: Ok, the measures for the alternate interior angle are 30 [writes 30 on the angle] (.). How's about this angle [points at the lower part of 70 degrees angle]?



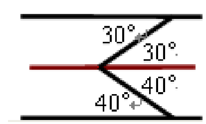
SG: The measures are 40.
T: How's about this angle? [points at the angle with a question mark]



SG: The measures are 40.
T: What geometric property can support this inference [points back and forth between the two angles]?



SG: The alternate interior angles property

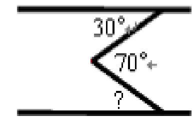


T: Ok. So we know that the measures for this angle are 40 [erases the question mark and writes 40 on the unknown angle]. Is that correct? (.) Do you have any alternative strategies to this task?

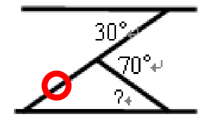
---

[24] The numbers used in the transcript refer to the sequence of the text demonstrated in the study and may not be the actually numbers in the transcript for the whole episode analyzed.
[25] Notes for all transcript in this study: (.) short pause; (number) longer pause lasts the number of seconds; (inaudible) talk can not be recognized clearly; _____emphasis of the words; [] writing or physical actions from Teacher Nancy or students; T: Teacher Nancy; SG: group of students; S: individual student.

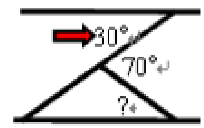
SG: We can extend the line down to the parallel line



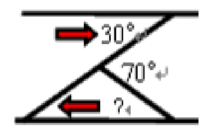
T: What do you mean [erases the auxiliary line and angle
measures discussed previously and puts the question
mark back to the task diagram]?

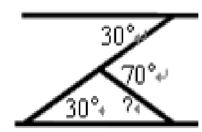


SG: Extend the middle segment to intersect the below line
[gesture the line extension virtually].
T: Extend the line [extends the line virtually]?
SG: Yes.
T: What else can we do after extending the line?
SG: The corresponding angles
SG: The alternate interior angles
T: Corresponding angles or alternate interior angles
(smiles)?
SG: Alternate interior angles

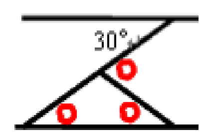


T: Alternate interior angles [points at the 30˚ angle].
Thus, this angle is the alternate interior angle
corresponding to the angle with 30˚ [points at the
alternate interior angle]. Right?(.) Ok, measures for
this angle are 30 [writes 30˚ on the angle]. And then?





SG: The exterior angle property



T: We can apply the exterior angle property [points at the
three angles corresponding to this property] and
calculate the angle measures to obtain the 40˚ answer.

The above transcript illustrates the discussion between Nancy and her students when working on the first GCN task. Her students proposed two different methods of constructing the auxiliary lines that can obtain the unknown angle measures; one involves drawing a line parallel to the given parallel lines and the other is to extend one of the segments in between the two given parallel lines so that the line drawn can intersect the bottom parallel line. After the students elaborated the methods regarding constructions of the auxiliary line, Nancy worked with them to figure out the geometric properties corresponding to each reasoning step.

Several learning opportunities were afforded by the task and related work in class. First, Nancy provided students the opportunity to visualize and operate the given diagram to decide where to add auxiliary lines as well as to figure out how the line additions could help them obtain the solutions, the work requiring the *operative apprehension* (Duval, 1995). The second learning opportunity was that the students could work on recognizing the sub-constructs in the diagram and corresponding geometric properties which can be used to support reasoning steps in the solution generations. The cognitive process of recognizing the sub-constructs in the diagram requires the *perceptive apprehension* and the process of identifying the geometric properties embedded in these sub-constructs further asks for the *discursive apprehension* of the diagram (Duval, 1995). For example, in lines 23 and 24, Nancy pointed back and forth at the angles of diagram, which allowed students to visualize the triangle shape with the three angles and identify they were corresponding to the alternate interior angles property. Another example can be seen in lines 50 and 51, which captures how Nancy pointed at the three angles corresponding to the exterior angle property on the diagram, which allowed students to correctly understand the geometric property embedded in the diagram. This is relevant because drawing auxiliary lines and identifying sub-constructs with corresponding geometric properties correctly on a diagram require different *diagram apprehensions* (*operative*, *perceptive*, and *discursive*) (Duval, 1995), which are fundamental competencies in constructing GP. The third learning opportunity was that students could apply different strategies with different sets of geometric properties to obtain a solution to this task (e.g., the alternate interior angles property for the first method and the exterior angle property for the second method).

After this first GCN task was solved, Nancy introduced another GCN task by adding complexity to the diagram of the first task, as shown in the middle of Figure 3.2. She then asked students to find the unknown measure. At this moment, one of her students immediately replied that the answer to the second GCN task was 40°. In response, Nancy decided to add the third GCN task (the one on the right side of Figure 3.2) to the associated diagram so that the resulting diagram was even more complex than those of the previous two GCN tasks. Nancy then discussed the second GCN task with students again, focusing how to draw auxiliary lines on the diagram and what geometric properties embedded in the diagram can be used to generate a solution. After figuring out the unknown measure to this task, Nancy hinted to the students that a pattern may possibly exist in the sequence of the three GCN tasks.

T: ….let's think how we solved these tasks. (1.0) Actually, my point does not ask you to memorize the finding, but you will find out a rule existing in these tasks. Later, when you encounter similar tasks again, you can obtain the answer quickly by applying the rule.

Nancy emphasized that the pattern can benefit students in their future problem solving, but she did not expect or want students to memorize the rule. Nancy wanted students to systematically think through the three GCN tasks and guess what pattern was embedded in the tasks. In a way, the explanation in the transcript allowed Nancy to manage the transition of *instructional situations* (Herbst, 2006). Originally, the didactical contract between Nancy and her students was to solve the GCN tasks and found the unknown measures. After providing the explanation as shown in the transcript, Nancy changed the *instructional situation* from solving the GCN tasks to conjecturing the patterns embedded in the sequence of the GCN tasks.

In addition, the students' inability to solve the third task with the most complex diagram also forced Nancy to change the classroom activity. She asked students to work in groups instead of in individual and walked around the classroom to check if they were able to solve the task. After making sure that the students knew how to solve it, Nancy re-directed their attention to the pattern embedded in the three GCN tasks again.

T: Ok, now we can see what conclusion we can obtain. We just solved several tasks and can anyone perceive the conclusion? (2.0)

SS: (inaudible)

T Ok, I re-write the measures again [re-labels the measures in the three GCN diagrams]. (.) When you solve the tasks and notice that you can apply the same properties and solution strategies to obtain the solutions. Do you think it is possible some patterns embedded in the tasks? (0.5) Actually, mathematical patterns or rules were discovered in history when similar cases appeared in different mathematics settings. Now, can you figure out the embedded patterns?

SG: Yes. (1.0) To add angle measures on both sides of the diagram.

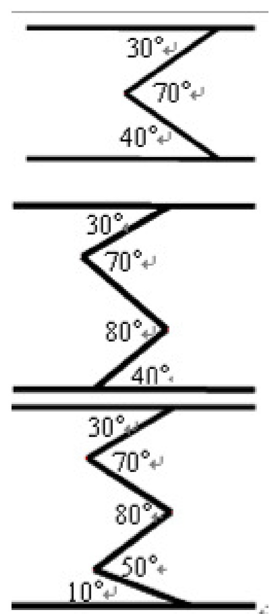
T: <u>What</u> do you mean? To <u>add</u> both sides?

SG: To add the angle measures on left side in the diagram

T: And then?

SG: That equals the sum of angle measures on the right side.

T: Sum of angle measures on the left side of the diagram equals that on the right side. Ok, let us check if the pattern is correct with the three tasks.



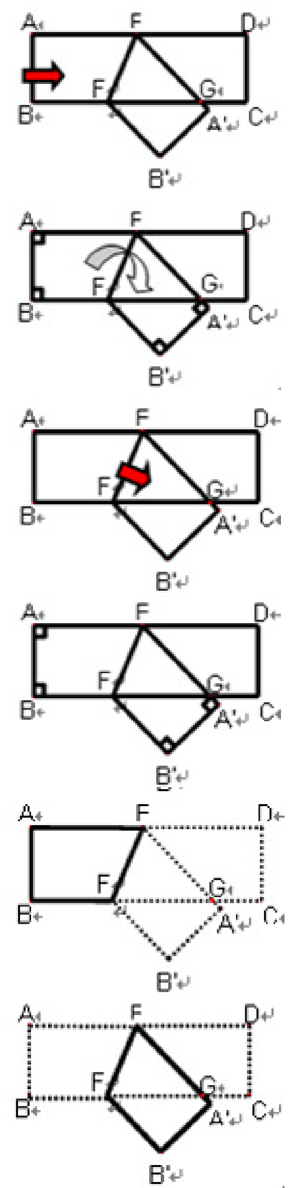
As can be seen in lines 57, 58, and 59 in the transcript, Nancy asked if students could identify the pattern in the three GCN tasks. Because none of her students responded to her immediately, she scaffolded students thinking as they tried to find the pattern. Specifically, Nancy re-labelled the measures of the angles in the three GCN diagrams and used the example of how mathematics has been discovered (i.e., observing similar examples that appear in different mathematics setting) to facilitate the learning. The action of re-labelling the measures directed students' attention to particular locations on the diagram and seemed to hint students that the pattern is related to these measures. Nancy's effort successfully brought students from measure calculations to pattern conjecturing. After the students figured out the pattern, Nancy confirmed it by checking the measures in the three GCN tasks with her students.

Episode Two

Episode Two uses another GCN task to show how the use of gestures is an important tool to scaffold students learning of geometry and to help students obtain solutions by visualizing geometric diagrams with relevant geometric properties. Specifically, because the GCN task diagram is complex and involves a transformation action, the analysis can illustrate the use of diverse types of gestures can scaffold students reasoning geometric relationship but not lower the level of cognitive demand of the task.

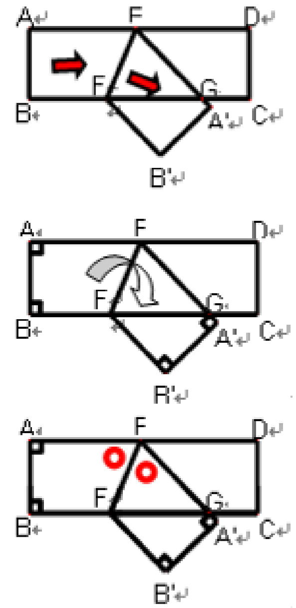
Similar to Episode One, the transcript in Episode Two also has three frames: the subjects, the verbal and non-verbal communication, and the diagram alterations or locations to which the physical actions refer.

T: Pay attention to here. Like this paper <u>folded</u> problem [gestures the action of folding virtually], there are many things we need to know. For example, we do not know any numerical information here [points to the diagram]. However, because this part [points at ABFE] is flipped and turned [gestures the flipping and turning actions] so this part becomes this piece [points at A’B’FE]. These angles should be right angles [puts right angle marks on ∠B’, ∠A’, ∠B and ∠A]. (.) What is the relation between the quadrilateral [outlines the four sides of the quadrilateral ABFE] and the quadrilateral [outlines the four sides of the quadrilateral A’B’FE]? (.)



SG: Congruence.

T: Congruence. Are these two congruent? (.) These two [points at ABEF and A’B’EF] are flipped [gestures the folding action]. So, these two shapes [points to ABEF and A’B’EF] are congruent. Because these two quadrilaterals are congruent [puts labels on ∠AEF], these two angles [points at ∠AEF and ∠FEG] should be congruent too [puts labels on ∠FEG].



The transcript shows how Nancy scaffolded her students to interpret the diagram configurations and the word “folded” as well as how the word “folded” influences the geometric properties embedded in the given diagram that can be used to obtain solutions. The word “folded” in the written givens implies that the two quadrilaterals ABFE and A’B’FE and corresponding angles and segments (e.g., ∠AFE=∠A’FE). To scaffold students to visualize the congruence of the quadrilaterals and the corresponding segments and angles, Nancy used different gestural moves.

First of all, in lines 83, 84, and 89, Nancy used a gesture, which can be considered a representing gesture, to demonstrate the folding action from quadrilateral ABFE to quadrilateral A’B’FE. Thus, students had the opportunity to understand the relationship between the flipping action and the diagram configurations (e.g., the two congruent quadrilaterals), which is not visible in a static geometric diagram. Secondly, the use of the pointing gesture revealed in lines 88, 90, and 91 also provides students an opportunity to make connection of the two congruent quadrilaterals. In addition, the use of pointing gestures also helped students follow Nancy's reasoning and explanation. For example, the pointing gesture (line 106) in relation to ∠AEF and ∠FEG helped students understand that these two angles are the corresponding angles of the congruent quadrilaterals.

Thirdly, Nancy used an outlining gesture to re-draw the two quadrilaterals ABFE and A'B'FE to further impress upon her students the shapes of the two congruent quadrilaterals. This outlining gesture can further facilitate students' ability to see the corresponding properties (e.g., AB=A'B'). Fourthly, the repeated representing gestures can also be viewed as a kind of emphasizing gesture because they highlight the importance of the folding action to the interpretation of the diagram configuration. Overall, the use of gestures to interact with the diagrams can convey the spatio-graphical meaning of the diagram (e.g., shape of quadrilateral ABFE) and provide students the opportunities to connect between spatio-graphical properties and theoretical properties (e.g., the congruence of the two quadrilaterals ABFE and A'B'FE). Outlining, representing, and pointing gestures also function as an informal proof that allows students to retrieve theoretical properties. For example, students inferred the congruence of ∠AEF and ∠FEG which are labeled with circles in the diagram in lines 104, 105, 107, and 108.

After using gestures to help students understand the congruence of the two quadrilaterals with the corresponding angles and segments, Nancy asked students to infer any other geometric properties embedded in the diagram based on the written given without considering the goal of this task. These goal-free inferences eventually led students to recognize the necessity of drawing auxiliary lines on the diagram because geometric properties embedded in the given diagram were not sufficient to obtain solutions. In the end, Nancy asked two of the students to find solutions themselves and then demonstrate their methods to construct the auxiliary lines and solutions generated based on the auxiliary constructed lines.

Discussion

Sustaining the Cognitive Demand of GCN tasks

The analysis of Episode One shows that the Nancy's setting up of a sequence of GCN tasks provided students diverse opportunities to learn geometry. Specially, they had opportunities to (1) operate the diagram to decide where to draw auxiliary lines; (2) visualize geometric properties embedded in the diagram that can be used to obtain solutions; (3) apply different strategies to solve the GCN tasks. While GCN tasks and GP tasks require similar knowledge and problem-solving processes (further analysis is given in the companion study in Chapter Four of this dissertation), students' engagement with GCN tasks can contribute to their potential competence in constructing GP tasks later.

In particular, setting up a sequence of tasks also allowed Nancy to scaffold students as they made conjectures about the pattern that could possibly be used for future problem solving. The conjectures about the pattern in the three GCN tasks directed students' attention to their underlying mathematical structures, which is the work that would not have been successful had the students been working on individual task. The ability to identify the pattern also can facilitate students' reification of the GCN tasks into an *object* (Sfard, 1991) on which they could draw when dealing with a more complex GCN tasks or GP tasks and more possibly manage the work.

The set up of a sequence of mathematical tasks can be accomplished by changing the elements of the tasks (e.g., different angle measures in a similar diagram configuration) (e.g., Bell, 1993). However, the analysis shows that Nancy did not do so. Rather, she carefully made the GCN task diagram more complex to sustain or increase the level of cognitive demand of the tasks. In this regard, students could not directly apply the experiences gained by solving the first GCN task diagram to obtain the solutions of the second and the third high-demand tasks. This kind of teaching that consistently maintains the cognitive demand can enable more learning to occur (Hiebert & Wearne, 1993; Stein & Lane, 1996; Stigler & Hiebert, 2004; Tarr et al., 2008) and possibly prevent shallow learning that occurs when students apply the superficial visual associations of the diagram to guess the answers to the GCN tasks without understanding (Aleven, Koedinger, Sinclair, & Snyder, 1998).

Sustaining or increasing the cognitive demand of the tasks in a classroom may also unfortunately mean that some students cannot solve the problems because they are too challenging. To address this potential problem, Nancy used different instructional strategies to help all students participate in reasoning through challenging tasks. For example, Nancy had students work in cooperative learning groups (Schoen, Cebulla, Finn, & Fi, 2003) and used peer discussion to facilitate learning. Furthermore, when students could not identify the pattern, Nancy re-labeled the measures and used mathematical discovery to scaffold students' work to produce the pattern. These efforts created a classroom environment that allowed Nancy to foster and monitor high-level learning by setting the bar high and expecting students to reach it.

In addition, the sequence of GCN tasks with different diagram configurations also enhances students' ability to make the generalizations from the pattern in one task to that of other tasks. The pattern in the GCN task described here can be applied more broadly than to just the geometric diagram settings as a pair of parallel lines with two segments in between. Although the instruction did not discuss the extent to which the derived pattern can be generalized, the classroom discussion did potentially provide students learning opportunities to do so to any number of angles formed by segments with two parallel lines.

Use of Gestures to Facilitate Student Learning

The analysis of Episode Two further shows how Nancy used different types of gestures to facilitate students' learning without lowering cognitive level of the task. As a semiotic mediator (Arzarello & Paola, 2007), Nancy's use of gestures scaffolded students mentally operating the diagrams, and enhancing their ability to recognize the congruence of the two quadrilaterals and relevant embedded geometric properties. The interplay between gestures and the diagram functioned as an informal proof for proving the congruence of the two quadrilaterals and activated relevant geometric properties. The activation of these geometric properties, Nancy expected, could help students infer other geometric properties and recognize that auxiliary lines are necessary to obtain a solution. In particular, this study shows the gesturing scaffold did not lower the cognitive demand level of the task, because students themselves had to work on recognizing the need to draw auxiliary lines on the diagram and retrieving sufficient geometric properties embedded in the diagram to obtain solutions.

This analysis of gestures is particularly important because it reveals how instructional intervention focusing on the use of non-verbal communication can possibly sustain the cognitive demand of tasks and facilitate learning, especially in the classroom in East Asian countries classrooms, which usually are described as teacher-centered because mathematics teachers in East Asian countries often talk more than students (Leung, 2002; 2005). Most studies have concentrated on investigating the routines in activity types such as reviewing previous lessons, teacher demonstrations, whole-class discussion, as well as student group or seat work (Fwu & Wang, 2006; Hiebert et al., 2003; Lemke, 1990; Shimizu, 2009; Stigler & Hiebert, 1999). This study, however, proposes an alternative approach to understanding teaching and learning in the classroom in Taiwan by observing the use of gestures, which is a type of embodied cognition sources that can construct mathematical meaning for students (Nemirovsky & Ferrara, 2009; Roth & Thom, 2009).

In addition, as the teaching in East Asian countries is often examination-driven, the analysis of a non-verbal communication way also suggests caution about focusing only on verbal utterances to understand the instructional practice of Asian teachers. Using non-verbal communication by teachers in East Asian countries' classrooms may also influence students' learning. For example, as students frequently work on tests, it is possible that a mathematics teacher can diagnose students' misconceptions by evaluating their responses to tests items and then use classroom lectures to help students understand the misconceptions. It is also possible that a skillful mathematics teacher can identify students' facial expressions to know if students follow the lectures and understand the mathematics without the need for any verbal communication. While classroom teachers may use non-verbal communications to facilitate students' learning, students in teacher-centered and examination-oriented classroom may also learn mathematics that cannot be identified by only examining the verbal communication between teacher and students. For example, it can be the case that students learn mathematics by practicing abundant mathematical tasks and evaluate their learning by checking the answers to the tasks. In this regard, the students do not need to participate in classroom discussion. When they have problems regarding the challenging tasks, the can listen to classroom lectures to obtain the solutions. These possibilities all can make learning occur in a teacher-centered and examination-driven classroom in East Asian countries, and should be carefully considered in interpreting and understanding the differences in students' performance in cross-national comparisons.

REFERENCES

Aleven, V., Koedinger, K., Sinclair, H., & Snyder, J. (1998). *Combatting shallow learning in a tutor for geometry problem solving.* Paper presented at the intelligent tutoring systems: Fourth International Conference, ITS '98, Berlin.

Arzarello, F., & Paola, D. (2007). Semiotic games: The role of the teacher. In J. H. Woo, H. C. Lew, K. S. Park, & D. Y. Seo (Eds.), *Proceedings of the 31st Conference of the International Group for the Psychology of Mathematics Education* (vol. 2, pp. 17-24). Seoul, Korea.

Bell, A. (1993). Principles for the design of teaching. *Educational Studies in Mathematics, 24*(1), 5-34.

Boaler, J., & Staples, M. (2008). Creating mathematical futures through an equitable teaching approach: The case of Railside school. *Teachers College Record, 110*(3), 608-645.

Brousseau, G. (1997). *Theory of didactical situations in mathematics: Didactique des mathématiques, 1970-1990*. Dordrecht: The Netherlands: Kluwer Academic Publishers.

Chen, C.-L., & Herbst, P. (2007). *The interplay among gestures, discourse and diagrams in students' geometrical reasoning.* Paper presented at the 29th PME-NA. Lake Tahoe, NV.

Doyle, W. (1983). Academic work. *Review of Educational Research, 53*(2), 159-199.

Doyle, W. (1988). Work in mathematics classes: The context of students' thinking during instruction. *Educational Psychologist, 23*(2), 167-180.

Duval, R. (1995). Geometrical pictures: Kinds of representation and specific processings. In R. Sutherland & J. Mason (Eds.), *Exploiting mental imagery with computers in mathematics education* (pp. 142-157). Berlin: Springer.

Erickson, F. (2004). *Talk and social theory: Ecologies of speaking and listening in everyday life*. Cambridge: Polity Press.

Erickson, F. (2006). Definition and analysis of data from videotape: Some research procedures and their rationales. In J. Green, G. Camilli & P. Ellmore (Eds.), *Handbook of complimentary methods in education research* (pp. 177-191). Mahwah, NJ: Lawrence Erlbaum Associates.

Fwu, B.-J., & Wang, H.-H. (2006). Practice makes perfect on the blackboard: A cultural analysis of mathematics instructional patterns in Taiwan. *ZDM-The International Journal on Mathematics Education, 38*(5), 368-375.

Henningsen, M., & Stein, M. (1997). Mathematical tasks and student cognition: Classroom-based factors that support and inhibit high-level mathematical thinking and reasoning. *Journal for Research in Mathematics Education, 28*(5), 524-549.

Herbst, P. (2003). Using novel tasks in teaching mathematics: Three tensions affecting the work of the teacher. *American Educational Research Journal, 40*(1), 197-238.

Herbst, P. (2006). Teaching geometry with problems: Negotiating instructional situations and mathematics tasks. *Journal for Research in Mathematics Education, 37*(4), 313-347.

