Proceedings of the 7th International Symposium on Cavitation
CAV2009 – Paper No. 145
August 17-22, 2009, Ann Arbor, Michigan, USA

A Simple Approach to Estimating Three-Dimensional Supercavitating Flow Fields

Ivan N. Kirschner, PhD
Alion Science and Technology Corporation
Middletown, RI, USA

Ryan E. Chamberlin, PhD
Alion Science and Technology Corporation
Middletown, RI, USA

Sevag H. Arzoumanian
Alion Science and Technology Corporation
Cambridge, MA, USA

ABSTRACT

A simple method is formulated for predicting three-dimensional supercavitating flow behind cavitators subject to gravitational acceleration and motion of the cavitator. The method applies slender-body theory in the context of matched asymptotic expansions to pose an inner problem for the cavity evolution downstream from the locus of cavity detachment. This inner problem is solved by means of a coupled set of equations for the Fourier coefficients characterizing the cavity radius and the velocity potential as a function of downstream location and circumferential location, thus resulting in a two-dimensional multipole solution at each station. For the lowest-order term in the Fourier expansion, it is necessary to match the parabolic inner solution to a fully elliptic outer solution. This step allows the application of any one of a number of methods to solve the axisymmetric problem, which serves as the base solution that is perturbed by the three-dimensional effects. The method is an attempt to formalize the Logvinovich principle of independent cavity section evolution. Results flow past a circular disk cavitator are presented for several values of the cavity Froude number.

Keywords: supercavitation, multipole, slender body theory, potential flow, drag reduction, matched asymptotic expansion.

INTRODUCTION

At sufficiently high-speeds, cavitation will occur on the surface of submerged bodies at the point where the local pressure drops to the value of the vapor pressure of the ambient fluid. If the cavitation number is sufficiently low, a supercavity will form that covers the entire vehicle, as is shown in figure 1. The effect can be enhanced by ventilating the cavity from a gas supply on board the vehicle. With proper design, a supercavitating body can have a much lower drag coefficient than that of a fully-wetted body due to the near elimination of skin friction drag. This topic has been explored by many researchers, and has gained interest among the international community over the last several years.

Since the boundary of the cavity is a free surface, a nominally axisymmetric supercavity surrounding a body moving horizontally in a gravity field will be distorted. As gravitational effects become important relative to the inertia of the surrounding liquid, the cavity tends to tail up, and, depending on the shape of the cavitator and other conditions, nominally circular cross sections develop a dimple along the ventral line of the cavity or other asymmetric features [1], [2].

If the effects of gravity are great enough, this dimple becomes so pronounced downstream that the topology of cavity closure is fundamentally altered, and the flow transitions from the *re-entrant jet regime* to the *twin-vortex regime*, in which a pair of counter-rotating vortices is observed downstream of the nominal cavity closure region [3]. If these vortices are strong enough, the velocity field they induce must have some influence on the shape of the downstream part of the cavity. A simple heuristic analysis of the circulation that develops around the tailed up cavity is sufficient to show that the circulation of the trailing vortices is such that they induce up-wash between them [1].

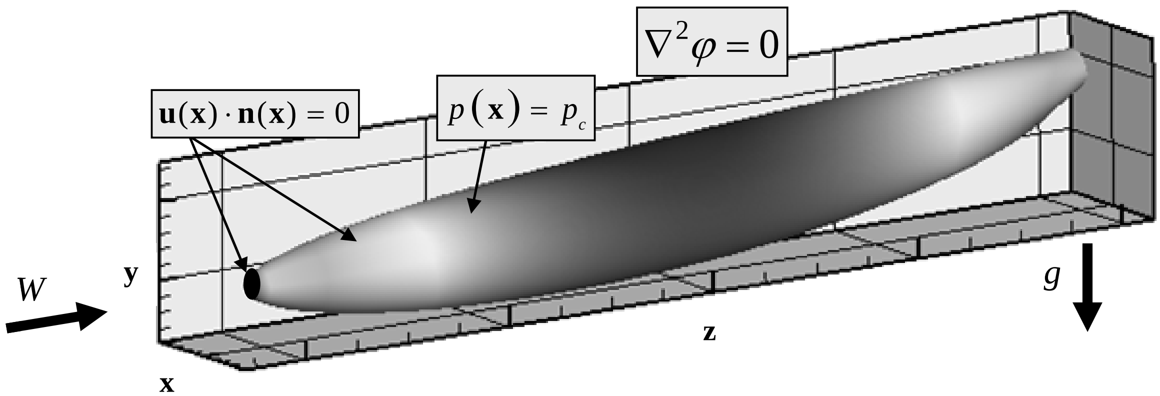

Figure 1. The problem of supercavitating flow past a vehicle moving horizontally in a gravity field.

If some means of generating lift is provided at the cavitator – for example, by inclining a disk or cone relative to the oncoming stream [4], [5] – or somewhere downstream along the cavity, the vortex system trailing from the lifting device also influences the cavity shape. If the lift opposes gravity, this trailing vortex system induces down-wash on the nominal center-plane of the cavity. Although this down-wash may counteract the tail-up due to gravity to some extent, in general the two effects do not cancel one another except perhaps at a single point along the length of the cavity [1], [6].

Such three-dimensional effects are important for many reasons. Firstly, the main value of supercavitation is drag reduction, which requires that most of the body be enveloped within the cavity. Thus, the distortion of the cavity must be considered in the hydrodynamical design of the cavitator-body system. Secondly, since buoyancy is lost in way of the cavity, some means must be provided to support the body. A lifting cavitator or canards may be considered in order to support the front end of the body. The afterbody may be supported by lifting surfaces or by allowing it to plane on the cavity boundary [7], [8], [9], [10], [11], [12]. In any of these cases, the design of the various lift-generating mechanisms must account for the three-dimensional distortion of the cavity, which itself is influenced not only by gravity, but also by the velocity field induced by the lifting devices. Thirdly, cavity distortion can have a significant effect on the rate at which gas is entrained downstream, especially if the transition to the twin-vortex regime occurs [3], [13], [14], [15], [16], [17], [18]. Since it is usually desirable to minimize the entrainment rate, its relationship to cavity distortion can be important. Finally, supercavities are subject to various flow instabilities, the most important of which is the effect known as *cavity auto-oscillation*, and the three-dimensional distortion has an influence on the cavity dynamics [19], [20], [11], [21], [18].

This article presents a relatively simple approach to estimating the three-dimensional shape of a slender cavity. Although the method is presently being extended to address the effects of lift, this article focuses on the effects of gravity on cavity shape. Three-dimensional effects of gravity and lift on the shape of nominally axisymmetric supercavities have been explored by various researchers. A useful overview of early work is presented in May [2], who presents a semi-empirical estimate of the distortion of the line of centers due to gravitational effects. Fundamental analyses by Logvinovich and the computational methods developed by his colleagues (including the work by Buyvol [22]) are presented in [1] and [6]. These references present other estimates – both theoretical and semi-analytical – of the distortion of the line of centers. A technique based on Fourier decomposition is also presented for predicting the distortion of the cavity in transverse planes due to gravity, cavitator projected shape, and cavitator lift. An alternative result for the distortion of the line of centers of a supercavity is presented by Semenenko [17]. A valuable discussion of the physics of slender, ventilated supercavities is presented in the textbook by Franc and Michel [23]. Grid-based computation capturing the three-dimensional shape of ventilated cavities has been performed by various researchers, as exemplified by the work of the team led by Kunz and Lindau [24], [25], [26], [27], which has been extended to the time domain. Kring, et al [28] present a preliminary version of a time-domain code for the fully three-dimensional problem.

