

# Verlinde $K$ -theory

by

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## CHAPTER I

### Introduction

#### 1.1 A Brief Survey of $K$ -theory

The set  $Vect(X)$  of isomorphism classes of finite dimensional complex vector bundles on a compact space  $X$  forms a monoid under fiberwise direct sum  $\oplus$ . We define the group  $K(X)$  to be the group completion of the monoid  $(Vect(X), \oplus)$ . In other words, elements of  $K(X)$  are (equivalence classes of) formal differences between vector bundles, in the same way that elements in the group of integers may be thought of as equivalence classes of formal differences between elements in the monoid of natural numbers under addition. What is more, this construction can be extended to a full  $\mathbb{Z}/2$ -graded generalized cohomology theory  $K^*(X)$ , called *the (complex) topological  $K$ -theory of  $X$* , for which  $K^0(X) \cong K(X)$  and  $K^1(X) \cong K(S^1 \wedge X)$ .

One natural context in which we might wish to work with formal differences of vector bundles is if our space  $X$  parametrizes a family of operators on a Hilbert space  $\mathcal{H}$ , i.e., if we have a map  $D : X \rightarrow End(\mathcal{H})$ . In this situation, the kernels  $\{Ker(D(x))\}$  stitch together to form a vector bundle over  $X$ , as do the cokernels, and we might naively like to consider the formal difference  $Ker(D) - Coker(D)$ . In order for this difference to be eligible as representing an element of  $K(X)$ , the kernels and cokernels of the  $D(x)$  must be finite dimensional, which is to say the

$D(x)$  must be Fredholm operators. As it turns out, restricting attention to the space of Fredholm operators  $Fred(\mathcal{H}) \subset End(\mathcal{H})$  is no restriction at all; by the Atiyah-Jänich theorem  $K(X) \cong [X, Fred(\mathcal{H})]$ , where  $[X, Y] = \pi_0 Map(X, Y)$  denotes the collection of homotopy classes of maps from  $X$  to  $Y$ .

Reinterpreting maps  $X \rightarrow Fred(\mathcal{H})$  instead as sections of the trivial bundle

$$X \times Fred(\mathcal{H}) \rightarrow X,$$

suggests a natural generalization of  $K(X)$ ; in particular, we may consider general  $Fred(\mathcal{H})$ -bundles over  $X$

$$Fred(\mathcal{H}) \hookrightarrow E \xrightarrow{\tau} X$$

and define the zeroth twisted  $K$ -group of  $X$  with respect to the twist  $\tau$  as the collection of homotopy classes of sections  $K^\tau(X) := \pi_0 \Gamma(X, E)$ . To be completely precise, there are some topological subtleties that must be dealt with; see Section 3 in [1] for details. Such twistings  $Fred(\mathcal{H}) \hookrightarrow E \xrightarrow{\tau} X$  are classified by elements of  $H^3(X; \mathbb{Z})$ . One may, more generally, twist by classes in  $H^1(X; \mathbb{Z}/2) \times H^3(X; \mathbb{Z})$ , which becomes relevant when considering Hilbert spaces which are  $\mathbb{Z}/2$ -graded.

A different generalization of  $K(X)$  comes by letting  $X$  be a  $G$ -space, for  $G$  a compact Lie group, and considering  $G$ -equivariant complex vector bundles on  $X$ . Recall that a  $G$ -equivariant vector bundle is simply a vector bundle  $E \xrightarrow{\pi} X$  in the category of  $G$ -spaces, which is to say  $E$  is a  $G$ -space and  $\pi$  commutes with the  $G$ -action,  $\pi(g \cdot e) = g \cdot \pi(e)$ . As in the non-equivariant case, the set of isomorphism classes of  $G$ -vector bundles  $Vect_G(X)$  forms a monoid under fiberwise direct sum. Mirroring the non-equivariant case, following Segal [13] we define the zeroth  $G$ -equivariant  $K$ -group of  $X$ ,  $K_G(X)$ , to be the group completion of  $Vect_G(X)$ .

