Second-order statistics and discrete-time detection modeling for partially saturated processes

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The basic problem in underwater detection is formulated under the general assumptions of partially saturated propagation of narrow-band and acoustic signals. Expressions for the joint probability density function (PDF) of ρ, the short-time average root-mean-square pressure at the receiver are obtained. This joint PDF is a general result reducing to the PDF's for the fully saturated and the unsaturated cases for limiting values of the appropriate variables. Subsequently, defining detection as occurring whenever ρ exceeds a specified threshold level ρ_0 and, using the above results, the upcrossing and downcrossing statistics of the envelope process are studied. Closed form expressions for the probability mass functions (PMF's) of the interarrival time (time between two successive detections) and holding time (time between an upcrossing and the first subsequent downcrossing) are obtained. Results using our partially saturated detection model reduce, in limiting cases, to results already obtained in the literature for fully saturated and unsaturated propagation.

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LIST OF SYMBOLS

a	probability ("down" at stage $n+1$/"up" at stage n)
A_0	amplitude of the ensemble of unmodulated sinusoids
b	probability ("up" at stage $n+1$/"down" at stage n)
$C_x(\tau)$	autocovariance of random variable X
$C_{xy}(\tau)$	crosscovariance of random variables X and Y
$C_{yx}(\tau)$	crosscovariance of random variables Y and X
$C_y(\tau)$	covariance of random variable Y
$f_{X_t,Y_t,X_{t+\tau},Y_{t+\tau}}(x_1,y_1,x_2,y_2)$	joint probability density function of random variables: $X(t)$, $Y(t)$, $X(t+\tau)$, $Y(t+\tau)$
$f_{\rho_t,\theta_t,\rho_{t+\tau},\theta_{t+\tau}}(\rho_1,\theta_1,\rho_2,\theta_2)$	joint probability density function of random variables: $\rho(t)$, $\theta(t)$, $\rho(t+\tau)$, $\theta(t+\tau)$
H	holding time (time between an upcrossing and the first subsequent downcrossing)
I	interarrival time (time between two successive detections)
J or $\left[\frac{\partial(x_1,y_1,x_2,y_2)}{\partial(\rho_1,\theta_1,\rho_2,\theta_2)}\right]$	Jacobian matrix of the transformation T
J_1	Jacobian matrix of the transformation $[X(t),Y(t)]^T \rightarrow [\rho(t),\theta(t)]^T$
J_2	Jacobian matrix of the transformation $[X(t+\tau),Y(t+\tau)]^T \rightarrow [\rho(t+\tau), \theta(t+\tau)]^T$, where T stands for transpose
M	number of independent paths
$N(t)$	Gaussian uncorrelated noise
$N_x(t)$	cosine component of $N(t)$
$N_y(t)$	sine component of $N(t)$
$N_x^{(n)}(t)$	cosine component of zero mean uncorrelated noise for nth path
$N_y^{(n)}(t)$	sine component of zero mean uncorrelated noise for nth path
$p(t)$	acoustic pressure as a function of time
PDF	probability density function
PMF	probability mass function
r_n	amplitude of nth path
$R_{\theta_n}(\tau)$	autocorrelation function of $\theta_n(t)$
$R_{N_x^{(n)}}(\tau)$	autocorrelation function of $N_x^{(n)}(t)$
$R_{N_y^{(n)}}(\tau)$	autocorrelation function of $N_y^{(n)}(t)$
$S(t)$	acoustic signal
$X(t)$	cosine component of the envelope of narrow-band process $p(t)$
$Y(t)$	sine component of the envelope of narrow-band process $p(t)$
$\mathbf{X}$	random vector $[X(t), Y(t), X(t+\tau), Y(t+\tau)]^T$
$\delta(x)$	unit impulse function or Delta function
ϵ_m	Newman factor ($\epsilon_0 = 1$, $\epsilon_m = 2 m > 0$)
θ_n	phase of the nth path
$\theta(t)$	phase of narrow-band process $p(t)$
λ_{ij}, $i,j = 1,2,3,4$	elements of the inverse of the covariance matrix $[\Lambda_x]$
$[\Lambda_x]$	covariance matrix of random vector $\mathbf{X}$
$\mu_\theta^{(n)}$	mean of the phase $\theta_n(t)$
μ_x	mean of random vector $\mathbf{X}$
$\rho(t)$	envelope of narrow-band process $p(t)$
ρ_0	correlation coefficient of random variables $N_c(t)$ and $N_c(t+\tau)$ or of

	$N_s(t)$ and $N_s(t+\tau)$
ρ_{xy}	correlation coefficient of random X variables and Y
σ_x^2	variance of random variable $X(t)$
σ_y^2	variance of random variable $Y(t)$
σ_{xy}	covariance of random variables $X(t)$ and $Y(t)$
σ_θ^2	variance of $\theta_n(t)$
σ_N^2	mean-square noise
ϕ_F	phase of the signal for unsaturated propagation
ψ	variance of $N_c(t)$ and $N_s(t)$
ω_0	central angular frequency of narrow-band process $p(t)$