SIMPLIFICATION OF THE FULL THREE-DIMENSIONAL PROBLEM

The model presented herein falls in the class of potential flow methods. The fully three-dimensional problem is governed by Laplace's equation in the field, a no-flux condition on the surfaces of any wetted portion of the body, and both no-flux and constant-pressure conditions on the cavity boundary. (See figure 1.)

The full three-dimensional problem may be stated as follows. Under the assumptions of ideal flow, the total velocity $\mathbf{u}(\mathbf{x}) = \left(W + w(\mathbf{x})\right)\mathbf{k} + u_r\mathbf{e}_r + u_\theta\mathbf{e}_\theta$ may be computed as the gradient of the *total* velocity potential, $\mathbf{u}(\mathbf{x}) = \nabla\varphi(\mathbf{x})$. Here W is the free stream velocity, and the stream-wise coordinate is in the z-direction (not to be confused with the general position and velocity vectors, $\mathbf{x}$ and $\mathbf{u}$).

Considering that for many problems of interest the cavity sections are approximately circular just downstream of the cavitator, it is natural to solve the problem in cylindrical-polar coordinates. For incompressible flow, the total potential satisfies Laplace's equation in the field:

$$\nabla^2\varphi = \frac{\partial^2\varphi}{\partial r^2} + \frac{1}{r}\frac{\partial\varphi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\varphi}{\partial\theta^2} + \frac{\partial^2\varphi}{\partial z^2} = 0. \qquad (1)$$

On the entire hydrodynamic envelope consisting of the cavitator surface S_n and the cavity surface S_c, the no-flux condition applies:

$$u_r - u_\theta\frac{1}{R}\frac{\partial R}{\partial\theta} - (W+w)\frac{\partial R}{\partial z} = 0 \quad \text{on } S_\mathrm{n}\cup S_\mathrm{c}, \qquad (2)$$

where $r = R(\theta, z)$ is the set of points defining the surface of the cavitator-cavity system. The surface of the cavity is unknown *a priori*, and an additional condition is required to obtain a unique solution of the problem. Specifically, the pressure on the cavity surface must equal the cavity pressure, $p = p_\mathrm{c}$ on S_c, which for the current simple case is considered to be constant in both space and time. Application of Bernoulli's equation provides the following dynamic condition on the cavity surface:

$$\frac{p_\mathrm{c}}{\rho} + \frac{1}{2}\left[u_r^2 + u_\theta^2 + (W+w)^2\right] + gR\sin\theta = \frac{p_\infty}{\rho} + \frac{1}{2}W^2 \quad \text{on } S_\mathrm{c}, \qquad (3)$$

where p_∞ is the pressure at upstream infinity and ρ is the density of the ambient liquid. The circumferential coordinate θ is measured from the horizontal plane passing through the origin.

For the current problem, it will be convenient to make quantities dimensionless with respect to the total length of the combined cavitator-cavity system, $\ell = \ell_\mathrm{n} + \ell_\mathrm{c}$, (where ℓ_n is the cavitator length and ℓ_c is the cavity length) and the free stream velocity, W. Then the field equation, equation (1), is left unchanged, and the boundary conditions become

$$u_r - u_\theta\frac{1}{R}\frac{\partial R}{\partial\theta} - (1+w)\frac{\partial R}{\partial z} = 0 \quad \text{on } S_\mathrm{n}\cup S_\mathrm{c} \qquad (4)$$

and

$$u_r^2 + u_\theta^2 + 2w + w^2 + \frac{2}{F^2}R\sin\theta = \sigma \quad \text{on } S_\mathrm{c}, \qquad (5)$$

where the cavitation number is defined as

$$\sigma = \frac{p_\infty - p_\mathrm{c}}{1/2\ \rho W^2} \qquad (6)$$

and the Froude number based on total system length is

$$F = \frac{W}{\sqrt{g\ell}}. \qquad (7)$$

The boundary conditions may be written in terms of the dimensionless total potential as

$$\frac{\partial\varphi}{\partial r} - \frac{1}{R^2}\frac{\partial\varphi}{\partial\theta}\frac{\partial R}{\partial\theta} - \frac{\partial\varphi}{\partial z}\frac{\partial R}{\partial z} = 0 \quad \text{on } S_\mathrm{n}\cup S_\mathrm{c} \qquad (8)$$

and

$$\left(\frac{\partial\varphi}{\partial r}\right)^2 + \frac{1}{R^2}\left(\frac{\partial\varphi}{\partial\theta}\right)^2 + \left(\frac{\partial\varphi}{\partial z}\right)^2 + \frac{2}{F^2}R\sin\theta = \sigma + 1 \quad \text{on } S_\mathrm{c}. \qquad (9)$$

Simplifying assumptions

The current model also falls under the category of slender-body theory, which allows the solution to be treated separately as those for longitudinal and transverse flow problems according to the classical theory presented (for example) by Van Dyke [29], Ashley and Landahl [30], Cole [31], Newman [32], and Kevorkian and Cole [33]. The longitudinal flow problem is solved as the classical matched asymptotic expansion for slender bodies, in which an inner solution valid very close to the body is matched to an outer solution valid in the far field. The variant of this problem applicable to axisymmetric supercavitating flows has been addressed analytically, to various levels of rigor, by many researchers, for example, Reichardt [34], Münzer and Reichardt [35], Garabedian [36], Cuthbert and Street [37], Brennan [38], Logvinovich [1], Chou [39], Logvinovich and Serebryakov [40], Vorus [41], Varghese, et al [42], Kuria, et al [43], Serebryakov [44], Tulin [45], [46], and others. Since the problem of primary interest at present is the distortion of this solution due to gravitational effects, a simple surrogate has been applied to serve as the axisymmetric solution about which the distortions occur. The surrogate axisymmetric solution has been selected for convenience in illustrating the current approach. Any other valid axisymmetric solution may be substituted, a step that may, in fact, improve accuracy. In fact, if additional fidelity is desired, more numerically intensive and (presumably) more accurate versions of the axisymmetric problem have been presented in many articles and texts, for example Guzevsky (in work dating back to the 1970s, exemplified in English by [47], [48]), Kirschner, et al [49], Uhlman, et al [50], Krasnov [51], Varghese, et al [52] (which addresses the partially cavitating axisymmetric problem), and others. The relative merits of each of the various approximate and high-fidelity approaches would make the subject of a lengthy review paper, especially the validity of the corresponding results via comparison with data.

As is justified below, the transverse flow problem is solved using a multipole expansion in each plane, wherein the lowest term in the series corresponds to the inner solution of the longitudinal flow problem, which is therefore treated as known, namely the surrogate axisymmetric solution mentioned above. The coefficients of the remaining terms in the multipole expansion are then determined such that the solution satisfies the dynamic and kinematic condition on the cavity boundary. The resulting method is cast as a marching problem, and, via application of the Galilean transformation, the solution of the transverse flow problem may be thought of as the evolution of a two-dimensional cavity under the influence of gravity. An extension of this approach is theoretically applicable to cases involving lift on the cavitator or on other appendages such as canards, and could also be extended to time-dependent problems.

The resulting method is similar to the Fourier decomposition described by Logvinovich [1], Buyvol [22], and Logvinovich, et al [6]. However, whereas those references state the problem as a geometric perturbation of the cavity geometry about the axisymmetric geometry, the current approach attempts to formalize and justify the basic technique in the context of asymptotic and multipole expansions. In this sense, the current approach is an attempt to link Logvinovich' *principle of independent expansion of cavity sections* [1], [6], [11], [16], [17], [18], [20], [21], [22], [40], [44] to the classical asymptotic expansions as originally developed for fully-wetted bodies, and to overlay a physically realistic asymmetric distortion in a simultaneous Fourier-type expansion for both the cavity geometry and the potential, resulting in a multipole representation of the latter. Rather than a geometric perturbation of the cavity geometry, the result is a more straightforward Fourier expansion of the in-plane potential, amenable to a simpler specification of classical mixed boundary conditions which are, in general of the Robin type. However, the two methods should be essentially equivalent, and it is expected that they should provide similar if not identical solutions.

