Proceedings of the 7th International Symposium on Cavitation
CAV2009
August 17-22, 2009, Ann Arbor, Michigan, USA

CAV2009 – Paper No. 151

NUMERICAL INVESTIGATION OF CAVITATING FLOW THROUGH THE CASCADE OF ARBITRARY FOIL

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ABSTRACT

Two methods are considered for computer modeling of cavitating flow through a cascade of any foils. The first method consists of numerical modeling of non-circulation flows of a cascade of foils with subsequent analytical solution of plate-cascades. The second method provides direct computer modeling using numerical algorithms for an isolated foil. It is shown, that both methods yield identical numerical results but the second one is more convenient for numerical algorithms and computing.

INTRODUCTION

Let a propeller rotates with angle velocity ω and moves along the x-axis at speed U. Consider cylindrical coordinates x, r, θ. The Laplace equation in these coordinates is

$$\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial r^2}+\frac{\partial\varphi}{r\partial r}+\frac{\partial^2\varphi}{r^2\partial\theta^2}=0\,. \tag{1}$$

If the propeller's blade width is not too large then the radial components of speed should be much smaller than the others, and so the second and third terms in equation (1) can be neglected. Denoting, $r\theta = y$, equation (1) is reduced to two-dimensional Laplace equation

$$\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}=0\,, \tag{2}$$

that is the flow on cylindrical surface of fixed radius r is similar to plane flow through a cascade of blade sections at that radius with inlet velocity $\vec{V}_1 = (U, \omega r)$. All blades are arranged periodically along the y-axis with a spacing between blades $T = 2\pi r / N$, where N is the number of propeller blades.

If the axial speed is much les than the others as in centrifugal pumps then equation (1) reduces to two-dimensional Laplace equation in polar coordinates

$$\frac{\partial^2\varphi}{\partial r^2}+\frac{\partial\varphi}{r\partial r}+\frac{\partial^2\varphi}{r^2\partial\theta^2}=0\,. \tag{3}$$

As the blades of a centrifugal pump are arranged periodically on a circle and the flow is radial from the x-axis, the only possible flow is a point vortex-source of the strength, $m = \Gamma + iq$, where Γ is the circulation, q is the amount of a fluid flowing through blades. Using transformation $z = e^{\zeta}$, one can reduce the flow through a ring cascade to the flow through a strip cascade with inlet velocity $v_1 e^{i\alpha_1} = (q - i\Gamma)/2\pi$ and outlet velocity $v_2 e^{i\alpha_2} = q/2\pi$.

Also, investigation of a flow through cascade has a wide range of applications in many practical problems. The beginning of analytical researches was done by S.A. Chaplygin in 1914 [1]. The regular statement of the theory of cascades was given in [2] - [4]. It should be noted that exact analytical solutions have been obtained only for few configurations of cascade blades, but now there are many numerical methods applicable for calculating a considerable range of problems. Some numerical methods for cavitating flow using the speed potential have been considered in [5], [6].

Below, the numerical method of direct iteration of single foil [7] – [9] is applied to a cascade of arbitrary foil with cavities. The numerical algorithm and calculation of the flow through a cascade of any blades, as well as for a single foil are, based on stream function and are very easy for computing.

1. PROBLEM STATEMENT

A sketch of the flow is shown in Fig. 1. Usual kinematical conditions for the stream function, $\psi = const$, or for speed potential, $\partial\varphi/\partial n = 0$, at the blade/cavity boundary should be satisfied; and for cavitating flow, the dynamic condition for pressure $p = p_0 = const$ at the cavity boundary must be fulfilled. As distinct from single foil, the outlet velocity in infinity, $V_2 e^{-i\alpha_2}$, differs from inlet speed $V_1 e^{-i\alpha_1}$ and could be found by solving the mathematical problem of flow through the cascade.

