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Design of cavitation-free hydrofoils by a given pressure envelope

Dmitri V. Maklakov
Professor of Kazan State University of Architecture and Engineering
Kazan, Tatarstan, Russia

Farit G. Avkhadiev
Professor of Kazan State University
Kazan, Tatarstan, Russia

ABSTRACT

In this paper we shortly describe basic aspects of the theory of pressure envelopes which in the frame work of potential flows allows one to design a wing section shape that generates exactly a specified pressure envelope. By means of this theory we analyze and modify a series of hydrofoils designed by Eppler. The modifications based on shifts and proportional stretches of the dependence of the maximum velocity on the angle of attack. Besides, applying the theory, we solve an optimal problem and design a series of optimal hydrofoils which have a maximal width of the pressure bucket. We present accurate estimates of the maximal width as a function of the cavitation number and angle of attack.

INTRODUCTION

In hydrofoil theory the pressure envelope means the dependence $F(\alpha)$ of the minimal pressure coefficient, taken with opposite sign, on the angle of attack α:

$$F(\alpha) = -C_{p\,\min}(\alpha) = 2\frac{p_\infty - p_{\min}(\alpha)}{\rho v_\infty^2},$$

where $p_{\min}$ is the minimal pressure on the profile surface, p_∞ is the pressure at infinity, v_∞ is the velocity at infinity, ρ is the density of the fluid.

The function $F(\alpha)$ is one of the main characteristics of hydrofoils, which allows the cavitation-free incidence range to be predicted in advance. The classical condition of noncavitating flow implies that the pressure p must be greater than the vapour pressure p_v everywhere in the fluid (see e.g. [1]). In terms of F(a) this condition can be written as

$$F(\alpha) < Q, \quad Q = 2\frac{p_\infty - p_v}{\rho v_\infty^2},$$

where Q is the vapour cavitation number.

In a seaway, the cavitation number Q and changes in angles of attack (the latter can be caused by a sea state or by control devices of incidence variations) depend on the craft's speed. Thus according to craft operating requirements various types of pressure envelopes can be desired to operate in a seaway without the danger of cavitation.

Let us introduce the function

$$f(\alpha) = \sqrt{1 + F(\alpha)}. \tag{1}$$

It follows from the Bernoulli equation that $f(\alpha)$ defines the dependence of the maximal velocity on the profile surface on the angle of attack. In a series of works by Avhadiev and Maklakov it has been developed a method of designing hydrofoils whose pressure envelopes coincide with a function specified in advance (see [2] -[4], [6]). A systematical presentation of the method can be found in the monographs [5], [7].

In this paper we analyze a series of hydrofoils designed by Eppler (e816, e817, e836, e837, e838, e874, see [8]) and demonstrate how it is possible to modify the series by means of proportional stretch (shrink) and shift of the function $f(\alpha)$ along the α-axis. Besides, we solve an optimal problem and design a series of optimal hydrofoils which have maximal widths of the pressure bucket. We present accurate estimates of the maximal width as a function of the cavitation number and angle of attack.

1. BASIC ASPECTS OF THE PRESSURE ENVELOPE THEORY

Consider a two-dimensional potential flow of an ideal incompressible fluid over a single profile. Let $z = z(t)$ be the conformal mapping of the domain exterior to the unit circle in the parametric t-plane onto the flow region in the z-plane. The correspondence of points is: $z(\infty) = \infty$, $z(1) = 0$ (see Fig. 1, a, b). The origin of the coordinate system in the z-plane is at the trailing edge. The mapping $z = z(t)$ matches in one-to-one manner the points on the parametric circumference and the points on the profile. Let γ be a polar angle in the parametric t-plane, α be an angle of attack relative to the zero-lift line. We denote by $v(\gamma,\alpha)$ the velocity distribution along the parametric circle at the angle of attack α. The velocity at infinity is taken to be unity.

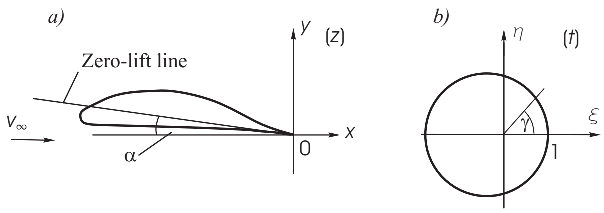

Figure 1. Physical z-plane; b) parametric t-plane

The complex potential w of the flow as a function of the parametric variable t is

$$w(t) = u_0(\mathrm{e}^{-\mathrm{i}\alpha}t + \mathrm{e}^{\mathrm{i}\alpha}/t + 2\mathrm{i}\sin\alpha\log t), \tag{2}$$

where $u_0 > 0$ is a constant which has the dimension of the velocity potential. Be means of the formula

$$\frac{\mathrm{d}w}{\mathrm{d}z} = \frac{\mathrm{d}w}{\mathrm{d}t}\Big/ z'(t) \tag{3}$$

and (2) we find

$$v(\gamma,\alpha) = u_0\left|\frac{(\mathrm{e}^{\mathrm{i}\gamma}-1)[\mathrm{e}^{\mathrm{i}\gamma} - \mathrm{e}^{\mathrm{i}(\pi+2\alpha)}]}{z'(\mathrm{e}^{\mathrm{i}\gamma})}\right|. \tag{4}$$

The function $z(t)$ can be represented in the form $z(t) = \mathrm{e}^{-\mathrm{i}\alpha}z_0(t)$, where $z_0(t)$ maps conformally the domain $|t| > 1$ onto the exterior of the hydrofoil located at the zero lift angle of attack. Then $\operatorname{Im} z_0'(\infty) = 0$. From equations (2), (3) and the condition $v_\infty = 1$ we find that $z_0'(\infty) = u_0$. From the relation (4) we deduce

$$v(\gamma,\alpha) = |\cos(\gamma/2-\alpha)|g(\gamma), \tag{5}$$

where the function

$$g(\gamma) = \frac{4u_0|\sin\gamma/2|}{|z_0'(\mathrm{e}^{\mathrm{i}\gamma})|} \tag{6}$$

is 2π-periodic and continuous.