Hiebert, J., Gallimore, R., Garnier, H., Givvin, K., Hollingsworth, H., Jacobs, J., et al. (2003). *Teaching mathematics in seven countries*. Washington: National Center for Education Statistics.

Hiebert, J., & Wearne, D. (1993). Instructional tasks, classroom discourse, and students' learning in second-grade arithmetic. *American Educational Research Journal, 30*(2), 393-425.

Hsu, H.-Y. (2007). Geometric calculations are more than calculations. In J. H. Woo, H. C. Lew, K. S. Park, & D. Y. Seo (Eds.), *Proceedings of the 31st Conference of the International Group for the Psychology of Mathematics Education* (vol. 3, pp. 57-64). Seoul, Korea.

Hsu, H.-Y. (2008). *Learning opportunity of reasoning: The interplay between gestures and diagrammatic properties.* Paper presented at the 11th ICME, Monterrey, Mexico.

Huang, R., & Li, Y. (2009). Pursuing excellence in mathematics classroom instruction through exemplary lesson development in China: A case study. *ZDM-The International Journal on Mathematics Education, 41*(3), 297-309.

Johnstone, B. (2002). *Discourse Analysis*. Malden, MA: Blackwell.

Kaur, B. (2009). Characteristics of good mathematics teaching in Singapore grade 8 classrooms: A Juxtaposition of teachers' practice and students' perception. *ZDM-The International Journal on Mathematics Education, 41*(3), 333-347.

Krainer, K. (1993). Powerful tasks: A contribution to a high level of acting and reflecting in mathematics instruction. *Educational Studies in Mathematics, 24*(1), 65-93.

Lakoff, G., & Núnez, R. (Eds.). (2000). *Where mathematics comes from: How the embodied mind brings mathematics into being*. New York: Basic Books.

Lampert, M. (2001). *Teaching problems and the problems of teaching*. New Haven: Yale University Press.

Lemke, J. (1990). *Talking science: Language, learning, and values*. Westport: Ablex Publishing Corporation.

Leung, F. K. S. (2002). In search of an East Asian identity in mathematics education. *Educational Studies in Mathematics, 47*, 35-51.

Leung, F. K. S. (2005). Some characteristics of East Asian mathematics classrooms based on data from the TIMSS video study. *Educational Studies in Mathematics, 60*, 199-215.

Lin, F.-L., & Tsao, L.-C. (1999). Exam math re-examined. In C. Hoyles, C. Morgan & G. Woodhouse (Eds.), *Rethinking mathematics curriculum* (pp. 228-239). London: Falmer Press.

Lin, P.-J., & Li, Y.-P. (2009). Searching for good mathematics instruction at primary school level valued in Taiwan. *ZDM-The International Journal on Mathematics Education, 41*(3), 363-378.

Ma, L. (1999). *Knowing and teaching elementary mathematics*. Mahwah, New Jersey: Lawrence Erlbaum Associates Inc.

Maschietto, M., & Bussi, M. G. B. (2009). Working with artefacts: Gestures, drawings and speech in the construction of the mathematical meaning of the visual pyramid. *Educational Studies in Mathematics, 70*, 143-157.

McNeill, D. (1992). *Hand and mind: What gestures reveal about thought*. Chicago: University of Chicago Press.

Merriam, S. B. (1998). *Qualitative research and case study applications in education*. San Francisco, CA: Jossey-Bass.

Mok, I. A. C. (2009). In search of an exemplary mathematics lesson in Hong Kong: An algebra lesson on factorization of polynomials. *ZDM-The International Journal on Mathematics Education, 41*, 319-332.

NCTM. (2000). *Principles and Standards for School Mathematics*. Reston, VA: National Council of Teachers of Mathematics.

Nemirovsky, R., & Ferrara, F. (2009). Mathematical imagination and embodied cognition. *Educational Studies in Mathematics, 70*, 159-174.

Pang, J. S. (2009). Good mathematics instruction in South Korea. *ZDM, 41*(3), 349-362.

Pozzer-Ardenghi, L., & Roth, W.-M. (2005). Photographs in lectures: Gestures as meaning-making resources. *Linguistics and Education, 15*, 275-293.

Radford, L. (2003). Gestures, speech, and the sprouting of signs: A semiotic-cultural approach to students' types of generalization. *Mathematical Thinking and Learning, 5*(1), 37-70.

Radford, L. (2009). Why do gestures matter? Sensuous cognition and the palpability of mathematical meanings. *Educational Studies in Mathematics, 70*, 111-126.

Roth, W.-M. (2001). Gestures: Their role in teaching and learning. *Review of Educational Research, 71*(3), 365-392.

Roth, W.-M. (2002). From action to discourse: The bridging function of gestures. *Cognitive Systems Research, 3*, 535-554.

Roth, W.-M., & Lawless, D. V. (2002). Signs, deixis, and the emergences of scientific explanation. *Semiotica, 138*(1), 95-130.

Roth, W.-M., & Thom, J. S. (2009). Bodily experience and mathematical conceptions: From classical views to a phenomenological reconceptualization. *Educational Studies in Mathematics, 70*, 175-189.

Sabena, C. (2004). *The role of gestures in conceptualisation: An exploratory study on the integral function.* Paper presented at the International Group for the Psychology of Mathematics Education.

Schoen, H., Cebulla, K., Finn, K., & Fi, C. (2003). Teacher variables that relate to student achievement when using a standard-based curriculum. *Journal for Research in Mathematics Education, 34*(3), 228-259.

Sfard, A. (1991). On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics, 22*, 1-36.

Shimizu, Y. (2009). Characterizing exemplary mathematics instruction in Japanese classrooms from the learner's perspective. *ZDM-The International Journal on Mathematics Education, 41*(3), 311-318.

Silver, E. (2009). Cross-national comparisons of mathematics curriculum materials: What might we learn? *ZDM-The International Journal on Mathematics Education, 41*, 827-832.

Silver, E., & Stein, M. (1996). The Quasar project: The "revolution of the possible" in mathematics instructional reform in urban middle schools. *Urban Education, 30*(4), 476-521.

Simon, M., & Tzur, R. (2004). Explicating the role of mathematical tasks in conceptual learning: An elaboration of the hypothetical learning trajectory. *Mathematical Thinking and Learning, 6*(2), 91-104.

Stein, M., Grover, B., & Henningsen, M. (1996). Building student capacity for mathematical thinking and reasoning: An analysis of mathematical tasks used in reform classrooms. *American Educational Research Journal, 33*, 455-488.

Stein, M., & Lane, S. (1996). Instructional tasks and the development of student capacity to think and reason: An analysis of the relationship between teaching and learning in a reform mathematics project. *Educational Reacher and Evaluation, 2*, 50-80.

Stein, M., & Smith, M. (1998). Mathematical tasks as a framework for reflection: From research to practice. *Mathematics Teaching in the Middle School, 3*, 268-275.

Stein, M., Smith, M., Henningsen, M., & Silver, E. (2000). *Implementing standards-based mathematics instruction: A casebook for professional development*. New York, NY: Teachers College Press.

Stigler, J., Gallimore, R., & Hiebert, J. (2000). Using video surveys to compare classrooms and teaching across cultures: Examples and lessons from the TIMSS video studies. *Educational Psychologist, 35*(2), 87-100.

Stigler, J., & Hiebert, J. (1999). *The teaching gap: Best ideas from the world's teachers for improving education in the classroom*. New York: Free Press.

Stigler, J., & Hiebert, J. (2004). Improving mathematics teaching. *Educational Leadership, 61*(5), 12-16.

Tarr, J., Reys, R., Reys, B., Chavez, O., Shih, J., & Osterlind, S. (2008). The impact of middle-grades mathematics curricula and the classroom learning environment on student achievement. *Journal for Research in Mathematics Education, 39*, 247-280.

Thibault, P. (2000). The multimodal transcription of a television advertisement: Theory and practice. In A. Baldry (Ed.), *Multimodality and multimediality in the distance learning age* (pp. 311-384). Campobasso: Palladino Editore.

van Boxtel, C., van der Linden, J., & Kanselaar, G. (2000). Collaborative learning tasks and the elaboration of conceptual knowledge. *Learning and Instruction, 10*, 311-330.

Williams, J. (2009). Embodied multi-modal communication from the perspective of activity theory. *Educational Studies in Mathematics, 70*, 201-210.

Yang, Y. (2009). How a Chinese teacher improved classroom teaching in Teaching Research Group: A case study on Pythagoras theorem teaching in Shanghai *ZDM-The International Journal on Mathematics Education, 41*(3), 279-296.

Appendix 3.1

Solution to the GCN Task Analyzed in Episode One

Table 3.2 Solution Steps with geometric reasons to GCN task in Table 3.1

Step	Calculating sentences	Geometric reasons
Step 1	AD//BC	Opposite sides of a rectangles are parallel because rectangle is a parallelogram
Step 2	$\angle A=\angle B=\angle C=\angle D=90^\circ$	Interior angles in a rectangle are 90˚
Step 3	AB=ES=8 cm	The distance between two parallel lines is constant.
Step 4	AB=TG=8 cm	The distance between two parallel lines is constant.
Step 5	$\angle EGB=\angle GED=45^\circ$	The alternate interior angles property
Step 6	$\angle TGE=180^\circ-90^\circ-45^\circ=45^\circ$	Angle sum property of a triangle
Step 7	TG=ET=8 cm	Property of an isosceles triangle
Step 8	EG=$8\sqrt{2}$ cm	The Pythagorean Theorem
Step 9	$\angle AEA'=180^\circ-45^\circ=135^\circ$	Property of linear pair
Step 10	$\angle AEF=\angle A'EF=67.5^\circ$	Symmetry property
Step 11	$\angle EFG=\angle AEF=67.5^\circ$	The alternate interior angles property
Step 12	EG=GF=$8\sqrt{2}$ cm	Properties of isosceles triangle
Step 13	Area of $\triangle EFG=32\sqrt{2}\,cm^2$	Triangle area formula

The minimum number of solution steps proposed in the above table is established on the basis of the drawing of two auxiliary lines, as shown in Figure 3.3. A total of 13 reasoning steps are required in this solution. The first step is to infer AD//BB on the basis of the property that “opposite sides of a rectangle are parallel because rectangle is a parallelogram.” Step Two is to infer that the four interior angles of the rectangle are all 90˚. Based on the first reasoning step that AD//BC, the third step can deduce that AB=ES=8 cm using the property “the distance between two parallel lines is constant.” The fourth step is to infer TG=8 applying the same geometric property as the third step does. The fifth step further deduces $\angle EGB=\angle GED=45^\circ$ on the basis of the alternate interior angles property and Step Six reasons $\angle TGE=45^\circ$ on the basis of the angle sum property of a triangle. Step Seven further infers TG=ET=8 cm based on the property of an isosceles triangle. Step Eight can conclude EG=8$\sqrt{2}$ cm because of the Pythagorean Theorem. Next, one can infer ∠AEA'= 135° using the property of linear pairs. Step Ten further reasons ∠AEF=°A'EF=67.5° because of the symmetry property that results from the folding action described in the givens. Step Eleven infer EG=GF=8$\sqrt{2}$ cm on the basis of an isosceles triangle property. Finally, the area of $\triangle$EFG=32$\sqrt{2}$ cm^2 can be concluded on the basis of triangle area formula.

CHAPTER FOUR

CONCEPTUALIZATION OF THE RELATIONSHIP BETWEEN GEOMETRIC CALCULATION WITH NUMBER (GCN) AND GEOMETRIC PROOF (GP)

Introduction

The value of teaching and learning proof at the secondary school level has been a matter of some disagreement in the field. On the one hand, some scholars have argued that proof and mathematical reasoning are fundamental to knowing and using mathematics and have claimed that even elementary students are capable of constructing a proof with the appropriate scaffolding (Ball & Bass, 2003; Maher & Martino, 1996; Stylianides, 2007; Zack, 1999). In line with this view the National Council of Teachers of Mathematics (2000) has identified proof and reasoning as an essential topic to be taught across all grade levels, and, as such, a central goal of mathematics education. On the other hand, the teaching of proof is often resisted by teachers and students who relegate it to a less important role in mathematics curriculum, because of challenges in teaching and learning of this topic not only in the U.S. but also in other countries with a tradition of teaching proof in the secondary curriculum (Chazan, 1993; Harel & Sowder, 1998; Mariotti, 2006). Mariotti (2006) highlights the concern of the idea of "proof for all" wondering whether

> "these words[26] would have been possible only a few years ago, and still now the idea of "proof for all" claimed in the quotation is not a view that most teachers hold, even in countries where there is a longstanding tradition of including proof in the curriculum. I'm thinking of my country, Italy, but also, as far as I know, France or Japan. In fact, the main difficulties encountered by most students have led many teachers to abandon this practice and prompted passionate debate amongst math educators" (Mariotti, 2006, p. 173).

When geometry proof is viewed as a difficult topic for middle school students, many attractive alternative options have been proposed (e.g., making reasonable conjectures; Herbst, 2006). One of such options can be geometric calculation with number (GCN), which generally involves numerical calculations within a mental or physical geometric diagram (e.g., calculating angle measures in a triangle by applying angle sum property) (Aleven, Koedinger, Sinclair, & Synder, 1998; Ayres & Sweller, 1990; Chinnapan, 2000; Lulu Healy & Celia Hoyles, 1998; Küchemann & Hoyles, 2002). GCN is an attractive alternative because it can provide opportunities for students to become familiar with and apply geometric properties, one of the aims in the secondary geometry curriculum (Schumann & Green, 2000). In this regard, the major link between GCN and GP seems to be the application of geometric properties. Another benefit of GCN comes from the perspective of cognitive development. The use of GCN to practice geometric properties aligns well with children's development of the concept of geometry and space. As Piaget, Inhelder, and Szeminska (1960) indicated, children at the elementary school level are capable of performing calculating tasks by applying

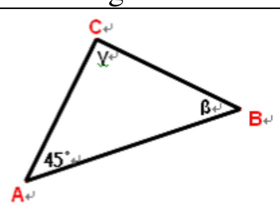
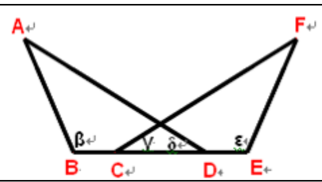
[26] [these words] refers to the citation from NCTM
"Reasoning and proof are not special activities reserved for special times or special topics in the curriculum but should be a natural, ongoing part of classroom discussions, no matter what topic is being studied (NCTM, 2000, p. 342).

geometric properties (e.g., size of angle, length of distance) and using calculation as a way to envisage the truth and validity of mathematics.

A more comprehensive theory specifying the relationship between proof and calculation is the three mathematics worlds model proposed by Tall and his colleagues (Tall, 2002; 2006a; 2006b; 2007). In this theory, calculation and proof are treated as two different mathematics worlds, each of which has its own way of thinking of and operating on mathematical objects as well as developing the standards of validity and mathematical truth. Even though Tall and his colleagues establish the theory by recognizing the conceptual and abstract entity that a calculation can possess because calculations are both the problem-solving actions and the thinking over these actions (Gray & Tall, 1994), Tall and his colleagues only treat calculation as a pre-stage of learning formal proof. This is because calculation can be a generic example contributing to the understanding of mathematical definition in the formal mathematics world. Regarding the development of geometry through the three mathematical worlds, Tall anchors van Hiele's theory (Tall, 2002; 2004), emphasizing that the operation of Euclidean proofs and theorems as well as the description of these geometric properties in the *embodiment* world can lead students to the understanding of mathematical formal proof and the relations among geometric properties.

Although Tall and his colleagues propose this theory to reveal the relations between calculation and proof in general as two different mathematics systems where calculations are treated as a pre-stage of learning formal proofs, they do not particularly deal with the relations between GCN and GP, nor the role that a geometric diagram can play in influencing the relations. Nor does the theory proposed by van Hiele (Fuys, Geddes, & Tischler, 1988). The link between GCN and GP also cannot be articulated in some empirical survey studies, which survey students' performance on GCN and GP (Healy & Hoyles, 1998; Lin, Cheng, & et al., 2003), because diverse factors can confound the comparisons (e.g., the geometric properties required to obtain a solution, the number of reasoning steps to calculate or prove the problems). Other studies have directly treated GCN as low-level cognitive demand tasks only for the application of properties and rules and have used this perspective to compare students' performance on the two types of tasks to assess the relations (Heinze, Cheng, Ufer, Lin, & Reiss, 2008; Heinze, Cheng, & Yang, 2004; Heinze, Ufer, Cheng, & Lin, 2008). For example, using the two tasks in Table 4.1, Heinze, Ufer, Cheng, and Lin (2008) characterized the GCN tasks (in above part of the table) as the one requiring the application of basic knowledge, and the other (in below part of the table) as GP tasks.

Table 4.1 The examples of GCN task (above) and GP task (below) used to survey students' performance between German and Taiwan (Heinze, Ufer et al., 2008)

Diagram	Givens
	The triangle is isosceles with ∣AC∣ = ∣BC∣. Calculate the missing angles.
	C and D are points on the line BE. We have ∣BD∣ = ∣EC∣, $\gamma = \delta$ and $\beta = \varepsilon$. Prove that ∣AB∣ = ∣EF∣ Give reasons for all steps of your proof

As shown in Table 4.1, the GCN task focuses on calculating unknown angle measures, whereas the GP task requires proving that two segments are congruent (AB=EF). These two tasks are different from each other in terms of the geometric properties needed to obtain the solutions, diagram visualization demands, and the number of proving or calculating steps. Solving the GCN task requires the triangle angle sum property and the properties related to an isosceles triangle to find the measures, whereas solving the GP task necessitates the use of the Angle-Side-Angle triangle congruence postulate to prove the conclusion. The visualization demands also differ between the two tasks. Proving the GP task requires one to recognize the sub-constructs (two overlapping triangles) in the given diagram, the work of which work can be more demanding than that needed to identify the diagram in the GCN task as an isosceles triangle. Moreover, the reasoning steps required to generate the solution in each of the two tasks are also different. For the GCN task, each of the unknowns can be inferred by applying a geometric property (e.g., measure of ∠ABC can be obtained by using isosceles triangle property). For the GP task multiple reasoning steps are needed: (1) steps of finding the needed conditional statements to conclude that two overlapping triangles are congruent; and (2) step of applying the result of congruent triangles to infer that segment AB = EF.

However, considering GCN as tasks of lower-level cognitive-demand ignores the complexity and particularity a task can be (Stein, Smith, Henningsen, & Silver, 2000), and may underestimate the link between both types of tasks. For instance, using the tasks in Table 4.1 as examples again, we see that a one-step GCN may be much easier to solve than constructing a multiple-step GP not because of their differences in task format, but because of the cognitive demand as determined by the number of reasoning steps needed to generate a solution. Thus, using only simple applications of basic knowledge to characterize GCN tasks as low demanding tasks, we may fail to see the relationship between GCN and GP. The argument proposed in this study is that the link between GCN and GP can be stronger than the application of geometric properties, because of the abstract nature of geometric diagrams.

To explore how diagram configurations create a stronger link between the two types of tasks, I hypothesize that the diagram and the geometric properties required to obtain a solution are keys to the link. Researchers have provided evidences of the various functions of diagram that can support this proposed hypothesis. Diagram is the milieu in which students can dynamically operate and anticipate a solution based on the dynamic operations (Duval, 1995; 1998; Harel & Sowder, 1998; Herbst, 2004; Hsu, 2007). Diagrams are also the scheme by which students remember the givens, the labels, and the reasoning steps needed to generate a proof solution (Lovett & Anderson, 1994) as well as the objects that can be parsed into chunks to cue the geometric knowledge needed to generate a proof solution (Koedinger & Anderson, 1990). Moreover, Larkin and Simon (1987) indicated that diagrams are the site where problem solving occurs so that they can function as artifact in scaffolding students in learning proofs. For instance, asking students themselves to read the given information and then color these properties on the diagram greatly can facilitate students' learning of constructing GP tasks (Cheng & Lin, 2006; 2007). Through the use of the color in a diagram students can visualize the useful geometric properties that can be used to obtain a proof solution. As a result, diagrams can be the key to establishing the relationship between GCN and GP, and create a closer connection of the two types of tasks, which is worthy of further investigation.

Furthermore, diagrams are relevant for solving both GCN and GP because for each type of tasks students need to visualize geometric diagram and identify what geometric properties embedded in the diagram that can be used to generate a solution, aligned with the concept of *descriptive mode* proposed by Herbst (2004). However, recognizing geometric properties had been reported as one of the main difficulties in learning geometry (Duval, 1995; Fischbein & Nachlieli, 1998; Zykova, 1975).

Given that geometric diagrams are common to both GCN and GP and geometric properties needed to obtain a solution, this study investigates the following research question:

To what extent is GCN similar to GP when controlling the diagram configurations and requirements of geometric properties necessary for a solution?

Theoretical Comparison between GCN and GP

The relationship between GCN and GP is roughly analogous to the distinction between *a problem to find* and *a problem to prove* (Pólya, 1945). In his seminal book, *How to solve it*, Pólya explicates the differences between the two kinds of problems from three perspectives: the aim of the problem, the principals, and the actions taken during the problem-solving process. In relation to the aims, Polya states that

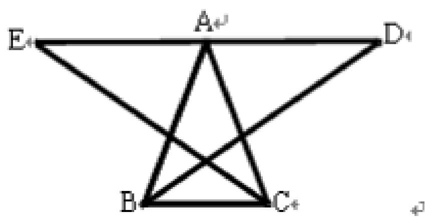
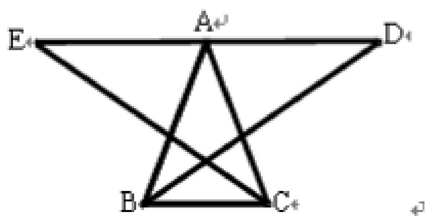
> "The aim of *a problem to find* is to find a certain object, the unknown of the problem…The aim of *a problem to prove* is to show conclusively that a certain clearly stated assertion is true, or else to show that it is false." (Polya, 1945, p. 154)

Accordingly, when GCN is viewed as *a problem to find*, GCN and GP have quite different aims. Specifically, GCN viewed from this perspective is directed at finding certain answers to a problem, whereas GP involves proving or disproving statements. This key difference in aims also impacts the nature of the inferences one can make from the problems. For instance, GP allows generalization to all cases that satisfy a given statement. GCN, on the other hand, is limited to only a particular instance of a problem, lacking the power to generalize to all cases.

Considering the second perspective, the principals, Polya asserts that the principals for *a problem to prove* are the hypothesis and the conclusion, whereas the principals for *a problem to find* are the data, the unknowns, and the conditions. To better understand the differences in the principals for GP and GCN, the following examples are useful here. A GP statement can be formulated as an if-then statement "if one of the angles of a parallelogram is 90°, then the parallelogram is a rectangle." In this statement, the hypothesis is a conditional statement starting with "if" and the conclusion is the second part starting with "then." The if-then formulation provides students guidance for organizing proving steps into a logic sequence. An example of a GCN can be to calculate the measure for $\angle$A in a triangle ABC given that the measures for $\angle$B and $\angle$C are 50° and 70°, respectively. An analysis of the principals for this GCN reveals that the data are the measures for $\angle$B and $\angle$C and the unknown is the measure for $\angle$A. The condition is the connection between the data and the unknown, which in this task is the angle sum property for a triangle. By applying the angle sum property, students can connect the relationship between the data and the unknown, and determine the answer to this task.