INTRODUCTION

In general the pressure for a narrow-band ocean acoustic multipath process is given by

$$p(t) = S(t) + N(t)$$
$$= \sum_{n=1}^{M} r_n \cos[\omega_0 t - \theta_n(t)] + N_x^{(n)}(t)\cos\omega_0 t + N_y^{(n)}(t)\sin\omega_0 t$$

or

$$p(t) = X(t)\cos\omega_0 t + Y(t)\sin\omega_0 t,$$

where

$$X(t) = \sum_{n=1}^{M} r_n \cos\theta_n(t) + N_x^{(n)}(t) \quad (1)$$

and

$$Y(t) = \sum_{n=1}^{M} r_n \sin\theta_n(t) + N_y^{(n)}(t).$$

M is the number of independent paths between the source and the receiver, r_n is the amplitude of the nth path and $N_x^{(n)}(t)$ and $N_y^{(n)}(t)$ are zero mean uncorrelated Gaussian additive noise for the nth path. In general $\theta_n(t)$ is slowly varying function of time. The envelope and phase of the pressure are defined as follows:

$$\rho(t) = [X^2(t) + Y^2(t)]^{1/2},$$
$$\theta(t) = \tan^{-1}[Y(t)/X(t)]. \quad (2)$$

At short ranges and low frequencies, or for stable channels, the propagation is said to be unsaturated and the PDF of the envelope $\rho(t)$ is Rician[1] and independent of the number of paths.[2] The second-order PDF of the envelope $\rho(t)$, at two different points in time has been already derived by Perakis and Psaraftis.[3]

At sufficiently long ranges and/or high frequencies, the propagation is fully saturated, which means that the single path phase of $\theta_n(t)$ can be characterized as a random variable uniformly distributed between 0 and 2π, or is normally distributed with a standard deviation $> 2\pi$. In this regime when $M > 4$ and the single path amplitudes r_n are approximately equal, phase random multipath propagation is obtained. It has been found[4] that the envelope ρ of a fully saturated phase random process with additive Gaussian noise obeys a Rayleigh PDF. Moreover, several other statistics and joint PDFs for the phase random process have been obtained, and are presented in a comprehensive summary by Mikhalevsky.[5]

At intermediate ranges, where the signal experiences enough perturbations in the channel so that each $\theta_n(t)$ can be characterized as a Gaussian random variable but with standard deviation $< 2\pi$, partially saturated propagation is obtained. The first-order PDF of the random variable $\rho(t)$ has been already determined.[2] As the variance of the single path phase goes to zero, or becomes larger, the PDFs converge to the unsaturated and fully saturated results, respectively.[2]

In previous publications,[6] continuous and discrete-time detection models using the results of phase random acoustic propagation have been formulated. "Detection" was defined as an upcrossing of random variable ρ over a specified threshold ρ_0. A continuous-time model was first developed for obtaining the PDFs of the interarrival time and of the holding time. The model was then compared with the extensively used (λ,σ) model and with available acoustic data. This model was seen to exhibit similar long-term behavior but markedly different short-term characteristics as compared with the (λ,σ) model a fact which is due to the memory of the process. Comparison with data has demonstrated in most cases, a significantly improved prediction capability over the (λ,σ) model.

Subsequently, a two-state model and a four-state discrete-time Markov detection model were developed and closed-form expressions for the probability mass functions of the corresponding interarrival and holding times were derived.

It is the purpose of this paper to develop an acoustic detection model for the partially saturated case. An expression for the joint PDF of the random variables $\rho(t)$ and $\rho(t+\tau)$, $f_{\rho_t\rho_{t+\tau}}(\rho_1,\rho_2)$ is first obtained. Under the proper assumptions this joint PDF reduces to the ones for the limiting cases of fully saturated and unsaturated propagation, respectively. Using the above results a two-state discrete-time Markov model is next derived. Finally, the PMFs of the interarrival time (time between two successive detections) and the holding time (time between an upcrossing and the first subsequent downcrossing) are obtained.

I. ASSUMPTIONS

The process $p(t)$ is assumed to be narrow band, and the number of independent paths M large enough so that processes $X(t)$ and $Y(t)$ can be assumed to be Gaussian. This follows from the central limit theorem under the assumption that the processes $\theta_1(t)\ldots\theta_M(t)$ are independent.

The mean values of the processes $X(t)$ and $Y(t)$ can be easily determined as a function of r_n and the mean and the autocorrelation of $\theta_n(t)$.[2] The autocovariances and the crosscovariances of the above processes can be also determined if the autocorrelations of the processes $\theta_n(t)$ are known [Eqs. (A1)–(A4), Appendix]. The autocorrelations of the Gaussian noise $R_{N_x}^{(n)}(\tau)$ and $R_{N_y}^{(n)}(\tau)$ are also assumed to be known.

Summarizing, the problem is to determine the joint PDF of the random variables

$$\rho(t) \equiv \rho_t \equiv [X^2(t) + Y^2(t)]^{1/2}$$

and

$$\rho(t+\tau) \equiv \rho_{t+\tau} \equiv [X^2(t+\tau) + Y^2(t+\tau)]^{1/2}$$

given that the mean, the autocovariances and the crosscovariance of the Gaussian processes $X(t)$ and $Y(t)$ are known.

II. DETERMINATION OF THE JOINT PDF OF $\rho(t)$ AND $\rho(t+\tau)$

The random variables $X(t), Y(t), X(t+\tau), Y(t+\tau)$ are jointly Gaussian. The random variables $\rho(t), \theta(t), \rho(t+\tau), \theta(t+\tau)$ are obtained by a transformation of $X(t), Y(t), X(t+\tau), Y(t+\tau)$. The joint PDF of $\rho(t), \theta(t), \rho(t+\tau), \theta(t+\tau)$ can be obtained from the joint PDF of $X(t), Y(t), X(t+\tau), Y(t+\tau)$. A double integration over $\theta(t)$ and $\theta(t+\tau)$ of the obtained PDF will give the joint PDF of $\rho(t)$ and $\rho(t+\tau)$.