It follows from (5) that for the potential flow the function $g(\gamma)$ completely defines the velocity distribution along the profile surface at any angle of attack α. If the function $g(\gamma)$ is known, then the shape of the profile can be easily restored. Indeed, from (6) we find

$$P(\gamma) = \log|z_0'(\mathrm{e}^{\mathrm{i}\gamma})/u_0| = \log\left|4\sin\frac{\gamma}{2}\right| - \log g(\gamma),$$

where $\gamma \in [-\pi,\pi]$. The function $P(\gamma)$ is a real part of the function $\chi(t) = \log(z_0'(t)/u_0)$, which is analytic in the exterior of the unit circle. Hence, $\chi(t)$ can be restored by means of the Schwarz integral:

$$\chi(t) = -\frac{1}{2\pi}\int_{-\pi}^{\pi} P(\gamma)\frac{\mathrm{e}^{\mathrm{i}\gamma}+t}{\mathrm{e}^{\mathrm{i}\gamma}-t}\,\mathrm{d}\gamma, \tag{7}$$

and here we take into account that $\operatorname{Im} z_0'(\infty) = 0$. Further, we deduce

$$z_0(t) = u_0\int_1^t \mathrm{e}^{\chi(t)}\,\mathrm{d}t, \tag{8}$$

and setting $t = \mathrm{e}^{\mathrm{i}\gamma}$, define thereby the contour of the profile in parametric form. The constant u_0 is a proportionality factor which should be chosen such that the profile has a given chord length.

Since $z_0'(\infty) = u_0$, we have $\chi(\infty) = 0$. By means of the Schwarz integral we deduce the condition, which provides the given (unit) velocity at infinity:

$$\int_{-\pi}^{\pi}\log g(\gamma)\,\mathrm{d}\gamma - 2\pi\log 2 = 0. \tag{9}$$

Tho contour of the profile has to be closed, which means that in the expansion of the function $\mathrm{d}z/\mathrm{d}t$ in powers of $1/t$ the coefficient of $1/t$ is zero. In terms of $g(\gamma)$ the conditions of closedness takes the form

$$\int_{-\pi}^{\pi}\log g(\gamma)\cos\gamma\,\mathrm{d}\gamma + \pi = 0, \quad \int_{-\pi}^{\pi}\log g(\gamma)\sin\gamma\,\mathrm{d}\gamma = 0. \tag{10}$$

Let us introduce the function

$$f(\alpha) = \max_{\gamma} v(\gamma,\alpha). \tag{11}$$

This function will be an envelope of the family of the functions $v(\gamma,\alpha)$, if γ is taken as a parameter of the family and α is taken as a variable. We have the same for the pressure envelope function. Thus $f(\alpha)$ can be called a velocity envelope. Bernoulli's integral relates the velocity envelope $f(\alpha)$ to the pressure envelope $F(\alpha)$ by the simple relation (1).

The mathematical formulation of the problem of hydrofoil design with a given pressure envelope consists in finding 2π-periodic, continuous function $g(\gamma)$ from the equation

$$\max_{\gamma\in\mathbb{R}} g(\gamma)|\cos(\gamma/2-\alpha)| = f(\alpha), \tag{12}$$

where $f(\alpha)$ is a given function. This equation is the basic equation of the pressure envelope theory.

Eq. (12) is neither integral, nor differential. It is a new type of equations that can be characterized as those of convolution type obtained by replacing integral by maximum (see [6]).

Theoretically the angles of attack relative to the zero-lift line can be changed in the range $-\pi/2 \le \alpha \le \pi/2$. A practical range of change of α for hydrofoils does not exceed several degrees. However, in solving Eq. (12) we assume the pressure envelope function $f(\alpha)$ to be given for $-\pi/2 \le \alpha \le \pi/2$. The matter is that the values of $f(\alpha)$ at the unrealisticly large angles of attack define the shape of the profile near the leading edge and it is well known that this shape is of great importance in hydrofoil design.

It follows from (12) that $f(\pi/2) = f(-\pi/2)$, hence the function $f(\alpha)$ can be continued periodically onto the entire α-axis by setting $f(\alpha+\pi) = f(\alpha)$. Thus we assume that the function $f(\alpha)$ is π-periodic and defined for any $\alpha \in (-\infty, +\infty)$. We shall call a point on the profile surface, where the maximum velocity locates at $\alpha = \pm\pi/2$, a hydrodynamic leading edge. Generally speaking, the hydrodynamic leading edge does not coincides with the geometric one except of symmetric profiles. Let γ_g be the image of the geometric leading edge on the parametric circle and γ_n be the image of the hydrodynamic leading edge. Our computations have demonstrated that for hydrofoils the ratio $|\gamma_n - \gamma_g|/\gamma_g$ is not more than several tenths of a percent. This is a hydrodynamic interpretation of the non-realistic angles of attack $\alpha = \pm\pi/2$.