Both of the above generalizations extend individually to full  $\mathbb{Z}/2$ -graded general-

ized cohomology theories, and in fact may be combined to form twisted equivariant  $K$ -theory, or *Verlinde  $K$ -theory*,  $K_G^{\tau+*}(X)$ . In this case, one may twist by classes in  $H_G^1(X; \mathbb{Z}/2) \times H_G^3(X; \mathbb{Z})$ , where  $H_G^*(X) := H^*(EG \times_G X)$ .

This thesis uses geometric and algebraic techniques to investigate naturally-arising algebraic structures on twisted equivariant  $K$ -theory functors. These functors have been of considerable interest since Freed, Hopkins, and Teleman [6] explained how, as special cases, they give rise to *Verlinde rings*, rings which describe the category of positive energy, projective representations of loop groups. More specifically, I analyze the nature of completions of general twisted equivariant  $K$ -groups.

## 1.2 Plan of the thesis

For the moment, let  $G$  denote a simple simply-connected Lie group. Let  $\tau^\vee := \tau + h^\vee$ , where  $h^\vee$  is the dual Coxeter number of  $G$ . Let  $R^\tau(LG) \cong K_G^{*+\tau^\vee}(G)$  be the Verlinde ring of level  $\tau$  positive energy representations of  $LG$ . Notice that the  $\tau^\vee$  on the  $K$ -theory side becomes  $\tau$  on the representation side. This is an artifact of passing between  $G//G$  on the  $K$ -theory side and  $L\mathfrak{g}^*//LG$  on the representation side.

The first major result of the thesis is published in a joint paper with Igor Kriz [9]. I describe the structure of the completed Verlinde rings  $(R^\tau(LG))_I^\wedge$  as abelian groups, where  $I$  is the augmentation ideal of the representation ring  $R(G)$ . Note that the  $R(G)$ -module structure on  $K_G^{*+\tau}(G)$  enabling the completion arises from the collapse map  $G \rightarrow *$ , after one observes that  $R(G) \cong K_G^*(*)$ . Then we have the theorem (stated more precisely in Theorem II.2 in Section 2.1 below):

**Theorem I.1.** *Let  $\langle \cdot, \cdot \rangle$  denote the unique Weyl group invariant inner product on the weight lattice of  $G$ , normalized so that long roots  $\ell$  satisfy  $\langle \ell, \ell \rangle = 2$ . Then, as*

abelian groups,

$$(R^\tau(LG))_I^\wedge \cong \bigoplus_{p \text{ prime}} \mathbb{Z}_p^{N(G,\tau,p)},$$

where  $N(G, \tau, p)$  is the number of regular weights  $\lambda$  such that for every weight  $w$ , the denominator of the rational number  $\langle w, \lambda \rangle / \tau$  is a power of  $p$ .

For each Lie type, I also describe the numbers  $N(G, \tau, p)$  more explicitly. The techniques used to obtain this result are classic commutative algebra along with a case-by-case consideration of weight diagrams, though we remain hopeful that we may in the future develop a more unified argument.

In the next part of the thesis (Chapter III), I extend the first result. In particular, let  $G$  be a general compact Lie group,  $X$  a compact  $G$ -space, and  $\tau \in H_G^3(X, \mathbb{Z})$  a twist of  $G$ -equivariant  $K$ -theory on  $X$ . One can use the twist  $\tau$  to produce a  $G$ -equivariant  $S^1$ -gerbe, to which there canonically corresponds a groupoid  $\Gamma$ . This passage from the homological datum  $\tau$  to the geometric object  $\Gamma$  is rather akin to, and motivated by, the relationship between complex line bundles and first Chern classes. When  $G$  is connected,  $\pi_1(G)$  is torsion free, and  $\tau$  (viewed as a bilinear form) is non-degenerate, one can show using the methods of [7] that

$$K^*(\Gamma) \cong R(S^1 \times LG^\tau)$$

where the right hand side is the representation ring of finite sums of positive energy lowest-weight irreducible representations of  $LG$  corresponding to  $\tau$ . Building from this we produce a completion theorem for affine groups:

**Theorem I.2.** *We have*

$$K^*(B(LG^{\tau^\vee})) \cong R(S^1 \times LG^\tau)_I^\wedge$$

where  $I$  is the augmentation ideal of  $R(S^1 \times G)$ . Here  $LG^\tau$  is a central extension of  $LG$  by the positive cocycle  $\tau$ .