The joint PDF of the random variables $X(t), Y(t), X(t+\tau), Y(t+\tau)$ is[7,8]

$$f_{X_t, Y_t, X_{t+\tau}, Y_{t+\tau}}(x_1, y_1, x_2, y_2) = \frac{1}{(2\pi)^2 |\Lambda_x|^{1/2}} \exp\left(-\frac{1}{2}(\mathbf{x} - \boldsymbol{\mu}_x)^T [\Lambda_x]^{-1} (\mathbf{x} - \boldsymbol{\mu}_x)\right), \tag{3}$$

where

$$\mathbf{x} \equiv (x_1, y_1, x_2, y_2)^T, \quad \boldsymbol{\mu}_x \equiv (\mu_x, \mu_y, \mu_x, \mu_y)^T,$$

and $[\Lambda_x]$ is the covariance matrix of the random vector $(X_t, Y_t, X_{t+\tau}, Y_{t+\tau})^T$ that is defined as follows:

$$[\Lambda_x] = E\,[(X_t - \mu_x, Y_t - \mu_y, X_{t+\tau} - \mu_x, Y_{t+\tau} - \mu_y)^T \times (X_t - \mu_x, Y_t - \mu_y, X_{t+\tau} - \mu_x, Y_{t+\tau} - \mu_y)]$$

or

$$[\Lambda_x] = \begin{bmatrix} C_x(0) & C_{xy}(0) & C_x(\tau) & C_{yx}(\tau) \\ C_{yx}(0) & C_y(0) & C_{xy}(\tau) & C_y(\tau) \\ C_x(\tau) & C_{xy}(\tau) & C_x(0) & C_{xy}(0) \\ C_{yx}(\tau) & C_y(\tau) & C_{xy}(0) & C_y(0) \end{bmatrix},$$

where

$$C_x(\tau) \equiv E(X_{t+\tau} - \mu_x)(X_t - \mu_x),$$
$$C_y(\tau) \equiv E(Y_{t+\tau} - \mu_y)(Y_t - \mu_y),$$
$$C_{xy}(\tau) \equiv E(X_{t+\tau} - \mu_x)(Y_t - \mu_y),$$
$$C_{yx}(\tau) \equiv E(Y_{t+\tau} - \mu_y)(X_t - \mu_x).$$

The elements of $[\Lambda_x]$ are derived as shown in the Appendix. Let

$$\rho(t) \equiv [X^2(t) + Y^2(t)]^{1/2},$$
$$\theta(t) \equiv \tan^{-1}[Y(t)/X(t)],$$
$$\rho(t+\tau) \equiv [X^2(t+\tau) + Y^2(t+\tau)]^{1/2},$$
$$\theta(t+\tau) \equiv \tan^{-1}[Y(t+\tau)/X(t+\tau)],$$

then

$$X(t) = \rho(t)\cos\theta(t),$$
$$Y(t) = \rho(t)\sin\theta(t),$$
$$X(t+\tau) = \rho(t+\tau)\cos\theta(t+\tau),$$
$$Y(t+\tau) = \rho(t+\tau)\sin\theta(t+\tau).$$

Then the PDF of $\rho(t), \theta(t), \rho(t+\tau), \theta(t+\tau)$ is (A2)–(A4)

$$f_{\rho_t, \theta_t, \rho_{t+\tau}, \theta_{t+\tau}}(\rho_1, \theta_1, \rho_2, \theta_2) = f_{X_t, Y_t, X_{t+\tau}, Y_{t+\tau}}(x_1, y_1, x_2, y_2) \left| \frac{\partial(x_1, y_1, x_2, y_2)}{\partial(\rho_1, \theta_1, \rho_2, \theta_2)} \right|, \tag{4}$$

where the right side is calculated at $x_1 = \rho_1 \cos\theta_1$, $y_1 = \rho_1 \sin\theta_1$, $x_2 = \rho_2 \cos\theta_2$, $y_2 = \rho_2 \sin\theta_2$ and

$$\left| \frac{\partial(x_1, y_1, x_2, y_2)}{\partial(\rho_1, \theta_1, \rho_2, \theta_2)} \right| = \det[J] = \det \begin{bmatrix} \frac{\partial x_1}{\partial \rho_1} & \frac{\partial x_1}{\partial \theta_1} & \frac{\partial x_1}{\partial \rho_2} & \frac{\partial x_1}{\partial \theta_2} \\ \frac{\partial y_1}{\partial \rho_1} & \frac{\partial y_1}{\partial \theta_1} & \frac{\partial y_1}{\partial \rho_2} & \frac{\partial y_1}{\partial \theta_2} \\ \frac{\partial x_2}{\partial \rho_1} & \frac{\partial x_2}{\partial \theta_1} & \frac{\partial x_2}{\partial \rho_2} & \frac{\partial x_2}{\partial \theta_2} \\ \frac{\partial y_2}{\partial \rho_1} & \frac{\partial y_2}{\partial \theta_1} & \frac{\partial y_2}{\partial \rho_2} & \frac{\partial y_2}{\partial \theta_2} \end{bmatrix} = \rho_1 \rho_2.$$