The function $g(\gamma)$ is connected with the derivative $z_0'(\gamma)$ of the conformal mapping by the formula (6). Since the profile is smooth, except of the trailing edge point, a solution to Eq. (12) belongs to the class of 2π-periodic, nonnegative, continuous functions which can vanish only at the points $\gamma = 2n\pi$, $n \in \mathbb{Z}$. The set of such functions we denote by G.

Denote by T the set of strictly positive, 2π-periodic and trigonometrically convex functions. The definition of the trigonometrical convexity is similar to that of the ordinary convexity (see [9]): A function $f(\alpha)$ is trigonometrically convex if for two arbitrary points α_1 and α_2, $0 < \alpha_2 - \alpha_1 < \pi$, the following inequality holds

$$f(\alpha) \le H(\alpha) \quad \alpha_1 < \alpha < \alpha_2, \tag{13}$$

where

$$H(\alpha) = \frac{f(\alpha_1)\sin(\alpha_2-\alpha) + f(\alpha_2)\sin(\alpha-\alpha_1)}{\sin(\alpha_2-\alpha_1)}. \tag{14}$$

Geometrically the inequality (13) means that the the graph of $f(\alpha)$ for $\alpha \in [\alpha_1, \alpha_2]$ lies not above the trigonometrical chord, determined by Eq. (14).

Let $f \in T$. We introduce the functions

$$q(\alpha) = 2\left[\alpha + \arctan\frac{f'(\alpha)}{f(\alpha)}\right], \tag{15}$$

$$g_m(\gamma; f) = \min_{\alpha\in\mathbb{R}} \frac{f(\alpha)}{|\cos(\gamma/2-\alpha)|}. \tag{16}$$

Besides, we define the following constants, which are the functionals depending on $f(\alpha) \in T$:

$$K_0 = \int_{-\pi}^{\pi} \log g_m(\gamma; f)\,\mathrm{d}\gamma - 2\pi\log 2, \tag{17}$$

$$K_1 = \int_{-\pi}^{\pi} \log g_m(\gamma; f)\cos\gamma\,\mathrm{d}\gamma + \pi, \tag{18}$$

$$K_2 = \int_{-\pi}^{\pi} \log g_m(\gamma; f)\sin\gamma\,\mathrm{d}\gamma, \tag{19}$$

where $g_m(\gamma; f)$ is defined by (16), and the constants are obtained by the substitution of $g_m(\gamma; f)$ for $g(\gamma)$ in the left hand sides of the conditions (9), (10).

We define a nose part of the profile as a a set of the points on the profile surface where the maximums of velocity are located as α changes in the range $-\pi/2 \le \alpha \le \pi/2$. The corresponding set on the parametric circumference we denote by N. In the general case the nose part may be disconnected and may consist of isolated points and arcs, Fig. 2.

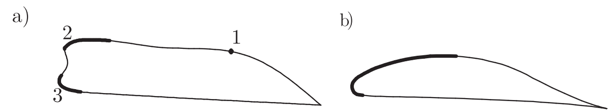

Figure 2. Profiles with disconnected (a) and connected (b) nose parts.

From the results of the works [2] – [6] it follows the following theorem.

Theorem 1 *1) For solvability of the basic equation (12) in the class G it is necessary and sufficient that $f(\alpha) \in T$. The function $g_m(\gamma; f)$ defined by the formula (16) belongs to the class G, strictly positive and satisfies Eq. (12); the function $1/g_m(2\gamma, f) \in T$.*

2) If the function $f(\alpha) \in T$ is the velocity envelope for a certain profile, then $f(\alpha) > 1$, and there exists such an angle of attack α_c that

$$q(\alpha_c - 0) < 0 < q(\alpha_c + 0) \tag{20}$$

where $q(\alpha)$ is defined by the formula (15); the constants K_0, K_1, K_2 satisfy the inequality

$$\Delta K = K_0 - \sqrt{K_1^2 + K_2^2} > 0. \tag{21}$$

3) If the function $f(\alpha) \in T$ is the velocity envelope for a certain profile with a connected nose part, then

$$\begin{array}{ll} g(\gamma) = g_m(\gamma; f) \quad \textit{for} \quad \gamma \notin [\gamma_0^+, \gamma_0^-], \\ g(\gamma) \le g_m(\gamma; f) \quad \textit{for} \quad \gamma \in [\gamma_0^+, \gamma_0^-], \end{array} \tag{22}$$

where $\gamma_0^+ = q(\alpha_c + 0)$, $\gamma_0^- = q(\alpha_c - 0)$, and the constants K_0, K_1 and K_2, besides (21), satisfy the condition

$$\alpha_e = \arg(K_1 + \mathrm{i}\,K_2) \in (\gamma_0^-, \gamma_0^+) \tag{23}$$

In this theorem we have tried to concentrate the basic properties of the velocity envelopes for realistic profiles without self-intersections.