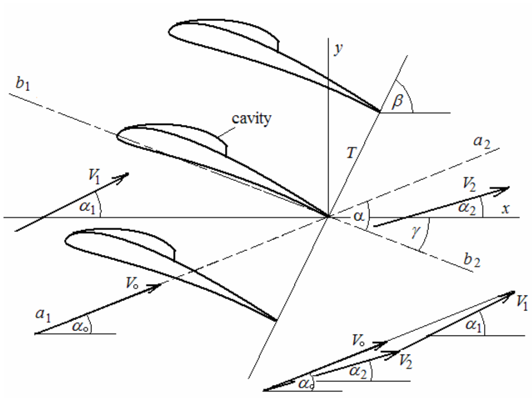


Figure 1: A foil-cascade with partial cavities: straight line $b_1 b_2$ is neutral; line $a_1 a_2$ is directed towards the average speed

The conservation law of fluid amount and circulation connect both speeds at the left and right sides:

$$V_2 e^{-\alpha_2 i} - V_1 e^{-\alpha_1 i} = \Gamma / T e^{\beta i}, \tag{4}$$

where Γ is the circulation around blade, $Te^{\beta i}$ is the vector of cascade period.

Hydrodynamic force is calculated as for a single foil by

$$X - iY = i\rho \Gamma V_0 e^{-\alpha_0 i}, \tag{5}$$

where $V_0 e^{i\alpha_0} = 0.5(V_1 e^{i\alpha_1} + V_2 e^{i\alpha_2})$ is the average speed.

3. APPLICATION OF NUMERICAL ALGORITHM FOR A FINITE NUMBER OF FOILS

A flow around a finite number of foils is determined by the integral equation for speed distribution on foils [7]

$$\oint_C v(s_\tau) G(z,\tau) ds_\tau + c = \mathrm{Im}(Ve^{-i\alpha} z), \tag{6}$$

where $G(z,\tau) = (-1/2\pi)\ln|z-\tau|$ is the Green's function for unbounded domain. The integral equation can be reduced using the boundary element method to a set of linear equations. Numerous examples of numerical calculations [9] show simplicity of computing and a sufficiently high precision.

The integral equation for a cascade can be found from equation (5) used for a finite number of foils by increasing that number to infinity. Since the blades of cascade arrange a system of periodically located foils, the integration over all foil boundaries can be reduced to integration over a single foil, C. The kernel of integral can be found as a limit of the sum

$$G_c = \frac{1}{2\pi} \lim_{n\to\infty} \left(\ln \frac{1}{\tau - z} + \sum_{\substack{k=-n \\ k\neq 0}}^{n} \ln \frac{Tk}{|z - \tau - Tk|} \right) =$$

$$= -\frac{1}{2\pi} \mathrm{Re} \ln \left((\tau - z) \prod_{k=1}^{\infty} \left(1 - \frac{(\tau - z)^2}{k^2 T^2} \right) \right)$$

In view of statement of the sine as infinite product [Gradstein &Ryzhik, 1962, 1.431,1], the later product can be written as

$$G_c = -\frac{1}{2\pi} \mathrm{Re} \ln \left[\frac{Te^{i\beta}}{\pi} \sin \left(\frac{\pi}{Te^{i\beta}} (\tau - z) \right) \right]. \tag{7}$$

However, as for single foil, the inlet velocity should be the same as outlet one, i.e. $Ve^{i\alpha} = V_1 e^{i\alpha} = V_2 e^{i\alpha_2}$. Hence the circulation around a foil should be zero and the inlet speed as well as outlet one is directed along neutral axis $b_1 b_2$. Though equation (6) allows considering another inlet speed, $V_1 e^{i\alpha_1}$, the outlet speed will differ from the real one. Two methods of numerical investigation of cavitating flow through a cascade of any foils are considered below: one is based on conformal mapping, another on modified integral equation like (6).

4. CONFORMAL MAPPING

The numerical algorithms should consist of two parts: numerical calculation of non-circulation flow through a cascade and analytical solution of flow through the plate-cascade.