One cost of the current approach is the appearance of the multipole singularity within the cavity contour in each plane. As will be illustrated below, if not treated carefully, this is associated with unrealistic distortion at the end of the cavity. In contrast, the Logvinovich approach appears to lack a singularity within the cavity contour. Since both methods hint at the twin-vortex regime as the computational plane approaches cavity closure for low values of the cavity Froude number, a more complex solution topology is suggested, further development of which will be deferred.

The slender system formed by the cavitator-cavity system allows for a simplification of the type presented for flows past fully-wetted bodies by the authors described above. With the total system length ℓ defined above, and the maximum diameter d_{m}, then the slenderness parameter, $\varepsilon = d_{\mathrm{m}}/\ell$, is small for the cases of interest. Although the formulation may be extended to more general cases, it will be assumed that the body is moving at constant forward speed. Cavity dynamics effects will be ignored for the present, such that the entire flow field is steady.

Roughly following Newman [32], the solution of the longitudinal flow problem proceeds as a matched asymptotic expansion, wherein an inner solution for the *disturbance* velocity potential $\Phi(r,\theta,z)$ is sought, valid near the surface of the cavitator-cavity system, satisfying the boundary conditions on that surface and the two-dimensional Laplace equation. This solution does not satisfy the condition at infinity in the outer field, and it contains an arbitrary additive constant. A corresponding outer solution for the *disturbance* velocity potential $\phi(r,\theta,z)$ with the proper behavior at infinity has the form of a three-dimensional source distribution along the axis of the cavitator-cavity system. The inner expansion of the outer solution is required to match the inner solution in a matching region $\varepsilon\ell \ll r \ll \ell$. After matching, resulting expressions for the inner and outer solutions of the longitudinal flow problem are given by

$$\Phi \simeq -\frac{1}{2\pi} S'(z)\log\frac{r}{\ell} + f(z), \qquad r \gg \varepsilon\ell, \quad (10)$$

and

$$\phi \simeq -\frac{1}{2\pi} S'(z)\log\frac{r}{\ell} + f(z), \qquad r \ll \ell, \quad (11)$$

where $S(z)$ is the sectional area of the combined cavitator-cavity system, and

$$\begin{aligned} f(z) &= \frac{1}{4\pi}\int_0^z S''(z')\log\frac{2(z-z')}{\ell}dz' \\ &\quad -\frac{1}{4\pi}\int_z^\ell S''(z')\log\frac{2(z'-z)}{\ell}dz' \qquad (12) \\ &= \frac{1}{4\pi}\int_0^\ell S''(z')\operatorname{sgn}(z-z')\log\frac{2|z-z'|}{\ell}dz'. \end{aligned}$$

The cavitator-cavity system need not be axisymmetric, but all of the asymmetry is reflected in the inner solution; its outer limit and the outer solution are axisymmetric. It is important that the argument of the logarithm appearing in the solution is made dimensionless with a quantity on the order of the length of the combined cavitator-cavity system, in order that proper matching is achieved in the overlap region.

For the supercavitation problem, $S'(z)$ is unknown in way of the cavity, and must be determined as part of the solution process. For purposes of simplicity in the current analysis (more than any consideration of accuracy), the solution of the axisymmetric longitudinal flow problem will be estimated as a Garabedian cavity, that is, a cavity profile for which the maximum cavity diameter d_{m} and the length of the combined cavitator-cavity system ℓ are approximated by Garabedian's formulae [36]:

$$\frac{d_{\mathrm{m}}}{d_{\mathrm{n}}} \simeq \sqrt{\frac{C_D}{\sigma}} \qquad (13)$$

and

$$\frac{\ell}{d_{\mathrm{n}}} \simeq \sqrt{\frac{C_D}{\sigma^2}\log\frac{1}{\sigma}}. \qquad (14)$$

Here d_{n} is the cavitator diameter at cavity detachment (or a characteristic diameter over the locus of detachment if it is not axisymmetric) and C_D is the cavity drag coefficient of the cavitator based on its projected area. It should be noted that the second of Garabedian's formulae, equation (14), is generally written for the length of the cavity, rather than the length of the combined system. However, since it is an expression for the asymptotic behavior as the cavity length increases, the distinction is negligible.

For the current illustration, the Garabedian cavity dimensions were applied to specify a cavity of ellipsoidal form. For other applications, the authors have often applied the Münzer-Reichardt profile [35] to estimate the cavity shape, but, since it violates the slender-body assumptions at the cavity endpoints and would thus require complicating corrections, the simpler shape was selected. Moreover, in general the Münzer-Reichardt profile does not match commonly applied cavitator profiles at the cavity detachment point, and modification of the cavity profile is required to improve the estimate of the geometry in that region.

For the present, this surrogate solution comprised of the Garabedian ellipsoid will serve as the known leading axisymmetric term in a multipole expansion, the shape of which is distorted asymmetrically by the remaining terms in the multipole expansion in order to satisfy the boundary conditions for the case of a cavitating body traveling horizontally in a gravitational field. Although reasonably straightforward in theory, the extension of this model to the case of a lifting body will be deferred to a future publication.

The inner problem

For the inner problem, the solution must satisfy the field equations very close to the surface of the combined cavitator-cavity system, along with the consistent boundary condition. Since a prerequisite to this assumption is the slenderness of the body, stream-wise gradients are dominated by gradients in the transverse plane, and the governing equation of the inner problem reduces to the two-dimensional Laplace equation:

$$\nabla^2_{\mathrm{2D}}\Phi = \frac{\partial^2\Phi}{\partial r^2} + \frac{1}{r}\frac{\partial\Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2} \simeq 0. \qquad (15)$$

Consistent with the slenderness approximation, the boundary conditions become

$$\frac{\partial R}{\partial z} \simeq \frac{\partial\Phi}{\partial r} - \frac{1}{R^2}\frac{\partial\Phi}{\partial\theta}\frac{\partial R}{\partial\theta} \quad \text{on } S_{\mathrm{n}} \cup S_{\mathrm{c}} \qquad (16)$$

and

$$\frac{\partial \Phi}{\partial z} \simeq \frac{\sigma}{2} - \frac{1}{F^2} R \sin\theta - \frac{1}{2}\left(\frac{\partial \Phi}{\partial r}\right)^2 - \frac{1}{2R^2}\left(\frac{\partial \Phi}{\partial \theta}\right)^2 \quad \text{on } S_c. \tag{17}$$

These boundary conditions have been rearranged to highlight the distinction between the axial and transverse aspects of the problem: from a numerical standpoint it can be seen that the solution can be treated as a nonlinear system of differential equations in the stream-wise direction in the form of successive transverse flow problems, although the nonlinearities inherent in the system must be addressed.