The discussion so far has shown that GCN and GP are quite different in terms of the aims and principals. However, with regard to the third perspective, actions taken during the problem-solving process, the two types of tasks are similar, especially when the diagram configurations and the geometric properties required to obtain solutions are controlled. To explain the similarity in the problem-solving actions for the two types of tasks, a pair of GCN and GP tasks is presented.

Table 4.2 GP (above) and GCN (below) with the same diagram configurations and the same requirements of geometric properties for obtain a solution

The given diagram	The written givens
	Triangle ABC in which AB and AC are the same lengths. Construct a line through point A so that the line is parallel to BC and the bisectors of angle B and angle C intersect the line at point D and point E. Prove AE=AC
	Triangle ABC in which AB and AC are the same lengths. Construct a line through point A so that the line is parallel to BC and the bisectors of angle B and angle C intersect the line at point D and point E. If AC=6 cm, measure of $\angle$ACB=75°. Find (1) the length of AE (2) Find the measure of $\angle$AEC.

As Table 4.2 displays that the diagrams attached to the GCN and GP tasks are the same. The written givens in the two tasks all specify the construction of diagrams which implicitly reveal what geometric properties can be used to generate a solution. For example, the sentence, "construct a line through point A that is parallel to BC," implies that segments ED and BC are parallel. The major difference in the two task is that the GP provides a to-be-proved statement, whereas the GCN offers the angle and segment measures and indicates what unknowns need to be found.

To further illustrate the similarities between GCN and GP, the concept of a *plan tree* is used to outline the actions regarding the knowledge structure needed during the problem-solving process. *Plan tree* have its basis in ACT-R theory and has been defined by Anderson et al. (1981) as "an outline for actions" (p. 193) that is generated based on the logically separate stages. Using geometry proofs as the subject content to explain the thinking mechanisms of a plan tree, Anderson et al. describe stages as a set of geometric rules that allow students to move from the givens of the task through intermediate levels of statements to the to-be-proved statements. The *plan tree* for proof is the knowledge structure generated by unpacking various links of relevant knowledge and re-organizing these links and knowledge into a logic sequence. This process results in reasoning solution paths that usually involve forward and backward searches. In a forward search, students must search from the givens to find sets of solution paths that can yield the to-be-proven statements. In a backward search, students also have to infer from the to-be-proven statements that may be related to the givens. The process of forward and backward reasoning gradually maps out a path containing relevant geometric knowledge that can connect and fit with both the givens and the conclusion statements.

To solve the GCN and prove the GP, given that multiple strategies can be applied to generate solutions, the *plan trees* in Figure 4.1 demonstrate one strategy with the same geometric properties that can be used to obtain the solutions for the two tasks.

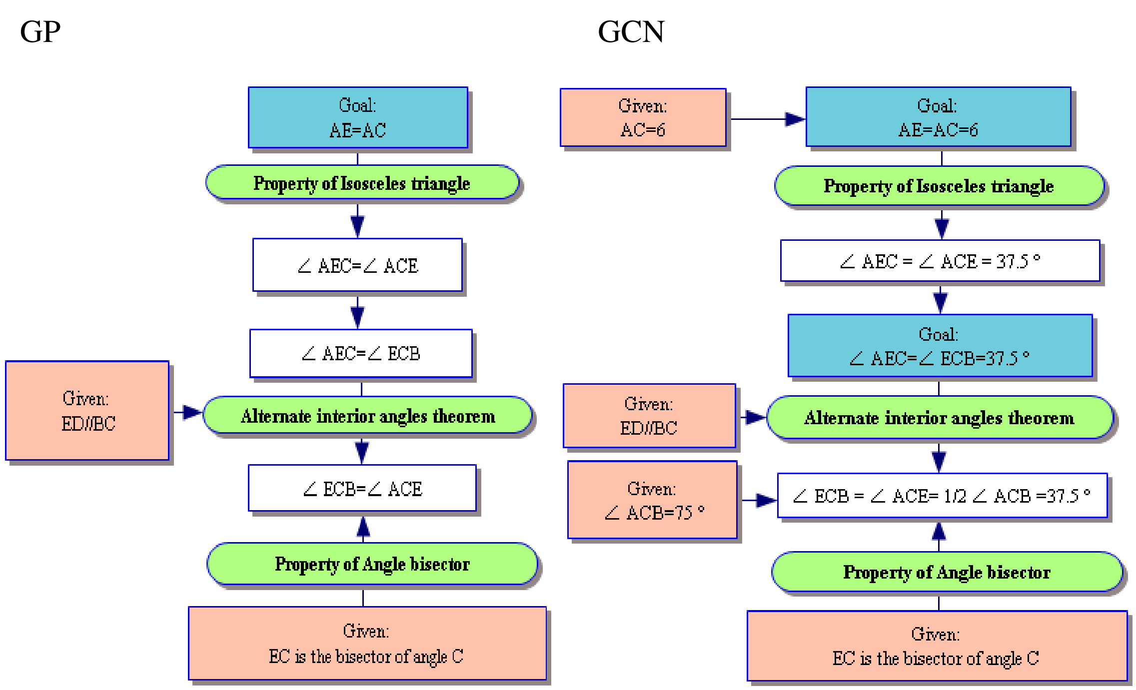


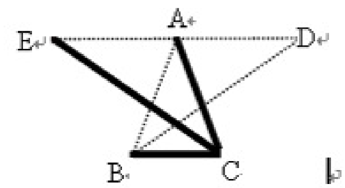
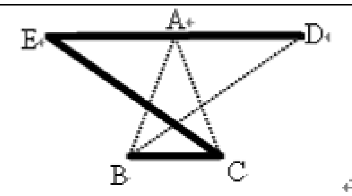
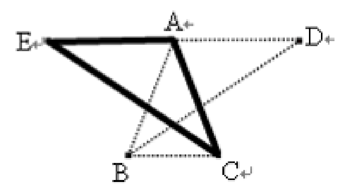
Figure 4.1 The *plan trees* of the GP (on the left side) and the GCN (on the right side)

According to the *plan trees* for both GP and GCN, the geometric properties required to obtain the solutions are the same: the property of angle bisector, the alternate interior angles property, and the isosceles triangle property. Meanwhile, the *plan trees* also demonstrate the similarity in sequencing these geometric properties to generate solutions for each task. As can be seen on the left side of Figure 4.1, the GP solution involves several reasoning steps to prove that AE=AC. The first step is to infer ∠ECB=∠ACE using the property of angle bisector. Then, ∠AEC =∠ECB can be obtained by applying the alternate interior angles property given that ED is parallel to BC. In the following, the inferences ∠AEC=∠ECB and ∠ECB=∠ACE together allow one to deduce that ∠AEC=∠ACE. The final step is to use this inferred result together with the property of an isosceles triangle to obtain the conclusion AC=AE, which is the goal of the task.

To find the length of AE and the measures of ∠AEC, the GCN also requires the same sequence of geometric properties as GP does. The first step is to infer that ∠ECB=∠ACE=37.5° because of the property of angle bisector. The second step further reasons ∠AEC=∠ECB=37.5° on the basis of the alternate interior angles theorem so that the answer to the first unknown can be obtained. The next step is to use the statement ∠AEC =∠ACE = 37.5°, the given AC=6 cm, and the property of an isosceles triangle to conclude that the length of AE is 6 cm, which is the answer to the second unknown.

Furthermore, the sequence of reasoning steps along with corresponding geometric properties for both tasks also requires students to visualize the same sub-constructs in the given diagram. Table 4.3 summarizes the steps together with the description of visualizing the sub-constructs of the given diagram and identifying the relevant geometric properties necessary to obtain solutions for the two tasks.

Table 4.3 The process of visualizing sub-constructs of the diagram for corresponding geometric properties

Steps with description of visualizing the diagram	Sub-construct in the diagram
Step One: Identify ∠ACB with its two sub-angles (∠ACE and ∠ECB) and infer that ∠ACE=∠ECB because EC is an angle bisector.	
Step Two: Identify lines ED, BC, and transversal EC and then infer that ∠AEC=∠ECB because of ED//BC and the alternate interior angles property.	
Step Three: Identify △AEC and infer that AE=AC because of the property of isosceles triangle and the inferred result ∠AEC=∠ACE.	

Here students need to identify and infer from the diagram: (1) the bottom ∠ACB in the △ABC and its relation to segment EC; (2) the relationship among the parallel lines ED, BC, and its transversal EC; and (3) the △AEC associated with ∠AEC and ∠ACE as well as segments AE and AC.

In sum, this section demonstrates that GCN and GP are similar to each other in terms of the actions taken during the problem-solving process, especially in relation to the actions of visualizing the sub-constructs in the diagram to retrieve relevant geometric properties. This analysis is consistent with the observation Kuchemann and Holyes (2002) who noted that the skills used to solve a GCN are also necessary to construct a GP because each step in a GCN involves a deduction. The analytical analysis shown above further elaborates the similarities in reasoning steps for both GP and GCN tasks and how these steps are strongly associated with the geometric diagram as well as the geometric properties necessary to obtain the solutions for both types of tasks.

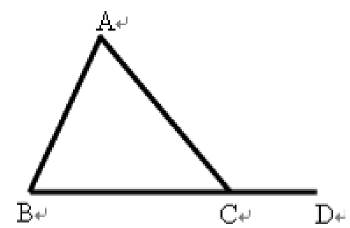
Methods

The study investigates the extent to which GCN is similar to GP when controlling the diagram configurations and the requirements of geometric properties necessary for a solution. To this end, I conducted a study to examine whether students' performance on GCN and GP is more dependent on variations in the format (proving or calculation), or on students' ability to visualize the diagram in such a way that allows them to retrieve the geometric properties necessary for a solution.

Rationale for Survey Design

Four pairs of GCN and GP items (see Appendix 4.1) were designed for the survey. To explore the visualization of the given diagram that may influence students' performance, a total of five criteria are considered in designing the items. The first criterion for designing the items is to control the configurations of diagram used in the items. Each pair of GCN and GP items in the survey employs the same diagram configurations, in which the labels are located in the same places, and both the size and orientation presented in the survey are identical. The second criterion has to do with the requirements of the geometric properties necessary to obtain a solution. Since a task usually can be solved by employing different solution strategies with different sets of geometric properties, the use of identical diagrams in a pair of GCN and GP items provide students the same opportunity to visualize the sub-constructs in the diagram along with corresponding geometric properties that can be used to obtain a solution.

Table 4.4 Pair 1 items included in the survey

Given diagram	GCN item	GP item
	$\triangle$ABC in which AC=BC and BCD is collinear. If the measure of $\angle$ACD is 130°, find the measure of $\angle$ABC.	$\triangle$ABC in which AC=BC and BCD is collinear. Prove $\angle$ACD=2$\angle$ABC

Taking the items in Table 4.4 as an example, we see that solutions to both GCN and GP items can be obtained by applying two sets of geometric properties. The first set includes the triangle sum property, the property of a linear pair, and the isosceles triangle property, and the second set contains the exterior angle property and the isosceles triangle property.

The third criterion in the survey design is to manage the sequence of the geometric properties in the solutions for a pair of GCN and GP items. If the diagrams are identical, the study can control how geometric properties embedded in the diagram are retrieved sequentially to obtain a solution.

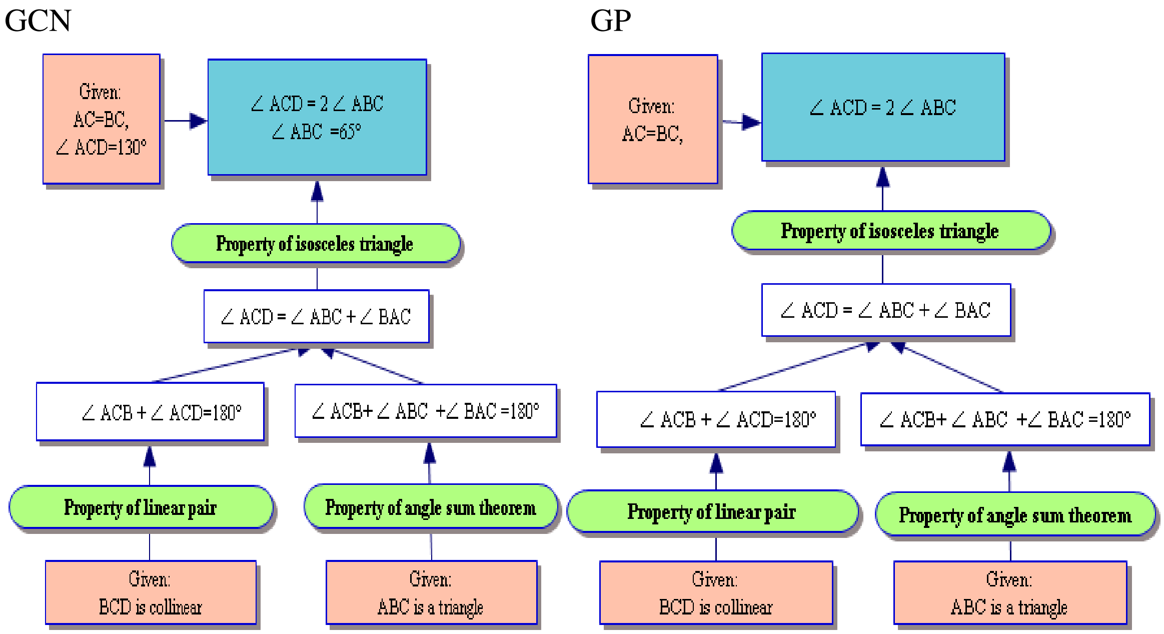


Figure 4.2 The plan trees of the Pair 1 GCN (left side) and GP (right side) items

Figure 4.2 shows the solution structure in the plan trees for both GCN and GP items, in which geometric properties necessary to obtain the solutions appear in the same sequence. According to the plan trees, students first apply the property of linear pairs and the triangle sum property to infer that ∠ACD=∠ABC+∠BAC. Next, students use the isosceles triangle property to obtain the answer to the GCN item, which is 65°, and to prove the statement that ∠ACD=2∠ABC. Sequencing the geometric properties in the same order for obtaining the solutions for the pairs of GCN and GP items can ensure that student's attention is directed to the same sub-constructs in the diagram, thus imposing the same cognitive demand on students when they work on the pairs of the items.

The fourth criterion in designing the survey items is to include different geometric diagrams for different pairs of GCN and GP items. While individual pairs of items differ in terms of diagram configuration and the geometric properties needed to obtain a solution, this study can prevent a confounding variable from being generated because of students' prior experiences working in a similar diagram setting.

In order to further probe how the given geometric diagram may influence students' performance on GCN and GP items, the fifth criterion is to design Pair 1 and Pair 4 items[27] so that their given diagrams can be further compared. The two pairs of items are designed to be similar in terms of the application of the same sets of geometric properties to obtain solutions. The major difference in the two pairs of items is the given diagrams. The given diagram in Pair 1 items is simpler than that in Pair 4 items. Figure 4.3 details the procedure used to make the Pair 4 diagrams more complex in comparison to the Pair 1 diagrams. This process includes adding two auxiliary segments, changing the measures of the angles and sides, and rotating the diagram to a new orientation.

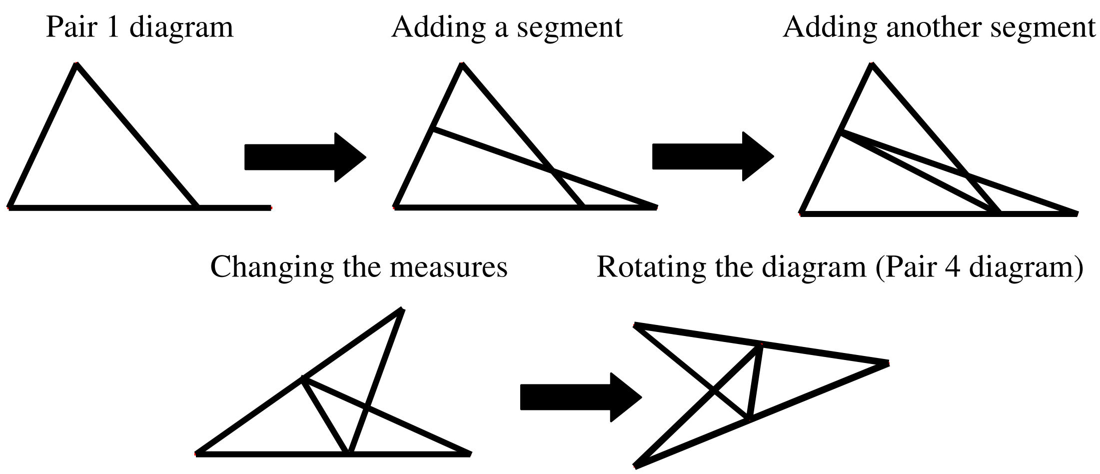
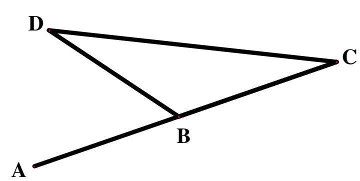
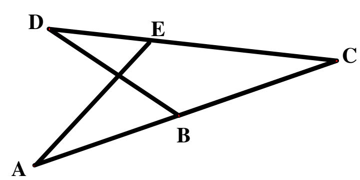


Figure 4.3 Procedure of complicating Pair 1 diagram into Pair 4 diagram

[27] I particularly arranged the Pair 1 in the beginning of the survey and the Pair 4 as the final items that students need to solve. Doing so is to minimize the influence caused by students' experiences in solving the Pair 1 items on their performance of solving Pair 4 items.

The major consideration leading to the decision of how to add greater complexity from Pair 1 to Pair 4 is based on pilot study results. The pilot study included two pairs of items (see Table 4.5) in which the givens for GCN and GP items as well as the geometric properties necessary to obtain solutions were identical. The only difference in the two pairs of GCN and GP items was the diagram configurations. Specifically, as shown in Table 4.5, the diagrams used in both pairs of items were identical except for segment AE added to Pair 2 diagrams. The statistical analysis from the pilot study showed no significant differences in the performance of 9th grade students on the Pair 1 and Pair 2 items for both types of tasks (for GCN: z=.735, p>.05; for GP: z=.925, p>.05).

Table 4.5 Pairs of items with diagrams designed in the pilot study

	Diagram configurations	GCN givens	GP givens
Pair 1		ABC is collinear and BD＝BC. If the measure of ∠ABD is 76°， find the measure of ∠BDC=_____.	ABC is collinear and BD＝BC. Prove ∠ABD=2∠BDC
Pair 2			

Although the pilot study was not rigorously designed in terms of the procedures to administer the items to students and the sample size (about 60 students solved each pair of items), which may weaken the interpretation of the results, the pilot study does raise two concerns with respect to the diagram configurations accompanying the items. The first concern is that adding only one segment to the diagram in Pair 2 items may not prevent students from recognizing the sub-constructs of the diagram and the corresponding geometric properties as they did when solving Pair 1 items. Thus, there was no significant difference in students' performance between the two pairs of items. This result is consistent with a subsequent research question. To what extent can the diagram complexity influence students' performance on each type of task when controlling the geometric properties necessary to obtain solutions? This research question also leads to the decision of complicating the diagram in the Pair 4 items in the formal survey, as shown in Figure 4.3.

The second concern in designing the task items is that the similarity of the givens in GCN and GP items may also become a confounding variable in interpreting the results. While the givens for both pairs of GCN and GP items are identical, practice on Pair 1 items should contribute to students' ability to figure out the solutions for Pair 4 items.

To address the limitations in the pilot study as a result of the added of one segment, the diagrams used in Pair 4 items in the formal survey were made more complex than those used in Pair 1 by total adding two segments, rotating the diagram position, and changing the given measures. In doing so, this study aims to confirm that the diagram complexity does influence students' performance on GCN and GP. Regarding the investigation of the extent to which the diagram complexity may influence students' ability to identify the relevant geometric properties in diagram configurations is beyond the scope of this dissertation. The main concern here is whether the addition of segments, which will increase the diagram complexity in terms of creating new sub-diagram configurations and new geometric properties embedded in the diagram, may prevent students from recognizing the correct diagram configurations to guess the correct geometric properties to obtain a solution.

According to these criteria and concerns in designing the GCN and GP items in the survey, this study investigates the extent to which GCN is similar to GP when controlling the diagram configurations and the requirements of the geometric properties necessary for a solution.

Survey Structure

Four pairs of GCN and GP items are included in the survey. Each item in the survey asks students to complete three sections, including writing down each proving or calculating step, labeling the given diagrams to indicate what actions with respect to geometric property that was worked, and providing geometric properties as reasons to support each calculating or proving step. An example of how to complete the item (see Appendix 4.3) is also given in the survey. Collecting information on how students label their problem-solving process on the diagrams as well as on the geometric reasons for each calculating or proving step can benefit this study by revealing students' visualization of the given diagram regarding the geometric properties needed for solutions. Furthermore, requiring students to complete these two sections on the survey also provides insights into how the superficial visual associations of the diagrams (Aleven et al., 1998) may influence student performance on answering both GCN and GP items.

Survey Procedure

This study employed two survey procedures to administer the items to students. As it is unclear how students experiences with GP items may influence their performance on the paired GCN items and vice versa, the use of different survey procedures can contribute to examining the influence of working on GCN items before or after working on GP items. For example, in Pair 1 items (see Table 4.4) students can calculate the answer for GCN directly (∠ABC=35°) by applying the conclusion statement of its paired GP (∠ACD=2∠ABC). The direct application of the GP conclusion can free students from making inferences based on the given information (e.g., AC=BC) or the geometric properties (e.g., exterior angle property) embedded in the diagram, the cognitive that constructing paired GP item requires. It is also possible that the conclusion statement in GP item also provides students a hint to the solution because they can reason backward from the conclusion (∠ACD=2∠ABC) to generate a solution plan.

In addition, another uncertainty is regarding whether asking students to solve GCN and GP items at different times also influences their performance on the two types of items. To further understand how students' work on both types of tasks may interrelate, this study creates a four-condition model. Central idea to the model is asking students to work on GCN and GP items in different orders and different time sequences. The four-condition model will provide rich data for investigating the interface of students' work on both GP and GCN items.