The condition $f(\alpha) \in T$, i.e. the condition of trigonometrical convexity, is of great importance since it throws away enormous quantity of functions which cannot be realized as velocity envelopes. Indeed, in [9] it is proved that if $f(\alpha)$ is continuous and has the first and second piecewise continuous derivatives, then such a function is trigonometrically convex if and only if

$$f''(\alpha) + f(\alpha) \ge 0, \quad f'(\alpha - 0) < f'(\alpha + 0), \tag{24}$$

where $f'(\alpha-0)$ and $f'(\alpha+0)$ are the derivatives of $f(\alpha)$ on the left and right respectively, the first inequality holds at the points of continuity of $f'(\alpha)$, the second holds at the points of discontinuity. The first inequality is analogous to the condition $f''(\alpha) \geq 0$ for ordinary convex functions. The inequalities (24) means, for example, that the function $f(\alpha) = a\cos\alpha + b\sin(\alpha) - \varepsilon$, where a and b are certain constants, cannot be a part of the velocity envelope no matter how small is $\varepsilon > 0$. But for any $\varepsilon \leq 0$ this function is admissible.

The inequality (20) is equivalent to the statement that the location of the maximal velocity cannot be at the point of the trailing edge. It follows from (20) that any velocity envelope (or pressure envelope) has at least one point of discontinuity of its derivative. When the angle of attack passes from $\alpha < \alpha_c$ to $\alpha > \alpha_c$, the location of the maximal velocity will jump from the lower surface of the profile to the upper one. At $\alpha = \alpha_c$ the maximal velocity lies on the upper and lower surfaces simultaneously. In what follows, the angle of attack α_c will be called a *central angle of attack*.

The parameters ΔK and α_e are the key parameters, responsible for geometry of hydrofoils. Roughly speaking, ΔK determines the thickness of the profile part near the trailing and α_e determines the curvature of this part. The parameter α_e we shall call an *eccentricity* of the profile. If $\alpha_e = 0$, the profile will be called *centered*.

Besides ΔK and α_e we introduce $k_w = \Delta K/K_0$. If k_w, ΔK and α_e are known the constants K_0, K_1, K_2 can be restored uniquely.

2. DESIGNING CLOSURE COMPONENT

As follows from the relation (22), for the profile with a connected nose part the function $g(\gamma)$ is known everywhere, except for the segment (γ_0^+, γ_0^-), where $g(\gamma) \leq g_m(\gamma; f)$. This segment defines the pressure recovery region of the profile. Let $g(\gamma) = g_m(\gamma)\exp[-m(\gamma)]$ on this segment, and $m(\gamma)$ is the desired function. This function must satisfy the conditions

$$m(\gamma) \geq 0, \quad m(\gamma_0^+) = 0, \quad m(\gamma_0^-) = 0 \tag{25}$$

$$\int_{\gamma_0^-}^{\gamma_0^+} m(\gamma)\,\mathrm{d}\gamma = K_0, \tag{26}$$

$$\int_{\gamma_0^-}^{\gamma_0^+} m(\gamma)\cos\gamma\,\mathrm{d}\gamma = K_1,\ \int_{\gamma_0^-}^{\gamma_0^+} m(\gamma)\sin\gamma\,\mathrm{d}\gamma = K_2. \tag{27}$$

On the segment $[\gamma_0^+, \gamma_0^-]$ the velocity

$$v(\gamma,\alpha) = g_m(\gamma; f)\,\mathrm{e}^{-m(\gamma}|\cos(\gamma/2-\alpha)|.$$

From the point of view of favorable development of the boundary layer on this segment a nonseparated flow will be provided if $v(\gamma,\alpha)$ is close as possible to a constant value. This desired closeness can be achieved if we minimize the functional

$$I = \int_{\gamma_0^-}^{\gamma_0^+} \left[\frac{\mathrm{d}}{\mathrm{d}\gamma}\log v(\gamma,\alpha)\right]^2 \mathrm{d}\gamma$$

with respect to $m(\gamma)$ and α. It is possible to prove that on the segment $[\gamma_0^+, \gamma_0^-]$ the function

$$g_m(\gamma; f) = f(\alpha_c)/\cos(\gamma/2-\alpha_c).$$

Then

$$I = \int_{\gamma_0^-}^{\gamma_0^+} \left[\frac{1}{2}\tan(\gamma/2-\alpha_c) - \frac{1}{2}\tan(\gamma/2-\alpha) - m'(\gamma)\right]^2 \mathrm{d}\gamma.$$

The difference $\frac{1}{2}\tan(\gamma/2-\alpha_c) - \frac{1}{2}\tan(\gamma/2-\alpha)$ has the order of $|\alpha-\alpha_c|$ and for hydrofoils is very small. Because of this we can omit it to get finally

$$I[m] = \int_{\gamma_0^-}^{\gamma_0^+} [m'(\gamma)]^2\,\mathrm{d}\gamma. \tag{28}$$

Thus, the problem of finding the closure component is reduced to the minimization of the functional (28) under the constrains (25)–(26). After discretization we obtain a quadratic programming problem whose solvability is provided by the conditions (21) and (23) of Theorem 1.