Galilean transformation of the inner solution

It is conceptually convenient to convert the boundary-value problem for the inner solution to the problem of the approximate Lagrangian "evolution" of a cavity section as it traverses the system from the cavity detachment locus on the cavitator to the point of cavity closure by applying the Galilean transformation:

$$z = Wt. \tag{18}$$

Under this transformation, the first partial derivative in the stream-wise direction is given by

$$\frac{\partial}{\partial z} = \frac{1}{W}\frac{\partial}{\partial t}. \tag{19}$$

In dimensionless form, these expressions are simply

$$z = t. \tag{20}$$

and

$$\frac{\partial}{\partial z} = \frac{\partial}{\partial t}. \tag{21}$$

Substituting these expressions into the time-independent form of the boundary conditions, equations (16) and (17), produces the following kinematic and dynamic conditions governing the "evolution" of a cavity section:

$$\dot{R} = \frac{\partial R}{\partial t} = \frac{\partial \Phi}{\partial R} - \frac{1}{R^2}\frac{\partial \Phi}{\partial \theta}\frac{\partial R}{\partial \theta} = u_r - \frac{1}{R} u_\theta \frac{\partial R}{\partial \theta} \quad \text{on } S_n \cup S_c \tag{22}$$

and

$$\dot{\Phi} = \frac{\partial \Phi}{\partial t} \simeq \frac{\sigma}{2} - \frac{1}{F^2} R \sin\theta - \frac{1}{2}\left(\frac{\partial \Phi}{\partial r}\right)^2 - \frac{1}{2R^2}\left(\frac{\partial \Phi}{\partial \theta}\right)^2 = \frac{\sigma}{2} - \frac{1}{F^2} R\sin\theta - \frac{1}{2}u_r^2 - \frac{1}{2}u_\theta^2 \quad \text{on } S_c. \tag{23}$$

It can be seen that the transformed dynamic condition, equation (23), has the same form as the dynamic condition – derived from the unsteady Bernoulli equation – governing the evolution of an unsteady two-dimensional cavity.

Multipole expansion

Separating the two-dimensional Laplace equation in cylindrical-polar coordinates leads to the classical two-dimensional multipole expansion for the velocity potential (see, for example, Newman [32]):

$$\Phi(r,\theta,t) = A_0(t)\ln r + \sum_{m=1}^{\infty}\frac{1}{r^m}\left[A_m(t)\cos(m\theta) + B_m(t)\sin(m\theta)\right], \tag{24}$$

where the "time" dependence of the coefficients has been retained in consideration of the results of the Galilean transformation presented above. The "time"-dependent cavity radius may also be expressed in terms of a Fourier series:

$$R(\theta,t) = R_0(t) + \sum_{m=1}^{\infty}\left[R_{cm}(t)\cos(m\theta) + R_{sm}(t)\sin(m\theta)\right] \tag{25}$$

The partial time derivatives of the cavity surface function and the potential are thus given by

$$\dot{R}(\theta,t) = \dot{R}_0(t) + \sum_{m=1}^{\infty}\left[\dot{R}_{cm}(t)\cos(m\theta) + \dot{R}_{sm}(t)\sin(m\theta)\right] \tag{26}$$

and

$$\dot{\Phi}(r,\theta,t) = \dot{A}_0(t)\ln r + \sum_{m=1}^{\infty}\frac{1}{r^m}\left[\dot{A}_m(t)\cos(m\theta) + \dot{B}_m(t)\sin(m\theta)\right] \tag{27}$$

Evaluating the potential on the cavity surface S_c, at $r = R(\theta)$, and substituting the resulting expression and equation (26) into the dynamic and kinematic conditions, equations (22) and (23), produces the following nonlinear system of dynamical equations governing the evolution of the Fourier coefficients of index greater than zero:

$$\sum_{m=1}^{\infty}\left[\dot{R}_{cm}\cos\left(m\theta\right)+\dot{R}_{sm}\sin\left(m\theta\right)\right] = -\dot{R}_0+u_r-\frac{u_\theta R'}{R} \quad \text{on } S_c, \tag{28}$$

and

$$\sum_{m=1}^{\infty}\frac{1}{R^m}\left[\dot{A}_m\cos\left(m\theta\right)+\dot{B}_m\sin\left(m\theta\right)\right] = -\dot{A}_0\ln R+\frac{\sigma}{2}-\frac{1}{F^2}R\sin\theta-\frac{1}{2}u_r^2-\frac{1}{2}u_\theta^2, \tag{29}$$

where primes denote differentiation with respect to θ, and where the leading Fourier coefficients for the cavity geometry and velocity potential are treated as known from the axisymmetric base solution. The local in-plane velocity components on the cavity surface are the appropriate in-plane derivatives of the velocity potential:

$$u_r\left(R,\theta,t\right) = \frac{A_0\left(t\right)}{R} -\sum_{m=1}^{\infty}m\frac{1}{R^{m+1}}\left[A_m\left(t\right)\cos\left(m\theta\right)+B_m\left(t\right)\sin\left(m\theta\right)\right] \tag{30}$$

and

$$u_\theta\left(R,\theta,t\right) = -\sum_{m=1}^{\infty}\frac{m}{R^{m+1}}\left[A_m\left(t\right)\sin\left(m\theta\right)-B_m\left(t\right)\cos\left(m\theta\right)\right]. \tag{31}$$

The angular derivative of the cavity surface function is computed as

$$R'\left(\theta,t\right)=-\sum_{m=1}^{\infty}m\left[R_{cm}\left(t\right)\sin\left(m\theta\right)-R_{sm}\left(t\right)\cos\left(m\theta\right)\right]. \tag{32}$$

The "time"-dependent area of the cavity is given by

$$A\left(t\right)=\frac{1}{2}\int_{-\pi}^{\pi}R^2\left(\theta,t\right)d\theta = \pi R_0^2\left(t\right)+\frac{\pi}{2}\sum_{m=1}^{\infty}\left[R_{cm}^2\left(t\right)+R_{sm}^2\left(t\right)\right]. \tag{33}$$

It can be seen that the area of the base solution is therefore a lower bound on the area of the distorted solution, a fact that might serve as the basis for a improved approximation via iteration on the base inner solution, equation (10).

Numerical implementation

The solution of the problem of three-dimensional flow past a slender cavitating body in a gravity field is thus reduced to determining the evolution of the coefficients describing the velocity potential and the geometry of each section as it evolves downstream of its generation at the cavitator. The leading coefficients of each are treated as known from the selected axisymmetric base solution. The initial values of the remaining coefficients for the cavity geometry and velocity potential are specified to properly represent the cavitator geometry and flow at cavity detachment. The solution then proceeds numerically by computing the evolution of the remaining coefficients in truncated forms of the two series. Solution of the evolution system has been performed using standard MATLAB routines.

Truncating the Fourier expansion at M terms, the kinematic and dynamic conditions, equations (28) and (29), may be approximately satisfied at K points as

$$M_{Rkm}\dot{R}_m=U_k;\quad k=1:K,\ m=1:2M, \tag{34}$$

and

$$M_{\Phi km}\dot{\Phi}_m=P_k;\quad k=1:K,\ m=1:2M, \tag{35}$$

where

$$\dot{R}_m=\dot{R}_m\left(t\right)=\begin{cases}\dot{R}_{cm}\left(t\right), & m=1:M\\ \dot{R}_{sm-M}\left(t\right), & m=M+1:2M\end{cases} \tag{36}$$

$$\dot{\Phi}_m=\dot{\Phi}_m\left(t\right)=\begin{cases}\dot{A}_m\left(t\right), & m=1:M\\ \dot{B}_{m-M}\left(t\right), & m=M+1:2M\end{cases} \tag{37}$$

$$U_k=U_k\left(R_m,\Phi_m\right) = u_r\left(R\left(\theta_k,t\right)\right) -\frac{u_\theta\left(R\left(\theta_k,t\right)\right)}{R\left(\theta_k,t\right)}\frac{\partial R\left(\theta_k,t\right)}{\partial\theta}-\dot{R}_0\left(\theta_k,t\right) \tag{38}$$

and

$$P_k=P_k\left(R_m,\Phi_m\right) = -\frac{1}{2}\left(u_r^2(R(\theta,t))+u_\theta^2(R(\theta,t))\right) -\frac{R(\theta,t)\sin\theta}{Fr_\ell^2}+\frac{1}{2}\sigma-\dot{A}_0\left(R(\theta,t)\right)\ln R(\theta,t) \tag{39}$$

The so-called *mass matrices* are defined as

$$M_{Rkm} = \begin{cases} \cos(m\theta_k), & m = 1:M \\ \sin((m-M)\theta_k), & m = M+1:2M \end{cases} \quad (40)$$

and

$$M_{\Phi km} = M_{\Phi km}(R_m) = \begin{cases} \dfrac{\cos(m\theta_k)}{R^m(\theta_k,t)}, & m = 1:M \\ \dfrac{\sin((m-M)\theta_k)}{R^{m-M}(\theta_k,t)}, & m = M+1:2M \end{cases}. \quad (41)$$

(These are not related to the actual mass of the fluid nor the added mass of the cavitator-cavity system, but are so denoted using the parlance of the theory of numerical solution of nonlinear ordinary differential equations.)