Denoting by λ_{ij}, $i,j = 1,\ldots,4$ the elements of $[\Lambda_x]^{-1}$ and integrating over θ_1 and θ_2 from 0 to 2π, we finally obtain

$$f_{\rho_t \rho_{t+\tau}}(\rho_1, \rho_2) = \frac{\rho_1 \rho_2}{4\pi^2 |\Lambda_x|^{1/2}} \int_0^{2\pi}\!\!\int \exp\left(-\frac{1}{2}(A + B\cos\theta_1 + C\sin\theta_1 + D\cos 2\theta_1 + E\sin 2\theta_1 + F\cos\theta_2 + G\sin\theta_2 + H\cos 2\theta_2 + I\sin 2\theta_2 + J\cos\theta_1\cos\theta_2 + K\sin\theta_1\cos\theta_2 + L\sin\theta_1\sin\theta_2 + M\cos\theta_1\sin\theta_2)\right) d\theta_1\, d\theta_2, \tag{5}$$

where

$$A = \lambda_{11}\rho_1^2/2 + \mu_x^2\lambda_{11} + 2\mu_x\mu_y\lambda_{12} + 2\lambda_{13}\mu_x^2 + 2\lambda_{14}\mu_x\mu_y + \lambda_{22}\rho_1^2/2 + \mu_y^2\lambda_{22}$$

$$+2\lambda_{23}\mu_x\mu_y+2\lambda_{24}\mu_y^2+\lambda_{33}\rho_2^2/2+\mu_x^2\lambda_{33}+2\lambda_{34}\mu_x\mu_y+\rho_2^2\lambda_{44}/2+\mu_y^2\lambda_{44},$$

$$B=-2\lambda_{11}\mu_x\rho_1-2\lambda_{12}\mu_y\rho_1-2\lambda_{13}\mu_x\rho_1-2\lambda_{14}\mu_y\rho_1,$$

$$C=-2\lambda_{12}\mu_x\rho_1-2\lambda_{22}\mu_y\rho_1-2\lambda_{23}\mu_x\rho_1-2\lambda_{24}\mu_y\rho_1,$$

$$D=\lambda_{11}\rho_1^2/2-\lambda_{22}\rho_1^2/2,\quad E=\lambda_{12}\rho_1^2,$$

$$F=-2\lambda_{13}\mu_x\rho_2-2\lambda_{23}\mu_y\rho_2-2\lambda_{33}\mu_x\rho_2-2\lambda_{34}\mu_y\rho_2,$$

$$G=-2\lambda_{14}\mu_x\rho_2-2\lambda_{24}\mu_y\rho_2-2\lambda_{34}\mu_x\rho_2-2\lambda_{44}\mu_y\rho_2,$$

$$H=\lambda_{33}\rho_2^2/2-\lambda_{44}\rho_2^2/2,\quad I=\lambda_{34}\rho_2^2,\quad J=2\lambda_{13}\rho_1\rho_2, K=2\lambda_{23}\rho_1\rho_2,\quad L=2\lambda_{24}\rho_1\rho_2,\quad M=2\lambda_{14}\rho_1\rho_2.$$

It is tedious but it can be shown that by integrating Eq. (5) over ρ_2 from 0 to $+\infty$ the first order PDF for ρ in the partially saturated case, as given by Eq. (21), Ref. 2 is recovered.

In the fully saturated case θ_n can be characterized as a Gaussian random variable with standard deviation tending to $+\infty$. In this case the PDF of the phase of $\rho(t)$ approaches the uniform PDF from 0 to 2π, and the quadrature components of $\rho(t)$ at times t and $t+\tau$ are zero mean Gaussian processes with covariance of the same form as that of pure noise. Under these assumptions Eq. (5) can be shown to reduce to the fully saturated PDF as given by Refs. 4, 9, and 10.

In the limiting case that the variance of the phases of the independent paths, σ_θ, is zero it can be shown that Eq. (5) reduces to the second-order PDF for an unsaturated acoustic process as given by Ref. 3.

From the above limiting cases it is concluded that Eq. (5) is a general result reducing to the PDFs of fully saturated and unsaturated process, when the appropriate variables take on their limiting values. A more complete discussion with description of the mathematical derivations of the limiting cases can be found in Ref. 11.

III. DISCRETE TIME DETECTION MODEL

A two-state discrete time Markov model for the partially saturated process similar to the model developed in Ref. 6 for the phase random process will be adopted in order to study the statistics of the upcrossings and downcrossings of the envelope process.

Define states U for "up" and D for "down" by the requirements $\rho > \rho_0$ and $\rho < \rho_0$, respectively, where ρ_0 a specified threshold level. Observations are taken at discrete points in time $0, \Delta t, 2\Delta t, \ldots, n\Delta t$, where Δt is a user-specified increment. Then the transition probabilities are[6]

$$P_{\rm UD}\equiv p[\rho<\rho_0 \text{ at time } (n+1)\Delta t/\rho>\rho_0 \text{ at time } n\Delta t]=a,$$

$$P_{\rm UU}\equiv p[\rho>\rho_0 \text{ at time } (n+1)\Delta t/\rho>\rho_0 \text{ at time } n\Delta t]=1-a,$$

$$P_{\rm DU}\equiv p[\rho>\rho_0 \text{ at time } (n+1)\Delta t/\rho<\rho_0 \text{ at time } n\Delta t]=b,$$

$$P_{\rm DD}\equiv p[\rho<\rho_0 \text{ at time } (n+1)\Delta t/\rho<\rho_0 \text{ at time } n\Delta t]=1-b.$$