3. SIMPLE TRANSFORMATIONS OF THE VELOCITY ENVELOPES

We investigate the question how the function $g_m(\gamma; f)$ and the constants K_0, K_1, K_2 will change if we stretch or shift the velocity envelope $f(\alpha)$. Let for a certain profile the functions $f(\alpha)$ and $g_m(\gamma; f)$ are known. It follows from (16) that, if A is a constant value, then $g_m(\gamma, Af) = Ag_m(\gamma, f)$, i.e. under the proportional stretch of $f(\alpha)$ the function $g_m(\gamma, f)$ also changes proportionally. We shall mark the new values of the constants by a star. From (17)–(19) we infer

$$K_0^* = 2\pi\log A + K_0, \quad \Delta K^* = 2\pi\log A + \Delta K$$
$$K_1^* = K_1, \quad K_2^* = K_2,$$

i.e. the constant K_0 changes, but K_1 and K_2 remain the same. Hence, by means of the stretch we always can satisfy the condition (21). To do so it is enough to choose $A > \exp[-\Delta K/(2\pi)]$. If we want to get a desired value of k_w^* by means of the stretch, we set

$$A = \exp\frac{k_w^*\,K_0 - \Delta K}{2\pi(1-k_w^*)}. \tag{29}$$

Now consider the function $f(\alpha - \alpha_s)$, whose graph is shifted with respect to the graph of $f(\alpha)$ by the angle α_s along the α-axis. From (16), (17)–(19) we deduce

$$g_m(\gamma; f(\alpha-\alpha_s)) = g_m(\gamma - 2\alpha_s; f(\alpha)),$$

$$\begin{gathered} K_0^* = K_0, \\ K_1^* = (K_1-\pi)\cos 2\alpha_s - K_2\sin 2\alpha_s + \pi \\ K_2^* = K_2\cos 2\alpha_s + (K_1-\pi)\sin 2\alpha_s. \end{gathered} \tag{30}$$

Thus, the shift of the function $f(\alpha)$ by the angle α leads to the shift of the function $g_m(\gamma; f)$ by 2α along the γ-axis. The constant K_0 remains the same, but the constants K_1 and K_2 change according to (30). By means of the shift we always can satisfy the condition (23).

For a centered profile with zero eccentricity ($\alpha_e = 0$) the constant $K_2 = 0$. Because of this any profile can be centered by means of the shift of $f(\alpha)$ by the angle $\alpha_s = \frac{1}{2}\arctan[K_2/(\pi - K_1)]$.

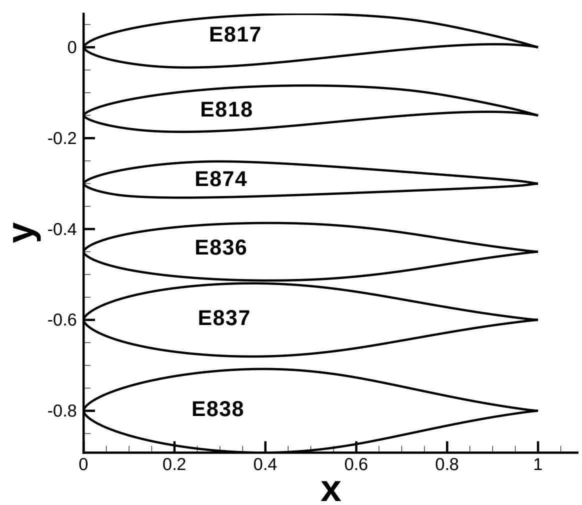

Figure 3. Eppler's series of hydrofoils.

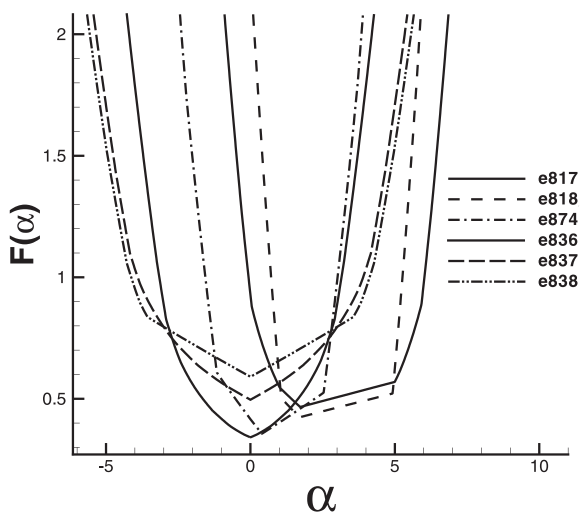

Figure 4. Pressure envelopes for Eppler's series

4. ANALYSIS OF EPPLER'S SERIES

Our aim now is to analyze the series of Eppler's profiles by computing for them the parameters ΔK, k_w and α_e. We believe that these parameters are of great importance for any hydrofoil design. The geometry of Eppler's series can be found in his monograph [8]. The series is shown in Fig. 3.

To determine the functions $f(\alpha)$ for these profiles we have used a very accurate method of conformal mappings based on solving an integral equation. In Fig. 4 we demonstrate the pressure envelope for the series. The functions $g_m(\gamma; f)$ have been found from Eq. (16) by solving the minimization problem. To define $g_m(\gamma; f)$ accurately we have developed a special numerical algorithm. The characteristics of the pressure envelopes of the series are shown in Table 1. In Table 1 all angular characteristics are presented in degrees. The angle α_0 is the zero lift angle of attack. The parameter u_0 defines the lift coefficient $C_y = 8\pi u_0 \sin\alpha$. The value $F(\alpha_c)$ is the minimal value of $F(\alpha)$.

Table 1.