This “by hand” selection of  $I$  is necessitated by the fact that  $R(\widetilde{LG}^\tau)$  itself does not have an appropriate augmentation.

For a non-equivariant (generalized) cohomology theory, it is a fundamental question what its value is when evaluated on an  $n$ -sphere. One natural analogue for an  $n$ -sphere in the context of a  $G$ -equivariant theory is a representation sphere  $S^V$ , where  $V$  is a finite dimensional representation of  $G$  and  $S^V$  is its one point compactification. In Chapter IV, I evaluate the Verlinde  $K$ -theory of representation spheres  $S^V$ . For the adjoint representation sphere  $S^{\mathfrak{g}}$ , I use the theory of buildings. For general representation spheres, I make use of results from [7].

## CHAPTER II

### A Specific Completion Theorem

#### 2.1 The Basic Computation

Let  $G$  be a simple, connected, simply connected, compact Lie group. Given a topological space  $X$ , we let  $LX = \text{Map}(S^1, X)$  denote the space of continuous maps from the circle to  $X$ , also called the free loop space of  $X$ . In this section, for a class  $\tau \in H_G^1(G; \mathbb{Z}/2) \times H_G^3(G; \mathbb{Z}) \cong \mathbb{Z}$ , we compute  $K^\tau(LBG)$ , the nonequivariant twisted  $K$ -theory of the loop space of the classifying space of  $G$ . We begin by observing the well-known result that  $LBG \simeq EG \times_G G$ , where  $G$  acts on itself by conjugation, so

$$K^\tau(LBG) \cong K^\tau(EG \times_G G) \cong K_G^\tau(EG \times G).$$

To this we may apply a twisted version of the Atiyah-Segal completion theorem of A. Lahtinen and C. Dwyer [10],

**Theorem II.1.** *Let  $X$  be a finite  $G$ -CW complex, where  $G$  is a compact Lie group.*

*Then the projection  $\pi : EG \times X \rightarrow X$  induces an isomorphism*

$$K_G^\tau(X)_{I_G}^\wedge \xrightarrow{\cong} K_G^{\pi^*(\tau)}(EG \times X)$$

*for any twist  $\tau$  corresponding to an element of  $H_G^1(X; \mathbb{Z}/2) \oplus H_G^3(X; \mathbb{Z})$ .*

In the above,  $I_G \subset R(G)$  is the augmentation ideal of the representation ring and  $(-)_{I_G}^\wedge$  indicates completion, as an  $R(G)$ -module, with respect to  $I_G$ . Thus  $K_G^\tau(EG \times$



$G) \cong K_G^\tau(G)_{I_G}^\wedge$ . Finally, as a consequence of the work of Freed, Hopkins, and Teleman [6], we know  $R^\tau(LG) := K_G^{\tau+\dim(G)+h^\vee}(G) \cong R(G)/J_\tau$ , where  $h^\vee$  is the dual Coxeter number of  $G$ , and the Verlinde ideal  $J_\tau$  is the kernel of the composite map

$$R(G) \xrightarrow{i} R(T) \xrightarrow{\varphi} \prod_{a \in A_\tau} \mathbb{C}.$$

Here,  $T$  is a maximal torus in  $G$ ,  $i$  is the inclusion  $R(G) \cong R(T)^W \subset R(T)$ , and  $\varphi = \prod_{a \in A_\tau} \varphi_a$  is defined on weights  $w$  by

$$\varphi_a(w) = e^{2\pi i \langle w, a \rangle / \tau},$$

where  $\langle \cdot, \cdot \rangle$  denotes the Killing form normalized so that long roots have square length 2,  $A_\tau$  is the collection of weights  $w$  in the interior of the fundamental Weyl chamber such that  $\langle w, \theta \rangle < \tau$ , and  $\theta$  is the highest root of  $G$ . The elements of  $A_\tau$  are *the regular weights at level  $\tau$* .