It is emphasized that the geometry mass matrix defined in equation (40) does not depend on the solution, and therefore is independent of time, whereas the potential mass matrix defined in equation (41) does depend on time via the geometric solution but not via the potential.

The system can be concatenated to form the following single partitioned matrix equation comprising the approximate system of linearly implicit, nonlinear ordinary differential equations governing the time-dependent behavior of a cavity in a gravity field:

$$M_{\alpha\beta}\dot{x}_\beta = b_\beta; \quad \alpha = 1:2, \ \beta = 1:2, \quad (42)$$

where

$$x_\beta = \begin{cases} R_m, & \beta = 1 \\ \Phi_m, & \beta = 2 \end{cases}, \quad (43)$$

$$b_\beta = \begin{cases} U_m, & \beta = 1 \\ P_m, & \beta = 2 \end{cases}, \quad (44)$$

and

$$M_{\alpha\beta}(x,t) = \begin{cases} M_{Rkm}, & \alpha = 1, \ \beta = 1 \\ M_{\Phi km}, & \alpha = 2, \ \beta = 2 \\ 0, & \text{otherwise} \end{cases}. \quad (45)$$

The structure of the concatenated solution system is shown in figure 2.

$$\left(\begin{array}{c|c} M_{11} = M_{Rkm} & M_{12} = 0 \\ \hline M_{21} = 0 & M_{22} = M_{\Phi km} \end{array}\right) \left\{\begin{array}{c} x_1 = \dot{R}_m \\ \hline x_2 = \dot{\Phi}_m \end{array}\right\} = \left\{\begin{array}{c} b_1 = U_m \\ \hline b_2 = P_m \end{array}\right\}$$

Figure 2. The structure of the partitioned matrix equation.

VERIFICATION AND PRELIMINARY VALIDATION

Verification and preliminary validation of the model for supercavitating flow in a gravity field has proceeded by exploring the shape of the cavity at a fixed cavitation number as the cavity Froude number varies. The predicted behavior is illustrated in the following sub-section, after which some issues associated with implementation of the code are discussed.

Horizontal flow past a non-lifting disk cavitator at non-zero Froude number

Logvinovich [1] provides several formulae for the distortion of the line of centers of the cavity as a function of cavity Froude number. The most reliable seems to be the following (here given in the “time”-dependent form):

$$h(t;F) = \frac{(1+\sigma)t^2}{6F^2}. \qquad (46)$$

Using this expression to distort a base solution comprised of a parabolic cavity contour defined by Paryshev [11] with minor corrections made to the cavity dimensions as described in Kirschner and Arzoumanian [18], it is possible to plot an estimated profile of the cavity as it intersects its plane of bilateral symmetry. The results of the current model may then be overlaid on that profile for purposes of comparison, as presented for a cavitation number of $\sigma = 0.030$ and for several values of the cavity Froude number. The results are presented in figure 3. On the left-hand side of this figure, the predicted profiles are compared with the distorted base solution. Bear in mind that the comparison is for the profile intersecting the plane of symmetry in both cases. The right-hand side of the figure shows transverse slices through the cavity, some of which have been labeled to facilitate correspondence with the associated location in the profile view. It can be seen that the profile comparisons are quite good. The model also predicts increasing distortion of the planar cavity sections as the downstream end of the cavity is approached. These effects are more pronounced at lower cavity Froude numbers, where the effect of gravity is more important.

Implementation issues

The method is computationally efficient: the results shown in figure 3, which were computed at 300 stations along the cavity length, required only a few seconds to compute on a modern laptop computer. Provided the multipole is allowed to migrate with the contour of the cavity sections as they evolve, the method appears to be robust for the first few terms of the multipole expansion. If too many terms are included, however, the computation becomes ill-conditioned. At a certain point the geometry predicted by the method becomes physically unrealistic. Beyond some number of terms, rank deficiencies are reported in the matrix solver.

These issues are illustrated in figure 4, which compares the physically realistic solution of figure 4a with the problematic solutions shown in figure 4b. The results of figure 4a (which are the same as those shown in figure 3a) were computed with the series truncated at $M = 6$ and the multipole specified to migrate upward at a rate proportional to the square of the time-like variable. The results of figure 4b were computed with the series truncated at $M = 9$. It can be seen that the downstream end of the cavity becomes unrealistically distorted. Further research is required to fully understand the numerical source of this problem, and the apparently related problem of the rank deficiencies that occur if even more terms are retained.

Figure 4c shows the results if the multipole is not allowed to migrate as the cavity rises. Although the upstream portion of the solution is remarkably robust to the choice of multipole location, as the solution proceeds the predicted cavity boundary comes so close to the multipole that the solution distorts very rapidly in a physically unrealistic way until, a short time later, the solution can no longer proceed numerically.