The transition probabilities (see Fig. 1) a and b can be evaluated as follows[6]:

$$a=\int_0^{\rho_0}\int_{\rho_0}^{\infty} f_{\rho_{n\Delta t}\rho_{(n+1)\Delta t}}(\rho_1,\rho_2)d\rho_1\,d\rho_2 \Big/ \int_{\rho_0}^{\infty} f_{\rho_{n\Delta t}}(\rho_1)d\rho_1,$$

$$b=\int_{\rho_0}^{\infty}\int_0^{\rho_0} f_{\rho_{n\Delta t}\rho_{(n+1)\Delta t}}(\rho_1,\rho_2)\,d\rho_1\,d\rho_2 \Big/ \int_0^{\rho_0} f_{\rho_{n\Delta t}}(\rho_1)d\rho_1,$$

where the second-order PDF can be taken from Eq. (5). The first-order PDF can be taken from[2]

$$f_{\rho_{n\Delta t}}(\rho_1)=\frac{\rho_1}{2\pi\sigma_x\sigma_y(1-\rho_{xy}{}^2)^{1/2}}\left[\exp\frac{-1}{2(1-\rho_{xy}{}^2)}\left(\frac{\rho_1^2+2\mu_x{}^2}{2\sigma_x{}^2}-\frac{2\rho_{xy}\mu_x\mu_y}{\sigma_x\sigma_y}+\frac{\rho_1^2+2\mu_y^2}{2\sigma_y{}^2}\right)\right]$$
$$\times\int_0^{2\pi}\exp\left(\frac{-1}{2(1-\rho_{xy}{}^2)}(C_1\cos 2\phi+C_2\sin 2\phi+C_3\cos\phi+C_4\sin\phi)\right)d\phi,$$

where

$$C_1=\rho_1^2\left(\frac{2}{2\sigma_x{}^2}-\frac{1}{2\sigma_y{}^2}\right),\quad C^2=-\rho_1^2\frac{\rho_{xy}}{\sigma_x\sigma_y},\quad C_3=\rho_1\left(\frac{2\rho_{xy}\mu_y}{\sigma_x\sigma_y}-\frac{2\mu_x}{\sigma_x{}^2}\right),\quad C_4=\rho_1\left(\frac{2\rho_{xy}\mu_x}{\sigma_x\sigma_y}-\frac{2\mu_y}{\sigma_y{}^2}\right),$$

$$\sigma_x\equiv\{E[X^2(t)]-E^2[X(t)]\}^{1/2}=C_x^{1/2}(0),\quad \sigma_y\equiv\{E[Y^2(t)]-E^2[Y(t)]\}^{1/2}=C_y^{1/2}(0),$$

$$\rho_{xy}=\frac{E(\{X(t)-E[X(t)]\}\cdot\{Y(t)-E[Y(t)]\})}{\sigma_x\sigma_y}=\frac{C_{xy}(0)}{[C_x(0)C_y(0)]^{1/2}}.$$

Expressions for the interarrival and holding time PDF can be easily found. Suppose the following sequence of transitions:

Discrete time									
	0	1	2	...	k	$k+1$	...	n	$n+1$
state	D	U	U		U	D		D	U

Let us define:

$P_H(k) \equiv P$(the first downcrossing between k and $k+1$/upcrossing between 0 and 1)$\equiv P$(holding time $= KT$)

and

$P_I(n) \equiv P$(upcrossing between n and $n+1$/upcrossing between 0 and 1) $= P$(interarrival time $= nT$)

Then the PMF of the holding and interarrival times is

$$P_H(k) = (1-a)^{k-1}a \quad \text{and} \quad P_I(n) = [ab/(a-b)][(1-b)^{n-1} - (1-a)^{n-1}].$$

IV. IMPLEMENTATION OF THE PARTIALLY SATURATED MODEL

Several holding and interarrival time PMFs were calculated and the corresponding histograms were plotted. In calculating the above PMF, the first- and the second-order PDFs of the envelope of the pressure were first obtained by using numerical integration. The holding time and the interarrival time PMFs were then calculated by using the transition probabilities $P_{\rm UD}$ and $P_{\rm DU}$ obtained by integrating the first- and the second-order PDFs of the envelope of the pressure.

The following values of the various parameters involved in the calculations were used

M (number of paths)	10
r_n (single-path amplitudes)	0.397
$\mu_\theta^{(n)}$ (single-phase amplitudes)	0.349,0.524,0.698,0.873,1.745,2.094,2.269,2.618,2.967,3.142
σ_N^2 (mean-square noise)	1.58
τ (time step)	0.4

The autocorrelation function of the cosine and sine components of the nth path noise process, $N_x^{(n)}$ and $N_y^{(n)}$ was assumed to be given by $R_{N_x}^{(n)}(\tau) = \sigma_N^2 e^{-2\pi^2 v^2 \tau^2}$ where $v = 0.2$ Hz. This equation has been used in Ref. 3 to derive the PMFs of the holding and the interarrival times for unsaturated propagation.