## 2.2 The result

Pulling this all together, we see that

$$K^{\tau^\vee}(LBG) \cong (R^\tau(LG))_{I_G}^\wedge := \varprojlim R^\tau(LG)/(I_G^n \cdot R^\tau(LG)).$$

The following result identifies this completion as an abelian group. The proof of this theorem is postponed to the end of the chapter.

### Theorem II.2.

$$(R^\tau(LG))_{I_G}^\wedge \cong \bigoplus_{p \text{ prime}} \mathbb{Z}_p^{N(G, \tau, p)}$$

where  $N(G, \tau, p)$  is the number of regular weights  $a$  such that for every weight  $w$ , the denominator of  $\langle w, a \rangle / \tau$  is a power of  $p$ .

The values of  $N(G, \tau, p)$  are described explicitly below, followed in the next subsection, Subsection 2.2.1, by their proofs. Notation: Write  $\tau = p^k \tau'$ , where  $p$  does

not divide  $\tau'$ . In the case of  $A_n$ , we will analogously need to write  $(n+1) = p^\ell(n+1)'$ , where  $p$  does not divide  $(n+1)'$ . Observe that, except for the  $A$  and  $D$  families and  $E_8$  (!), every  $N(G, \tau, p)$  description splits into two parts: one a description for a single “bad” prime  $p \in \{2, 3\}$ , the other a description for all the other primes. The prime 3 is only bad for  $G_2$  and  $E_6$ .

**Type A:** The number  $N(A_n, \tau, p)$  is the number of tuples

$$(b_1, \dots, b_n) \in \mathbb{Z}^n$$

such that

$$(2.1) \quad p^k > b_1 > \dots > b_n > 0,$$

$$(2.2) \quad (n+1)' | (b_1 + \dots + b_n).$$

**Type B:** Assume  $n > 1$ .

- For  $p = 2$ , the number  $N(B_n, \tau, 2)$  is the number of tuples

$$(b_1, \dots, b_n) \in \mathbb{Z}^n \cup (\mathbb{Z} + \frac{1}{2})^n$$

such that

$$(2.3) \quad b_1 > \dots > b_n > 0,$$

$$(2.4) \quad p^k > (b_1 + b_2).$$

- For  $p > 2$ , the number  $N(B_n, \tau, p)$  is the number of tuples

$$(b_1, \dots, b_n) \in \mathbb{Z}^n$$

which satisfy (2.3), (2.4), and

$$(2.5) \quad 2 | (b_1 + \dots + b_n).$$

**Type C:** Assume  $n > 1$ .

- For  $p = 2$ , the number  $N(C_n, \tau, 2)$  is the number of tuples

$$(b_1, \dots, b_n) \in \mathbb{Z}^n$$

such that

$$(2.6) \quad p^k > b_1 > \dots > b_n > 0.$$

- For  $p > 2$ , the number  $N(C_n, \tau, p)$  is the number of tuples

$$(b_1, \dots, b_n) \in \mathbb{Z}^n$$

which satisfy (2.6), and

$$(2.7) \quad 2|b_i \text{ for all } i$$

It is apparent that an explicit description may be given for  $N(C_n, \tau, p)$ . In particular,

$$N(C_n, \tau, p) = \begin{cases} \binom{2^k-1}{n} & \text{if } p = 2 \\ \binom{\frac{p^k-1}{2}}{n} & \text{if } p > 2. \end{cases}$$

The numbers for other types do not appear so simple to analyze.

**Type D:** Assume  $n > 2$ .

- For  $p = 2$ , the number  $N(D_n, \tau, 2)$  is the number of tuples

$$(b_1, \dots, b_n) \in \mathbb{Z}^n \cup (\mathbb{Z} + \frac{1}{2})^n$$

such that

$$(2.8) \quad b_1 > \dots > b_{n-1} > |b_n|,$$

$$(2.9) \quad p^k > (b_1 + b_2).$$

- For  $p > 2$ ,  $N(D_n, \tau, p)$  is the number of tuples

$$(b_1, \dots, b_n) \in \mathbb{Z}^n$$

which satisfy (2.8), (2.9), and

$$(2.10) \quad 2|(b_1 + \dots + b_n).$$

$G_2$ :