Table 4.6 Details of the four survey conditions

Survey Condition	Day 1	Day 2
Condition 1: GP first and GCN later	4 GP	4 GCN
Condition 2: GCN first and GP later	4 GCN	4 GP
Condition 3: GP first and GCN later	2 GP first and 2 paired GCN later	2 GP first and 2 paired GCN later
Condition 4: GCN first and GP later	2 GCN first and 2 paired GP later	2 GCN first and 2 paired GP later

All participating students completed four pairs of GP and GCN items on two consecutive days. As Table 4.6 shows, Condition 1 refers to the situation in which students solved all four GP items on Day 1 and solved paired GCN items on Day 2. Opposite to Condition 1, Condition 2 asked students to solve GCN items on Day 1 and paired GP items on Day 2. For Condition 3, students were required to solve two GP items first and then two paired GCN items on Day 1. These students worked on another two pairs of items in the same way on Day 2. Condition 4 is also opposite to Condition 3 and asked students to work on two GCN items first and then two paired GP items on Day 1 as well as the other two pairs of GCN and GP items in the same sequence on Day 2. Because of the study design, Condition 1 and Condition 2 allow a direct comparison of order effects (GP before or after GCN), and Condition 3 and Condition 4 also allow a comparison of the same order effect. In addition, conditions 1 and 3, as well as conditions 2 and 4, also allow a direct comparison of timing effects (i.e., paired GP/GCN items solved close in terms of time versus separated by one day).

Students were not informed that they needed to work on GCN or GP items on Day 2 when they worked the items on Day 1. Students were also not allowed to revise their previous work while they worked on the paired items. Before formally surveying the items, students were well informed about how to work on the items in survey.

Student Sample and Procedure to Answer the Survey Items

8th and 9th grade students from Taiwan participated in this study. By selecting these two grades of students, this study can compare how students' experiences in learning GP may influence their performance on solving GCN and GP items. 8th grade students in Taiwan do not have to learn how to construct GP but 9th graders do. The Taiwanese curriculum standard of mathematics introduces mostly geometry content in 8th grade and 9th grade. In the 8th grade, the curriculum standard uses the geometric content of the congruence of triangles and the properties related to parallel lines and quadrilaterals to gradually introduce the proof concept to students (Ministry of Education, 2003). In this stage, students are required to fill out only one of the steps in a proof or provide the corresponding geometric properties for a proof step. In 9th grade, GP is formally introduced and students are required to learn to construct proofs by themselves.

The survey was administrated to 9th grade students during May and June 2009 after the students had completed all the proof lessons presented at the middle school level. Survey for the 8th grade students was administrated in June 2009 after students had learned the geometric content of the properties related to triangles and quadrilaterals. Thus, students had enough geometric knowledge to answer the pairs of items in the survey.

A total of 483 9th grade students from three middle schools and 509 8th grade students from two middle schools in Taiwan answered the survey items. In order to prevent the competence disparity among classes and schools from being a confounding variable in the comparison of students' responses, student participants were assigned to condition treatments on the basis of their class. In other words, students in a class were equally and randomly assigned to one of the four condition treatments and completed the required survey items in different orders and time sequences on two consecutive days.

Because some of the participating students only answered the survey items on one of the two consecutive testing days, their responses were excluded from this study. Finally, a total of 413 9th grade students and 502 8th grade students constituted the valid sample for this study. The following table summarizes the distributions of students in the two grade levels assigned to the four condition treatments.

Table 4.7 Distributions of students in two grade levels assigned to condition treatments

	9^{th} grade	8^{th} grade	Number of students
Condition 1: GP first and GCN later (separated a day)	100	124	224
Condition 2: GCN first and GP later (separated a day)	103	122	225
Condition 3: GP first and GCN later (solved at a close proximity)	107	128	235
Condition 4: GCN first and GP later (solved at a close proximity)	103	128	231
Total number of students	413	502	915

Coding Scheme Used to Analyzed the Response on the GCN and GP Items

An elaborated coding scheme developed by Cheng and Lin (2005; 2006) was employed to evaluate students' responses on GP items. In the scheme, Cheng and Lin categorized students' responses on GP on the basis of crucial geometric properties that are necessary to obtain a proof solution. They proposed four levels in describing student performance on constructing GP items: acceptable proof, incomplete proof, improper proof, and intuitive responses. Their clarification in relation to geometric properties necessary to generate proof solution is aligned with the study, which aims to explore how the geometric properties embedded in diagram configurations can determine student performance on GCN and GP. In order to compare GCN and GP items, levels of students' responses on the GCN items in correspondence with these on the GP items were established (see Appendix 4.4) in the following manner: correct calculation with reasons, incomplete calculation, improper calculation, and intuitive response.

In addition, researchers have reported that students are better at providing numerical answers than articulating the geometric properties as reasons for their calculations because they can rely on the superficial visual associations of geometric diagrams as a way to guess answers in GCN items (Aleven et al., 1998; Ayres & Sweller, 1990). This study also assesses the influence of superficial visual associations of geometric diagrams in guessing GCN answers by analyzing the percentages of students who can obtain correct answer in each of the five coding levels. The analysis can contribute to (1) understanding students' use of superficial visual associations of diagram in relation to their ability to identify the geometric properties necessary to obtain the solutions; and to (2) examine if the use of superficial visual associations of diagrams to guess the answers of GCN is task-dependent. If guessing numerical answers to GCN items is task-dependent, it may bring up a follow-up research question: is it possible for certain GCN tasks in which students can not rely on superficial visual associations of diagrams to obtain the correct numerical answers? Although investigating this research question is beyond the scope of this study, it should be carefully considered as one of key factors in outlining the relationship between GCN and GP. Moreover, as proof is often regarded as a hard topic (Chazan, 1993; Cheng & Lin, 2005; 2006; 2008; Harel & Sowder, 1998; Heinze, Cheng, & Yang, 2004; Senk, 1989; Weber, 2002), this study also investigates the percentages of students who did not provide any response to the GCN and GP items respectively.

To analyze students' overall performance on the four pairs of items, points for the categories in the coding framework were assigned for the statistical purposes. The criterion to determining the points to students' responses is aligned with the proposed hypothesis: diagram and geometric properties necessary to obtain solutions are keys to students' performance on GCN and GP items. In this regard, the better ability to visualize the geometric properties from the diagrams and then use these geometric properties to generate a solution, the more points that students can obtain. If students could not recognize any geometric properties in the diagram, even though they can provide some intuitive responses to the GCN and GP items, they still can't get any point. As a result, three points were assigned to responses consisting of an acceptable proof or a correct calculation with reasons. Two points were given to responses coded as an incomplete proof or incomplete calculation. One point was assigned to responses coded as "improper proofs or improper calculations". Students who provided intuitive responses or did not answer the survey items received no points.

The statistical methods applied to analyze students' performance on GCN and GP items included descriptive analysis, independent-samples T-tests, paired-samples T-tests, correlation tests, Chi-square tests, and nonparametric related-samples tests. Furthermore, this study also applied a grounded-theory approach to investigate students' error responses on both types of items. The analysis focused on students' written responses including calculating or proving sentences, labels or re-drawing on the given diagram configurations, and geometric properties used to support the corresponding calculating or proving sentence. The in-depth qualitative analysis can reveal how students visualize the diagram configurations and identify the corresponding geometric properties needed for their solutions as well as the connection between diagram visualization and identification of geometric properties.

List of Sub-research Questions for the Study

To explore the extent to which GCN can be similar to GP when controlling the diagram configurations and the geometric properties necessary to obtain a solution, this study implemented a survey study. The goal is to investigate students' performance on the designed four pairs of GCN and GP items and to evaluate if increasing the complexity of the diagram configurations impacts students' performance on both types of items. Selecting two grade levels of students participating in this study also provides the data source to examine students' experiences of learning proofs in influencing their performance on both types of tasks. While it is unclear how students' experiences in solving the pairs of GCN and GP items in survey relates their performances on both types of items, two survey procedures and two ways to solve the GCN and GP items in different times are also examined.

In addition, this study explores how students rely on superficial visual associations of diagrams to guess the GCN answers by means of comparing the percentages between correct calculations only and correct calculation with reasons. As GP is often viewed as a difficult topic, this study also checks the percentages of students who did not provide an answer between the GCN and GP items. Finally, the qualitative analysis of students' error responses related to the diagram and geometric properties needed for solutions can provide rich information in conceptualizing the relationship between GCN and GP.

The following summarizes sub-research questions examined in this study.

1. *When controlling the given geometric diagrams and the geometric properties required to obtain a solution, do students perform differently on the GCN and GP items?*
2. *If the geometric properties necessary to obtain solutions are identical, do different diagram shapes influence students' performance on the GCN and GP items?*

3. *When their performance on the GCN and GP items is compared, do students at different grade levels perform differently?*
4. *Do the survey procedures (GCN before or after GP) influence students' performance on the GCN and GP items?*
5. *Do the timing effects (GCN/GP solved close in terms of time vs. separated by one day) influence students' performance on the GCN and GP items?*
6. *Do the percentages of students who provide correct answers with reasons differ from those of students who provide correct answers only in GCN items? If so, what are the distributions of students, who can obtain correct answers, in relation to the four coding categories?*
7. *Do the percentages of students who did not provide any responses differ between the GCN and GP items?*
8. *What types of error responses in relation to diagram configurations and corresponding geometric properties can be found?*

Findings

Comparison between GCN and GP Items

As shown in Table 4.8, the mean score on GCN items for both grades of students is 6.33 and the mean score on GP items is 6.03. The difference in the mean scores of the two types of items is statistically significant ($t = -4.508$, $p < .05$). This finding shows that GCN items are easier for students than GP items when controlling the given diagrams and the requirements of geometric properties for solutions in the survey.

Table 4.8 Comparison of students' performance on GCN and GP in general

T test and P value						
	Mean	Mean Difference	Std. Error Mean	T	df	Sig.
GCN	6.33	-.305	.068	-4.508	914	.000*
GP	6.03					

*: A significant difference at the .05 level

A closer examination of the responses of students in each grade reveals a different result. Table 4.9 shows that the difference in the mean scores of GCN items and GP items for 8th grade are still significant (GCN=6.05, GP=5.65, t=-4.289, p<.05). However, different from that for 8th grade, the results for 9th grade students are not significantly different for both types of items (GCN=6.68, GP=6.49, t= -1.945; p>.05).

Table 4.9 Comparison of GCN and GP responses between two grade levels

T tests and p values								
		Mean	Mean difference	Std. Error Deviation	T	df	Sig.
8th grade	GCN	6.05	-.398	.093	-4.289	501	.000*
	GP	5.65					
9th grade	GCN	6.68	-.191	.098	-1.945	412	.052
	GP	6.49					

*: A significant difference at the .05 level

Figure 4.4 provides a graph of the difference in the mean scores of GCN and GP items for the two grade levels. As shown in the figure, the mean difference for 8th grade is much greater than that for 9th grade. Although for 9th grade students the mean score of the GCN items is still slightly higher than that of GP items, the statistical result confirms that the GCN and GP tasks impose the same level of cognitive demand on 9th students. In other words, for 9th grade students, the given diagram and the requirements of geometric properties for the solutions are the key to determining their performance on both types of tasks as opposed to the difference in the format between a calculation task and a proof task.

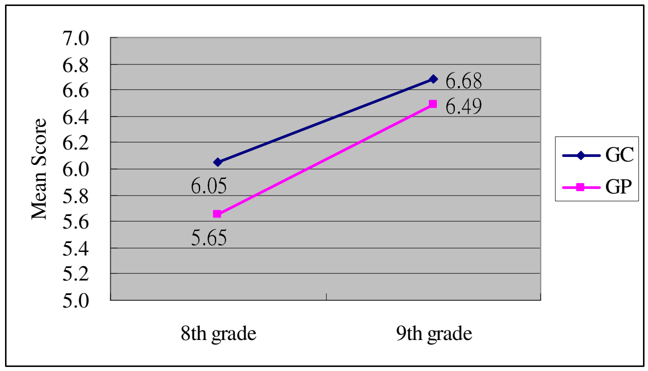


Figure 4.4 Mean scores of GCN and GP for both grades

The results from correlation analysis also show that students' performance on the GCN items is highly related to that on the GP items. The coefficients (see Table 4.10) for both grades together, 8th grade, and 9th grade are 0.874, 0.864, and 0.884 respectively, all of which lead to the conclusion that students' GCN item responses are significantly dependent on GP items. The correlation results also indicate the size of the effect of this dependence is large (Cohen, 1988; 1992).

Table 4.10 Correlation analysis on GCN and GP for both grades together, 8th grade, and 9th grade

Correlation tests and p values		
	Pearson correlation	Sig.
Both grades	.874	.000*
8th grade	.864	.000*
9th grade	.884	.000*

*: A significant different at the .05 level

Comparison on Individual Pair Items

This study also investigates students' responses on each pair of items. This was done to determine whether students' performance varied depending on the task setting, especially for the items included in the survey that also vary in terms of the diagram composition and the requirements of geometric properties.

The first step here is to check the dependency between GCN and GP items for two grade levels by applying a chi-square test method.

Table 4.11 Chi-square tests and p values of comparing GCN and GP items for two grades

	χ^2 values and p value in parenthesis			
	Pair 1	Pair 2	Pair 3	Pair 4
8^{th} grade	435.189 (.000*)	522.365 (.000*)	606.766 (.000*)	353.861 (.000*)
9^{th} grade	327.145 (.000*)	487.845 (.000*)	529.080 (.000*)	388.530 (.000*)

*: A significant difference at the .05 level

As revealed by the results in Table 4.11, student performance in each grade on the GCN items is significantly dependent to its paired GP items. For 8^{th} grade, the values of χ^2 test for the four pairs of items all result in statistical significances at the .05 level. For 9^{th} grade students, the values of χ^2 test also cause significances at the .05 level for the four pairs of items.

After recognizing student performance on GCN and GP items are significantly dependent, the second step is to analyze whether students' performance differs significantly for individual pair of GCN and GP items. Table 4.12 demonstrates that 8^{th} grade students performance on GCN items of Pair 1, Pair 2, and Pair 4 was significantly better than that on the paired GP items (Z=-2.150, p<.05 for Pair 1; Z=-3.305, p<.05 for

Pair 2; Z=-3.651, p<.05 for Pair 4). For Pair 3, 8th grade students' performance was not significantly different between the two types of tasks.

Opposed to the results of 8th grade students, 9th graders performed no significant differences in GCN and GP items for all four pairs of items.

Table 4.12 Paired-samples tests and p values for individual pair of items for two grade levels

Paired-samples tests and p values							
		8th grade			9th grade		
		Mean	Z value	Sig.	Mean	Z value	Sig.
Pair 1	GCN	1.81	-2.150	.032*	1.89	-1.230	.219
	GP	1.71			1.82		
Pair 2	GCN	1.47	-3.305	.001*	1.67	-1.819	.069
	GP	1.35			1.60		
Pair 3	GCN	1.13	-.248	.804	1.35	-.941	.347
	GP	1.14			1.38		
Pair 4	GCN	1.64	-3.651	.000*	1.78	-1.850	.064
	GP	1.45			1.69		

*: A significant difference at the .05 level

<u>Comparison of Students' Responses Between Pair 1 and Pair 4 Items</u>

In order to further investigate how a geometric diagram impacts student performance on GCN and GP items, the study designed two pairs of items, Pair 1 and Pair 4, in both which the geometric properties necessary for solutions are parallel, but the diagrams given in the two pairs of GCN and GP items are different.

As Table 4.13 shows, students in each grade performed significantly better on Pair 1 items than on Pair 4 items. For 8th grade students, the mean scores for GCN (1.85) and GP (1.76) for Pair 1 are higher than the mean scores for GCN (1.76) and GP (1.56) for Pair 4. This finding is statistically significant (Z= -4.175, p<.05 for GCN items; Z=-7.155, p<.05 for GP items). A similar result can be found for 9th grade students; for this group the mean scores for GCN (1.81) and GP (1.71) for Pair 1 are greater than that for GCN (1.64) and GP (1.45) for Pair 4 items. This finding is also statistically significant (Z=-3.570, $p<.05$ for GCN; Z=-6.653, $p<.05$ for GP). Figure 4.5 provides two graphs showing the differences in mean scores for both types of items in Pair 1 and Pair 4. The above findings confirm that the given diagrams do influence students' performance on GCN and GP items, even though the sequence of the geometric properties necessary to obtain a solution are in the same order.

Table 4.13 Related-samples tests for comparing on Pair 1 and Pair 4 items

Related-sample Z tests and p values				
		Mean value	Z value	Sig.
8^{th} grade	Pair 1 GCN	1.81	-3.570	.000*
	Pair 4 GCN	1.64		
	Pair 1 GP	1.71	-6.653	.000*
	Pair 4 GP	1.45		
9^{th} grade	Pair 1 GCN	1.89	-2.274	.023*
	Pair 4 GCN	1.78		
	Pair 1 GP	1.82	-3.242	.001*
	Pair 4 GP	1.69		

*: A significant difference at the .05 level

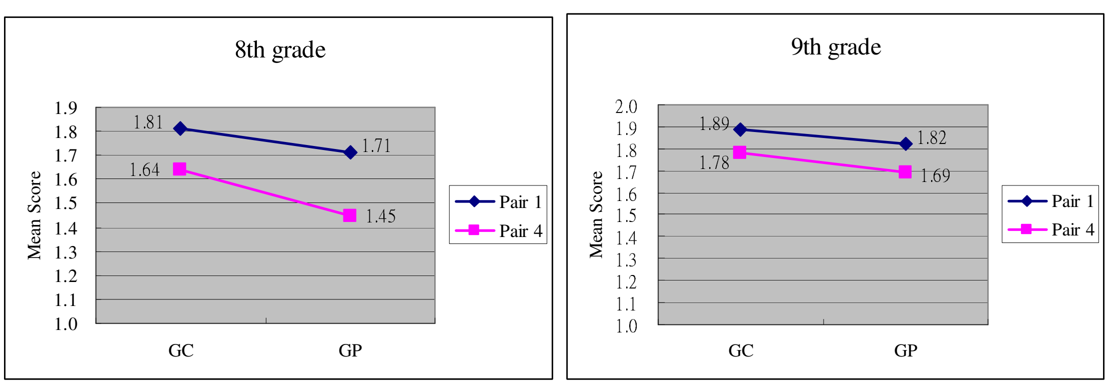


Figure 4.5 Mean scores for Pair 1 and Pair 4 items

Comparison between Two Grade Levels

Table 4.14 shows that the mean difference in 8th grade and 9th grade students' scores on GCN items is -.631, which causes a significance (t= -3.019, p<.05). The mean difference in GP performance between the two grades is -.838, which is also statistically significant (t= -2.451, p<.05). The results indicate that 9th grade students performed significantly better than 8th grade students did for both GCN items and GP items. Figure 4.6 is a graph showing the significant differences between both grade levels for GCN and GP items.

Table 4.14 T tests for comparing grade differences for GCN and GP items

		T tests and p values				
		Mean difference	Std. Error Difference	T	df	Sig.
GCN	8th grade / 9th grade	-.631	.278	-3.019	913	.003*
GP	8th grade / 9th grade	-.838	.257	-2.451	913	.014*

*: A significant difference at the .05 level

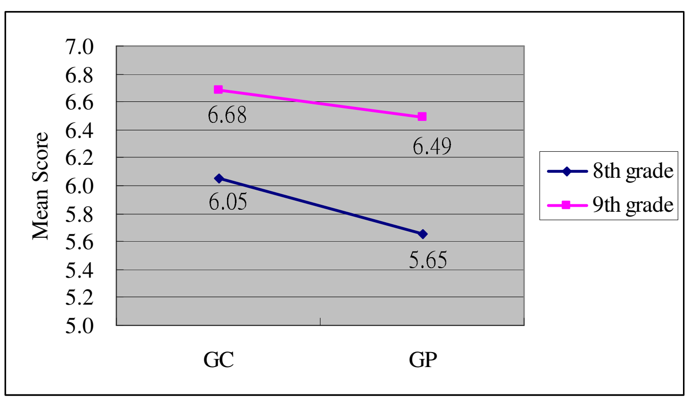


Figure 4.6 Mean scores for 8th grade and 9th grade on GCN and GP items

Comparison between Two Survey Procedures

As it is unclear if the order of GCN and GP items in which students solve problems influences their performance on the two types of tasks, this study examines two survey procedures (GCN first or after GP) on students performance on both types of items. Illustrated in Table 4.15, for GCN items the mean score for the procedure that students solved GCN first and GP later is 6.06, whereas the mean score when GP was solved first and GCN later is 6.61. The mean difference in GCN scores in relation to the two survey treatments is significant ($t=2.156$, $p<.05$). In other words, the experience of proving GP items does influence students' performance on solving paired GCN items. For GP items, the means score for the procedure when students solved GCN first is 6.08, whereas the mean score of GP when students solved GP first is 5.98. The mean difference in GP item scores in the two survey procedures is not statistically significant ($t= -.339$, $p>.05$). Thus, the experience of solving GCN items first does not influence student performance on proving the paired GP items.

Table 4.15 T tests for comparing survey procedures on GCN and GP items

T tests and p values						
		Mean	Std. Error Differences	t	df	Sig.
GCN	GCN first GP later	6.06	.278	2.156	913	.031 [a]
	GP first GCN after	6.61				
GP	GCN first GP later	6.08	.257	-.339	913	.734
	GP first GCN later	5.98				

*a: A significant difference at the .05 level

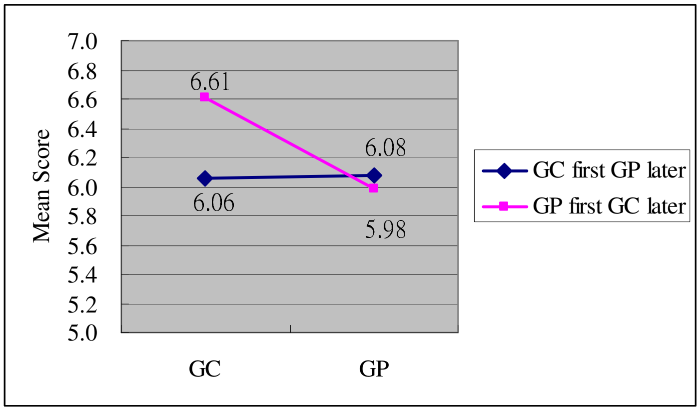


Figure 4.7 Mean scores for GCN and GP under two survey procedures

Figure 4.7 further shows the difference in mean scores for GCN items, which is greater than that for GP items. The intersection of the two lines also indicates that students still performed better as a result of their practice on the paired items for both GCN and GP items.

<u>Comparison on the Timing Effect</u>

This study also examines the timing effect on students' performance on GCN and GP items. As shown in Table 4.16, students' performance on GCN for the two conditions (GCN/GP solved closely together vs. separated by one day) is not significant ($t=-.020$, $p>.984$). A similar result is obtained for GP items ($t= -.508$, $p>.05$). Figure 4.8 displays the slight differences in mean scores for GCN and GP under the timing effect treatment. The result demonstrates that solving paired GCN and GP items at different times does not influence students' performance on the two types of items.