The autocorrelation function of the nth path phase process was assumed to be given by: $R_\theta^{(n)}(\tau) = \sigma_\theta^2 e^{-2\pi\mu\tau} + \mu_\theta^{(n)2}$ where $\mu = 0.02$. The above equations are plotted in Fig. 2.

The holding time and the interarrival time histograms are plotted in Figs. 3 and 4 with a continuous line through the top of each bar for various values of the standard deviation of the phase, σ_θ, in the range from 0.0 (unsaturated case) to 3.142 (fully saturated case) and threshold equal to 2.37. The same histograms are plotted in Figs. 5 and 6 for threshold equal to 1.58. It is observed that as $\sigma_\theta \to 0$ the histograms of the holding and the interarrival time converge to the corresponding histograms for the unsaturated case obtained in Ref. 3. This was expected, since it has been theoretically proved that as $\sigma_\theta \to 0$ the second-order PDF of the envelope of the partially saturated process converges to the corresponding fully saturated PDF.

It is observed from Figs. 3 and 5 that the distribution of the holding time becomes narrower as we move from the unsaturated propagation ($\sigma_\theta = 0$) to the fully saturated ($\sigma_\theta = \pi$). It is more likely to observe shorter holding times for fully saturated propagation than for unsaturated. This can be explained from the fact that the transition probability $P_{\rm UD}$ is higher for σ_θ close to π than for σ_θ close to zero. In other words it is easier to pass from the "up" state to the "down" state for cases close to the fully saturated than for cases close to the unsaturated. The shape of the histogram

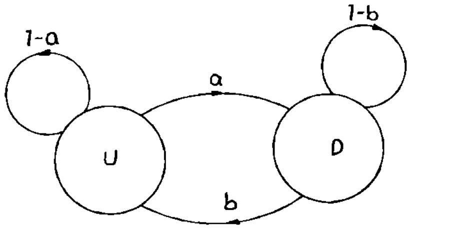

FIG. 1. The two-state Markov model.

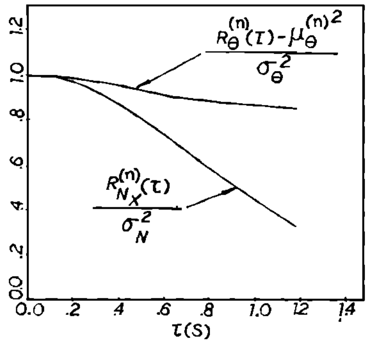

FIG. 2. Autocorrelation functions of the nth path phase process and the noise process.

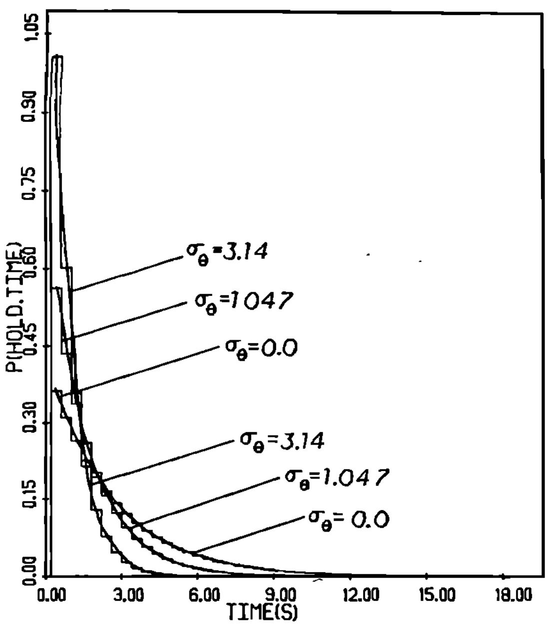

FIG. 3. Holding time histograms for threshold equal to 2.37.

for the interarrival time also depends on the value of σ_θ. It is more likely to have shorter interarrival times for σ_θ close to π than for σ_θ close to zero.

As it has been noted in Ref. 3 the timing of the detection events (i.e., the PDF of the interarrival times) is almost independent of the detection threshold for values of σ_θ close to zero. This property of the interarrival time PDF is not observed for values of σ_θ close to π and it is more likely to observe shorter interarrival times for higher thresholds. The same observation can be made for the holding time dependence on the value of the threshold.

V. CONCLUSIONS AND FUTURE RESEARCH

In this paper an analytical model for the partially saturated acoustic detection process was presented and probability mass functions for the interarrival and holding times were derived. The solution of the fully saturated and the unsaturated model are limiting cases of the solution of the most general partially saturated model developed here.

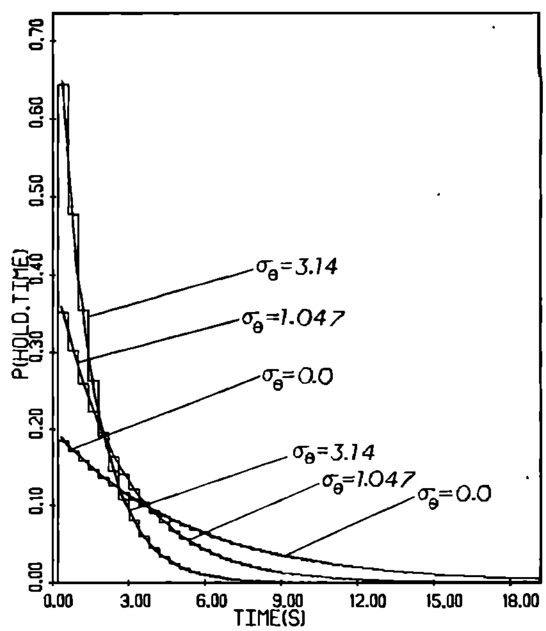

FIG. 5. Holding time histograms for threshold equal to 1.58.