- For  $p = 3$ , the number  $N(G_2, \tau, 3)$  is the number of tuples

$$(b_1, b_2) \in \mathbb{Z}^2$$

such that

$$(2.11) \quad 2b_2 > b_1 > b_2 > 0,$$

$$(2.12) \quad p^k > b_1.$$

- For  $p \neq 3$ ,  $N(G_2, \tau, p)$  is the number of tuples

$$(b_1, b_2) \in \mathbb{Z}^2$$

which satisfy (2.11), (2.12), and

$$(2.13) \quad 3|(b_1 + b_2).$$

$F_4$ :

- For  $p = 2$ , the number  $N(F_4, \tau, 2)$  is the number of tuples

$$(b_1, b_2, b_3, b_4) \in \mathbb{Z}^4 \cup (\mathbb{Z} + \frac{1}{2})^4$$

such that

$$(2.14) \quad b_2 > b_3 > b_4 > 0,$$

$$(2.15) \quad b_1 > b_2 + b_3 + b_4,$$

$$(2.16) \quad p^k > b_1 + b_2.$$

- For  $p > 2$ ,  $N(F_4, \tau, p)$  is the number of tuples

$$(b_1, b_2, b_3, b_4) \in \mathbb{Z}^4$$

which satisfy (2.14), (2.15), (2.16), and

$$(2.17) \quad 2|(b_1 + b_2 + b_3 + b_4).$$

$E_8$ : The number  $N(E_8, \tau, p)$  is the number of tuples

$$(b_1, \dots, b_8) \in \mathbb{Z}^8 \cup (\mathbb{Z} + \frac{1}{2})^8$$

such that

$$(2.18) \quad b_1 > \dots > b_7 > |b_8|,$$

$$(2.19) \quad b_1 > b_2 + \dots + b_7 - b_8,$$

$$(2.20) \quad p^k > b_1 + b_2.$$

and

$$(2.21) \quad 2|(b_1 + \dots + b_8).$$

.

$E_7$ :

- For  $p = 2$ , the number  $N(E_7, \tau, 2)$  is the number of tuples

$$(b_1, \dots, b_7) \in \frac{1}{\sqrt{2}}\mathbb{Z} \times (\mathbb{Z}^6 \cup (\mathbb{Z} + \frac{1}{2})^6)$$

such that

$$(2.22) \quad b_2 > \dots > b_6 > |b_7|,$$

$$(2.23) \quad \sqrt{2}b_1 > b_2 + \dots + b_6 - b_7,$$

$$(2.24) \quad p^k > \sqrt{2}b_1,$$

$$(2.25) \quad 2|(\sqrt{2}b_1 + \dots + b_6 - b_7).$$

- For  $p > 2$ ,  $N(E_7, \tau, p)$  is the number of tuples

$$(b_1, \dots, b_7) \in \frac{1}{\sqrt{2}}\mathbb{Z} \times (\mathbb{Z}^6 \cup (\mathbb{Z} + \frac{1}{2})^6)$$

with

$$(2.26) \quad 2b_i \equiv \sqrt{2}b_1 \pmod{2} \text{ for } i = 2, \dots, 7$$

such that (2.22), (2.23), (2.24), and

$$(2.27) \quad 2|(b_2 + \dots + b_7).$$

$E_6$ :

- For  $p = 3$ , the number  $N(E_6, \tau, 3)$  is the number of tuples

$$(b_1, \dots, b_6) \in \left(\frac{1}{\sqrt{3}}\mathbb{Z} \times \mathbb{Z}^5\right) \cup \left(\frac{1}{\sqrt{3}}\left(\mathbb{Z} + \frac{1}{2}\right) \times \left(\mathbb{Z} + \frac{1}{2}\right)^5\right)$$

such that

$$(2.28) \quad b_2 > \cdots > b_5 > |b_6|,$$

$$(2.29) \quad \sqrt{3}b_1 > b_2 + \cdots + b_5 - b_6,$$

$$(2.30) \quad p^k > (\sqrt{3}b_1 + b_2 + \cdots + b_6)/2$$

$$(2.31) \quad 2|(\sqrt{3}b_1 + b_2 + \cdots + b_6)|.$$

- For  $p \neq 3$ ,  $N(E_6, \tau, p)$  is the number of tuples

$$(b_1, \dots, b_6) \in (\sqrt{3}\mathbb{Z} \times \mathbb{Z}^5) \cup (\sqrt{3}(\mathbb{Z} + \frac{1}{2}) \times (\mathbb{Z} + \frac{1}{2})^5)$$

such that (2.28), (2.29), (2.30), (2.31).

