STOCHASTIC LEADTIMES IN TWO-LEVEL DISTRIBUTION TYPE NETWORKS

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ABSTRACT

Leadtime uncertainty occurs in both manufacturing and distribution systems and can cause difficult coordination problems. We analyze a simple three-location, two-level distribution-type system with stochastic leadtimes with the objective of determining planned leadtimes which minimize the sum of expected inventory holding and tardiness costs. In a manufacturing context, the system can be viewed as one in which common processing or procurement is done first, whereupon another manufacturing stage differentiates this common product. Within a distribution framework, the system is one in which material is transported to a central facility (such as a regional warehouse) and subsequently transported to smaller local distributors or retailers.

We investigate two heuristic policies which are simple adjustments to optimal solutions for serial systems resulting from a decoupling of the distribution network. We also report computational experience which indicates that the optimal "location" and quantity of safety time depends largely upon the relationship among the due dates and average leadtimes for the final stages of processing or delivery.

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1.0 INTRODUCTION

Leadtime uncertainty causes significant problems in both manufacturing and distribution systems. In manufacturing environments, early completion of a stage of processing results in idle inventory while tardy completion of an order may result in a lost sale or additional expediting costs. In distribution environments, early deliveries lead to unnecessary congestion in storage areas as well as inventory holding costs. Late deliveries generally result in lost sales or backordering expenses.

Most existing work on uncertain leadtimes has dealt with single stage systems. Whybark and Williams (1976) report on simulation studies which indicate that safety time is a more efficient and effective buffer than safety stock when timing is uncertain. Weeks (1981) develops a method for setting optimal (customer) due dates when leadtimes are uncertain. Grasso and Taylor (1984) have found in simulation studies that leadtime variability, quantity and type of buffers used, lot sizing rules, and the cost structure have statistically significant effects on system performance.

In an earlier paper (1985), we studied the problem of determining optimal planned leadtimes in serial systems where the processing, procurement, or transport time at each stage may be stochastic. The objective considered both tardiness costs and inventory holding costs incurred when orders are completed early. This paper addresses the same problem for two-level distribution-type manufacturing and distribution environments.

Our objectives in this research are to develop approaches to the problem which provide insight, not just computational techniques. Because of the complexity of the problem (as will be apparent later), we study a simple three-location system (depicted in Figure 1) in an attempt to develop a fundamental understanding of the critical issues and the interactions among the various factors. This understanding will provide for improved decision-making in more complex systems, and provide a foundation for workable approaches to more realistic problems.

We note that the literature to date on optimizing planned leadtimes is limited to the cases of single-stage and serial systems. Thus, what follows represents an initial investigation for a more general system, albeit with strong assumptions. In particular, we do not consider the effects of scheduling and queueing within the system. The principal reason for this is that few descriptive models of queues with inserted idle time exist, and normative approaches to job shop scheduling problems with both earliness and tardiness considerations are limited to single-machine cases (e.g., Lakshiminarayan et al., 1978). By examining a simple normative model, we hope to obtain an indication of whether development of such detailed models for multi-stage systems with inserted idle time would be worthwhile. Ultimately, it would be desirable to marry a normative model with the corresponding descriptive model. Nevertheless, this study provides an indication of what may or may not be feasible to attack analytically, and the computational results provide insight which may lead to development of more comprehensive approaches.

In section 2 we describe two application areas briefly to explain the motivation for our research. A detailed description and formulation of the problem appear in section 3. The heuristic procedures are discussed in section 4. Section 5 summarizes results of computational work that provides results on performance of the heuristics and some insight into the characteristics of optimal solutions in various systems. We conclude with a summary and discussion in section 6.

2.0 TWO APPLICATIONS

2.1 Warehouse-Retailer System

A common approach to distribution in integrated warehouse-retailer distribution systems is for the warehouse to place orders on vendors periodically, and for mixed truckloads of goods to be distributed to retail stores on either a fixed or variable ("as needed") schedule.

Vendor leadtimes are almost always uncertain because of product availability considerations or transport times. Although models of scenarios with major product availability problems should consider inventory levels explicitly, minor product availability problems might be modeled using stochastic leadtimes. The effective transport time to the retailer may also be random because of variable transport times or because a shipment may be held until a full truckload is accumulated.