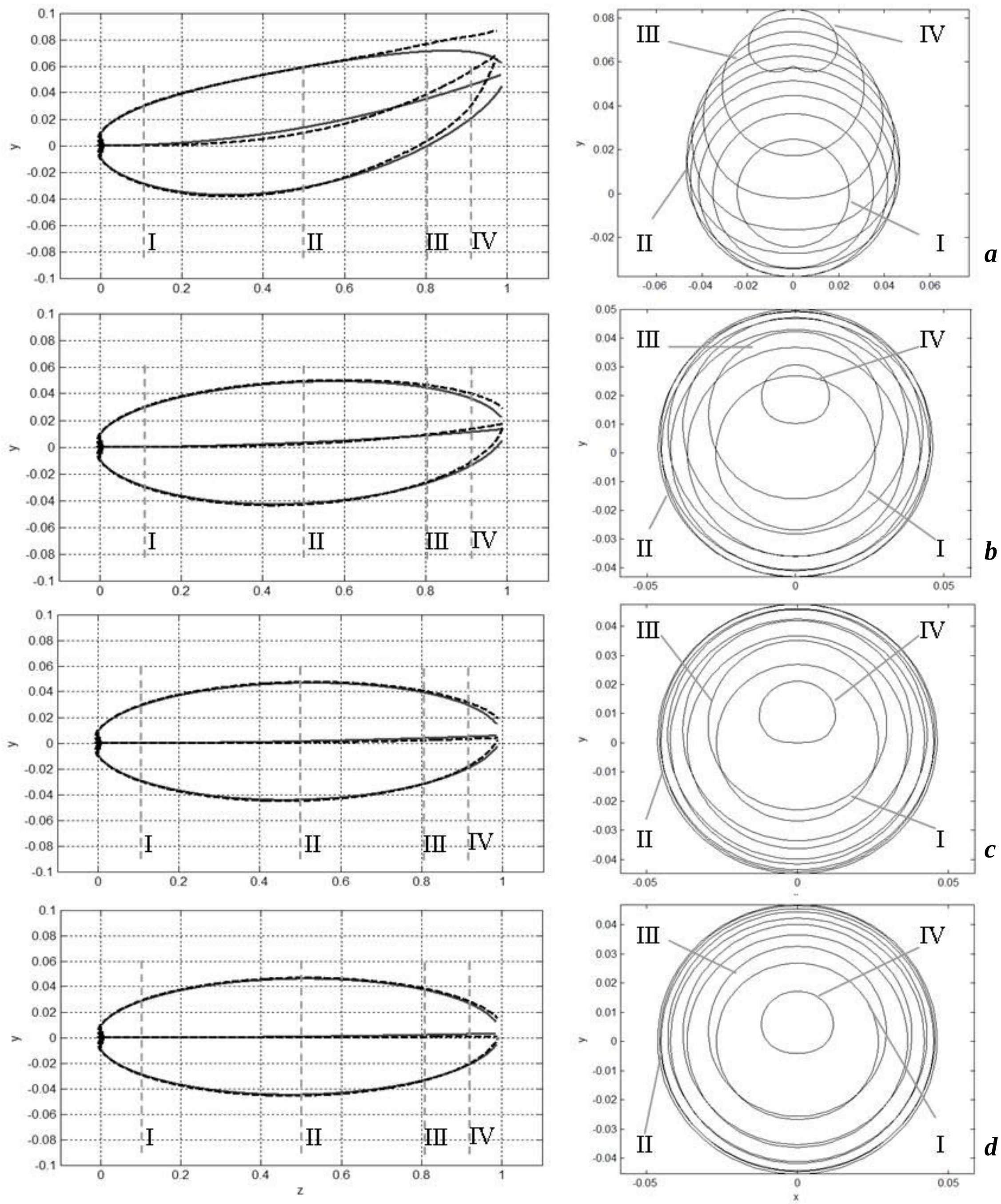

Figure 3. Results of the current model (dashed lines in plots on left; all contours on right) compared with low-order estimates of cavity distortion based on the Logvinovich formula (solid lines in plots on left) for flow past a non-lifting supercavitating disk for a cavitation number of $\sigma = 0.030$ and for several values of the cavity Froude number based on total system length: a - $F = 2.5$; b - $F = 5.0$; c - $F = 7.5$; d - $F = 10.0$. Note that the horizontal and vertical scales do not match in the profile views shown on the left-hand side of the figure.

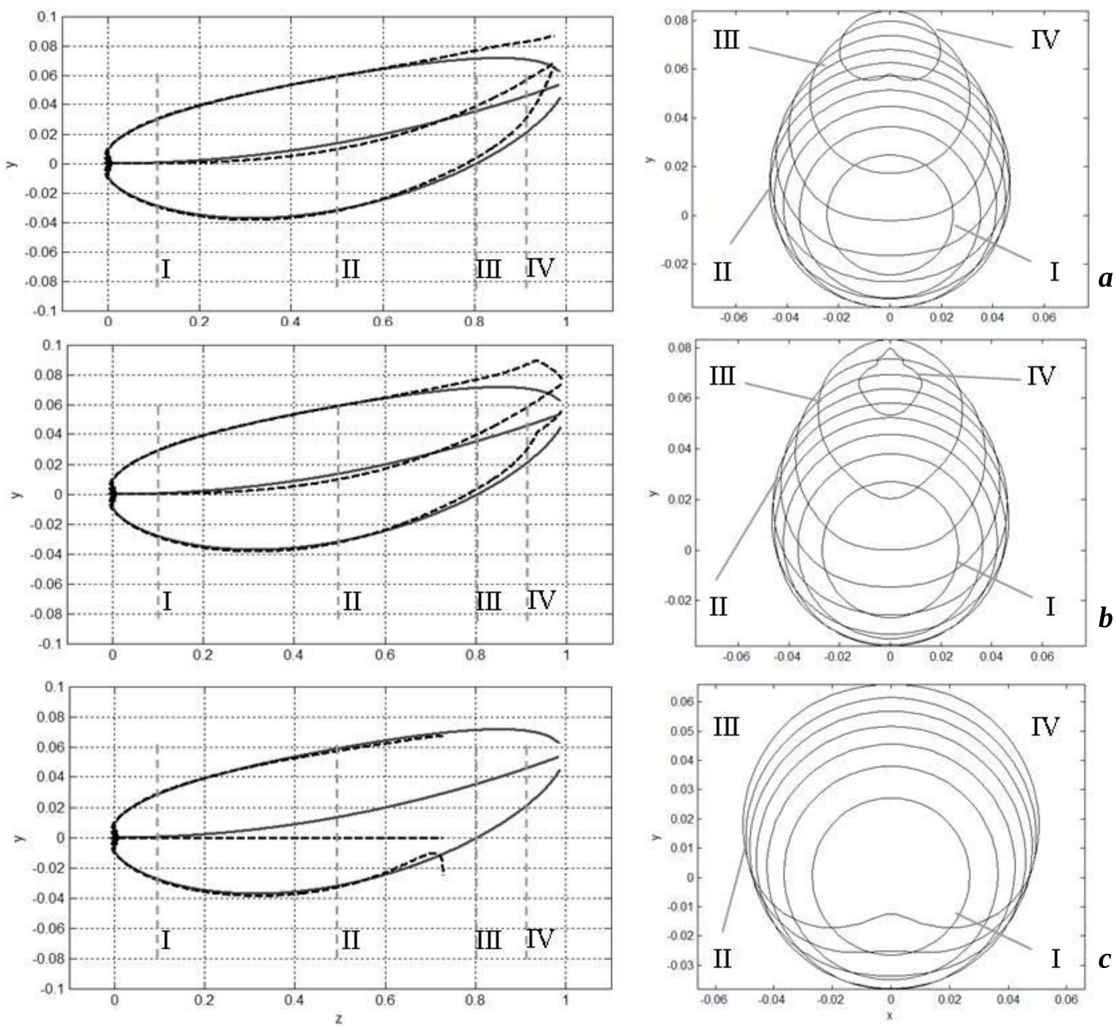

Figure 4. Some issues associated with numerical implementation of the current model (example computations performed for a cavitation number of $\sigma = 0.030$ and a cavity Froude number of $F = 2.5$): a – the physically reasonable result previously presented in figure 3; b – illustration of numerical problems that arise if too many terms of the multipole expansion are retained; c – illustration of the need to allow the multipole to migrate with the cavity sectional contour. Note that the horizontal and vertical scales do not match in the profile views shown on the left-hand side of the figure.

SUMMARY AND CONCLUSIONS

A simple method has been formulated and implemented to estimate the effects of gravity on a supercavity. The method extends classical slender-body theory to allow computation of the terms of a multipole expansion for the inner solution of the potential flow problem. The lowest-order terms for the axisymmetric flow are assumed known from some other solution method. The resulting method allows for numerical marching of the non-axisymmetric distortion the cavity sectional geometry and concurrent evolution of the velocity potential in the stream-wise coordinate, which is treated as a time-like variable.

Preliminary validation against an independent formula for the tail-up of the line of cavity centers shows promising results. Work remains to improve the numerical method to allow computation for higher terms of the multipole expansion, and to provide for a rational approach to moving the multipole as the cavity contour deforms. Most importantly, future efforts must address the effects of lift, maneuvering, and afterbody planing interaction on the cavity geometry.