The first part is identical to a conformal map of a foil-cascade onto the plate-cascade. Since the circulation around a foil is zero, the speeds at infinity on both sides of cascade are equal, and so design equations can be obtained from equation (6) with kern function (7). The inlet function should be directed along neutral line $b_1 b_2$ but the value can be arbitrary. Let the *x*-component of inlet speed be equal to the unity, $V_x = 1$, then the *y*-component $V_y = \cot\gamma$ is unknown and should be calculated from equation (6)

$$\oint_C v(s_\tau) G_c(z,\tau) ds_\tau + c + y\cot\gamma = x, \quad x, y \in C. \tag{8}$$

The latter equation using BEM can be written in matrix form

$$\mathbf{B} \cdot \mathbf{V}^* = \mathbf{Y}, \tag{9}$$

where *N*-first components are the values of speed V_k^* and ordinates Y_k at the nodal points; the last two components of the vector $\mathbf{V}^*$ are constants, V_{N+1}^* and $V_{N+2}^* = \tan\gamma$, respectively; the last two components of the vector $\mathbf{Y}$ are equal to zero. The matrix-vector $\mathbf{V}^* = \mathbf{B}^{-1} \cdot \mathbf{Y}$.

The domain of complex potential ζ represents the plane with periodical horizontal slits of period $T' e^{\beta_0 i} = TV_1^* e^{(\beta-\gamma)i}$, Fig. 2; the slit length is $l = 0.5\sum_k |V_k^*| l_k$.

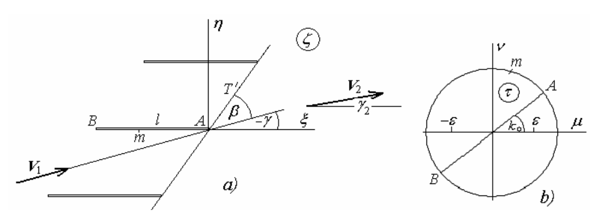


Figure 2: The flow through plate cascade: the sketch of flow on ζ -plane (a); unit circle on the parametrical τ -plane (*b*)

A conformity of slit abscissa ξ and curvilinear abscissa of blade S can be found from equations $d\xi = V^* dS$, or in discrete form, $\xi_{k+1} - \xi_k = V_k^* l_k$.

4. FLOW THROUGH PLATE-CASCADE

The second part of the method is a flow through plate-cascade, Fig. 2a, with inlet speed

$$\left(\frac{dw}{d\zeta}\right)_1 = \left(\frac{dw}{dz}\right)_1 \left(\frac{dz}{d\zeta}\right)_1 = e^{i\gamma}\cos\gamma\,. \tag{10}$$

The analytical solution of the flow problem is well-known and can be found in many books [Kochin at al 1955, Sedov 1966 and others]. Here it is presented briefly for further application. The analytical solution can be found by conformal mapping the flow domain on the ζ-plain onto inside unit circle of the parametric τ -plane, Fig. 2b. The mapping function is

$$\zeta = -\frac{T'}{2\pi i}\left(e^{i\beta'}\ln\frac{\tau-\varepsilon}{\tau+\varepsilon} - e^{-i\beta'}\ln\frac{\varepsilon\tau-1}{\varepsilon\tau+1}\right), \tag{11}$$

where $T' = TV_1^*$, $\beta' = \beta - \gamma$; unknown parameter ε is found from equation for the split length

$$l = \frac{T'}{\pi}\left(\begin{array}{l} \sin\beta' \ln\dfrac{\sqrt{1-2\varepsilon^2\cos 2\beta' + \varepsilon^4} + 2\varepsilon\sin\beta'}{\sqrt{1-2\varepsilon^2\cos 2\beta' + \varepsilon^4} - 2\varepsilon\sin\beta'} + \\ + 2\cos\beta' \arg\tan\dfrac{2\varepsilon\cos\beta'}{\sqrt{1-2\varepsilon^2\cos 2\beta' + \varepsilon^4}} \end{array}\right). \tag{12}$$