In this framework, the problem involves determining when to place an order from the vendor so that the goods are available when desired (e.g., for a promotional sale), and, if the choice is available, when to release product for shipment to the stores from the warehouse.

2.1 Product with Multiple Options

Many products are an option or customization of a "standard" semifinished product or raw material. Examples of customization include upholstery covering on furniture, painting or finishing of various products, and sizes of clothing. Because of economies of scale in both purchasing and production, the "standard" product is produced or procured in (large) batches. The time required to complete the batch may be uncertain because of raw material availability, for example. There are other circumstances where the setup time is uncertain because of difficult calibrations. If quality is not perfect, the time required to produce a specified number of acceptable units will be uncertain. If the product is purchased, the "batch" procurement leadtime may be highly variable. The time required to customize the product may differ among versions, and for a given version, the processing time may vary as well because of machine breakdowns, rework, or because of the various factors discussed earlier.

In this type of manufacturing environment, the primary timing decisions relate to the start of the batch production process and to the start of the processes for each of the customization options.

3.0 DESCRIPTION AND FORMULATION OF THE PROBLEM

We use the following notation throughout the paper

α = fraction of batch at stage 3 to be used for batch at stage 1

$(1-\alpha)$ = fraction of batch at stage 3 to be used for batch at stage 2

h_i = inventory holding cost per batch per period at stage i

p_i = penalty per batch per period at stage i, i = 1,2

d_i = due date at stage i, i = 1,2

τ_i = actual leadtime at stage i (random variable)

X_i = planned leadtime at stage i (decision variable)

$f_i(\cdot)$ = density of leadtime for stage i

$F_i(\cdot)$ = leadtime distribution for stage i

Safety time at stage i is defined as $X_i - E(\tau_i)$ where this value may be positive or negative.

The system operates as follows. The common processing or procurement activity (stage 3) is started at time

$$\min(d_1 - X_1, d_2 - X_2) - X_3$$

If stage 3 requires less than X_3 time units, the entire batch is held until the X_3 time units expire. Then a fraction α of the common batch begins specialized processing (or transport to a retail store) at time $d_1 - X_1$ and the remaining units at time $d_2 - X_2$. If stage 3 requires greater than X_3 units of time, then stages 1 and 2 are begun at max $(\tau_3, d_1 - X_1)$ and max $(\tau_3, d_2 - X_2)$, respectively. Therefore, stage 3 may be completed "late" relative to one successor stage, and not the other, or it may be completed late relative to both. We assume that there is adequate time between now and the due dates to execute the schedule according to the planned leadtimes.

The reason for considering the possibility of "holding back" items which have completed stage 3 early is that immediate dispatching may not be optimal. Yano (1985a) has shown that it is usually not optimal to use unconditional immediate dispatching in serial systems, and we conjecture that the same phenomenon may exist in distribution networks. If it is optimal to dispatch immediately to stages 1 and 2, the solution will have $X_3 = 0$, but we do not constrain it to be so _a priori_. Systems with such "forbidden early departure" are discussed in more detail in Kanet and Christy (1984). We note also that just-in-time inventory systems frequently have policies that prohibit or limit early delivery of goods.

The problem is to determine planned leadtimes, X_i, which minimize total expected inventory holding costs and shortage costs. An inventory holding cost per batch per period, h_i, is charged only on product that has finished a stage and is waiting either to be dispatched to the next stage of processing or to satisfy the requirement at the due date. A linear penalty cost per batch, p_i, is charged for each period that each "specialized" batch is tardy. Throughout the paper we assume that the various leadtime distributions are mutually independent, continuous and differentiable. Extension to discrete distributions is straightforward. A formulation of the problem appears in Appendix A.

4.0 APPROACH TO THE PROBLEM

We derived first order conditions which appear in Appendix B. These first partial derivatives are <u>not</u> monotonic in the decision variables, making it difficult even to bound the solution space. Furthermore, establishing convexity of the objective function and/or determining simply stated conditions for convexity is not possible because the cost function can be discontinuous, and its derivatives are discontinuous at points where $d_1 - X_1 = d_2 - X_2$. (The Hessian matrix appears in Appendix C for the interested reader). It was possible to derive some analytic bounds on functions of sums of decision variables. However, these inequalities were not specific enough to provide much help in obtaining solutions.