ACKNOWLEDGMENTS

This research was funded in part by the Defense Advanced Research Projects Agency, under the Underwater Express Program, Ms. Khine Latt, Program Manager, via a subcontract to General Dynamics Electric Boat (GDEB), Ms. Jennifer Panosky and Mr. Joseph Porcaro, Program Managers, DARPA Contract N66001-047-C-2002 ATAR 780, Subcontract SNS 194-082. Thanks are in order to the Underwater Express Government Review Team who provided valuable technical comments during the course of this research, to Mr. John Chapman (GDEB) and the GDEB-led project team, including participants from the Naval Undersea Warfare Center Division, Newport, RI, the Applied Research Laboratory/The Pennsylvania State University, Vehicle Control Technologies, and Alion Science and Technology for their valuable technical comments. Special thanks are in order to Dr. Rudolph Martinez and Ms. Carina Ting for their contributions to the scheme for migration of the multipole, and to Drs. James Uhlman and John Grant for there mathematical insight concerning potential flow and slender-body theory.

NOMENCLATURE

A	sectional area of the cavitator-cavity system
A_m, B_m	multipole coefficients of the inner solution disturbance velocity potential
C_D	cavity drag coefficient of the cavitator
d_n	cavitator diameter at cavity detachment
d_m	maximum cavity diameter
F	Froude number based on total system length
g	gravitational acceleration
h	distortion of the line of centers of the cavity
ℓ	total length of the cavitator-cavity system
ℓ_c, ℓ_n	cavity length, cavitator length
$\mathbf{n}$	surface normal vector
$M_{Rkm}, M_{\Phi km}$	"mass" matrices of the nonlinear ordinary differential equations
p_c, p_∞	cavity pressure, pressure at upstream infinity
p	pressure
r, θ, z	cylindrical-polar coordinates
R	local radius of the cavitator-cavity system
R_{cm}, R_{sm}	multipole coefficients of the local radius of the cavitator-cavity system
S_c	cavity surface
S_n	cavitator surface
S	sectional area of the cavitator-cavity system
t	dimensionless "time"
U_k, P_k	Right hand sides of the nonlinear ordinary differential equations
$\mathbf{u}$	total velocity vector
u_r, u_θ, w	disturbance velocity components
W	free stream velocity in the z -direction
$\mathbf{x}$	general position vector
ε	slenderness parameter
φ	three-dimensional total velocity potential
Φ	inner solution disturbance velocity potential
ϕ	outer solution disturbance velocity potential
ρ	density of the ambient liquid
σ	cavitation number

REFERENCES

[1] Logvinovich, G.V., (1969) *Hydrodynamics of Free-Boundary Flows,* Naukova Dumka, Kiev, Ukraine; translated from the Russian by the Israel Program for Scientific Translations, Jerusalem (1972).

[2] May, A., (1975) "Water Entry and the Cavity-Running Behavior of Missiles," SEAHAC Technical Report 75-2, Naval Surface Weapons Center, White Oak Laboratory, Silver Spring, MD.

[3] Campbell, I.J., and D.V. Hilborne (1958) "Air Entrainment behind Artificially Inflated Cavities," *Proceedings of the Second Symposium on Naval Hydrodynamics,* Washington, DC.

[4] Kiceniuk, T., (1954) "An Experimental Study of the Hydrodynamics Forces Acting on a Family of Cavity-Producing Conical Bodies of Revolution Inclined to the Flow," CIT Hydrodynamics Report E-12.17, California Institute of Technology, Pasadena, CA. (Cited in [2].)

[5] Waid, R.L., (1957) "Cavity Shapes for Circular Disks at Angles of Attack," CIT Hydrodynamics Report E-73.4, California Institute of Technology, Pasadena, CA. (Cited in [2].)

[6] Logvinovich, G.V., V.I. Buyvol, A.S. Dudko, S.I. Putilin, and Yu.R. Shevchuk (1985) *Free-Surface Flows,* Naukova Dumka, Kiev, Ukraine.

[7] Logvinovich, G.V., (1980) "Some Problems in Planing Surfaces," translated from "Nekotoryye voprosy glissirovaniya i kavitatsii [Some Problems in Planing and Cavitation]," *Trudy TsAGI,* No. 2052, Moscow, Russia.

[8] Vasin, A.D., and E.V. Paryshev (2001) "Immersion of a Cylinder in a Fluid through a Cylindrical Free Surface," *Mekhanika Zhidkosti i Gasa,* **2**, pp 3-12. (In Russian, cited in [11]).

[9] Vasin, A.D., and E.V. Paryshev (2001) "Immersion of a Cylinder in a Fluid through a Cylindrical Free Surface," *Fluid Dynamics,* **36**, pp 169-177. (Cited in [11].)

[10] Paryshev, E.V., (2002) "The Plane Problem of Immersion of an Expanding Cylinder through a Cylindrical Free Surface of Variable Radius," *Proceedings of the 2002 International Summer Scientific School on High-Speed Hydrodynamics*, Chuvash National Academy of Science and Art, Cheboksary, Russia.

[11] Paryshev, E.V., (2006) "Approximate Mathematical Models in High-Speed Hydrodynamics," *Journal of Engineering Mathematics,* 55, pp 41-64.

[12] Kirschner, I.N., D.C. Kring, A.W. Stokes, N.E. Fine, and J.S. Uhlman (2002) "Control Strategies for Supercavitating Vehicles," *J. Vibration and Control,* 8 2, Sage Publications, London, England, UK. (Also presented at the Eighth International Symposium on Nonlinear Dynamical Systems, Blacksburg, VA.)

[13] Cox, R.N., and W.A. Clayden (1955) "Air Entrainment at the Rear of a Steady Cavity," *Proceedings of the Symposium on Cavitation in Hydrodynamics,* National Physics Laboratory, London, United Kingdom.

[14] Epshteyn, L.A., (1961) "Determination of the Amount of Gas Needed to Maintain a Cavity behind a Body Moving at Low Fr

Number," *Trudy TsAGI,* No. 824, Moscow, Russia. (In Russian, cited in [11].)

[15] Epshteyn, L.A., (1970) *Methods of the Dimensional Analysis and Similarity Theory in Problems of Ship Hydrodynamics,* Sudostroyeniya, Leningrad (now St. Petersburg), translated from the Russian by the Naval Intelligence Support Center.

[16] Savchenko, Y.N., V.N. Semenenko, S.I. Putilin, Y.D. Vlasenko, V.T. Savchenko, and Y.I. Naumova (2000) "Theory of Stable Model Motion with Ventilated and Unventilated Supercavities," IHM-NASU Review prepared for the Office of Naval Research - Europe, Institute of Hydromechanics, National Academy of Sciences of Ukraine, Kiev, Ukraine.

[17] Semenenko, V. N., (2001) "Artificial Supercavitation: Physics and Calculation," *Lecture Notes for the RTO AVT/VKI Special Course on Supercavitating Flows,* von Karman Institute for Fluid Dynamics, Rhode Saint Genèse, Belgium.

[18] Kirschner, I.N., and S.H. Arzoumanian (2008) "Implementation and Extension of Paryshev's Model of Cavity Dynamics," *Proceedings of the International Conference on Innovative Approaches to Further Increase Speed of Fast Marine Vehicles, Moving Above, Under And In Water Surface,* Saint-Petersburg, Russia.

[19] Silberman, E., and C.S. Song (1961) "Instability of Ventilated Cavities," *J. Ship Research,* **5**, pp 13-33.

[20] Paryshev, E.V., (1978) "A System of Nonlinear Differential Equations with a Time Delay, Describing the Dynamics of Non-Stationary, Axially Symmetric Cavities," *Trudy, TsAGI,* No. 1907, Moscow, Russia; translated from the Russian.

[21] Kirschner, I.N., (2006) "A Robust Solver for Delay Differential Equations," *Proceedings of the 2006 Conference on High-Speed Hydrodynamics & Numerical Simulation,* Kemerovo State University, Kemerovo, Russia.

[22] Buyvol, V.N.,(1980) "Slender Cavities in Flows with Perturbations," *Naukova Dumka,* Kiev, Ukraine. (In Russian, cited in [16].)