Name	α_0	u_0	α_c	γ_0^+	γ_0^-	$F(\alpha_c)$	ΔK	k_w	α_e
E817	-4.35	0.271	1.72	68	-126	0.47	0.13	0.20	-26.5
E818	-4.34	0.268	1.70	68	-132	0.42	0.14	0.22	-29.9
E874	-0.66	0.266	0.40	117	-144	0.36	0.25	0.41	-9.64
E836	0	0.274	0	96	-96	0.34	0.08	0.17	0
E837	0	0.279	0	104	-104	0.50	0.15	0.22	0
E838	0	0.283	0	99	-99	0.59	0.14	0.21	0

According to criteria (21) and (23) the limiting values of the parameters ΔK, k_w are zeroes. As one can see from Table 1, the symmetric profile E836 has the closest to zero parameters. For this reason we choose E836 for the stretch and shift modifications of its function $f(\alpha)$.

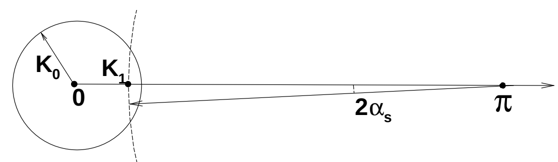

Figure 5.

We should note that pure shift modifications are dangerous. Indeed, profile E836 is symmetric and, hence, centered with $K_2 = 0$, α_e=0. For this profile $K_0 = \Delta K/k_w = 0.494$, $K_1 = K_0 - \Delta K = 0.417$. From formulas (30) one can deduce that with the shift by the angle α_s the complex point $K_1^* + iK_2^*$ moves along the circumference of radius $(\pi - K_1)$ with the center at the point $(\pi, 0)$ (see Fig. 5). As one can see from Fig. 5, the point $K_1^* + iK_2^*$ leaves the trapping circle of radius K_0 even for very small α_s, and the criterion (21) will be violated. A simple geometric reasoning show that for a centered profile the maximal possible shift without changes of K_0 is defined by the formula

$$\max(\alpha_e) = \frac{1}{2}\arccos\frac{K_0^2 + K_1^2 + 2K_1\pi - 2\pi^2}{2\pi(K_1 - \pi)}.$$

For E836 the maximal shift angle is only $2.59°$ and for this shift the profile will be nonrealistic necessarily. A bigger shift can be obtained if we allow changes of K_0. So, in modifications of E836 we conserve the parameter $k_w = 0.1$ and together with the shift make the corresponding stretch of $f(\alpha)$ by the formula (29). The results of such modifications are shown in Fig. 6 and in Table 2.

As one can see from Fig. 6 and Table 2, maintaining the parameter $k_w = 0.1$, we are able to make the shift of $f(\alpha)$ by the angle $\alpha_e = 3°$. For the angle $\alpha_e = 4°$ the profile is already nonrealistic and has a point of self-intersection. The figure and table demonstrate the critical values of the eccentricity angle α_e. We come to the conclusion that for the velocity envelopes with $|\alpha_e| > 30°$ the profiles are either nonrealistic at all (with self-intersections), or they have a very thin trailing edge as the profile with $\alpha_e = 3°$ in Fig. 6.

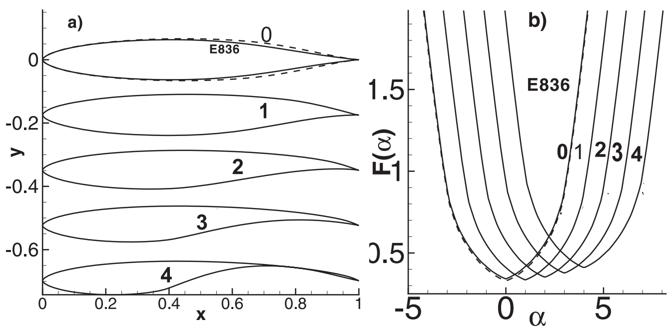

Figure 6. Modifications of profile e836.

Table 2.

Name	α_0	u_0	α_c	γ_0^+	γ_0^-	$F(\alpha_c)$	ΔK	k_w	α_e
E836	0	0.274	0	96	-96	0.34	0.08	0.17	0
$\alpha_s = 0$	0	0.277	0	96	-96	0.33	0.05	0.1	0
$\alpha_s = 1$	-1.78	0.276	1.0	98	-94	0.34	0.05	0.1	-12.7
$\alpha_s = 2$	-3.63	0.274	2.0	100	-92	0.35	0.05	0.1	-24.1
$\alpha_s = 3$	-5.13	0.270	3.0	102	-90	0.38	0.06	0.1	-37.1
$\alpha_s = 4$	-6.32	0.266	4.0	104	-88	0.41	0.07	0.1	-40.4

5. ESTIMATES OF THE WIDTH OF PRESSURE ENVELOPE BUCKETS

Let $f_0 > 1$, α_w, $\Delta K_0 > 0$ and $\gamma_0 \in [0, \pi]$ be given values. We formulate the following optimization problem: find the maximal $d > 0$ and a corresponding realistic profile such that the velocity envelope for this profile satisfies the conditions

$$f(\alpha) \leq f_0, \quad \alpha \in [\alpha_w - d, \alpha_w + d] \tag{31}$$

$$f(\alpha_w - d) = f(\alpha_w + d) = f_0 \tag{32}$$

$$\gamma_0^+ \geq \gamma_0, \quad \gamma_0^- < -\gamma_0, \quad \Delta K \geq \Delta K_0. \tag{33}$$

We shall solve the problem under a natural assumption that the central angle of attack $\alpha_c \in (\alpha_w - d, \alpha_w + d)$. In this problem the parameter f_0 is connected with the cavitation number Q by the simple relation $f_0 = \sqrt{1+Q}$. The intervals $[0, \gamma_0]$ and $[-\gamma_0, 0]$ define the dimensions of the pressure recovery regions on the upper and lower surfaces of the desired profiles, respectively. The value $2d$ is the width of the bottom of the pressure bucket.