Empirical investigations of the total cost function reveal a trough-shaped reponse surface with steep sides for X_1 and X_2 with X_3 fixed. It may be possible to construct a gradient procedure to find X_1 and X_2 for a given X_3, but since even the gradients are discontinuous at $d_1 - X_1 = d_2 - X_2$, the procedure would be extremely complex, if workable at all. We therefore chose an alternate approach which builds upon the results of Yano (1985a) for serial systems.

The algorithm developed in the paper referenced above finds optimal solutions to the same problem for two-stage (and multiple-stage) serial systems. We first briefly review these results and then turn to an examination of the distribution network problem.

For the two-stage serial problem, it was shown that:

(i) if a solution to the first order necessary conditions exists with $X_1 = F_1^{-1}[(h_2+p)/(h_1+p)]$ and $F_2(X_2) > 0$, it is optimal and the solution is unique; and

(ii) if a solution of the type decribed in (1) does not exist, then one (non-unique) optimal solution is $X_2 = 0$ and $X_1 = G_{12}^{-1}[p/(p+h_1)]$, where $G_{12} = F_1 * F_2$ and * denotes a convolution; furthermore this value of X_1 is strictly less than $F_1^{-1}[(h_2+p)/(h_1+p)]$

Observe that if it were possible to separate the stage 3 batch into two parts, a fraction α for stage 1, and fraction $1-\alpha$ for stage 2, it would be possible to obtain solutions for the two split serial systems independently using the algorithm in (Yano, 1985). However, we have assumed that because of economies of scale at stage 3, it is desirable to keep the batch intact. Nevertheless, the solutions for the serial systems can provide useful starting points for a solution procedure for the problem at hand.

Suppose that we were to obtain solutions for the two split serial systems described above. Let X_3^1 denote the solution for X_3 in the system with stage 1 as the final production stage (and similarly for stage 2). If $d_1 - X_1 - X_3^1 = d_2 - X_2 - X_3^2$, then the solutions to the hypothetical serial systems can be implemented since both solutions indicate the same starting time for stage 3. The solution, of course, is not optimal because it ignores the interaction between stages 1 and 2, but it is nonetheless feasible and we might even expect such a solution to provide good results. We will evaluate the quality of such solutions later in the paper. If, however, $d_1 - X_1 - X_3^1 \neq d_2 - X_2 - X_3^2$, the solution is not even feasible because the separate serial system do not agree on a single starting time for stage 3. The question thus becomes one of how the solutions might be adjusted to good feasible solutions.

Observe that if $d_1 - X_1 - X_3^1 > d_2 - X_2 - X_3^2$, then we have two simple strategies for making them equal. The first is to decrease the sum of X_2 and X_3^2 --that is to provide less total protection for product 2. The second strategy would be to increase $X_1 + X_3^1$, providing more total protection to product 1. Since shortage penalties generally are larger than inventory holding costs, it would appear that using the first strategy exclusively is unlikely to provide satisfactory results. Thus, exclusive use of the second simple strategy or a mixture of the two strategies might be desirable.

Throughout the remainder of this paper, the indices have been set so that in all cases, the optimal starting time at stage 3 for product 1 is later than that of product 2 (i.e., $d_1 - X_1 - X_3^1 > d_2 - X_2 - X_3^2$). We first note that for the serial systems, the optimality of $X_1 = F_i^{-1}[(h_3+p_1)/(h_1+p_1)]$ depends only upon the condition $F_3(X_3^1) > 0$, which is true in a majority of cases. Thus, changing X_3 is not likely to affect the optimality of X_1. On the other hand, changing X_1 but not X_3 will cause X_1 to be non-optimal (for nearly any value of X_3) and X_3 to be (conditionally) non-optimal for the new value of X_1. Thus, it appears that if a simple strategy is to be used, X_3^1 should be increased so as to equate the stage 3 starting times.

Now consider a mixed strategy. For the same reasons stated above, rather than to violate all first order conditions for the two hypothetical serial systems, we choose to leave X_1 and X_2 and to conditionally optimize X_3. This requires only a simple search procedure using equation B-1.

We have identified two heuristic strategies:

Strategy 1: Increase X_3^1 to equate stage 3 starting times.

Strategy 2: Conditionally optimize X_3 with X_1 and X_2 set according to solutions for split serial systems.