### 2.2.1 Proofs of the $N(G, \tau, p)$

Recall  $\tau = p^k \tau'$ ,  $(n+1) = p^\ell (n+1)'$ , where  $p$  divides neither  $\tau'$  nor  $(n+1)'$ .

**Type A:** The number  $N(A_n, \tau, p)$  is the number of tuples

$$(b_1, \dots, b_n) \in \mathbb{Z}^n$$

such that

$$(2.32) \quad p^k > b_1 > \cdots > b_n > 0,$$

$$(2.33) \quad (n+1)' | (b_1 + \cdots + b_n).$$

*Proof.* The weight lattice  $\Pi^*$  for  $A_n$  is  $\mathbb{Z}\{L_1, \dots, L_{n+1}\}/(\sum L_i) \subset \mathfrak{t}^*$ . A typical weight in the weight lattice for  $A_n$  is  $\mathbf{a} = [(a_1, \dots, a_n, 0)] = \sum_i a_i L_i$ , where the  $a_i$  are integers. The roots are the pairwise differences  $L_i - L_j$  where  $i \neq j$ . The dominant

weights (weights in the Weyl chamber) are given by the representative  $(n+1)$ -tuples  $[(a_1, \dots, a_n, 0)]$  such that

$$a_1 \geq \dots \geq a_n \geq 0.$$

The regular dominant weights at twisting level  $\tau$ , which we denote  $A_\tau$ , are given by representative  $(n+1)$ -tuples  $[(a_1, \dots, a_n, 0)]$  such that

$$\tau > a_1 > \dots > a_n > 0.$$

The normalized Killing form for  $A_n$  is

$$\langle \mathbf{a}, \mathbf{b} \rangle = \left( \sum_{i=1}^{n+1} a_i b_i \right) - \frac{1}{n+1} \left( \sum_{i=1}^{n+1} a_i \right) \left( \sum_{i=1}^{n+1} b_i \right).$$

Now,  $N(A_n, \tau, p)$  is the number of elements  $\mathbf{a} \in A_\tau$  such that for all  $\mathbf{w} \in \Pi^*$ , the denominator of

$$\frac{\langle \mathbf{a}, \mathbf{w} \rangle}{\tau}$$

is a power of  $p$ . Clearly,  $\mathbf{a}$  satisfies this condition for all  $\mathbf{w}$  if and only if it satisfies the condition for weights which generate  $\Pi^*$ . In particular, we will use the weights  $L_1, L_2, \dots, L_{n+1}$ . First, notice that,

$$\frac{\langle \mathbf{a}, L_i - L_{n+1} \rangle}{\tau} = \frac{a_i}{\tau},$$

so, in order for the denominator to be a power of  $p$ ,  $\tau' | a_i$  for  $i = 1 \dots n$ , and we can define integers  $b_i = a_i / \tau'$ . Also notice that

$$\frac{\langle \mathbf{a}, -L_{n+1} \rangle}{\tau} = \frac{\sum_{j=1}^{n+1} a_j}{\tau(n+1)} = \frac{\sum_{j=1}^{n+1} b_j}{p^{k+\ell}(n+1)'},$$

so it must also be the case that  $(n+1)' | b_1 + \dots + b_n$ . This shows the given conditions are necessary. To show they are sufficient, observe that

$$\frac{\langle \mathbf{a}, L_i \rangle}{\tau} = \frac{a_i - \frac{1}{n+1} \sum a_j}{\tau} = \frac{b_i}{p^k} - \frac{\sum b_j}{p^{k+\ell}(n+1)'} = \frac{b_i}{p^k} - \frac{c}{p^{k+\ell}}$$