The probability mass function for the interarrival and holding time can be used in the sequential hypothesis testing, optimal stopping problem in underwater detection as it has been formulated and solved in Ref. 12 and in the resource allocation algorithms developed in Ref. 13. The expression for the joint PDF of the envelope at two points in time can be used in the solution of the first passage problem for a narrowband Gaussian process by approximating the local maxima of the original process with the corresponding values of the envelope process. Unfortunately the expression for the joint PDF of the envelope process contains two integrations that can not be performed analytically. A closed form analytic

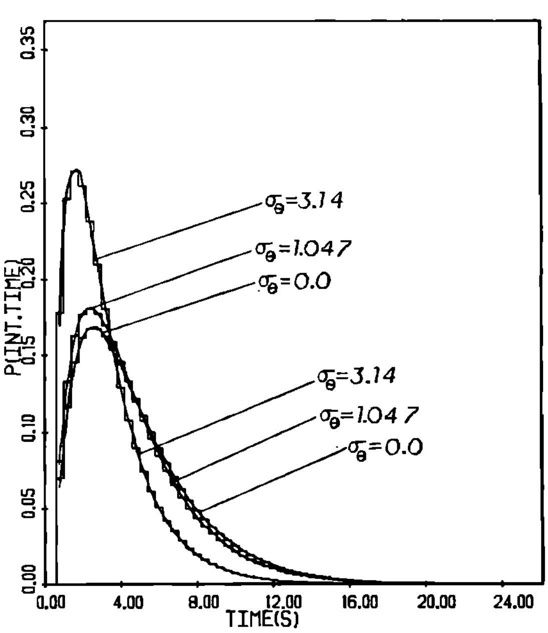

FIG. 4. Interarrival time histograms for threshold equal to 2.37.

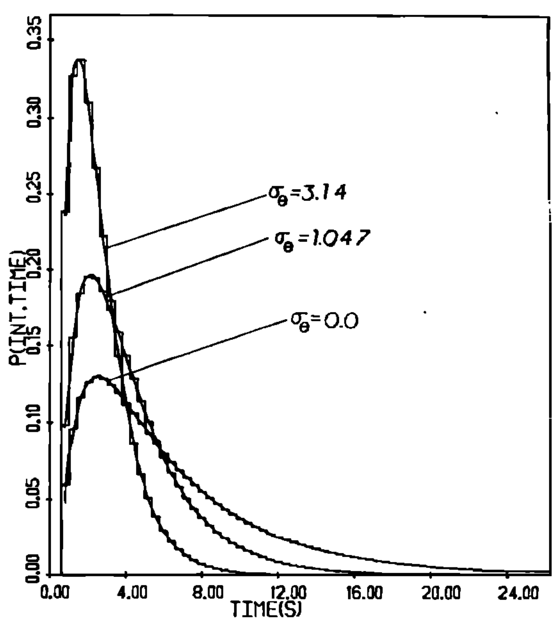

FIG. 6. Interarrival time histograms for threshold equal to 1.58.

expression for the above PDF would be very useful for reducing the computational cost and would provide insight into the limiting behavior of the joint PDF. The derivation of such a closed form analytical expression will be a topic of near-future research for the authors.

APPENDIX: DETERMINATION OF THE ELEMENTS OF THE COVARIANCE MATRIX $[\Lambda_x]$ FROM THE AUTOCORRELATION OF THE RANDOM PROCESSES $\theta_n(t)$

It is useful to determine the autocovariances and cross-covariance of the random processes $X(t)$ and $Y(t)$ as a function of the mean and the autocorrelation of $\theta_n(t)$. In other words it is necessary to determine $C_x(\tau)$, $C_y(\tau)$, and $C_{xy}(\tau)$ as a function of $R_{\theta_n}(\tau)$, $R_{N_x}^{(n)}(\tau)$, and $R_{N_y}^{(n)}(\tau)$ where $C_x(\tau)$, $C_y(\tau)$, and $C_{xy}(\tau)$ are the abovementioned autocovariances and autocovariance, and $R_{\theta_n}(\tau)$, $R_{N_x}^{(n)}(\tau)$, $R_{N_y}^{(n)}(\tau)$ are the autocorrelations of $\theta_n(t)$, $N_x^{(n)}(t)$, $N_y^{(n)}(t)$, respectively, that are assumed to be known. The single phase amplitudes r_n will be assumed to be deterministic and constant with time.

$$C_x(\tau) \equiv E[X(t+\tau)X(t)] - E^2[X(t)],$$

$$E[X(t+\tau)X(t)] = E\left(\sum_{n=1}^{M} r_n \cos\theta_n(t+\tau) + N_x^{(n)}(t+\tau)\right) \times \left(\sum_{m=1}^{M} r_m \cos\theta_m(t) + N_x^{(m)}(t)\right)$$

$$= \sum_{\substack{n=1\\m=1}}^{M} r_n r_m E[\cos\theta_n(t+\tau)\cos\theta_m(t)] + \sum_{n=1}^{M} R_{N_x}^{(n)}(\tau).$$