The parameter ΔK_0 is introduced to get meaningful, geometrically realistic profiles. Indeed, according to the criterion (21) $\Delta K > 0$. At $\Delta K = 0$ the profile is necessarily nonrealistic. But if ΔK is very small, the profile will be either with self-intersection or its trailing edge will be very thick. Hence we need to have some positive reserve for the value of ΔK for geometrically realistic profiles.

So, if we know the cavitation number Q and specify the middle α_w of the bottom of the pressure bucket, the solution to the problem will give us the maximal width of this bottom under the restriction (33) on dimensions of the pressure recovery region.

To solve the problem we need the following comparison theorem.

Theorem 2 *Let $f(\alpha)$ and $f^*(\alpha)$ be two trigonometrically convex, strictly positive, π-periodic functions. If $f(\alpha) \leq f^*(\alpha)$, then $\Delta K \leq \Delta K^*$, and the equality is possible if and only if $f(\alpha) \equiv f^*(\alpha)$.*

Thus, any decrease of $f(\alpha)$ leads to decrease of the parameter ΔK. This theorem has been proved only recently, and not to overloud the paper by pure mathematical reasoning we omit its proof.

Consider now any profile with the velocity envelope $f(\alpha)$ which satisfies the constrains (31)–(33). Let α_c be the central angle of attack of this profile. We denote $f_c = f(\alpha_c)$, $\alpha_1 = \alpha_w - d$, $\alpha_2 = \alpha_w + d$.

Now we join sequently five points $(\alpha_2 - \pi, f_0)$, (α_1, f_0), (α_c, f_c), (α_2, f_0), $(\alpha_1 + \pi, f_0)$ in the plane (α, f) by trigonometrical chords. The equations for these chords we find from (14). As a result we obtain a trigonometrically convex, strictly positive, π-periodic function:

$$f^*(\alpha) = \begin{cases} -\frac{f_0[\sin(\alpha-\alpha_2)+\sin(\alpha-\alpha_1)]}{\sin(\alpha_2-\alpha_1)}, & \pi/2 \leq \alpha < \alpha_1 \\ \frac{f_c \sin(\alpha-\alpha_1) - f_0 \sin(\alpha-\alpha_c)}{\sin(\alpha_c-\alpha_1)}, & \alpha_1 \leq \alpha < \alpha_c \\ \frac{f_c \sin(\alpha_2-\alpha) + f_0 \sin(\alpha-\alpha_c)}{\sin(\alpha_2-\alpha_c)}, & \alpha_c \leq \alpha < \alpha_2 \\ \frac{f_0[\sin(\alpha-\alpha_2)+\sin(\alpha-\alpha_1)]}{\sin(\alpha_2-\alpha_1)}, & \alpha_2 \leq \alpha < \pi/2 \end{cases} \tag{34}$$

The parameters related to the function $f^*(\alpha)$ we shall mark by a star. Since the function $f^*(\alpha)$ is formed by trigonometrical chords of $f(\alpha)$, we get $f(\alpha) < f^*(\alpha)$ and by Theorem 2 $\Delta K^* \geq \Delta K \geq \Delta K_0$. Moreover, at the point α_c we have

$$f^{*\prime}(\alpha_c + 0) \geq f'(\alpha_c + 0), \quad f^{*\prime}(\alpha_c - 0) \leq f'(\alpha_c - 0).$$

But the boundaries γ_0^+ and γ_0^- of the pressure recovery region are defined by the formula $\gamma_0^+ = q(\alpha_c + 0)$, $\gamma_0^- = q(\alpha_c - 0)$, where $q(\alpha)$ is determined by (15). Hence, $\gamma_0^{*+} > \gamma_0^+$, $\gamma_0^{*-} < \gamma_0^-$. So, the restrictions (33) for the function $f^*(\alpha)$ are fulfilled.

Now we formulate the following auxiliary optimization problem: maximize d for the function $f^*(\alpha) > 1$, defined by (34), under the restrictions (33).

If a solution to the auxiliary problem possesses the property $\Delta K = \Delta K_0$, then it will be the solution to the initial problem. Indeed, let $f^{**}(\alpha)$ be such a solution to the auxiliary problem with $d = d_{\max}$ and $f(\alpha)$ be any function, different from $f^{**}(\alpha)$, that satisfies the restrictions (31)–(33) for some d. Construct for $f(\alpha)$ the majorant function $f^{*}(\alpha)$ by means of (34). The function $f^{*}(\alpha)$ satisfies the restrictions (33). If $f^{*}(\alpha)$ is different from $f^{**}(\alpha)$, then $d \leq d_{\max}$. But the case $f^{*}(\alpha) \equiv f^{**}(\alpha)$ is only possible if $f(\alpha) \equiv f^{**}(\alpha)$, because otherwise $\Delta K[f] < \Delta K[f^{**}] = \Delta K_0$ and the restriction $\Delta K[f] \geq \Delta K_0$ will be violated.