Table 4.16 T tests for comparing the timing effect

T tests and p values						
		Mean	Std. Error Differences	t	Df	Sig.
GCN	A close proximity	6.34	.255	-.020	913	.984
	Separated by one day	6.33				
GP	A close proximity	6.10	.278	-.508	913	.612
	Separated by one day	5.96				

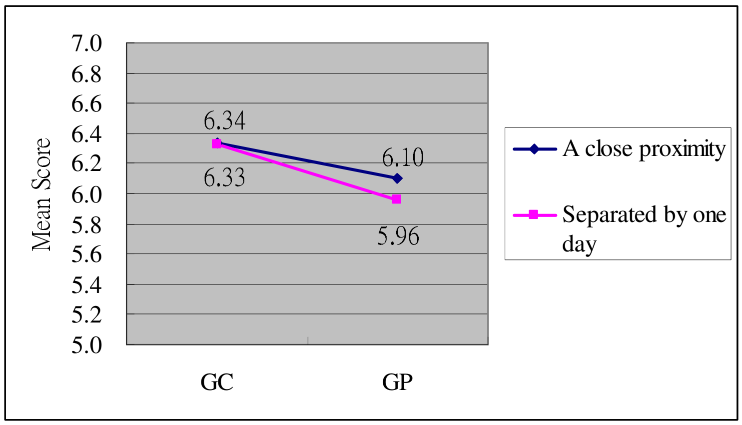


Figure 4.8 Mean scores for two time arrangement conditions

<u>Comparison between Correct Calculations and Correct Calculation with Reasons</u>

Another aim of this study is to understand how superficial visual associations of the diagram plays a role in influencing student perception to obtain the numerical answers without understanding what geometric properties account for the numerical answers. To achieve this aim, this study compares the differences between percentages of students who can obtain correct calculations and that of students who can obtain both correct calculations and geometric reasons.

Table 4.17 summarizes the percentages of correct calculation and correct calculation with reasons for each GCN item. The table shows that the percentages of obtaining correct answers are much higher than those of finding correct answers with supportive reasons. For Pair-1 GCN item, the percentage of students who can successfully obtain correct calculations is 73.7%, whereas that of correct calculation with reasons is 45.2%. For Pair-2 GCN item, the percentage of correct calculations is 70.1% and for correct calculations with reasons the percentage is 29.1%. For Pair-3 GCN item, even though the percentages are relatively lower than those for the other three pairs, 52.9% of students did obtain correct answers but only 13.7% of students could articulate their corresponding reasons. For Pair-4 GCN item, 68.9% students found the correct answers, but only 40.4% probed the corresponding geometric properties. Figure 4.9 provides a graph showing the performance differences in the percentages of correct calculation and those of correct calculation with reasons.

Table 4.17 Percentages of correct calculation and correct calculation with reasons

	Pair 1	Pair 2	Pair 3	Pair 4
Correct calculation	73.7%	70.1%	52.9%	68.9%
Correct calculation with reasons	45.2%	29.1%	13.7%	40.4%

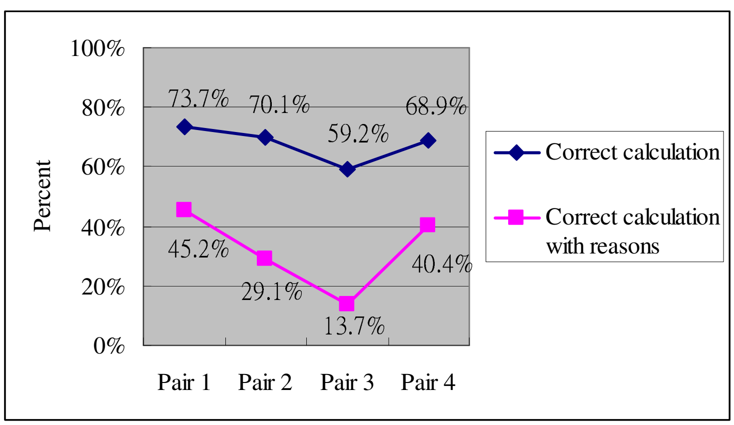


Figure 4.9 Percentage of correct calculation and correct calculation with reasons

The distributions of students who provided correct answers within the five coding categories were also analyzed. The goal here is to (1) understand what is meant by superficial visual associations in the geometric diagram with relation to the crucial geometric properties required for a solution; and to (2) investigate if using superficial visual associations to guess the answers of GCN is task-dependent. Table 4.18 summarizes the distributions of the percentages of students providing correct answers among the five coding categories. Figure 4.10 also provides bar charts of the distributions for the four GCN items.

Table 4.18 Percentages of correct calculations within the five coding categories

	Pair 1	Pair 2	Pair 3	Pair 4
Correct calculations with reasons	61.4%	41.5%	25.8%	58.7%
Incomplete calculation	17.2%	29.2%	32.6%	17.8%
Improper calculation	15.7%	18.6%	37.6%	18.4%
Intuitive response	5.6%	10.8%	3.9%	5.1%
No response	0%	0%	0%	0%
Total	100	100	100	100

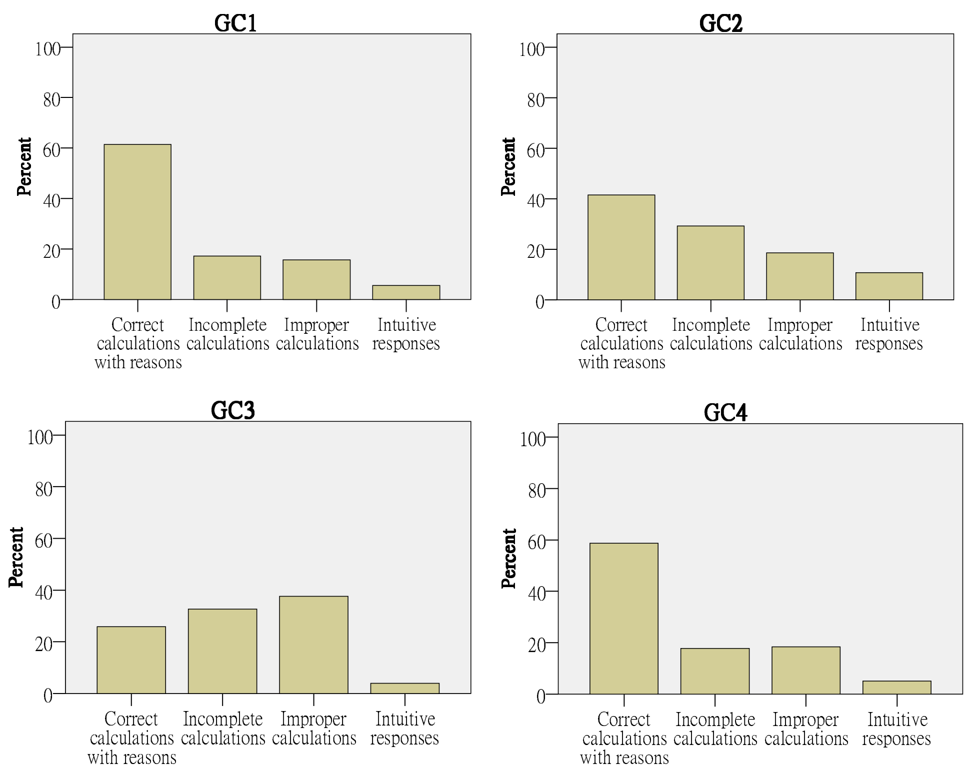


Figure 4.10 Distributions of correct calculation

Several observations can be made from the above table and bar charts. First of all, students who could guess the correct answers, but could not articulate the correct geometric properties underlying their calculations are distributed among the three coding categories: incomplete calculations, improper calculations, and intuitive responses. The distributions show that students may, on the one hand, use superficial association of diagram to guess the answer (e.g., quantities of segments look equal), but, on the other hand, also are able to retrieve some geometric properties embedded in the diagram although these properties may not contribute to the solution generations. For example, student answers coded as "improper calculation" revealed that students could visualize the diagram configurations to retrieve some geometric properties correctly, but these geometric properties do not contribute to solution generation.

Secondly, the distributions shown in Figure 4.10 differ among the four GCN items. This finding implies that using the superficial visual associations of diagrams as an approach to guess the answer is task-dependent. This result may be attributable to the specific characteristics of diagrams which can prompt students' different superficial connections between the given diagrams and the answers to the items as well as the design of the tasks. Thirdly, for GCN1, GCN2, and GCN4 students who could obtain the correct answer and could also recognize the geometric properties necessary for solutions occupy the highest percentages among five categories (61.4% for GCN1, 41.5% for GCN2, 58.7% for GCN 4). However, this is not the case for GCN 3 item. The highest percentage of students obtaining correct answers is located in the category of improper calculations, the situation in which students did not figure out the geometric properties were necessary to obtain the GCN solutions. The final observation made from Figure

4.10 is that the distributions are similar for GCN 1 and GCN 4, items which have similar calculation requirements and the geometric properties for solutions.

Following up the analysis of distributions for correct calculations, this study also examines if grade difference also influences the distributions for GCN items.

Table 4.19 Percentages of correct calculation and of correct calculation with reasons for two grade levels

	Pair 1		Pair 2		Pair 3		Pair 4	
	8th	9th	8th	9th	8th	9th	8th	9th
Correct calculation	68.1%	80.4%	67.9%	72.6%	51.2%	55.0%	64.3%	74.3%
Correct calculation with reasons	43.4%	47.5%	25.9%	32.9%	10.6%	17.4%	39.0%	42.1%

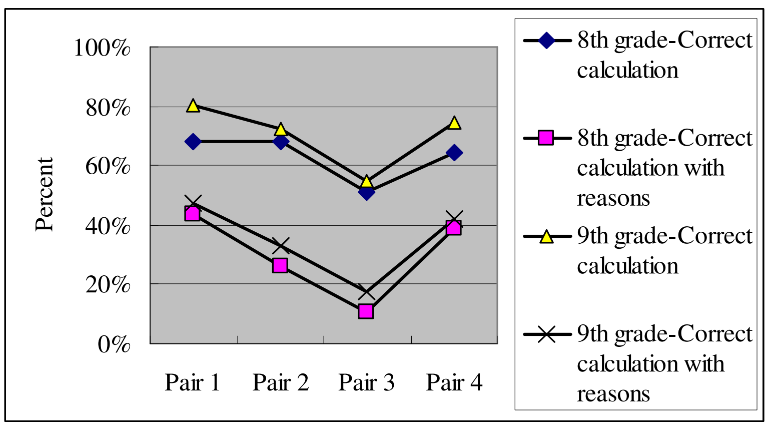


Figure 4.11 Percentages of correct calculation and correct calculation with reasons for two grades

According to Figure 4.11, the percentages of 9^{th} grade students who provided correct answers are higher than those for 8^{th} grade students for the four pairs of items. Similar results of correct calculations with reasons can also be found in Table 4.19. Another important observation that can be made from Figure 4.11 is that the percentage differences between 8^{th} grade and 9^{th} grade for correct calculation are much smaller compared to the differences between correct calculations and correct calculations with reasons in general. A similar result can be identified by observing the percentage differences in 8th grade and 9th grade for correct calculation with reasons, which is also smaller than the differences between correct calculations and correct calculations with reasons.

Further examination of the distributions of the correct answers for individual grade level is shown in Table 4.20 and Figure 4.12. Both the table and bar charts indicate that the percentage distributions of 8th grade and 9th grade students are similar for Pair 1, Pair 2, and Pair 4. However, the distributions for Pair 3 for the two grade levels are quite different. Higher percentage of 8th grade students than 9th grade students could not provide corresponding geometric properties as reasons but could obtain the correct answers.

Table 4.20 Distributions of correct answers for four GCN items between two grade levels

	Pair 1		Pair 2		Pair 3		Pair 4	
	8th	9th	8th	9th	8th	9th	8th	9th
Correct calculations with reasons	63.7%	59.0%	38.1%	45.3%	20.6%	31.7%	60.7%	57.6%
Incomplete calculation	18.7%	15.7%	29.0%	29.3%	27.2%	38.8%	15.2%	20.5%
Improper calculation	13.2%	18.4%	21.1%	15.7%	47.9%	26.0%	19.2%	17.6%
Intuitive response	4.4%	6.9%	11.7%	9.7%	4.3%	3.5%	5.0%	5.2%
No response	0%	0%	0%	0%	0%	0%	0%	0%

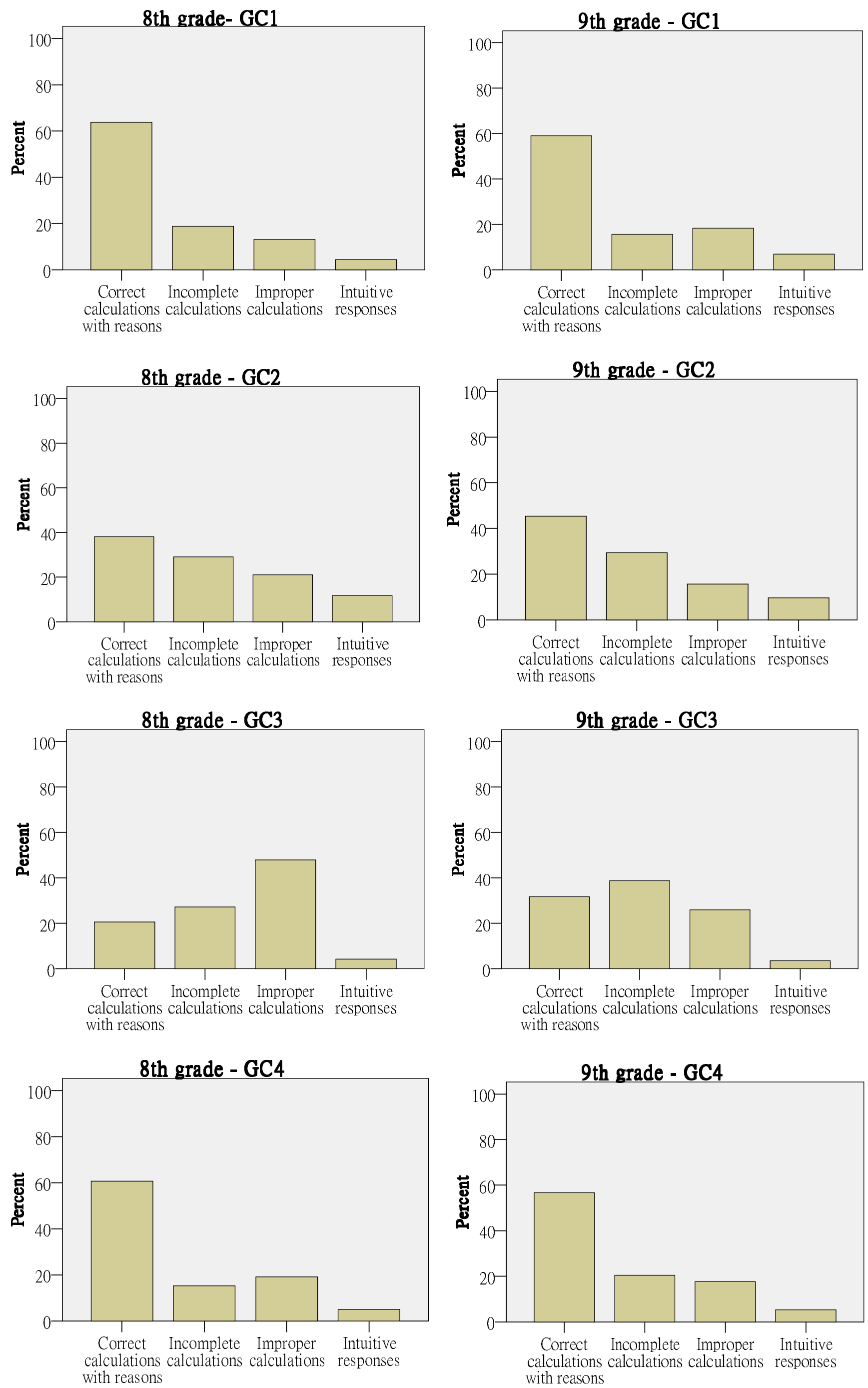


Figure 4.12 Distributions of the two grade students who can provide correct calculations in the four GCN items

Comparison of "No Responses"

Table 4.21 summarizes the percentages of students who provided no responses to GCN and GP items. As shown in Figure 4.13, the percentages of no responses on GP items are all higher than those on GCN items. The percentages for GCN are 4.5%, 8.0%, 6.2%, and 7.9%, while those for GP are 6.7%, 12.6%, 10.2%, and 13.6% for Pair 1, Pair 2, Pair 3, and Pair 4 respectively. These results suggest that students are more likely to give up on trying to solve GP items than to solve GCN items.

Table 4.21 Percentage of "No responses" on GCN and GP items

	GCN	GP
Pair 1	4.5%	6.7%
Pair 2	8.0%	12.6%
Pair 3	6.2%	10.2%
Pair 4	7.9%	13.6%

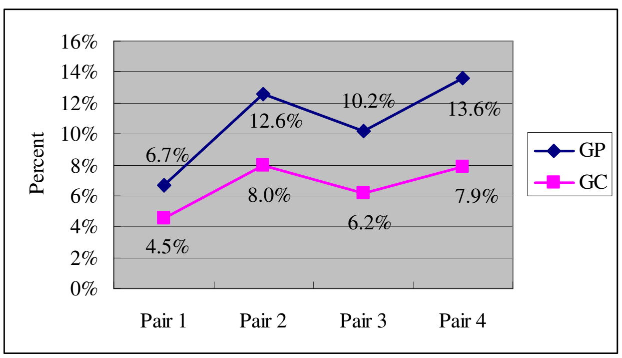


Figure 4.13 Percentage of No Responses

Analysis of Students' Error Responses

In line with the hypothesis that diagram and geometric properties necessary to obtain solution are central to the relationship between GCN and GP, I further inspected students' error responses produced because they incorrectly interpreted the given diagrams or retrieved the geometric properties required to obtain solutions.

Error Responses Related to the Given Diagram and the Geometric Properties Embedded in the Diagram Configurations

Four different types of error responses related to the given diagram and the geometric properties embedded in the diagram configurations are reported. They are (1) using superficial visual associations of diagram to guess the solution, (2) applying correct geometric property but interpreting the property wrongly on the diagram, (3) labeling the geometric property correctly on the diagram but naming the geometric property incorrectly, and (4) the retrieved geometric properties cannot form a successful solution plan. These four types of error responses are described as follows.

Type 1: Using superficial visual associations of diagram to guess the solutions

The first kind of error refers to those in which students relied on the superficial visual associations of diagram (Aleven et al., 1998) to guess the solution. In such errors, the visual associations of diagram become the warrant for students to believe that there are geometric properties in the diagram that they can use. However, these geometric properties can not be inferred or are not given in the tasks. For instance, students may infer two segments or two angles congruently based on their perceptual images of the diagram (e.g., they look the same lengths) but not based on the geometric properties or given information that would support the inference. In addition, the perceptual images based on the visual associations of diagram may also lead to an incorrect solution.

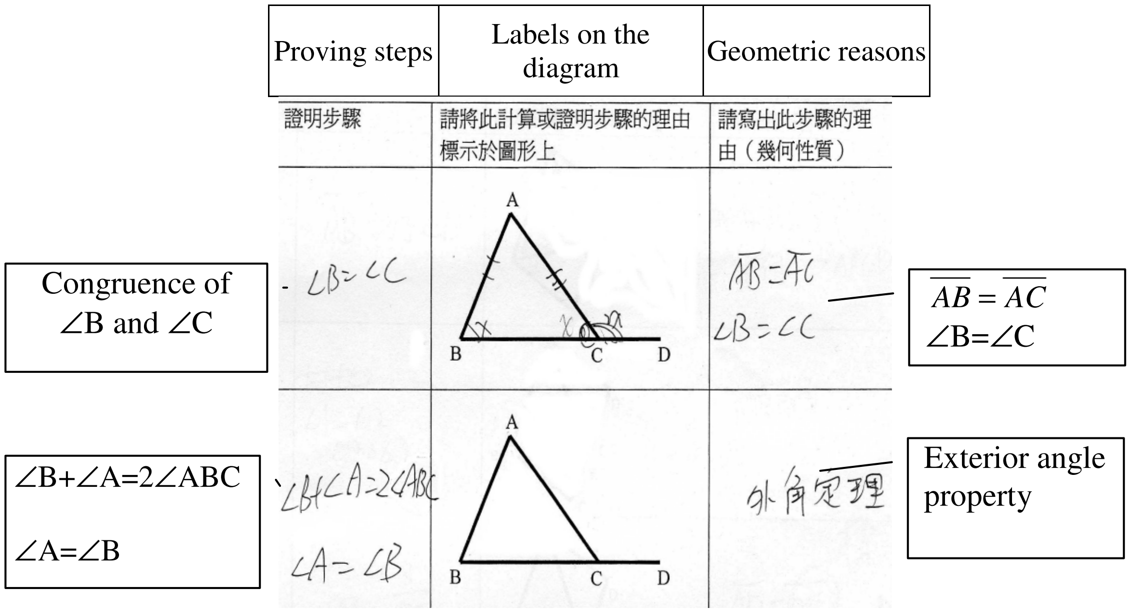


Figure 4.14 Using perceptual images on the diagram to find the solution

Figure 4.14 shows the error in relation to perceptual images by using an example selected from the responses to GP 1 item, the goal of which is to prove ∠ACD=2∠ABC given that $\overline{AC}=\overline{BC}$ and ABC is a triangle. Proving this GP item can be completed by two inferring steps. The first step is to infer ∠A=∠B on the basis of the given $\overline{AC}=\overline{BC}$ and the property of isosceles triangle. The second step has to reason ∠ACD=∠CAB+∠ABC using the exterior angle property. Combining these two proving steps one can conclude that ∠ACD=2∠ABC, which is the goal of this proving task. The student example is selected because it shows the incorrect reasoning in the first step which may be made based on the superficial visual associations of the diagram. Instead of deducing ∠A=∠B, the student inferred ∠B=∠C by the reason that $\overline{AB}=\overline{AC}$, which was incorrect for this task. The congruent labels on segments AB and AC in the given diagram also shows how the student visualized the diagram configurations for this inferring step. To explain the error with regard to the superficial visual associations of the diagram, two possibilities are proposed. The first possibility for the error of viewing $\overline{AB} = \overline{AC}$ may be that the segments AB and AC look as having the same lengths in the diagram so that the student directly assumed these two segments were congruent. The second possibility can be attributable to the prototype of an isosceles triangle, the congruent angles of which are often the base angles on the bottom. These two possibilities associated with an isosceles triangle may be the reasons that can be used to interpret the incorrect inference. As a result, the mis-understanding of the diagram hindered the student from generating a successful solution since the incorrect proving step could not lead to the proving conclusion, $\angle$ACD=2$\angle$ABC.

Interestingly, the example analyzed here shows that using visual associations of diagram to guess solutions is not specific to GCN items but also can happen when constructing GP items.

Type 2: Applying the correct geometric property, but interpreting it wrongly on the diagram

In the second error type, students applied correct geometric properties which were necessary for a solution, but incorrectly interpreted the geometric properties on the diagram.