Clearly, neither strategy is guaranteed to provide optimal solutions. We next evaluate these strategies and in so doing, attempt to obtain some insight into the characteristics of optimal solutions.

5.0 COMPUTATIONAL RESULTS

The primary objectives of the computational work are to evaluate the strategies described above, to characterize optimal solutions, and to develop an understanding of critical factors and tradeoffs which are important for future analyses of more general systems.

We constructed a set of 292 problems using parameters indicated in Table 1. For simplicity, we normalized h_3 to 1.0 and assume that the leadtimes have Poisson distributions (requiring only a single parameter, λ). We chose three values of $\boldsymbol{\lambda}$: (1,1,5) and (1,3,5) and (5,5,1). Typically, the common process is long in comparison to the customization processes, which, in turn, usually are nearly equal in duration. Similarly, procurement from a supplier usually takes much longer than delivery of the goods to stores. The first combination reflects such situations. However, the average time for the specialized processes may differ, as in the second combination. In other cases (usually manufacturing situations), purchased materials are readily available or basic fabrication is relatively quick, but customization processes may be lengthy (third combination). We limited the magnitude of the λ_i to reduce the computational effort of finding optimal solutions.

TABLE 1

We solved each problem using each applicable strategy and then used an extensive grid search to locate the globally optimal solution. The first strategy (increasing X_3^1) did not perform well. It consistently found solutions which were much more costly than the cost of the optimal solution, principally because of high work-in-process holding costs. The strategy of fixing X_1 and X_2 and conditionally optimizing X_3 performed extremely well overall. A summary of the results appears in Table 2.

TABLE 2

We analyzed the results further to determine why the second strategy performed poorly in the last set of problems. The common characteristic of these problems was that $d_1 - X_1$ and $d_2 - X_2$ were disparate (i.e., differing by more than a few time units) and the conditionally optimal value of X_3 (as well as X_3^1 and X_3^2) was equal to zero. Recall that the heuristic solved for X_1 and X_2 independently. Yet, when $d_1 - X_1$ and $d_2 - X_2$ are disparate, the interactions are likely to be important because of the cost of work-in-process inventory. (A portion of the stage 3 batch must wait to begin processing). In fact, this conjecture was borne out by the characteristics of the cases in which the second strategy performed well: they were much more symmetric ($d_1 - X_1$ much closer to $d_2 - X_2$, if not equal).

The problems associated with imbalance (or asymmetry) are less noticeable when the algorithm finds $X_3 > 0$, possibly because the procedure for finding X_3 considers the effects on work-in-process inventory in selecting X_3. On the other hand, when the conditionally optimal value of $X_3 = 0$, the resulting work-in-process inventory costs cannot be fully considered since such a solution merely indicates that X_1 and X_2 are more than adequate. In such situations it may be advisable to set $X_3 = 0$ and jointly optimize X_1 and X_2. Generally, this would lead to smaller values of X_1 and X_2 than those found on the first pass.

We make a few comments here on the characteristics of the optimal solutions. For problems in which $\lambda_1 = \lambda_2$ and $d_1 = d_2$, nearly all solutions had some safety time at the final stages and values of X_3 in the vicinity of λ_3. Problems with $\lambda_1 = \lambda_2$ and $d_1 \neq d_2$ also have the primary buffering locations at stages 1 and 2, but have less safety time at stage 3. It appears that the inventory cost associated with a portion of the stage 3 batch is sufficiently high to reduce the extent of buffering at stage 3. The differing due dates may lead to some "forced" waiting at stage 3 for a portion of the stage 3 batch.

Thus, while a portion of the batch is provided protection at the level of X_3, the remainder generally is provided with much more protection. Hence, the need for safety time at stage 3 is reduced.

When $\lambda_1 \neq \lambda_2$ and $d_1 \neq d_2$, the characteristics of X_3 depend strongly upon the "balance" of the system. The more disparate the values of $d_1 - \lambda_1$ and $d_2 - \lambda_2$, the smaller the value of X_3. These patterns were observed for the cases with unequal λ_i and unequal d_1, but the same statements can be made for the situations discussed above with $\lambda_1 = \lambda_2$. Thus, it appears that "balance" of the system may be a critical factor in characterizing optimal solutions.