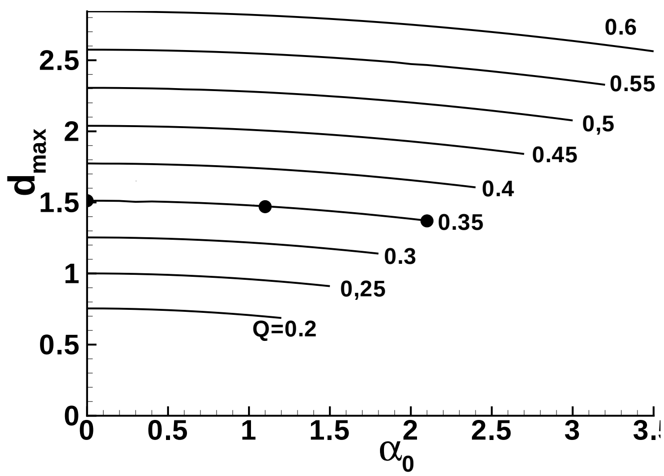

Figure 7. The graphs of the functions $d_{\max}(\alpha_w)$ for different Q.

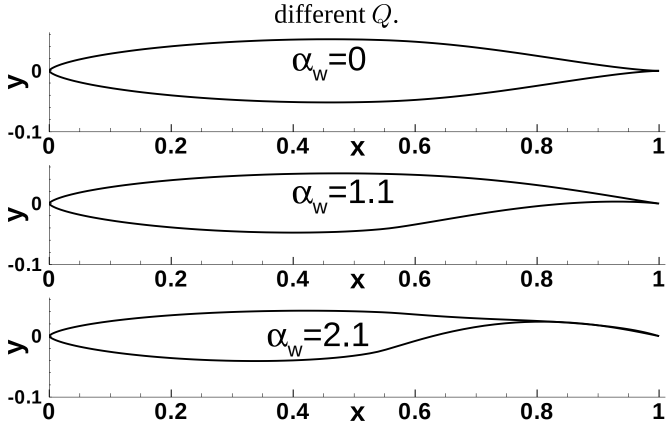

Figure 8. The shapes of optimal profiles at $Q = 0.35$.

We have investigated the auxiliary problem numerically and come to the conclusion that for all its solutions $\Delta K = \Delta K_0$. So, for the given cavitation number Q, the medial angle α_w of the pressure bucket and the angle γ_0, which defines the dimensions of the pressure recovery region, we are able to design a series of optimal profiles. In Fig. 7 we demonstrate the functions $d_{\max}(\alpha)$ for different values of Q. In these computations we choose $\Delta K_0 = 0.06$, $\gamma_0 = 80°$. We stop the increase of α_w when the eccentricity α_e of the obtained profiles exceeds $30°$, since further increase leads to nonrealistic profiles. So, every point on the graphs of Fig. 7 associates with some geometrically realistic optimal profile. For the points $(\alpha_w, d_{\max})$, marked by circles in Fig. 7, the shapes of such profiles are shown in Fig. 8. Their pressure envelopes one can see in Fig. 9.

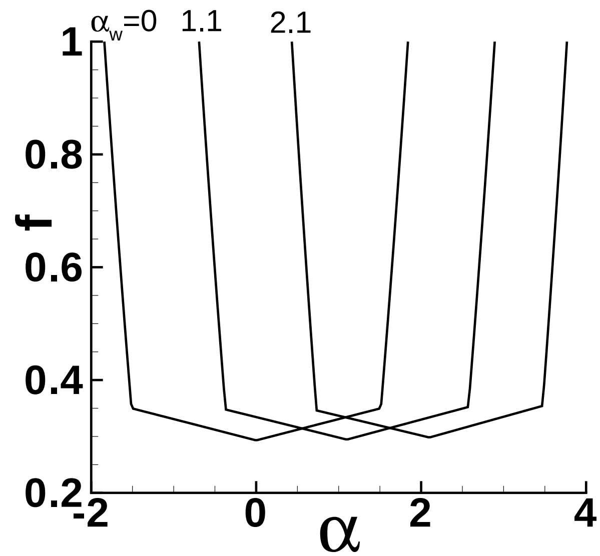

Figure 9. The pressure envelopes for optimal profiles at $Q = 0.35$.

CONCLUSIONS

In this paper we have worked out some practical criteria for designing hydrofoils by a given pressure envelope. In particular, the parameters ΔK and α_e, which are the functionals of the pressure envelope, have been introduced, and it has been demonstrated that for geometrically realistic profiles $\Delta K \approx 0.6$ and the eccentricity angle $|\alpha_e| \leq 30°$. In working out these criteria the series of Eppler's hydrofoile turns out to be helpful. We have shown how it is possible to stretch and shift the pressure envelopes and presented the results of such modifications.

Besides, we have given accurate estimates of the widths of pressure buckets and designed a series of optimal geometrically realistic profiles, which realize these estimates. So far our attention have been mainly stressed on geometrical requirements and the development of the boundary layer have not been considered. The velocity envelopes for the optimal profile have the simplest shape: they consist of four sinusoidal functions and have angular points. But we have strictly proved that, if the boundary layer is not taken into account, the obtained envelopes are optimal. It is clear that restrictions of nonseparated boundary layer should be included in the design procedure. Certainly these restrictions will make the pressure envelopes smoother, but at the same time will lead to decrease of the width of pressure buckets.

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