Since $\theta_n(t)$ and $\theta_m(t)$ have been assumed to be statistically independent for $n \neq m$ $\cos\theta_n(t)$ and $\cos\theta_m(t)$ are also statistically independent for $n \neq m$. Therefore

$$E[X(t+\tau)X(t)]$$

$$= \sum_{\substack{n=1\\m=1\\n\neq m}}^{M} r_n r_m E[\cos\theta_n(t+\tau)] E[\cos\theta_m(t)] + \sum_{n=1}^{M} \frac{r_n^2}{2}\{E\cos[\theta_n(t+\tau)-\theta_n(t)] - E\cos[\theta_n(t)+\theta_m(t+\tau)]\} + \sum_{n=1}^{M} R_{N_x}^{(n)}(\tau).$$

It is assumed[2] that

$$\theta_n(t) \sim N(\mu_\theta^{(n)}, \sigma_\theta^2) \text{ and } E[\theta_n(t+\tau)\cdot\theta_n(t)] = R_{\theta_n}(\tau).$$

Therefore,

$$\theta_n(t+\tau) - \theta_n(t) \sim N\{0, 2[\sigma_\theta^2 + \mu_\theta^{(n)2} - R_{\theta_n}(\tau)]\}$$

and

$$\theta_n(t+\tau) + \theta_n(t) \sim N\{2\mu_\theta^{(n)}, 2[\sigma_\theta^2 - \mu_\theta^{(n)2} + R_{\theta_n}(\tau)]\}$$

and

$$E[X(t+\tau)X(t)]$$

$$= \sum_{\substack{n=1\\m=1\\n\neq m}}^{M} r_n r_m E[\cos\theta_n(t+\tau)] E[\cos\theta_m(t)] + \sum_{n=1}^{M}\left(\frac{r_n^2}{2}\{\exp[-\sigma_\theta^2 - \mu_\theta^{(n)2} + R_{\theta_n}(\tau)] + \cos 2\mu_\theta^{(n)} \exp[-\sigma_\theta^2 + \mu_\theta^{(n)2} - R_{\theta_n}(\tau)]\}\right) + \sum_{n=1}^{M} R_{N_x}^{(n)}(\tau)$$

and finally

$$C_x(\tau) = \sum_{n=1}^{M} \frac{r_n^2}{2}\{\exp[-\sigma_\theta^2 - \mu_\theta^{(n)2} + R_{\theta_n}(\tau)] + \cos 2\mu_\theta^{(n)} \exp[-\sigma_\theta^2 - \mu_\theta^{(n)2} - R_{\theta_n}(\tau)]\} - \sum_{n=1}^{M} r_n^2 \cos^2\mu_\theta^{(n)} \exp(-\sigma_\theta^2) + \sum_{n=1}^{M} R_{N_x}^{(n)}(\tau), \tag{A1}$$

since

$$E[\cos\theta_n(t)] = \cos(\mu_\theta^{(n)}) \exp(-\tfrac{1}{2}\sigma_\theta^2)$$

and

$$E[\sin\theta_n(t)] = \sin(\mu_\theta^{(n)}) \exp(-\tfrac{1}{2}\sigma_\theta^2).$$

These expectations are obtained by integrating $\cos\theta_n$ over θ_n with its normal density.

The elements $C_y(\tau)$ and $C_{xy}(\tau)$ can be similarly determined.

$$C_y(\tau) = \sum_{n=1}^{M} \frac{r_n^2}{2}\{\exp[-\sigma_\theta^2 - \mu_\theta^{(n)2} + R_{\theta_n}(\tau)] - \cos 2\mu_\theta^{(n)} \exp[-\sigma_\theta^2 + \mu_\theta^{(n)2} - R_{\theta_n}(\tau)]\} - \sum_{n=1}^{M} r_n^2 \sin^2\mu_\theta^{(n)} \exp(-\sigma_\theta^2) + \sum_{n=1}^{M} R_{N_y}^{(n)}(\tau) \tag{A2}$$

and

$$C_{xy}(\tau) = \sum_{n=1}^{M} \frac{r_n^2}{2}\{\sin 2\mu_\theta^{(n)} \exp[-\sigma_\theta^2 + \mu_\theta^{(n)2} - R_{\theta_n}(\tau)]\} - \sum_{n=1}^{M} r_n^2 \cos\mu_\theta^{(n)} \sin\mu_\theta^{(n)} \exp(-\sigma_\theta^2). \tag{A3}$$

Finally since the processes $X(t)$ and $Y(t)$ are jointly stationary

$$C_{yx}(\tau) = C_{xy}(-\tau). \tag{A4}$$

In conclusion the elements of the covariance matrix $[\Lambda_x]$, $C_x(\tau)$, $C_y(\tau)$, $C_{xy}(\tau)$, etc. can be easily determined if the autocorrelation functions of the random processes $\theta_n(t)$ for $n = 1...M$ and the noise $N_x^{(n)}(t)$, $N_y^{(n)}(t)$ are known.

The mean of the vector **X** is[2]

$$\exp\left(-\frac{1}{2}\sigma_\theta^2\right)\left(\sum_{n=1}^{M} r_n \cos\mu_\theta^{(n)}, \sum_{n=1}^{M} r_n \sin\mu_\theta^{(n)}, \sum_{n=1}^{M} r_n \cos\mu_\theta^{(n)}, \sum_{n=1}^{M} r_n \sin\mu_\theta^{(n)}\right)^T.$$

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