In many situations where $d_1 - \lambda_1$ and $d_2 - \lambda_2$ were disparate, there was <u>negative</u> safety time at stage 3 (i.e, $X_3 < \lambda_3$). There was substantial compensation for this, however, by a long planned leadtime for the item with the earlier due date, and resulting slack for the other item as well. As an example, for $\boldsymbol{\lambda} = (1,3,5)$, $\mathbf{d} = (15,13)$, the solutions for $\alpha = .8$, $\mathbf{h} = (1.0,.25)$, $\mathbf{p} = (1,2.25)$ is $X = (2,6,3)$. Stage 3 is begun 9 (6+3) periods prior to the due date for item 2. For simplicity, let us refer to this as time zero. Now stage 3, having $\lambda_3 = 5$, will exceed its planned leadtime of 3 a significant portion of the time. However, there is sufficient slack in the planned leadtime at stage 2 ($X_2 = 6$) to compensate for this. Moreover, stage 1, which need not be started until time 9, can be started on time even if stage 3 is 4 periods late. This example demonstrates that negative safety time can be economically justified at some stages of production or delivery in some circumstances.

Using 8 of the problems, we performed a more detailed analysis of symmetric systems, which are most comparable to serial systems. We selected systems with $\alpha = .5$, $h_1 = h_2$, $p_1 = p_2$ and $\lambda_1 = \lambda_2$ and compared the results with hypothetical serial systems which collapsed the distribution network into a serial system.

Specifically, we set $h_1 = h_1 + h_2$, and $p = p_1 + p_2$ and solved the problem for the serial system. The results are listed in Table 3. There is a minor (and perhaps not significant) increase in total buffering (as measured by $X_1 + X_3$) for one problem. We had anticipated that there would be a greater amount of safety time at the "common" process. The relatively smaller inventory holding costs at this stage and the possibilitv of reducing penalties for more than one finished product might be expected to

the common process, but the results d

needed to ascertain the magnitude of

systems.

TABLE 3

6.0 SUMMARY AND CONCLUSIONS

We have developed and evaluated heuristic procedures for determining planned leadtimes in a simple distribution-type network. We found that an adaptation of procedures for serial systems combined with a search procedure using a first order condition for the distribution network provided excellent results. The procedure is hierarchical in nature: it first determines the planned leadtime for each of the non-common processes as if it were the last stage in a serial system and then determines the conditionally optimal planned leadtime for the common process.

Computational experience with the algorithm suggests that some type of mechanism for considering the interaction between the products is necessary, principally because of work-in-process inventory considerations. Yet, such a mechanism could be quite complex when there are more than two finished products.

This difficulty might be circumvented in an entirely different way. Recall that the solutions for planned leadtimes at the non-common stages are obtained from hypothetical split serial systems and do not depend upon the planned leadtime for the common process except to the extent that it is assumed to be positive. This condition is almost always satisfied, so we could use the adapted serial procedure for any number of finished products. Then, we could choose to batch those items with equal or nearly equal implied due dates for the common process for simultaneous processing at that stage. Having equalized the start times of the "specialization" processes, we might be able to find a reasonably good value of of the planned leadtime for the common process by only considering inventory holding costs of the common process batch (if it finishes early), inventory holding costs of the finished products, and shortage costs. This would permit us to avoid consideration of the complex interactions, essentially by eliminating or minimizing the interaction effects. It would appear that such a procedure would perform well provided setup costs at the common process do not dominate the batching decision.

Qualitatively, the computational results indicate that in "balanced" systems where the optimal start times of the specialization processes are nearly equal, the planned leadtime for the common process is relatively large, providing safety time generally at each stage. On the other hand, in many unbalanced systems, negative safety time for the common process is preferred, while the final stages maintain significant amounts of explicit or implicit safety time. The results also indicate that immediate dispatching to the specialization processes is usually not optimal.

Some recent research which evaluates the merit of shifting safety stock toward common components has been done by Baker, et al. (1986) and McClelland (1984). However, little work has been done on similar issues relating to safety time. Thus, it appears that further research is needed on both descriptive and normative models with the possibility of inserted idle time.

Additional research is also needed to generalize this model to larger distribution networks, in particular, those with three or more levels. The results here indicate that a procedure which determines planned leadtimes for operations nearest to the customer first and conditionally optimizes predecessor activities can perform well in a two-level system. It is possible that approaches which are similar in spirit will perform well on larger systems.