[23] Franc, J.-P. and J.-M. Michel (2004) *Fundamentals of Cavitation,* Kluwer Academic Publishers, Boston, MA.

[24] Lindau, J.W., R.F. Kunz, S. Venkateswaran, and D.A. Boger (2001) "Application of Preconditioned, Multiple-Species, Navier-Stokes Models to Cavitating Flows," *Proceedings of the Fourth International Symposium on Cavitation,* California Institute of Technology, Pasadena, CA.

[25] Lindau, J.W., R.F. Kunz, D.A. Boger, D.R. Stinebring, and H.J. Gibeling (2002) "High Reynolds Number, Unsteady, Multiphase CFD Modeling of Cavitating Flows," *J. Fluids Engineering,* **124**, 3, pp 607-616.

[26] Kinzel, M.P., J.W. Lindau, J. Peltier, F. Zajaczkowski, T. Mallison, R.F. Kunz, R. Arndt, and M. Wosnik (2007) "Computational Investigations of Air Entrainment, Hysteresis, and Loading for Large-Scale, Buoyant Cavities," *Proceedings of the DoD High Performance Computing Modernization Program Users Group Conference,* Pittsburgh, PA.

[27] Lindau, J.W., M.P. Kinzel, D.R. Stinebring, R.F. Kunz, D.A. Boger, and R.W. Noack (2006) "The Effects of a Free-Surface on Computed Turbulent, Super and Partially Cavitating Flows," *Proceedings of the Sixth International Symposium on Cavitation,* MARIN, Wageningen, The Netherlands.

[28] Kring, D.C., N.E. Fine, J.S. Uhlman, and I.N. Kirschner (2000) "A Time-Domain Cavitation Model using a Three-Dimensional, Boundary-Element Method," *Applied Hydromechanics,* **2**, 74-3, Institute of Hydromechanics, Ukrainian National Academy of Sciences, Kiev, Ukraine. (In Russian.)

[29] Van Dyke, M., (1964) *Perturbation Methods in Fluid Mechanics,* The Parabolic Press, Stanford, CA.

[30] Ashley, H., and M. Landahl (1965) *Aerodynamics of Wings and Bodies,* Dover Publications, New York, NY.

[31] Cole, J.D., (1968) *Perturbation Methods in Applied Mathematics,* Blaisdell Publishing Company, Waltham, MA.

[32] Newman, J.N., (1977) *Marine Hydrodynamics,* The MIT press, Cambridge, MA.

[33] Kevorkian, J., and J.D. Cole (1996) *Multiple Scale and Singular Perturbation Problems,* Springer-Verlag, New York, NY.

[34] Reichardt, H., (1946) "The Laws of Cavitation Bubbles at Axisymmetric Bodies in a Flow," Ministry of Aircraft Production Volkenrode, MAP-VG, Reports and Translations 766, Office of Naval Research, Arlington, VA. (Cited in [2].)

[35] Münzer, H., and H. Reichardt (1950) "Rotational Symmetrical Source-Sink Bodies with Predominantly Constant Pressure Distributions," A.R.E. Translation 1/50. (Cited in [2].)

[36] Garabedian, P. R., (1956) "Calculation of Axially Symmetric Cavities and Jets," *Pac J Math,* **6**.

[37] Cuthbert, J., and Street, R., (1964) "An Approximate Theory for Supercavitating Flow About Slender Bodies of Revolution," LMSC Report, TM81-73/39, Lockheed Missiles and Space Co., Sunnyvale, CA.

[38] Brennan, C., (1969) "A Numerical Solution of Axisymmetric Cavity Flows," *J Fluid Mechanics,* **37,** 4.

[39] Chou, Y. S. (1974) "Axisymmetric Cavity Flows Past Slender Bodies of Revolution," *J Hydronautics,* **8,** 1.

[40] Logvinovich, G.V., and V.V. Serebryakov (1975) "On Methods of Calculation of Slender Axisymmetric Cavity Shapes," *J. Hydromechanics,* **32**, pp 47-54, Kiev (in Russian).

[41] Vorus, W. S. (1991) "A Theoretical Study of the Use of Supercavitation/Ventilation for Underwater Body Drag Reduction," VAI Technical Report, Vorus & Associates, Inc., Gregory, MI.

[42] Varghese, A. N., Uhlman, J. S., and Kirschner, I. N., (1997) "Axisymmetric Slender-Body Analysis of Supercavitating High-Speed Bodies in Subsonic Flow," *Proceedings of the Third International Symposium on Performance Enhancement for Marine Applications,* T. Gieseke, editor, Newport, RI.

[43] Kuria, I. M., Kirschner, I. N., Varghese, A. N., and Uhlman, J. S., (1997) "Compressible Cavity Flows Past Slender Non-Lifting Bodies of Revolution," *Proceedings of the ASME & JSME Fluids Engineering Annual Conference & Exhibition, Cavitation and Multiphase Flow Forum,* FEDSM97-3262, Vancouver, BC.

[44] Serebryakov, V.V., (1998) "Asymptotic Approach for Problems of Axisymmetric Supercavitation Based on the Slender Body Approximation," *Proceedings of the Third International Symposium on Cavitation,* **2**, J.-M. Michel and H. Kato, editors, Grenoble, France, pp 61-70.

[45] Tulin, M.P., (1998) "Cavity Shapes in Three Dimensions," *Proceedings of the Fluid Dynamics Panel Workshop on High Speed Body Motion in Water,* Kiev, Ukraine, published as AGARD-R-827, Advisory Group for Aerospace Research & Development, North Atlantic Treaty Organzation, Neuilly-sur-Seine, France.

[46] Tulin, M.P., (1999) "On the Shape and Dimensions of Three-Dimensional Cavities in Supercavitating Flows," *Fluid Mechanics and its Applications: In Fascination of Fluid Dynamics,* A. Biesheuvel and G.F. Van Heijst, editors, Kluwer Academic Publishers, Dordrecht, The Netherlands.

[47] Guzevsky, L.G., (1979) "Numerical Analysis of Cavity Flows," pre-print, Report 40-79, Institute of Thermophysics, Siberian Branch of the Russian Academy of Sciences, Novosibirsk, Russia (in Russian).

[48] Guzevsky, L.G., (2002) "Analytical Approximations of Riabouchinsky Cavity Flows past a Disk," *Proceedings of the 2002 International Summer Scientific School on High-Speed Hydrodynamics,* Chuvash National Academy of Science and Art, Cheboksary, Russia.

[49] Kirschner, I. N., Uhlman, J. S., Varghese, A. N., and Kuria, I. M., (1995) "Supercavitating Projectiles in Axisymmetric Subsonic Liquid Flows," *Proceedings of the ASME & JSME Fluids Engineering Annual Conference & Exhibition, Cavitation and Multiphase Flow Forum,* FED 210, J. Katz and Y. Matsumoto, editors, Hilton Head Island, SC.

[50] Uhlman, J. S., Varghese, A. N., and Kirschner, I. N., (1998) "Boundary Element Modeling of Axisymmetric Supercavitating Bodies," *Proceedings of the 1st Symposium on Marine Applications of Computational Fluid Dynamics,* Hydrodynamic/Hydroacoustic Technology Center, McLean, VA.

[51] Krasnov, V. K., (2002) "The Movement of an Axisymmetrical Solid with Formation of a Cavity," *Proceedings of the 2002 International Summer Scientific School on High-Speed Hydrodynamics,* Chuvash National Academy of Science and Art, Cheboksary, Russia.

[52] Varghese, A.N., J.S. Uhlman, and Kirschner, I.N., (2005) "High-Speed Bodies in Partially Cavitating Axisymmetric Flow," *J. Fluids Eng,* 127, pp 41-54.