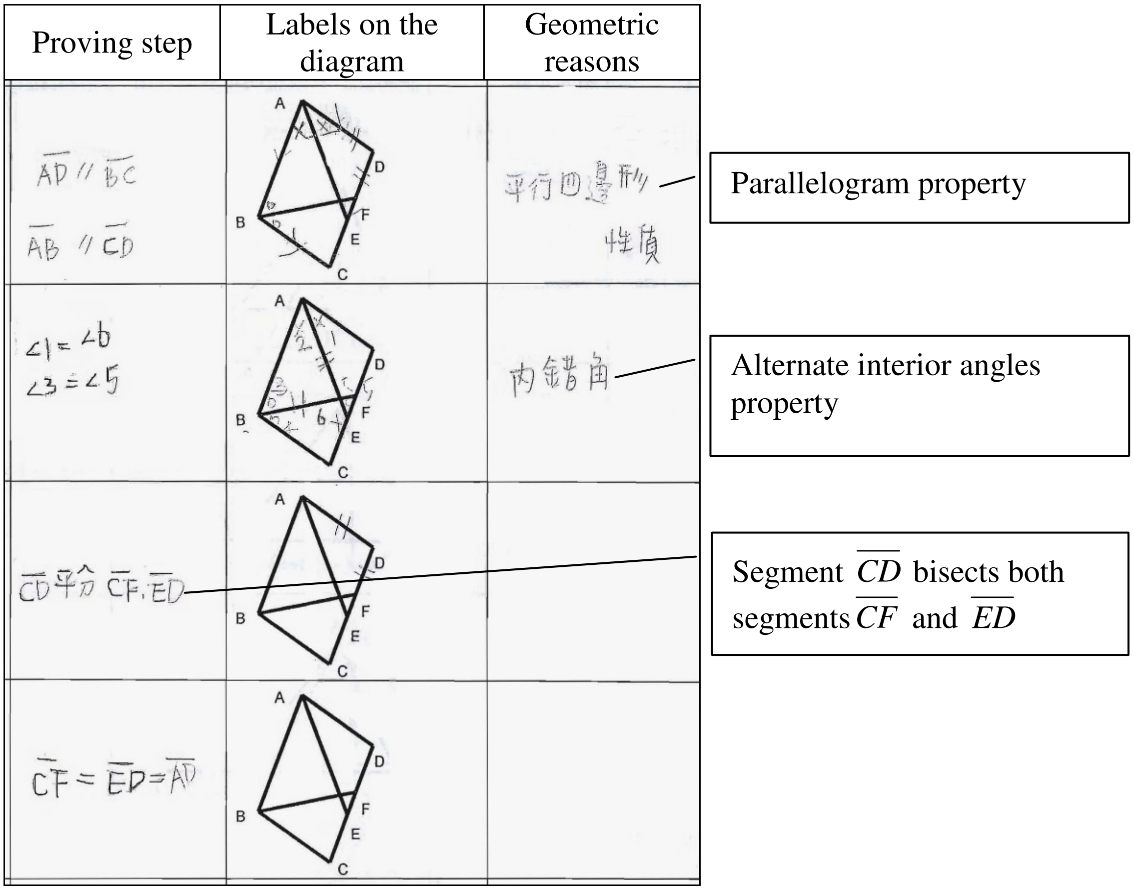


Figure 4.15 Misinterpreting the geometric property on the diagram

Figure 4.15 provides an example of this type of error response. The task in this example is GP 2 item with the goal of proving that ∠DAE=∠DEA given that ABCD is a parallelogram and AE and BF are angle bisectors. As shown in Figure 4.15, the student's first step was to infer that the opposite sides of a parallelogram are parallel (AD//BC and AB//CD) based on the parallelogram properties. Next, the student attempted to prove this task by applying the alternate interior angles property, one of the properties that can be used to obtain a solution. However, the student interpreted the alternate interior angles property incorrectly on the diagram (see Figure 4.16). She labeled the alternate interior angles property on the diagram as ∠1 and ∠6, which were not the angles ∠BAE and

∠AED corresponding to the alternate interior angles property. Nor were these two angles congruent to each other. This misapprehension of the angles corresponding to the geometric property on the diagram also prevented the student from achieving a successful solution because the incorrect inference ∠1=∠6 could not help the student to reason the next step associated with the congruent angles ∠BAE and ∠EAD because of the angle bisector property. As a result, the student did not obtain a proof solution to this task.

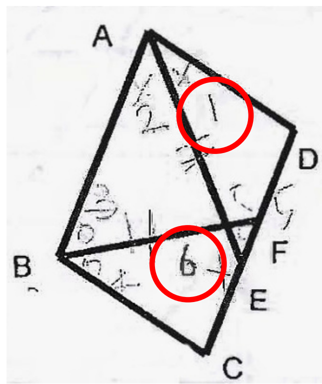


Figure 4.16 Using correct geometric property but labeling the property incorrectly on the diagram

Type 3: Labeling the geometric property correctly on the diagram but naming the property incorrectly

This error type is related to the mismatch between the name of geometric properties and its embedment in a diagram. The difference between this type error and the previous one is that the error here involves the correct interpretation of the diagram, but with the wrong name assigned to the geometric property.

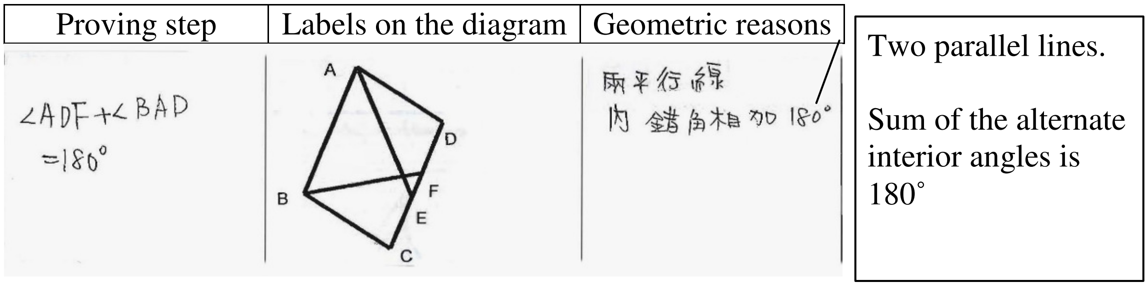


Figure 4.17 Error response of correct interpretation of configurations of the diagram but with incorrect geometric property

The response in Figure 4.17 is part of the proof construction of the GP 2 item from another student. As can be seen, the proving step did not match the corresponding geometric property. The proving sentence indicates that the student recognized that the sum of ∠ADF and ∠BAD was equal to 180° and could also recognize that parallel lines were the one of the geometric reasons for this. However, instead of using the consecutive interior angles property, the student wrote the alternate interior angles property as the reason for the step, which was incorrect.

Type 4: The retrieved geometric properties are not sufficient to generate a solution plan

Another general type of error response related to the geometric diagram involves the use of geometric properties that are not enough to generate a solution to this task.

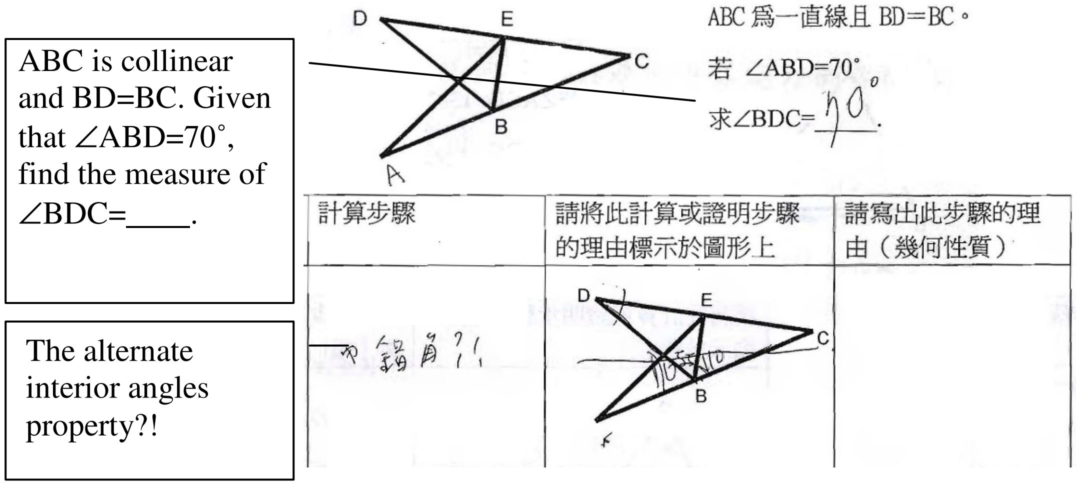


Figure 4.18 Example of type 4 error response

Figure 4.18 gives an example for this type of error response from the GCN 4 item, the goal of which is to calculate the measure of ∠BDC given that BD=BC and the measure of ∠DBA=70˚. The figure shows that the student intended to use the alternate interior angles property to solve this task by drawing a horizontal line that intersects the triangle in the given diagram. Given the measures on the diagram as shown in Figure 4.18, it seems that the student was able to calculate some angle measures (e.g., ∠DBC=110˚). Nevertheless, even so the drawing did not help the student retrieve enough geometric properties from the diagram configurations to generate a solution. The writing of the alternate interior angles property in the task may imply that the students thought the property is crucial to the solution, but, however, the student did not know how this property can help find the answer to the task.

Summary

1. When the given diagram and geometric properties needed to find solutions are controlled, 9th grade students performed equally well on both GCN items and GP

items. But 8th grade students performed better on GCN items than on GP items.

2. When the geometric properties needed to obtain solutions are controlled, diagram configurations did influence students' performance. Students performed significantly better on both GCN and GP items when the diagram had a simple configuration than when the diagram was complex.
3. 9th grade students performed better than 8th grade students on both GCN and GP items.
4. The treatment of survey procedures (GCN first or after GP) significantly influenced students' performance on GCN items, but not on GP items. Students performed better on GCN items when they solved GP items first and then GCN items later. But students performed similarly on the GP items regardless of which item type was presented first.
5. The treatment of the time effect (GCN/GP solved closely together vs. separated by one day) did not cause a significant difference in student performance on GCN and GP items.
6. The percentage of correct calculations for each GCN item was much higher than that for correct calculations with reasons. Students who obtained correct answers, but could not explain the correct geometric properties leading to their calculations were distributed among the three coding categories: incomplete calculations, improper calculations, and intuitive responses. The analysis also indicates that using the superficial visual associations of diagram as an approach to guess the answer is task-dependent.
7. The percentage of students who did not provide any response to GP items was much higher than that for GCN items.

8. Four types of students' error responses specific to the diagram and geometric properties are reported as (1) using superficial visual associations of the diagram to guess the solutions; (2) applying the correct geometric property, but interpreting it wrongly on the diagram; (3) labeling the geometric property correctly on the diagram, but naming the geometric property incorrectly; (4) The retrieved geometric properties are not enough to generate a solution plan. In particular, the error type caused by using superficial visual associations of diagram to guess the solutions can also occur when constructing GP items.

Discussion

Based on the theoretical analysis regarding the analogous distinction between "a problem to prove" and "a problem to find" (Polya, 1945) and comparisons of survey responses of two grade levels of students, this study, from a problem-solving perspective, conceptualizes the relationship between GCN and GP taking two major perspectives into the consideration. The first perspective has to do with the role of geometric diagram and geometric properties necessary to obtain a solution. The 9th grade students' responses showed that both the diagram and the required geometric properties are keys to determining the cognitive demand of geometric tasks given to students. This result highlights the weakness and limitation in theories describing the relationship between calculation and proof (Tall, 2002; 2005; 2007), which do not consider how the diagram might influence the relationship between the two types of tasks. The result also points out the limitations of studies, which have treated GCN as tasks of lower-level cognitive demand useful only for practicing geometric properties (Heinze, Cheng, Ufer, Lin, & Reiss, 2008; Heinze, Cheng, & Yang, 2004; Heinze, Ufer, Cheng, & Lin, 2008), and did not carefully consider the complexity and specificity of a mathematical task (Stein, Smith, Henningsen, & Silver, 2000).

This finding of survey responses of 9^{th} grade students also leads to an underlying assumption regarding the cognition associated with the difference between calculating specific measures and proving general statements. A common clarification between GCN and GP is that GCN tasks are often viewed as special cases because of the characteristics of calculating certain measures, whereas GP involves general statements that cannot be replaced by solving a specific case. For example, Kuchemann and Hoyles (2002) state

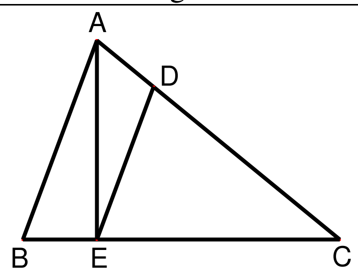
> "…the task [GCN] is only concerned with finding a specific value of an angle, rather than a general relationship between angles, which is more characteristic of a proof (p. 48)"

As indicated by the quote, Kuchemann and Hoyles treat GCN and GP as different types of tasks because GCN deals with certain measures, and not general relationships among angles, which GP does. Other researchers have argued the inappropriateness of using this binary categorization to describe students' cognitive behaviors and beliefs in GCN and GP. For instance, Chazan (1993) pointed out the complexity of students' beliefs in using measurements and deductive proofs as means to verify an argument. When students examine empirical examples for their justification, they can recognize the limitations of using this approach and may develop strategies for minimizing these limitations. Balacheff (1988) also stated that the use of representations (e.g., numerical calculating sentences vs. formal proving sentence) should not be treated as the central in conceptualizing students' proof conceptions. The more importance is how students can internalize the mathematical ideas and detach them from particular representations. In line with this viewpoint, GCN can be more than a naïve example and involve a proof conception.

This study further elaborates the abstract nature of a diagram (Duval, 1995; Fischbein, 1993; Fischbein & Nachlieli, 1998) can influence the relationship between both types of tasks. The argument proposed in this study is that, during the problem-solving process, visualizing the diagram configurations to retrieve relevant geometric properties forces students to see a geometric diagram as an abstract object and to make general inferences regarding the relationships between/among different angles or segments in the diagram (e.g., angle sum theorem involving three angles in a triangle), regardless of whether a student is solving a GCN task or engaging in proving a GP. Even though the measures of angles or segments are provided in a GCN task, these measures might not help or provide hints to students as they try to figure out the diagram configurations with corresponding geometric properties and generate a solution plan. In this regard, measures are used only to execute the calculations for the unknowns if students recognize the geometric properties that they can use to set up the calculating sentence. The results of the 9th grade students' performance on GCN and GP items provides strong evidence to support the argument that geometric diagram and geometric properties needed to obtain solutions are keys to the link between the two types of tasks.

This study also reveals that GCN are not tasks of lower cognitive demand. This finding indicates the limitation of the conclusions shown in other studies (Heinze, Cheng, Ufer, Lin, & Reiss, 2008; Heinze, Cheng, & Yang, 2004; Heinze, Ufer, Cheng, & Lin, 2008). One possible reason for this incorrect conclusion is that the GCN tasks designed and implemented in previous studies did not, as this study does, ask students to provide geometric reasons to account for each calculating step. If explanations are not required, students' performance on GCN will appear to be significant better than their performance on GP because of the *shallow learning* related to solving GCN tasks (Aleven, Koedinger, Sinclair, & Synder, 1998). Here, *shallow learning* of GCN refers to a situation in which students use the superficial visual associations of the diagram to obtain answers to GCN tasks rather reasoning logically from geometric properties. *Shallow learning*, therefore, becomes a confounding variable in conceptualizing the relationship between GCN and GP since the differences in the percentages of correct responses for GCN and GP should be significant when GCN does not require students to provide geometric reasons. Concerning the shallow learning in relation to the relationship between GCN and GP, this study proposes two perspectives that should be carefully considered and further examined. First of all, relying on superficial visual associations of the diagrams is not specific to GCN items. This can also occur in the construction of GP, as shown in the example used to illustrate Type 1 error response in the findings section, in which the student relied on superficial visual associations of the diagram to set up the proving sentences. If the use of superficial visual associations of the diagram is not specific to GCN items, how *shallow learning* influences the relationship between GCN and GP should be further investigated and should be carefully considered in articulating the relationship between GCN and GP. Secondly, I maintain that not every GCN task allows students to use superficial visual associations of diagrams as an approach to obtaining answers because this is task-dependent. To elaborate this viewpoint, I offer the following GCN example in Table 4.22.

Table 4.22 The GCN example used to elaborate shallow learning

Diagram	GCN task description
	In triangle ABC, AE⊥BC, AB//DE, and CD=CE. Given that measure of ∠DCE=40°, find the measure of ∠BAE=_____.

The written information details the properties that can be used to generate a solution, including ABC is a triangle; AE is perpendicular to BC; AB are DE are parallel; and CD=CE. The given also indicates that ∠DCE=40° and the goal is to find the measure of ∠BAE. Table 4.23 demonstrates a solution plan of reasoning the answer to this task.

Table 4.23 A solution to the GCN example in Table 4.22

Steps	Calculating sentence	Reason for the calculating sentence
Step One	∠CDE+∠CED=180°-40°=140°	Triangle sum property
Step Two	∠CDE=∠CED=70°	Properties of isosceles triangle
Step Three	∠EAC=180°-90°-40°=50°	Triangle sum property
Step Four	∠DAB=∠CDE=70° So ∠BAE=70°-50°=20°	Corresponding angles property

As shown in the above table, the solution plan requires four reasoning steps with corresponding geometric properties. The first step is to apply triangle sum property to calculate the sum of ∠CDE and ∠CED which is 140°. The next step involves inferring the congruence angles of ∠CDE and ∠CED based on the properties of an isosceles triangle in order to obtain ∠CDE =70°. The third step is to apply the angle sum property again to compute the measure of ∠EAC, which is 50°. Finally, the measure of ∠BAE can be calculated by reasoning the congruent angles of ∠DAB and ∠CDE based on corresponding angles property.

Using superficial visual associations of the diagram would not help students figure out the answer to this GCN task for two reasons. First of all, no obvious hints are available in the diagram (e.g., two segments or angles appear to have the same measures) that can be directly used to obtain the answer. Although the given indicates that CD=CE, this may not be so visible to students since the diagram orientation is different from the prototype image of an isosceles triangle in which the congruent legs often stand on the two sides and the base segment is horizontal on the bottom (see Figure 4.19).

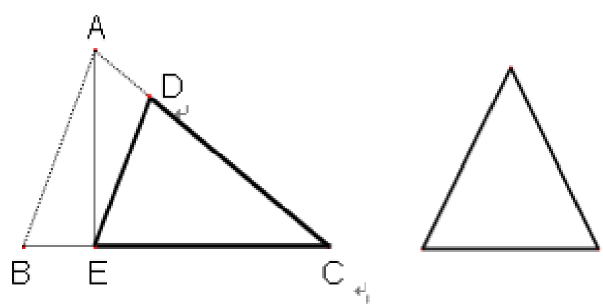


Figure 4.19 Sub-construct of an isosceles triangle in the given diagram (on the left side) and prototype of an isosceles triangle (on the right side)

Secondly, the given diagram is cognitively complex (Duval, 1995) because of the demand to identify the superficial visual associations of diagram to guess the numerical answer. For example, recognizing the sub-construct for the corresponding angles property requires students to suppress the segments AE and BC in order to visualize the sub-construct (see Figure 4.19) (Gal & Linchevski, 2010) as well as recognize what segments and angles in the diagram constitute this property. As a result, students cannot guess the answer to the GCN by relying only on the superficial visual associations of the diagram.

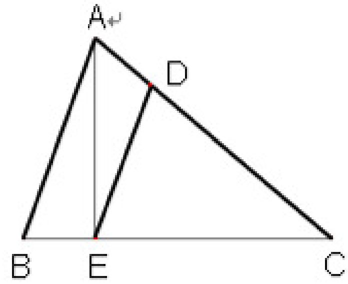


Figure 4.20 Sub-construct of the diagram for the corresponding angles property

After elaborating the roles of both the diagram and geometric properties necessary to obtain the solutions as key to the relation between GCN and GP, I now return to the second perspective associated with conceptualization of the relationship between the two types of tasks, which has to do with the difference in performance of two grade students on GCN and GP tasks. The eighth grade students performed significantly better on GCN items than on GP items, which is a different result as opposed to the 9th grade students. One possible reason to this finding can be the competence of doing calculations that is developed early than constructing proofs. As children at the elementary school level are capable of performing calculations tasks by applying geometric properties (e.g., size of angle, length of distance) (Piaget, Inhelder, & Szeminska, 1960), it is reasonable that, before formally learning GP in schools, students have better competence in solving GCN tasks. But, one may question what kinds of knowledge specific to GP that 8th grade students could not develop from experiences with solving GCN tasks so that their performance on GP is significant lower than that on GCN. For example, combining the proving sentences into a deductive chain is important in proof construction, but students may be lack of this kind of knowledge (Heinze, Reiss, & Rudolph, 2005; Lin, 2005; Ufer & Heinze, 2008). However, students may not develop this kind of knowledge from their experiences with solving GCN tasks. To solve a multiple-step GCN task, each calculating sentence uses some quantities from the given and produces new measures. The new measures can later be used to set up a new calculating sentence based on the corresponding geometric property. The recycling of using quantities from the given and new quantities ends when students finally obtain answers to the task. In this regard, the recycling in a way already organizes the solution steps into a logic sequence, which does not require additional cognitive work as GP does.

Combining students' performances of both grade levels on GCN and GP seems to suggest a learning trajectory to facilitate students' proficiency in creating GP because of the diagram complexity and problem-solving complexity GCN tasks used and enacted in Taiwanese classroom (see Chapter Two and Three). When solving GCN tasks, the complexity of diagram provides students opportunities to learn strategic knowledge, which is a key for effective problem solvers who need to recall actions that are likely to be useful when choosing which actions to apply among several alternatives (Weber, 2002; 2005). Hence, class experiences in solving GCN tasks prior to formally learning GP can nurture students' strategic knowledge in terms of visualizing diagram configurations so that they can retrieve relevant geometric properties and combine different geometric properties to generate a valid solution strategy in novel ways. This training with working on GCN tasks with diagram complexity can later contribute to the learning of GP.

Given that much research has reported that students have difficulties in learning proofs (Fuys, Geddes, & Tischler, 1988; Harel & Sowder, 1998; Healy & Hoyles, 1998; Heinze, Cheng, & Yang, 2004; Heinze, Reiss, & Rudolph, 2005; Li, 2002; Mariotti, 2006; Miyazaki, 2000; Senk, 1989), the learning trajectory proposed here can also be used as a framework to develop instructional approaches that emphasize diagrams and requirements of geometric properties to improve students' learning of GP. This possibility is very worthy of further investigation.