The speed on the ζ -plane is determined by

$$\frac{dw}{d\zeta} = iAe^{-i\frac{(k_2-k_0)}{2}}\frac{\tau - e^{ik_0}}{\tau + e^{ik_0}}, \tag{13}$$

where constants A and k_0 are obtained satisfying the condition for inlet speed,

$$\left(\frac{dw}{d\zeta}\right)_{\zeta=-\varepsilon} = e^{i\gamma}\cos\gamma\,. \tag{14}$$

The circulation around the plate is expressed as

$$\Gamma = 2\frac{\varepsilon T'(1-2\varepsilon\cos k_0 + \varepsilon^2)\sqrt{1-2\varepsilon^2\cos 2\beta + \varepsilon^4}}{(1-\varepsilon^2)(1-\varepsilon^4)}\sin 2\gamma\,. \tag{15}$$

A speed distribution on the plate is found in parametric form

$$V_a(t) = \left|\frac{dw}{d\zeta}\right|_{\zeta=e^{it}}; \quad \xi(t) = \operatorname{Re}\zeta(e^{it})\,. \tag{16}$$

This is completely enough for determination of flow characteristics of foil cascade on the z-plane. The circulation around the given foil is the same as (15). The outlet speed and hydrodynamic forces are determined from equation (4) and (5), respectively.

A speed distribution on foil can be calculated by

$$v(x) = \left[\frac{dw}{d\zeta}\right]_{\zeta=e^{it}}\left|\frac{d\zeta}{dz}\right|_{z\in C} = V_a V^*, \tag{17}$$

where V_a is an analytical speed on the flat plate, Eq. (16); V^* is the velocity on a foil in non-circulation flow and found numerically using the BEM-method.

5. CAVITATING FLOW

For cavitating flow, the velocity at cavity boundary should be constant, $V_a V^* = V_0 = const$. The ordinate of cavity is included in velocity V^* only, and so the following condition on the boundary corresponded to a cavity should be satisfied:

$$V^* = V_0 / V_a\,. \tag{18}$$

Now, a numerical algorithm of iteration manner can be used for calculating cavity ordinates and cavitating number at the given cavity length L_c and fixed abscissa of nodal points X_k. An initial boundary of the cavity for partial cavitation may be used as the boundary of the foil or any other curves. Iteration should be in following way:

$$\begin{array}{l} \mathbf{Y}^{(n-1)} \xrightarrow{(9)} \mathbf{V}^{*(n-1)}, \zeta^{(n-1)} \xrightarrow{(16)} \mathbf{V}_a^{(n)} \rightarrow \\ \xrightarrow{(18)} \mathbf{V}^{*(n)} \xrightarrow{(9)} \mathbf{Y}^{*(n)} \xrightarrow{\text{correcting}} \mathbf{Y}^{(n)}\,. \end{array} \tag{19}$$

It is to note that the analytical solution should be used at each iteration step only for correcting a speed at the cavity boundary. Besides, information about the velocity and boundary of the cavity is calculated at each step, and so an alteration between two steps can be graphically seen. Coincidence of these curves, as well as fulfilling the accuracy of cavitation number, $|\sigma^{(n)} - \sigma^{(n-1)}| < \varepsilon$, can be used to stop of iteration.

7. MODIFIED INTEGRAL EQUATION AND DIRECT ITARATION METHOD

The above-mentioned method of conformal map is quite complicated for numerical investigation although the well known analytical solution is used. Simpler is the method based on modified integral equation. As mentioned above (Sec. 3) integral equation (6) with kern function (7) can be used by another inlet speed. Inasmuch as the inlet and outlet speeds are peer entities then two integral equations could be considered

$$\oint_C v(s_\tau)G(z,\tau)ds_\tau + c_1 = \operatorname{Im}(V_1 e^{-i\alpha_1} z)$$

and

$$\oint_C v(s_\tau)G(z,\tau)ds_\tau + c_2 = \operatorname{Im}(V_2 e^{-i\alpha_2} z)\,.$$

Averaging both integral equations and taking into account equation (4), one can obtain required equation

$$\oint_C v(s_\tau)G(z,\tau)ds_\tau + c + \Gamma\operatorname{Im}\left(\frac{e^{-\beta i}z}{2T}\right) = \operatorname{Im}(V_1 e^{-i\alpha_1} z)\,. \tag{20}$$

Equation (20) differs from (9) by last item in the left part only, and so the both integral equations are solved numerically by the same algorithm. The second method allows calculating speed values at nodal points and the circulation simultaneously.