where  $c = (\sum b_j) / (n+1)' \in \mathbb{Z}$ . □



**Type B:** Assume  $n > 1$ . For  $p = 2$ , the number  $N(B_n, \tau, 2)$  is the number of tuples

$$(b_1, \dots, b_n) \in \mathbb{Z}^n \cup (\mathbb{Z} + \frac{1}{2})^n$$

such that

$$(2.34) \quad b_1 > \dots > b_n > 0,$$

$$(2.35) \quad p^k > (b_1 + b_2).$$

For  $p > 2$ , the number  $N(B_n, \tau, p)$  is the number of tuples

$$(b_1, \dots, b_n) \in \mathbb{Z}^n$$

which satisfy (2.34), (2.35), and

$$(2.36) \quad 2|(b_1 + \dots + b_n).$$

*Proof.* The weight lattice  $\Pi^*$  for  $B_n$  is  $(\mathbb{Z}^n \cup (\mathbb{Z} + 1/2)^n) \subset \mathfrak{t}^*$ . A typical weight in the weight lattice for  $B_n$  is  $(a) = (a_1, \dots, a_n) = \sum a_i L_i$ , where the  $a_i$  are either all integers or all half integers. The roots are the pairwise sums  $\pm L_i \pm L_j$  when  $i \neq j$ , as well as  $\pm L_i$ . The dominant weights are given by the  $n$ -tuples  $(a_1, \dots, a_n)$  such that

$$a_1 \geq \dots \geq a_n \geq 0.$$

The regular dominant weights at twisting level  $\tau$  are given by  $n$ -tuples  $(a_1, \dots, a_n)$  such that

$$a_1 > \dots > a_n > 0 \text{ and } \tau > a_1 + a_2.$$

The normalized Killing form is

$$\langle \mathbf{a}, \mathbf{b} \rangle = \sum_{i=1}^n a_i b_i.$$

As generators of the weight lattice, we may take  $L_1, \dots, L_n, (\sum L_i)/2$ . Let  $\mathbf{a} \in A_\tau$  be a regular dominant weight, then

$$\frac{\langle \mathbf{a}, L_i \rangle}{\tau} = \frac{a_i}{\tau}.$$

If we write  $a_i = A_i/2$ , where  $A_i \in \mathbb{Z}$ , then

$$\frac{a_i}{\tau} = \frac{A_i}{2p^k \tau'}.$$

If  $p = 2$ , then  $\tau' | A_i$ , and  $b_i = a_i/\tau'$  are either all integers or half integers, satisfying the given conditions. If  $p > 2$ , then  $2\tau' | A_i$ , so all the  $a_i$  must be integers, likewise for the  $b_i$ . Finally, examining

$$\frac{\langle \mathbf{a}, (\sum L_i)/2 \rangle}{\tau} = \frac{\sum b_i}{2p^k},$$

we see we get no new conditions if  $p = 2$ , but if  $p > 2$ , we must include the condition that  $2 | \sum b_i$ . □

**Type C:** Assume  $n > 1$ . For  $p = 2$ , the number  $N(C_n, \tau, 2)$  is the number of tuples

$$(b_1, \dots, b_n) \in \mathbb{Z}^n$$

such that

$$(2.37) \quad p^k > b_1 > \dots > b_n > 0.$$

For  $p > 2$ , the number  $N(C_n, \tau, p)$  is the number of tuples

$$(b_1, \dots, b_n) \in \mathbb{Z}^n$$

which satisfy (2.37), and

$$(2.38) \quad 2 | b_i \text{ for all } i$$

*Proof.* The weight lattice  $\Pi^*$  for  $C_n$  is  $\mathbb{Z}^n \subset \mathfrak{t}^*$ . A typical weight in the weight lattice for  $C_n$  is  $\mathbf{a} = (a_1, \dots, a_n) = (\sum a_i L_i)$ , where the  $a_i$  are all integers. The roots are the pairwise sums  $\pm L_i \pm L_j$  when  $i \neq j$ , as well as  $\pm 2L_i$ . The dominant weights (weights in the Weyl chamber) are given by  $n$ -tuples  $(a_1, \dots, a_n)$  such that