7.0 ACKNOWLEDGMENTS

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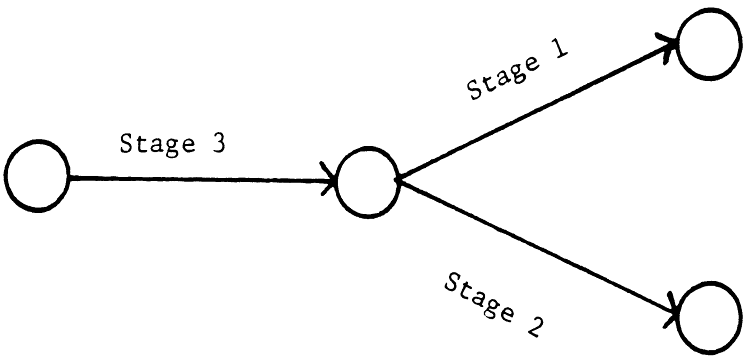

Figure 1

Distribution Network

TABLE 1

Problem Data

Parameter	Values
α	.2 or .5; also .8 if the system is not symmetric
h_1	1.25 or 2.0 times αh_3
h_2	1.25 or 2.0 times $(1 - \alpha)h_3$
p_i	1 or 9 times h_i
d	(15,15), (13,15); also (15,13) if the system is not symmetric

TABLE 2

Performance of Heuristic Procedure
(Leadtimes with Poisson Distributions)

Leadtime Parameters	Due Dates	No. of Problems	No. Optimal	Average % Deviation from Optimal
(1,1,5)	(15,15)	26	19	0.28%
(1,1,5)	(13,15)	48	37	0.53%
(1,3,5)	(13,15)	48	33	0.91%
(1,3,5)	(15,15)	48	19	1.63%
(1,3,5)	(15,13)	48	43	0.16%
(5,5,1)	(15,15)	26	20	0.47%
(5,5,1)	(13,15)	48	8	7.84%

TABLE 3

Comparison of Solutions for Distribution Systems (common batch split equally) and Similar Serial Systems (Leadtimes with Poisson Distributions; Common component holding cost = 1.0)

Leadtime Parameters	Finished Product Holding Cost ($h_1=h_2$)	Penalty to Holding Ratio	SOLUTIONS Distribution System	Serial System
(1,1,5)	0.625	1	(2,2,4)	(2,4)
		9	(3,3,6)	(3,6)
	1.0	1	(2,2,4)	(2,4)
		9	(3,3,7)	(3,6)
(5,5,1)	0.625	1	(6,6,0)	(6,0)
		9	(9,9,0)	(9,0)
	1.0	1	(6,6,0)	(6,0)
		9	(9,9,0)	(9,0)

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Appendix A

The two-product (or two-retailer) problem can be formulated as follows:

$$
\begin{aligned}
\text{minimize}\quad & h_3 \int_0^{X_3} (X_3-t_3)f_3(t_3)dt_3 \\
& +\alpha h_3\,[d_1-X_1-(d_2-X_2)]^+\ F_3(X_3+[d_1-X_1-(d_2-X_2)]^+) \\
& +(1-\alpha)h_3\,[d_2-X_2-(d_1-X_1)]^+F_3(X_3+[d_2-X_2-(d_1-X_1)]^+) \\
& +\alpha h_3 \int_{X_3}^{X_3+[d_1-X_1-(d_2-X_2)]^+} (X_3-t_3)f_3(t_3)dt_3 \\
& +(1-\alpha)h_3 \int_{X_3}^{X_3+[d_2-X_2-(d_1-X_1)]^+} (X_3-t_3)f_3(t_3)dt_3 \\
& +h_1F_3(X_3+[d_1-X_1-(d_2-X_2)]^+) \int_0^{X_1} (X_1-t_1)f_1(t_1)dt_1 \\
& +h_1 \int_{X_3+[d_1-X_1-(d_2-X_2)]^+}^{X_1+X_3+[d_1-X_1-(d_2-X_2)]^+} \int_0^{X_1+X_3+[d_1-X_1-(d_2-X_2)]^+-t_3} (X_1+X_3+[d_1-X_1-(d_2-X_2)]^+-t_1-t_3) \\
& \qquad f_3(t_3)f_1(t_1)dt_1dt_3 \\
& +h_2\ F_3(X_3+[d_2-X_2-(d_1-X_1)]^+) \int_0^{X_2} (X_2-t_2)f_2(t_2)dt_2 \\
& +h_2 \int_{X_3+[d_2-X_2-(d_1-X_1)]^+}^{X_2+X_3+[d_2-X_2-(d_1-X_1)]^+} \int_0^{X_2+X_3+[d_2-X_2-(d_1-X_1)]^+-t_3} (X_2+X_3+[d_2-X_2-(d_1-X_1)]^+-t_2-t_3) \\
& \qquad f_3(t_3)f_2(t_2)dt_2dt_3 \\
& +p_1\ F_3(X_3+[d_1-X_1-(d_2-X_2)]^+) \int_{X_1}^{\infty} (t_1-X_1)f_1(t_1)dt_1 \\
& +p_2\ F_3(X_3+[d_2-X_2-(d_1-X_1)]^+) \int_{X_2}^{\infty} (t_2-X_2)f_2(t_2)dt_2
\end{aligned}
$$