The analysis of Pair 1 and Pair 4 with different diagram configurations also shows that diagrams significantly influence students' performance on both GCN and GP tasks. When the requirements of geometric properties are controlled, students performed significantly better on both types of items that had simple diagram configurations as opposed to complex ones. This result confirms that the diagram configurations have an influence on the underlying cognitive complexity of solving geometric tasks (Duval, 1995) and play an important role in students' ability to recognize relevant geometric properties. In this regard, Zykova (1975) used the term "reinterpreting diagrams" (p. 94) to describe geometric tasks that require identifying different geometric concepts from the same diagram configurations. Reinterpretation of the diagram with different geometric properties can be psychologically complex and requires students to understand all of the geometric properties in the diagram configurations and make transitions among these properties. In addition, diagram complexity due to extra lines and angles may also become visual obstacles (Yerushalmy & Chazan, 1990) that prevent students from correctly identifying the corresponding geometric properties needed to obtain solutions. However, the results from the pilot study suggest that this view may not be entirely correct. In the pilot study when diagram configurations were made more complex by adding one segment, students' performance on the two types of tasks did not change. This phenomenon suggests the need for further investigation. For instance, a follow-up study could examine the extent to which the complexity of diagram configurations influences students' performance on GCN and GP.

The analysis of the percentage of students who provided no responses to either type of item shows that students are more likely to give up on a GP task than on a GCN task. The phenomenon may be due to students' conceptions and beliefs of the tasks. As Chazan (1993) points out, high school students prefer empirical arguments over deductive arguments when working on mathematical problems. Since students often face difficulties in constructing GP, they may lose their motivation to work on the tasks, even though they could infer some geometric properties from the diagram configurations. Further investigation of students' conceptions and beliefs regarding both types of tasks would be useful to understand how they perceive the relationship between GCN and GP. One example of a research question to explore this could be: what are the underlying reasons for student behaviors and cognition on both types of tasks when they provide a partial solution or try to solve a GCN as opposed to working on a GP?

Finally, another important contribution of this study is the design of pairs of GCN and GP items which are identical in terms of the diagram configurations and required geometric properties need to obtain solutions. The investigation of such parallel tasks has not been undertaken by other researchers, but is crucial to the ability to elaborate the relationship between GCN and GP.

REFERENCES

Aleven, V., Koedinger, K., Sinclair, H., & Synder, J. (1998). *Combatting shallow Learning in a tutor for geometry problem solving*. Paper presented at the Intelligent tutoring systems: Fourth International Conference, ITS '98, Berlin.

Anderson, J., Greeno, J., Kline, P., & Neves, D. (1981). Acquisition of problem-solving skill. In J. Anderson (Ed.), *Cognitive skills and their acquisition* (pp. 191-230). New Jersey: Lawrence Eblbaum.

Ayres, P., & Sweller, J. (1990). Locus of difficulty in multistage mathematics problems. *The American Journal of Psychology, 103*(2), 167-193.

Ball, D., & Bass, H. (2003). Making mathematics reasonable in school. In J. Kilpatrick, W. Martin & D. Schifter (Eds.), *A research companion to principals and standards for school mathematics* (pp. 27-44). Reston: National Council of Teachers of Mathematics.

Ballacheff, N. (1988). Aspects of proof in pupils' practice of school mathematics. In D. Pimm (Ed.) *Mathematics, teachers and children* (pp. 216-235). London: Hodder & Stoughton.

Chazan, D. (1993). High school geometry students' justification for their views of empirical evidence and mathematical proof. *Educational Studies in Mathematics, 24*, 359-387.

Cheng, Y.-H., & Lin, F.-L. (2005). One more step toward acceptable proof in geometry, *Symposium presented 11th EARLI Conference*. Nicosia, Cyprus.

Cheng, Y.-H., & Lin, F.-L. (2006). Using reading and coloring to enhance incomplete prover's performance in geometry proof. In J. Novotna, H. Moraova, M. Kratka, & N. Stehlikova (Eds.), *Proceedings of the 30th Conference of the International Group for the Psychology of Mathematics Education* (vol. 2, pp. 289-296). Prague, Czech Republic.

Cheng, Y.-H., & Lin, F.-L. (2007). The effectiveness and limitation of reading and coloring strategy in learning geometry proof. In J. H. Woo, H. C. Lew, K. S. Park, & D. Y. Seo (Eds.), *Proceedings of the 31st Conference of the International Group for the Psychology of Mathematics Education* (vol. 2, pp. 113-120). Seoul, Korea.

Cheng, Y.-H., & Lin, F.-L. (2008). A study on left behind students for enhancing their competence of geometry argumentation. In O. Figueras, J. L. Lortina, S. Alatorre, T. Rojano, & A. Sepulveda (Eds.), *Proceedings of the Joint Meeting of 32nd Conference of the International Group for the Psychology of Mathematics Education (PME) and PME-NA XXX* (vol. 2, pp. 305-312). Morelia, Mexico.

Lawson, M., & Chinnapan, M. (2000). Knowledge connectedness in geometry problem solving. *Journal for Research in Mathematics Education, 31*(No. 1), 26-43.

Cohen, J. (1988). *Statistical power analysis for the behavioral sciences* (2 ed.). Hillsdale, NJ: Lawrence Earlbaum Associates.

Cohen, J. (1992). "A power primer". *Psychological Bulletin, 112*(1), 155-159.

Duval, R. (1995). Geometrical pictures: Kinds of representation and specific processings. In R. Sutherland & J. Mason (Eds.), *Exploiting mental imagery with computers in mathematics education* (pp. 142-157). Berlin: Springer.

Duval, R. (1998). Geometry from cognitive point of view. In C. Mammana & V. Villani (Eds.), *Perspectives on the teaching of geometry for 21st century* (pp. 37-52). Dordrecht: Kluwer.

Fischbein, E. (1993). The theory of figural concepts. *Educational Studies in Mathematics, 24*(2), 139-162.

Fischbein, E., & Nachlieli, T. (1998). Concepts and figures in geometrical reasoning. *International Journal of Science Education, 20*(10), 1193-1211.

Fuys, D., Geddes, D., & Tischler, R. (1988). *The van Hiele model of thinking in geometry among adolescents* (Vol. 3). Reston: National Council of Teachers of Mathematics.

Gal, H., & Linchevski, L. (2010). To see or not to see: Analyzing difficulties in geometry from the perspective of visual perception. *Educational Studies in Mathematics, 74*, 163-183.

Gray, E., & Tall, D. (1994). Duality, ambiguity and flexibility: A proceptual view of simple arithmetic. *Journal for Research in Mathematics Education, 26*(2), 115-141.

Hanna, G. (2000). Proof, explanation and exploration: An overview. *Educational Studies in Mathematics, 44*, 5-23.

Harel, G., & Sowder, L. (1998). Students' proof schemes: Results from exploratory studies. *CBMS Issues in Mathematics Education, 7*, 234-283.

Healy, L., & Hoyles, C. (1998). *Justifying and proving in school mathematics.* London: Institute of Education.

Heinze, A., Cheng, Y.-H., Ufer, S., Lin, F.-L., & Reiss, K. (2008). Strategies to foster students' competencies in constructing multi-steps geometric proofs: Teaching experiments in Taiwan and Germany. *ZDM—The International Journal on Mathematics Education, 40*(3), 443-453.

Heinze, A., Cheng, Y.-H., & Yang, K.-L. (2004). Students' performance in reasoning and proof in Taiwan and Germany: Results, paradoxes and open questions. *ZDM—The International Journal on Mathematics Education, 36*(5), 162-171.

Heinze, A., Reiss, K., & Rudolph, F. (2005). Mathematics achievement and interest in mathematics from a differential perspective. *ZDM—The International Journal on Mathematics Education, 37*(3), 212-220.

Heinze, A., Ufer, S., Cheng, Y.-H., & Lin, F.-L. (2008). *Geometric proof competence and its individual predictors: A Taiwanese-German study*. Paper presented at ICME 11. Mexico.

Herbst, P. (2004). Interactions with diagrams and the making of reasoned conjectures in geometry. *ZDM—The International Journal on Mathematics Education, 36*(5), 129-139.

Herbst, P. (2006). Teaching geometry with problems: Negotiating instructional situations and mathematics tasks. *Journal for Research in Mathematics Education, 37*(4), 313-347.

Hsu, H.-Y. (2007). Geometric calculations are more than calculations. In J. H. Woo, H. C. Lew, K. S. Park, & D. Y. Seo (Eds.), *Proceedings of the 31st Conference of the International Group for the Psychology of Mathematics Education* (vol. 3, pp. 57-64). Seoul, Korea.

Koedinger, K., & Anderson, J. (1990). Abstract planning and perceptual chunks: Elements of expertise in geometry. *Cognitive Science, 14*, 511-550.

Küchemann, D., & Hoyles, C. (2002). The quality of students' reasons for the steps in a geometric calculation. *Proceedings of the British Society for Research into Learning Mathematics, 22*(2), 43-48.

Larkin, J. H., & Simon, H. A. (1987). Why a diagram is (sometimes) worth ten thousand words. *Cognitive Science, 11*, 65-99.

Li, Y.-F. (2002). *9th grade students' learning of geometry proofs: The obstacles caused by diagrams.* Unpublished master thesis, National Taiwan Normal University, Taipei. (In Chinese)

Lin, F.-L. (2005). Modeling students' learning on mathematical proof and refutation. In H. L. Chick, & J. L. Vincent (Eds.), *Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education* (vol. 1, pp. 3-18). Melbourne, Australia.

Lin, F.-L., Cheng, Y.-H., & et al. (2003). The competence of geometric argument in Taiwan adolescents, *International Conference on Science & Mathematics Learning*. Taipei, Taiwan.

Lovett, M., & Anderson, J. (1994). Effects of solving related proofs on memory and transfer in geometry problem solving. *Journal of Experimental Psychology: Learning, Memory, and Cognition, 20*(2), 366-378.

Maher, C., & Martino, A. (1996). The development of the idea of mathematical proof: A 5-year case study. *Journal for Research in Mathematics Education, 27*(2), 194-214.

Mariotti, M. A. (2006). Proof and proving in mathematics education. In Á. Gutiérrez & P. Boero (Eds.), *Handbook of research on the psychology of mathematics education* (pp. 173-204). Rotterdam: Sense Publishers.

Ministry of Education. (2003). *Grades 1-9 curriculum guidelines*. Taipei, Taiwan: Author.

Miyazaki, M. (2000). Levels of Proof in Lower Secondary School Mathematics. *Educational Studies in Mathematics, 41*, 47-68.

NCTM. (2000). *Principles and Standards for School Mathematics*. Reston: National Council of Teachers of Mathematics.

Piaget, J., Inhelder, B., & Szeminska, A. (1960). *The child's conception of geometry* (E. A. Lunzer, Trans.). New York: W.W. Norton & Company.

Pólya, G. (1945). *How to solve it*. Princeton, NJ: Princeton University Press.

Schumann, H., & Green, D. (2000). New protocols for solving geometric calculation problems incorporating dynamic geometry and computer algebra software. *International Journal of Mathematical Education in Science and Technology, 31*(3), 319-339.

Senk, S. (1989). Van Hiele levels and achievement in writing geometry proofs. *Journal for Research in Mathematics Education, 20*(3), 309-321.

Stein, M., Smith, M., Henningsen, M., & Silver, E. (2000). *Implementing standards-based mathematics instruction: A casebook for professional development*. New York: Teachers College Press.

Stylianides, A. (2007). The notion of proof in the context of elementary school mathematics. *Educational Studies in Mathematics, 65*, 1-20.

Tall, D. (2002). *Differing modes of proof and belief in mathematics.* Paper presented at the International Conference on Mathematics: Understanding proving and proving to understand, Taipei, Taiwan.

Tall, D. (2004). *Thinking through three worlds of mathematics.* In M. J. Hoines, A. B. Fuqlestad (Eds.), *Proceedings of the 28th Conference of the International Group for the Psychology of Mathematics Education* (vol. 4, pp. 281-288). Melbourne, Australia.

Tall, D. (2005). *The transition from embodied thought experiment and symbolic manipulation to formal proof.* Paper presented at the Kingfisher Delta'05, Fifth Southern Hemisphere Symposium on Undergraduate Mathematics and Statistics Teaching and Learning, Fraser Island, Australia.

Tall, D. (2006a). *A life-times' journey from definition and deduction to ambiguity and insight.* Paper presented at the Retirement as Process and Concept: A festschrift for Eddie Gray and David Tall, Prague.

Tall, D. (2006b). A theory of mathematical growth through embodiment, symbolism and proof, *International Colloquium on Mathematical learning from early childhood to adulthood*. Nivelles, Belgium: Centre de Recherche surl'Enseignement des Mathematiques.

Tall, D. (2007). *Embodiment, symbolism, argumentation and proof*, Paper presented at *t*he Conference on Reading, Writing and Argumentation

Ufer, S., & Heinze, A. (2008). *Development of geoemtric proof competence from grade 7 to 9: A longitudinal study*, Paper presented at ICME11, Mexico.

Weber, K. (2002). Student difficulty in constructing proofs: The need for strategic knowledge. *Educational Studies in Mathematics, 48*, 101-119.

Weber, K. (2005). Problem-solving, proving, and learning: The relationship between problem-solving processes and learning opportunities in the activity of proof construction. *Journal of Mathematical Behavior, 24*, 351-360.

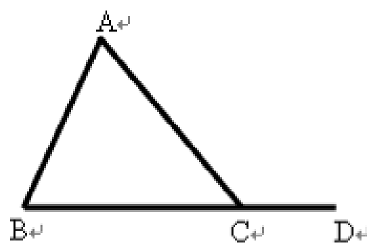
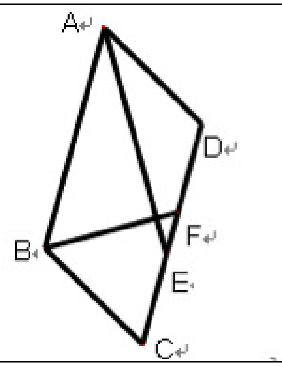
Yerushalmy, M., & Chazan, D. (1990). Overcoming visual obstacles with the aid of the Supposer. *Educational Studies in Mathematics, 21*, 199-219.

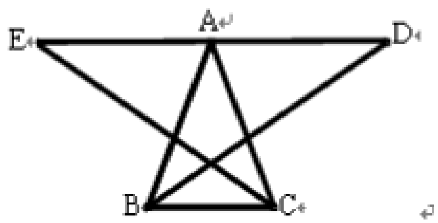
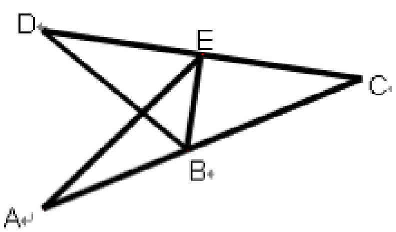
Zack, V. (1999). Everyday and mathematical language in children's argumentation about proof. *Educational Review, 51*(2), 129-146.

Zykova, V. I. (1975). Operating with concepts when solving geometry problems. In J. Kilpatrick & I. Wirszup (Eds.), *The learning of mathematical concepts* (Vol. 1, pp. 93-148). Chicago: University of Chicago.

Appendix 4.1

Summary of the GCN and GP Items Included in the Survey

Item	The given diagram	The givens for GCN	The givens for GP	List of geometric properties for solutions
Pair 1		$\triangle$ABC in which AC=BC and BCD is collinear. If the measure of ∠ACD is 130°, find the measure of ∠ABC.	$\triangle$ABC in which AC=BC and BCD is collinear. Prove ∠ACD=2∠ABC	Solution 1: The triangle sum property, the property of linear pair, the properties of an isosceles triangle Solution 2: The exterior angle property, the properties of an isosceles triangle
Pair 2		ABCD is a parallelogram. AE and BF are angle bisectors of ∠A and ∠B. If ∠AED=25° , find the measure of ∠EAD.	ABCD is a parallelogram. AE and BF are angle bisectors of ∠A and ∠B. Prove ∠AED=∠EAD.	Properties related to parallelogram (e.g., opposite sides of a parallelogram are parallel), the alternate interior angles property, the angle bisector property.

Pair 3		Triangle ABC in which AB and AC are the same lengths. Construct a line through point A so that the line is parallel to BC where the bisectors of angle B and angle C intersect the line at point D and point E. If AC=6 cm and ∠ACB=75°, find (1) the length of AE; (2) the measure of ∠AEC.	Triangle ABC in which AB and AC are the same lengths. Construct a line through point A so that the line is parallel to BC and the bisectors of angle B and angle C intersect the line at point D and point E. Prove AC=AE	Properties of an isosceles triangle , angle bisector property, properties related to parallel lines (e.g., the alternate interior angles property, the consecutive interior angles property)
Pair 4		△BDC in which BD=BC and ABC is collinear. If the measure of ∠ABD＝70°, find the measure of ∠BDC.	△BDC in which BD=BC and ABC is collinear. Prove ∠ABD=2∠BDC	Solution 1: The triangle sum property, property of linear pair, properties of an isosceles triangle Solution 2: The exterior angle property, properties of an isosceles triangle

Appendix 4.2

Plan Trees for Each Pair of GCN and GP Items in the Survey

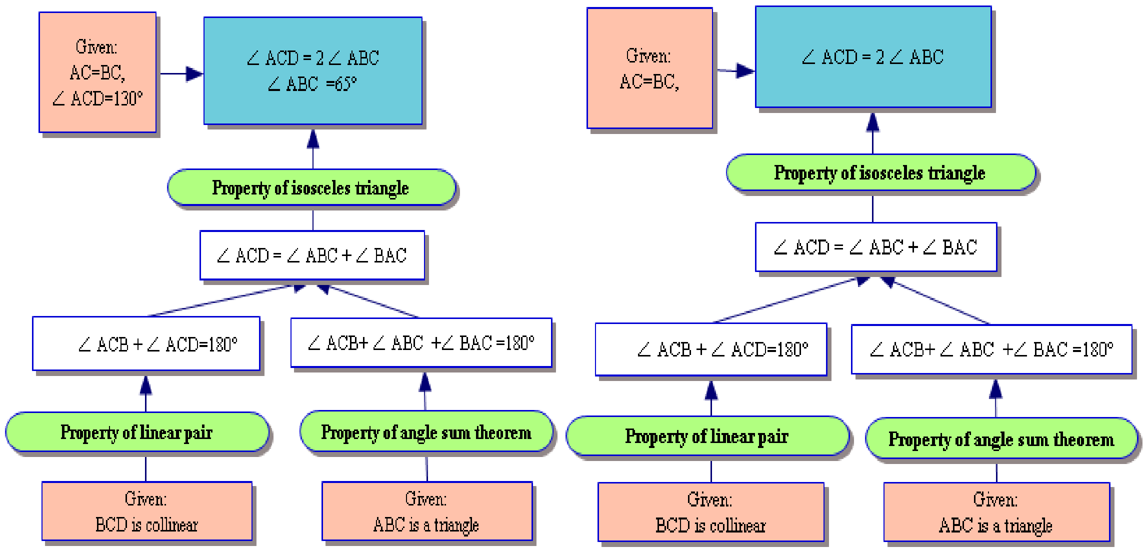


Figure 4.21 Plan trees for Pair 1 items (GCN on the left side and GP on the right side)

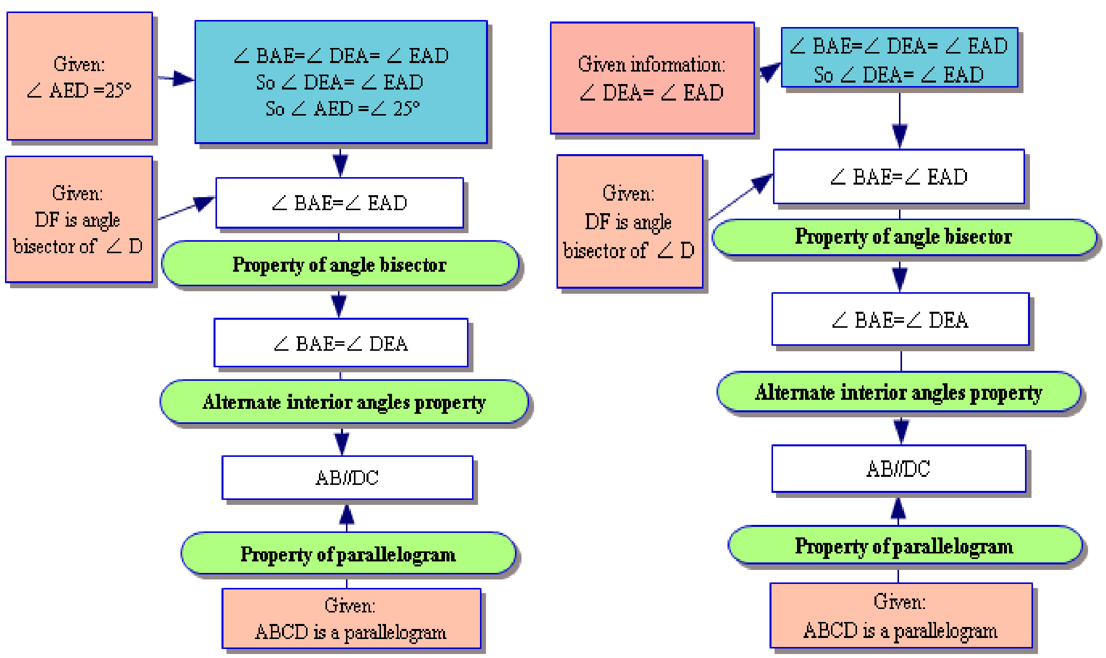


Figure 4.22 Plan trees for Pair 2 items (GCN on the left side and GP on the right side)

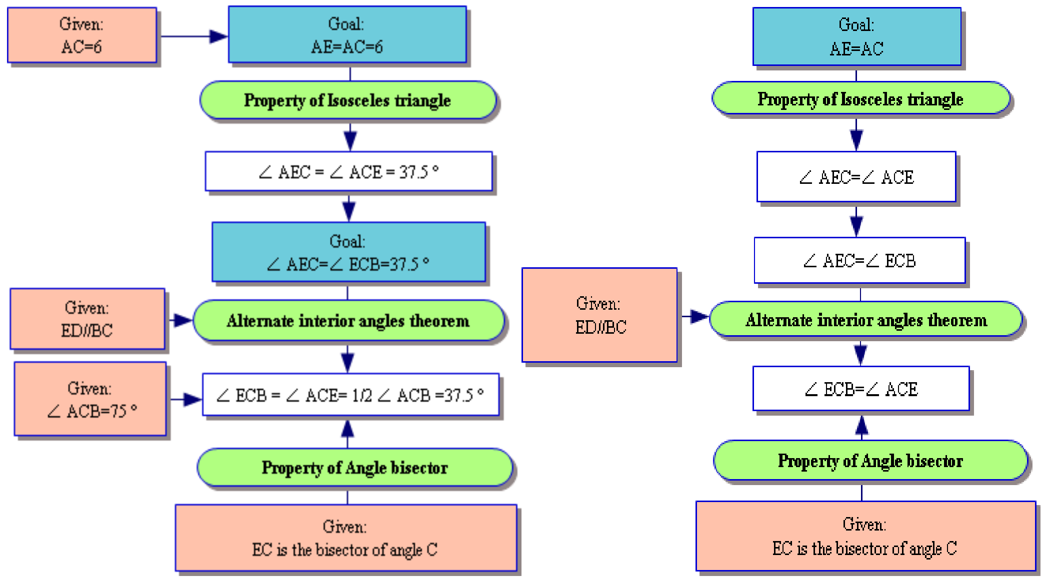


Figure 4.23 Plan trees for Pair 3 items (GCN on the left side and GP on the right side)