6. NUMERICAL RESULTS

For example, the cascade of Zhukovsky foil (h = 0.1, d = 0.05, c = 0) is considered; angle of attack $\alpha = 10^\circ$. The Riabouchinski model has been used by calculation; the trailing plate was supposed to be an element.

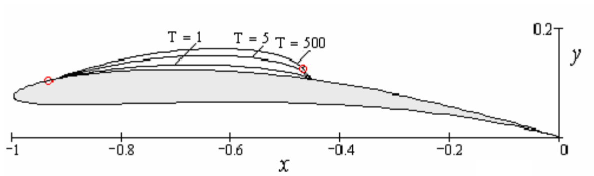
Some numerical results for cascade are presented in table 1. and in Figs. 3 – 5. The angle of attack (the angle between inlet

speed and neutral axis of single foil), $\alpha = 10^\circ$; the detachment point as well as the length of the cavity are fixed, $x_a = -0.933$ and $L_c = 0.465$. All computations have been fulfilled for the number of elements, *N*=200.
The influence of the grid spacing on main parameters is shown in table 1. The last array corresponds to a single foil.

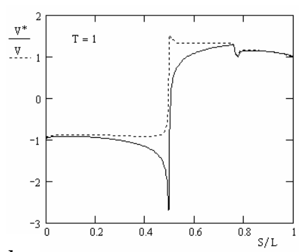
Table 1: Influence of grid spacing on main parameters.

T	σ	$-\Gamma$	$-x_0$	C_X	C_Y	ε
1	0.758	0.368	0.607	0.135	0.736	0.0008
2	1.131	0.584	0.624	0.170	1.168	0.0008
3	1.403	0.713	0.637	0.169	1.425	0.0009
4	1.596	0.796	0.642	0.159	1.593	0.0007
5	1.734	0.855	0.645	0.146	1.709	0.0008
10	2.085	0.994	0.651	0.099	1.989	0.0009
500	2.574	1.171	0.660	0.002	2.341	0.0004

The cavities for three greed spacing, *T* = 1, 5 and 500, as well as speed distributions over foil for *T* = 1, are shown in Fig. 3.



a



b

Figure 3: The cascade of Zhukovsky hydrofoil (*h*=0.1, *d*=0.05, *c*=0): the cavities for different spacing (a); the speed distribution on foil (b); $\alpha = 10^\circ$, $\beta = 90^\circ$

The dotted line on Fig. 3b is the velocity distribution on the foil, and the horizontal straight line corresponds to cavity boundary. The solid line is the speed distribution for non-circulation flow around the same foil with cavity.
One can see that the dotted line has a small height close to leading edge though the Villat condition is fulfilled. If the detachment point moves to the leading edge then the boundary of cavity intersects the foil boundary.

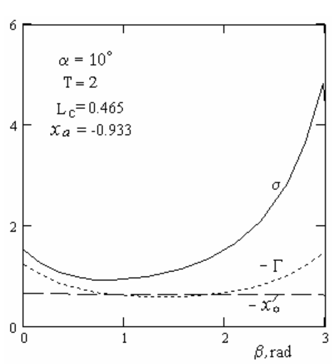


Figure 4: Dependence of the cavitation number, σ, the circulation, Γ, and the center of pressure, x_a, on the angle, β

An influence of angle β is shown in Fig. 4. It is interesting to notice that two modes of computing are possible: one is for interval $0 \le \beta < \pi + \gamma$, another for $\pi + \gamma < \beta \le \pi$. The angle of inclination of neutral axis, γ, is calculated by numerical algorithm and in that case $\gamma \approx -0.166$.
Both methods give the same results, but the second method is easier and has been used for the most part.