$$+p_1 \int_{X_3+[d_1-X_1-(d_2-X_2)]^+}^{\infty} \int_{X_1+X_3+[d_1-X_1-(d_2-X_2)]^+-t_3)}^{\infty} (t_1+t_3-X_1-X_3-[d_1-X_1-(d_2-X_2)]^+) f_3(t_3)f_1(t_1)dt_1dt_3$$

$$+p_2 \int_{X_3+[d_2-X_2-(d_1-X_1)]^+}^{\infty} \int_{X_2+X_3+[d_2-X_2-(d_1-X_1)]^+-t_3}^{\infty} (t_2+t_3-X_2-X_3-[d_2-X_2-(d_1-X_1)]^+) f_3(t_3)f_2(t_2)dt_2dt_3$$

s.t. $X_i \geq 0, \quad \forall \; i$

Some like terms have been combined and simplified. The terms represent: (1) holding costs at stage 3 when it is early; (2) + (4) other holding costs for one portion of the stage 3 batch (to be used in product 1); (3) + (5) other holding costs for the remaining portion of the stage 3 batch; (6) holding costs for stage 1 when stage 3 is on time relative to the stage 1 planned start time; (7) holding costs for stage 1 when stage 3 is tardy relative to the stage 1 planned start time; (8) parallel of (6) for stage 2; (9) parallel of (7) for stage 2; (10) shortage costs at stage 1 when stage 3 is on time relative to the planned stage 1 start time; (11) shortage costs at stage 2 paralleling (10); (12) shortage costs at stage 1 when stage 3 is tardy; and (13) shortage costs at stage 2 when stage 3 is tardy.

Appendix B

The first order necessary conditions are as follows:

$$\partial TC/\partial X_3 = (h_1+p_1)\int_{X_3+[d_1-X_1-d_2+X_2]^+}^{X_1+X_3+[d_1-X_1-d_2+X_2]^+} f_3(t_3)F_1(X_1+X_3+[d_1-X_1-d_2+X_2]^+-t_3)dt_3]$$

$$+ (h_2+p_2)\int_{X_3+[d_2-X_2-d_1+X_1]^+}^{X_2+X_3+[d_2-X_2-d_1+X_1]^+} f_3(t_3)F_2(X_2+X_3+[d_2-X_2-d_1+X_1]^+-t_3)dt_3]$$

$$+(p_1+\alpha h_3)F_3(X_3+[d_1-X_1-d_2+X_2]^+)$$

$$+(p_2+(1-\alpha)h_3)F_3(X_3+[d_2-X_2-d_1+X_1]^+)$$

$$-p_1 - p_2 = 0 \tag{B-1}$$

If $d_1-X_1 > d_2-X_2$ then

$$\partial TC/\partial X_1 = (h_1+p_1)[F_3(X_3+d_1-X_1-d_2+X_2)F_1(X_1)$$

$$+ \int_{X_3+d_1-X_1-d_2+X_2}^{X_3+d_1-d_2+X_2} f_3(t_3)F_1(X_3+d_1-d_2+X_2-t_3)dt_3]$$