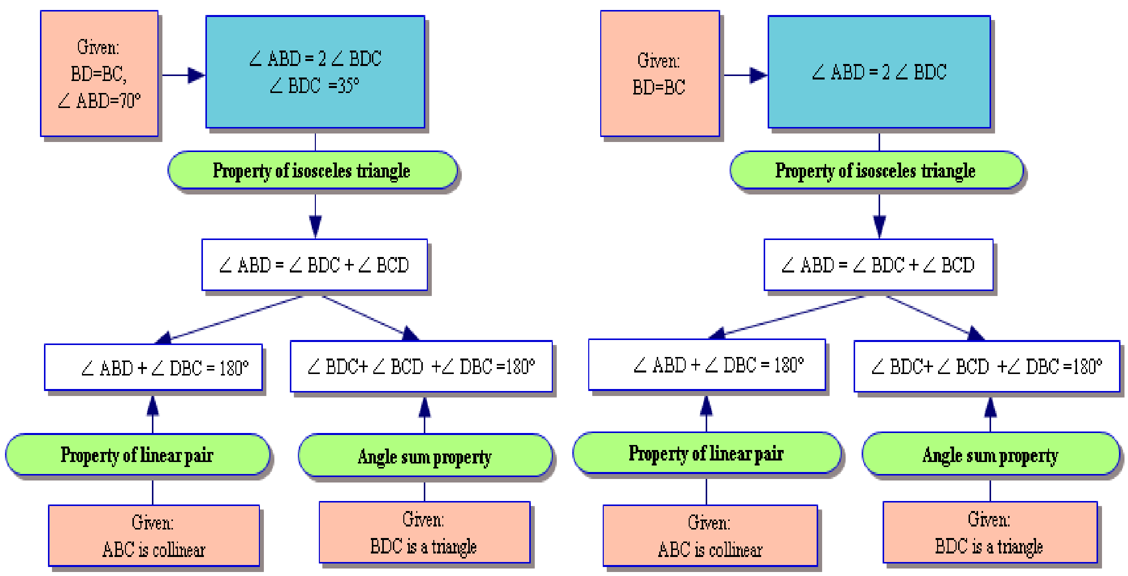
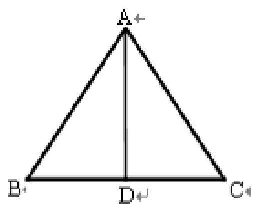


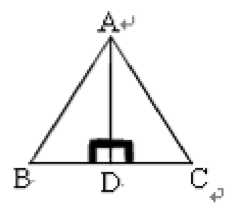
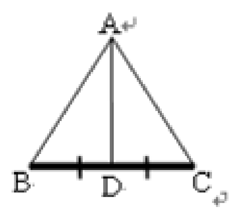
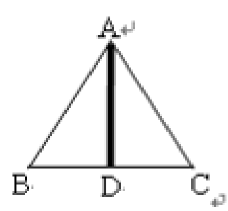
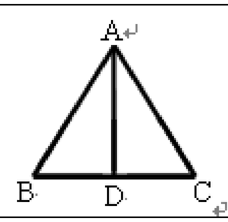
Figure 4.24 Plan trees for Pair 4 items (GCN on the left side and GP on the right side)

Appendix 4.3

Solution Example Included in the Survey

Please clearly write down your calculating or proving steps and indicate what geometric properties support your steps. Please also label your steps and geometric properties on the given diagram so that we can understand more about how you think of the task.

	In triangle ABC, if line AD is a perpendicular bisector of segment BC. Prove $\triangle ABD \cong \triangle ACD$

	Proving Steps	Labels on diagram	Reasons
Step 1	$\angle ADB = \angle ADC = 90^\circ$		Properties of perpendicular bisector ($AD \perp BC$)
Step 2	BD=DC		Properties of perpendicular bisector (BD=DC)
Step 3	AD＝AD		Shared segment for both $\triangle ABD$ and $\triangle ACD$
Step 4	$\triangle ABD \cong \triangle ACD$ (RHS)		Triangle congruence postulate

Appendix 4.4

Scheme for Coding GP and GCN Items

Coding	Proof scheme	Explanation of the scheme	coding	Calculation scheme	Explanation of the scheme
1	Acceptable proof	The proof is acceptable by authority (e.g., teachers or examiners). The proof is a valid deductive process from given premises to the wanted conclusion. In addition, students must provide correct geometric properties as reasons to support their proving steps.	1	Correct calculation with supportive reasons	The answer is correct and the calculations are a valid deductive process with correct geometric properties as reasons to support each calculating steps.
2	Incomplete proof	Crucial elements/properties are retrieved from the diagram to construct a proof solution but the solution has logical errors or a gap. For example, students may miss step(s) in their proving or show an inversive/reversive inference (such as using a latter property in a logic relation to prove a former proposition).	2	Incomplete calculation	Crucial elements/properties are retrieved from the diagram to form a solution but the solution has some errors. For example, students may mis-understand the given information and produce the wrong answer even though the inferring process from premises to conclusion is valid. Another example is that students made some mistakes when calculating the measures that lead to a wrong answer.
3	improper proof	Non-deductive approaches, using incorrect geometric properties, or using properties inappropriately. Students producing such solutions usually have difficulties with the required geometric knowledge or	3	Improper calculation	Non-deductive approaches, using incorrect geometric properties, or using properties inappropriately. Students producing such solutions usually have difficulties with the required geometric knowledge or they have an inadequate idea of mathematical reasoning even though they

		they have an inadequate idea of mathematical reasoning even though they still can make inferences based on non-crucial geometric properties			can make inferences based on non-crucial geometric properties. In GCN, it is possible that students guess the final answer even though they could not identify crucial properties required in this task.
4	Intuitive response	Students just work on the surface level without any inferring process of the task. For example, make a visual judgment or just give an intuitive response.	4	Intuitive response	Students work on the surface level without any inferring process of the task. For example, make a visual judgment or just give an intuitive response. In GCN, it is possible that students guess the final answer without any calculating sentences or reasoning steps.
99	No response		99	No response	

CHAPTER FIVE

GENERAL DISCUSSION

Using GCN Tasks to Facilitate Taiwanese Students' Competence in Constructing GP

This dissertation reports a research on geometric calculation with number (GCN) in Taiwan by conducting three sequential studies, corresponding roughly to key aspects of the Mathematical Tasks Framework (MTF); namely, tasks as found in curriculum materials and other instructional resources, tasks as enacted by students and their teacher, and student learning from and through mathematical tasks. In particular, the attentions of the dissertation focus on how GCN tasks were used and enacted in the classroom of one Taiwanese teacher and the extent to which evidence of performance of a sample of Taiwanese students on such tasks can be similar to that on geometric proof (GP) tasks. Although this dissertation only examined one classroom with one Taiwanese mathematics teacher, taken together the three studies offer a comprehensive glimpse at how Taiwanese students' experiences with GCN tasks might contribute to their high levels of competence with geometric proving and reasoning, areas that are generally quite difficult for students to master (Chazan, 1993; Harel & Sowder, 1998; Mariotti, 2006).

Study One (Chapter Two) presented an analysis of tasks as appeared in curricular/instructional materials; that is, the GCN tasks included in the curricular/instructional materials used by one Taiwanese mathematics teacher, Nancy. The analysis revealed that the Taiwanese teacher used tasks drawn from several sources, including not only the class textbook series, but also tasks of tests, tasks created by herself, and some in the supplemental materials that were collaboratively developed by her school colleagues. The analysis also revealed that these GCN tasks varied with respect to cognitive complexity, but that a substantial number of the tasks situated in auxiliary instructional materials, especially those for use in tests, were cognitively demanding because they required students to (1) visualize the sub-constructs in complex diagram configurations; (2) check if auxiliary lines are needed to solve the tasks; (3) generate solutions with multiple reasoning steps that use multiple geometric properties; and (4) apply multiple transformations to map corresponding reference diagrams onto the GCN task diagrams. The high demand tasks, in particular, afford opportunities for Taiwanese students to master the types of knowledge and the reasoning and problem-solving skills that are essential in creating GP.

Study Two (Chapter Three) presented an analysis of tasks as enacted in the classroom; that is, how Nancy set up and enacted these GCN tasks with her students. Of particular interest were that ways that Nancy sustained the engagement of her students with cognitively demanding GCN tasks involving complex diagram configurations. When solving these tasks with complex diagrams, students cannot merely apply information from geometric statements listed in the textbooks and corresponding reference diagrams as well as their previous experiences solving simple problems; instead they need to combine multiple facets of their geometric knowledge and their reasoning and problem-solving skills in novel ways. Through scaffolded experiences with GCN tasks containing complex diagrams, the teacher appeared to nurture students' competence in constructing and reasoning about geometric relationships in ways that are very likely to support their later work with GP. Another facet of the analysis in this study focused on the teacher's use of gestures to facilitate students' sustained engagement with complex GCN tasks. For example, Nancy used a range of gestural moves to scaffold students visualizing the complex diagram configurations. The ability to visualize the diagram configurations allowed students to keep working on the task and figure out the solutions themselves. Taken together, the findings of Studies One and Two offer a glimpse at classroom instruction in Taiwan involving GCN tasks and sketch a plausible pathway for Taiwanese students to gain competence through their experiences with GCN tasks in curricular/instruction materials in ways that are highly likely to support their later success with GP.

Study Three (Chapter Four) presented an analysis of student learning related to GCN tasks; that is, Taiwanese students' performance on a carefully constructed set of GCN and GP tasks, which use the same diagrams and require the same geometric properties to obtain solutions. The result shows that performance between GCN and GP for 9^{th} grade students in Taiwan is similar when the diagram configurations and the geometric properties necessary to obtain solutions are controlled. In other words, Study Three confirms that Taiwanese students' prior experiences in solving GCN tasks have a significant impact on their competence in creating GP.

Taken together the three studies that comprise this dissertation suggest why it might be the case that Taiwanese students would both develop potential competence in constructing GP before formal instruction to do so in schools and out-perform students in other countries on geometric proving and reasoning, as has been shown in the particular case of Germany (Heinze, Cheng, & Yang, 2004). In addition, the three studies taken together appear to have implications for other research that might seek to understand cross-national differences in mathematics performance.

Investigations of Students' Performance in East Asian Countries

Numerous cross-national comparisons of mathematics performance have been conducted in recent years (Heinze, Cheng, & Yang, 2004; Mullis, Martin, & Foy, 2008; Mullis, Martin, Gonzalez, & Chrostowski, 2004; Mullis et al., 2000; OECD, 2004) to examine differences in students' performance among countries and to identify factors associated with those differences. Researchers have focused on a number of different aspects of macro-level and micro-level factors in these investigations, ranging from country-level differences in curriculum, school organization and management and economic prosperity to school-level and classroom-level differences in teacher qualifications, instructional practices, and the enacted curriculum. One of the major thrusts is this body of work has been to examine the curriculum, usually the official textbooks, attempting to understand the relationship between curriculum and students' learning (Charalambous, Delaney, Hsu, & Mesa, 2010; Fan & Zhu, 2007; Fujita & Jones, 2002; Fuson, Stigler, & Bartsch, 1988; Howson, 1995; Li, Chen, & An, 2009; Lo, Cai, & Watanabe, 2001; Mayer, Sims, & Tajika, 1995; Son & Senk, 2010; Stevenson & Bartsch, 1992; Stigler, Fuson, Ham, & Kim, 1986; Valverde, Bianchi, Wolfe, Schmidt, & Houang, 2002; Zhu & Fan, 2006). Another major thrust of many of these investigations has been to examine classroom instruction to elaborating how instruction influences student learning outcomes (Hiebert et al., 2003; Jacobs & Morita, 2002; Leung, 1995; 2005; Santagata, 2005; Santagata & Barbieri, 2005; Santagata & Stigler, 2000; Stigler & Hiebert, 1999; 2004).

Especially popular have been studies examining performance differences between students in Western countries and their counterparts in East Asian countries. These studies have identified a number of similarities and differences in mathematics education between the two sets of countries and have used these similarities and differences to explain the diversity in students' mathematics performance. However, these studies are also limited in comprehensively outlining the whole story of how learning occurs in East Asian countries so that the students can consistently perform so highly in assessments of mathematical proficiency.

Most studies might be called vertical-dimension investigations, exploring a single factor (e.g., textbooks) that can partially explain the differences in students' learning outcomes. In contrast the three studies that comprise this dissertation might be seen as a horizontal-dimension approach that examines a sequence of factors in the teaching-learning trajectory with respect to geometric tasks in East Asian countries, an approach that might have the potential to explain more fully the observed high levels of performance in specific areas of mathematics (geometry reasoning and proof, in this case). Although this is a case study of only one Taiwanese classroom with one mathematics topic, geometric proving and reasoning, the findings do provide an insightful picture portraying how students in East Asian countries might learn mathematics through enacting abundant mathematical tasks situated in both official and auxiliary curricular/instructional materials provided by their teacher.

In addition, though instruction in East Asian countries, especially at the middle school levels, can be described as teacher-centered, in that teachers usually talk more than students (Leung, 2002; 2005), the analysis of gestures, a non-verbal communication way, also suggest caution about focusing only on verbal utterances to understand the instructional practice of Asian teachers; non-verbal communication by teachers in East Asian countries' classrooms may also influence students' learning. For example, as students frequently work on tests, it is possible that a mathematics teacher can diagnose students' misconceptions by evaluating their responses to test items and then use classroom lectures to help students understand the misconceptions. It is also possible that a skillful mathematics teacher can identify students' facial expressions to know if students follow the lectures and understand the mathematics without the need for any verbal communication. While classroom teachers may use non-verbal communications to facilitate students' learning, students in teacher-centered and examination-oriented classroom may also learn the mathematics that cannot be identified by examining the verbal communication between teacher and students. For example, it can be the case that students learn mathematics by practicing abundant mathematical tasks and evaluate their learning by checking the answers to the tasks. In this regard, the students do not necessitate to participating in classroom discussion. When they have problems regarding the challenging tasks, they can listen to classroom lectures to know the solutions. These possibilities all can make learning occur in a teacher-centered and examination-oriented classroom in East Asian countries, and should be carefully considered in interpreting and understanding the differences in students' performance in cross-national comparisons.

Using the MTF Framework to Investigate Student Learning Outcomes

In a recent paper commenting on cross-national comparison studies involving East Asian countries, Silver (2009) suggested that differences in students' learning across countries might be understood by analyzing mathematical tasks together with their cognitive demand levels using the three-stage MTF: tasks as they appear in curricular/instructional materials; tasks as set up by classroom teachers; and tasks as enacted by classroom teacher and students. These three stages have clear consequences for students' learning outcomes. In this regard, this dissertation provides a good example to support Silver's assertion that the MTF could be a core frame for an investigation of factors accounting for the differences in students' performance in cross-national comparisons.

First, this dissertation provides the evidence that auxiliary curricular/instructional materials can have strong influence on students' learning outcomes. In line with Stein and Smith (1998), they clarified that the pages in official printed textbooks are not the only curricular materials considered in influencing students' learning outcomes but also the auxiliary materials that a teacher may use in the classroom. Examination of official curriculum, usually the textbooks, among countries like most studies did is limited in exploring students' learning opportunities because auxiliary curricular/instructional materials can possess a high portion of mathematical tasks with diverse opportunities in East Asian countries.

Second, this dissertation also anchors the importance of sustaining the cognitive demand levels when setting up and enacting mathematical tasks with classroom students. The dissertation presents how Taiwanese mathematics teacher carefully sustained the cognitive demand of tasks and used different instructional strategies to make students' learning occur; these efforts can have significant impact on students' learning outcomes.

Finally, as the first three stages in the MTF framework have strong consequence on students' learning, the survey study also shows how including abundant GCN tasks, setting up these tasks, and enacting them with students can greatly influence students' competence not only in solving GCN but also in constructing GP. Taken together with the three studies, this dissertation provides an example to illustrate the influence of mathematical tasks as one of key factors to East Asian students' out-performance in cross-national comparisons.

REFERENCES

Charalambous, C., Delaney, S., Hsu, H.-Y., & Mesa, V. (2010). A comparative analysis of the addition and subtraction of fractions in textbooks from three countries. *International Journal for Mathematical Thinking and Learning, 12*(2), 117-151.

Chazan, D. (1993). High school geometry students' justification for their views of empirical evidence and mathematical proof. *Educational Studies in Mathematics, 24*, 359-387.

Fan, L., & Zhu, Y. (2007). Representation of problem-solving procedures: A comparative look at China, Singapore, and US mathematics textbooks. *Educational Studies in Mathematics, 66*, 61-75.

Fujita, T., & Jones, K. (2002). Opportunities for the development of geometrical reasoning in current textbooks in the UK and Japan. *Proceedings of the British Society for Research into Learning Mathematics, 22*(3), 79-84.

Fuson, K., Stigler, J., & Bartsch, K. (1988). Grade placement of addition and subtraction topics in Japan, Mainland China, the Soviet Union, Taiwan, and the United States. *Journal for Research in Mathematics Education, 19*(5), 449-456.

Harel, G., & Sowder, L. (1998). Students' proof schemes: Results from exploratory studies. *CBMS Issues in Mathematics Education, 7*, 234-283.

Heinze, A., Cheng, Y.-H., & Yang, K.-L. (2004). Students' performance in reasoning and proof in Taiwan and Germany: Results, paradoxes and open questions. *ZDM, 36*(5), 162-171.

Hiebert, J., Gallimore, R., Garnier, H., Givvin, K., Hollingsworth, H., Jacobs, J., et al. (2003). *Teaching mathematics in seven countries*. Washington: National Center for Education Statistics.

Howson, G. (1995). *TIMSS monograph No. 3: Mathematics textbooks: A comparative study of grade 8 texts*. Vancouver, BC: Pacific Educational Press.

Jacobs, J., & Morita, E. (2002). Japanese and American teachers' evaluations of videotaped mathematics lessons. *Journal for Research in Mathematics Education, 33*(3), 154-175.

Leung, F. K. S. (1995). The mathematics classroom in Beijing, Hong Kong and London. *Educational Studies in Mathematics, 29*, 297-325.

Leung, F. K. S. (2002). In search of an East Asian identity in mathematics education. *Educational Studies in Mathematics, 47*, 35-51.

Leung, F. K. S. (2005). Some characteristics of East Asian mathematics classrooms based on data from the TIMSS video study. *Educational Studies in Mathematics, 60*, 199-215.

Li, Y., Chen, X., & An, S. (2009). Conceptualizing and organizing content for teaching and learning in selected Chinese, Japanese and US mathematics textbooks: The case of fraction division. *ZDM-The International Journal on Mathematics Education, 41*, 809-826.

Lo, J.-J., Cai, J., & Watanabe, T. (2001). *A comparative study of the selected textbooks from China, Japan, Taiwan and the United States on the teaching of ratio and proportion concepts.* Paper presented at the Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education.

Mariotti, M. A. (2006). Proof and proving in mathematics education. In Á. Gutiérrez & P. Boero (Eds.), *Handbook of research on the psychology of mathematics education* (pp. 173-204). Rotterdam: Sense Publishers.

Mayer, R., Sims, V., & Tajika, H. (1995). A comparison of how textbooks teach mathematical problem solving in Japan and the United States. *American Educational Research Journal, 32*(2), 443-460.

Mullis, I. V. S., Martin, M. O., & Foy, P. (2008). *TIMSS 2007 international mathematics report*. Chestnut Hill, MA: TIMSS & PIRLS International Study Center, Boston College.

Mullis, I. V. S., Martin, M. O., Gonzalez, E. J., & Chrostowski, S. J. (2004). *TIMSS 2003 international mathematics report: Findings from the IEA's Trends in International Mathematics and Sciences Study at the fourth and eighth grades*. Chestnut Hill, MA: TIMSS & PIRLS International Study Center.

Mullis, I. V. S., Martin, M. O., Gonzalez, E. J., Gregory, K. D., O'Connor, K. M., Chrostowski, S. J., et al. (2000). *International mathematics report: Findings from IEA's repeat of the third international mathematics and science study at the eighth grade*. Chestnut Hill, MA: International Study Center.

OECD. (2004). *Learning for tomorrow's world-First results from PISA 2003*. Paris: Organization for Economic Co-operation and Development.

Santagata, R. (2005). Practices and beliefs in mistake-handling activities: A video study of Italian and US mathematics lessons. *Teaching and Teacher Education, 21*, 491-508.

Santagata, R., & Barbieri, A. (2005). Mathematics teaching in Italy: A cross-cultural video analysis. *Mathematical Thinking and Learning, 7*(4), 291-312.

Santagata, R., & Stigler, J. (2000). Teaching mathematics: Italian lessons from a cross-cultural perspective. *Mathematical Thinking and Learning, 2*(3), 191-208.

Silver, E. (2009). Cross-national comparisons of mathematics curriculum materials: What might we learn? *ZDM-The International Journal on Mathematics Education, 41*, 827-832.

Son, J.-W., & Senk, S. (2010). How reform curricula in the USA and Korea present multiplication and division of fractions. *Educational Studies in Mathematics, 74*(2), 117-142.

Stein, M., & Smith, M. (1998). Mathematical tasks as a framework for reflection: From research to practice. *Mathematics Teaching in the Middle School, 3*, 268-275.

Stevenson, H., & Bartsch, K. (1992). *An analysis of Japanese and American textbooks in mathematics*. Ann Arbor: Center for Japanese Studies, University of Michigan.

Stigler, J., Fuson, K., Ham, M., & Kim, M. (1986). An analysis of addition and subtraction word problems in American and Soviet elementary mathematics textbook. *Cognition and Instruction, 3*(3), 153-171.

Stigler, J., & Hiebert, J. (1999). *The teaching gap: Best ideas from the world's teachers for improving education in the classroom*. New York: Free Press.

Stigler, J., & Hiebert, J. (2004). Improving mathematics teaching. *Educational Leadership, 61*(5), 12-16.

Valverde, G., Bianchi, L., Wolfe, R., Schmidt, W., & Houang, R. (2002). *According to the book: Using TIMSS to investigate the translation of policy into practice through the world of textbooks*. Dordrecht: Kluwer.

Zhu, Y., & Fan, L. (2006). Focus on the representation of problem types in intended curriculum: A comparison of selected mathematics textbooks from mainland China and The United States. *International Journal of Science and Mathematics Education, 4*, 609-626.