7. SUPERCAVITATING FLOW

A cavity by full cavitating flow has two separate boundaries at which the same value of velocity is necessary to satisfy. For this purpose it is necessary to enter, in addition, a certain hypothesis at the end of a cavity; for example, to displace a critical point at a trailing plate. The cavities past a flat plates for three values, T = 200, 10 and 7, are presented on Fig. 5.

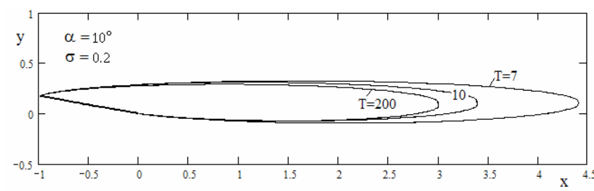


Figure 5: The cavity shapes past plate in a cascade.

Fig. 6 shows the cavity past Zhukovsky foil for one position in a cascade.

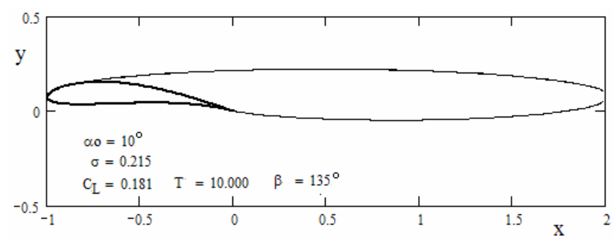


Figure 6: The cavity past Zhukovsky foil in a cascade

Zhukovsky foil has been considered by fixed distance, T = 10, and some angles but results were almost the same.

CONCLUSION

The basic in the given work is the numerical methods which allow to investigate cavitating flow of a cascade of foils of any configuration. Numerical results are of a preliminary character and can be slightly changed at more detailed calculations. The numerical algorithm, especially for full cavitation requires additional research. Anyway, each foil demands individual research which is easier for carrying out by dialogue.

ACKNOWLEDGMENTS

The work has been supported by the Russian Fund of Basic Researches, project number 08-01-97012p_Povolgie_a.

REFERENCES

[1] Chaplygin, S.A. 1914, “Theory of grid wing”, Mathematical Annual, Vol. XXIX, No. 2, 220-233 (in Russian).

[2] Weining, F. 1935, “Die Stromung um die Schaufeln von Turbomaschinen”, Springer.

[3] Kochin, E.A. 1944, “Hydrodynamical theory of cascade”, Gostechisdat (in Russian).

[4] Stepanov, G.Yu. 1962, “Hydrodynamics of turbomachine cascade”, Gosizdat, FML (in Russian).

[5] Achkinadze, A.S. and Krasilnikov, V.I. 2001, “A velocity based boundary element method with modified trailing edge for prediction of the partial cavities on the wing and propeller blades”, Cav2001 Proceedings, Pasadena, California USA.

[6] Vaz, G. and Bosschers, J. 2006, “Modeling three dimensional sheet cavitation on marine propellers using a boundary element method” CAV2006 Proceedings, Wageningen, the Netherlands.

[7] Terentiev, A.G. 1994, “Numerical investigation in hydrodynamics”, J. Izvestia AN ChR, v. 1, No. 2, 61-84 (in Russian).

[8] Terentiev, A.G. 2005, “Iteration method in the numerical hydrodynamics”, Proc. Conference “Education. Science. Innovation aspect”, Print Moscow State Open Univ., v. 3, No. 1, 238-243.

[9] Terentiev, A.G. and Pavlova N.A. 2006, “Numerical analysis of cavitating flows by direct iterative manner”, Proc. CAV2006, Wageningen, the Netherlands.

[10] Gradstein, I.S. and Ryzhik, I.M. 1962, “Tables of integrals, sums, series and products”, GIFML, M. 1962.