$$- \alpha h_3F_3(X_3+d_1-X_1-d_2+X_2)$$

$$-p_1 = 0 \tag{B-2}$$

and

$$\partial TC/\partial X_2 = \alpha h_3F_3(X_3+d_1-X_1-d_2+X_2)$$

$$+(h_1+p_1)\int_{X_3+d_1-X_1-d_2+X_2}^{X_3+d_1-d_2+X_2} f_3(t_3)F_1(X_3-d_1-d_2+X_2-t_3)dt_1dt_3$$

$$+(h_2+p_2)[F_2(X_2)F_3(X_3) + \int_{X_3}^{X_2+X_3} f_3(t_3)F_2(X_2+X_3-t_3)dt_3]$$

$$- p_1 - p_2 = 0 \tag{B-3}$$

If $d_2 - X_2 > d_1 - X_1$ then $\partial TC/\partial X_1$ and $\partial TC/\partial X_2$ have similar forms (interchanging subscripts 1 and 2 and α for $1-\alpha$).

In the special case where $d_1-X_1=d_2-X_2$ we have

$$\partial TC/\partial X_1 = (h_1+p_1)G_{13}(X_1+X_3)-p_1 = 0$$

$$\partial TC/\partial X_2 = (h_2+p_2)G_{23}(X_2+X_3)-p_2 = 0$$

where G_{ij} denotes a convolution of F_i and F_j.

APPENDIX C

For $d_1-X_1>d_2-X_2$, the elements of the Hessian matrix are:

$$\partial^2TC/\partial X_1{}^2 = (h_1+p_1)F_3(X_3+d_1-X_1-d_2+X_2)f_1(X_1)$$

$$+ \alpha h_3F_3(X_3+d_1-X_1-d_2+X_2)$$

$$\partial^2TC/\partial X_1\partial X_2 = (h_1+p_1)\int_{X_3+d_1-X_1-d_2+X_2}^{X_3+d_1-d_2-X_2} f_3(t_3)f_1(X_3+d_1-d_2+X_2-t_3)dt_3$$

$$- \alpha h_3f_3(X_3+d_1-X_2-d_2+X_2)$$

$$\partial^2TC/\partial X_1\partial X_3 = (h_1+p_1)\int_{X_3+d_1-X_1-d_2+X_2}^{X_3+d_1-d_2+X_2} f_3(t_3)f_1(X_3+d_1-d_2+X_2-t_3)dt_3$$

$$- \alpha h_3f_3(X_3+d_1-X_1-d_2+X_2)$$

$$\partial^2TC/\partial X_2{}^2 = \alpha h_3f_3(X_3+d_1-X_1-d_2+X_2)$$

$$+(h_1+p_1)\int_{X_3+d_1-X_1-d_2+X_2}^{X_3+d_1-d_2+X_2} f_3(t_3)f_1(X_1-d_1-d_2+X_2-t_3)dt_3$$

$$-(h_1+p_1)f_3(X_3+d_1-X_1-d_2+X_2)F_1(X_1)$$

$$+(h_2+p_2)[f_2(X_2)F_3(X_3)+\int_{X_3}^{X_2+X_3} f_3(t_3)f_2(X_2+X_3-t_3)dt_3]$$

$$\partial^2 TC/\partial X_2 \partial X_3 = \alpha h_3 f_3(X_3+d_1-X_1-d_2+X_2)$$

$$+(h_1+p_1)\int_{X_3+d_1-X_1-d_2+X_2}^{X_3+d_1-d_2+X_2} f_3(t_3)f_1(X_3-d_1-d_2+X_2-t_3)dt_3$$

$$-(h_1+p_1)f_3(X_3+d_1-X_1-d_2+X_2)F_1(X_1)$$

$$\partial^2 TC/\partial X_3{}^2 = (h_1+p_1)\int_{X_3+d_1-X_1-d_2+X_2}^{X_3+d_1-d_2+X_2} f_3(t_3)f_1(X_3+d_1-d_2+X_2-t_3)dt_3$$

$$-(h_1+p_1)f_3(X_3+d_1-X_1-d_2+X_2)F_1(X_1)$$

$$+(h_2+p_2)[\int_{X_3}^{X_2+X_3} f_3(t_3)f_2(X_2+X_3-t_3)dt_3 - f_3(X_3)F_2(X_2)]$$

$$+(p_1+\alpha h_3)f_3(X_3+d_1-X_1-d_2+X_2)$$

$$+(p_2+(1-\alpha)h_3)f_3(X_3)$$