$$a_1 \geq \dots \geq a_n \geq 0.$$

The regular dominant weights at twisting level  $\tau$  are given by  $n$ -tuples  $(a_1, \dots, a_n)$  such that

$$\tau > a_1 > \dots > a_n > 0.$$

The normalized Killing form is

$$\langle \mathbf{a}, \mathbf{b} \rangle = \frac{1}{2} \sum_{i=1}^n a_i b_i.$$

As generators of the weight lattice, we may, of course, take  $L_1, \dots, L_n$ . Let  $\mathbf{a} \in A_\tau$  be a regular dominant weight, then

$$\frac{\langle \mathbf{a}, L_i \rangle}{\tau} = \frac{a_i}{2p^k \tau'}.$$

As usual, for the denominator to be a power of  $p$ , it must be the case that  $\tau' | a_i$ , so define  $b_i = a_i / \tau' \in \mathbb{Z}$ . If  $p = 2$ , this gives our condition on the  $b_i$ . If  $p > 2$ , it must further be the case that  $2 | b_i$  for  $i = 1, \dots, n$ .  $\square$

**Type D:** Assume  $n > 2$ . For  $p = 2$ , the number  $N(D_n, \tau, 2)$  is the number of tuples

$$(b_1, \dots, b_n) \in \mathbb{Z}^n \cup (\mathbb{Z} + \frac{1}{2})^n$$

such that

$$(2.39) \quad b_1 > \dots > b_{n-1} > |b_n|,$$

$$(2.40) \quad p^k > (b_1 + b_2).$$

For  $p > 2$ ,  $N(D_n, \tau, p)$  is the number of tuples

$$(b_1, \dots, b_n) \in \mathbb{Z}^n$$

which satisfy (2.39), (2.40), and

$$(2.41) \quad 2|(b_1 + \dots + b_n).$$

*Proof.* The weight lattice  $\Pi^*$  for  $D_n$  is  $(\mathbb{Z}^n \cup (\mathbb{Z} + 1/2)^n) \subset \mathfrak{t}^*$ . A typical weight in the weight lattice for  $D_n$  is  $\mathbf{a} = (a_1, \dots, a_n)$ , where the  $a_i$  are either all integers or all half integers. The roots are the pairwise sums  $\pm L_i \pm L_j$  when  $i \neq j$ . The dominant weights (weights in the Weyl chamber) are given by  $n$ -tuples  $(a_1, \dots, a_n)$  such that

$$a_1 \geq \dots \geq a_{n-1} \geq |a_n|.$$

The regular dominant weights at twisting level  $\tau$  are given by  $n$ -tuples  $(a_1, \dots, a_n)$  such that

$$a_1 > \dots > a_{n-1} > a_n > -a_{n-1} \text{ and } \tau > a_1 + a_2.$$

The normalized Killing form is

$$\langle \mathbf{a}, \mathbf{b} \rangle = \sum_{i=1}^n a_i b_i.$$

The proof for the case of  $B_n$  carries over to this context, mutatis mutandis.  $\square$

$G_2$ : For  $p = 3$ , the number  $N(G_2, \tau, 3)$  is the number of tuples

$$(b_1, b_2) \in \mathbb{Z}^2$$

such that

$$(2.42) \quad 2b_2 > b_1 > b_2 > 0,$$

$$(2.43) \quad p^k > b_1.$$

For  $p \neq 3$ ,  $N(G_2, \tau, p)$  is the number of tuples

$$(b_1, b_2) \in \mathbb{Z}^2$$

which satisfy (2.42), (2.43), and

$$(2.44) \quad 3|(b_1 + b_2).$$

*Proof.* The weight lattice  $\Pi^*$  for  $G_2$  is  $\mathbb{Z}\{L_1, L_2, L_3\}/(L_1 + L_2 + L_3)$ . In other words this weight lattice is the same as for  $A_2$ . A typical weight in the weight lattice for  $G_2$  is  $\mathbf{a} = [(a_1, a_2, 0)]$ , where the  $a_i$  are integers. The roots are  $\pm\{L_1, L_2, L_1 + L_2, L_1 - L_2, 2L_1 + L_2, L_1 + 2L_2\}$ . The dominant weights are given by representative 3-tuples  $[(a_1, a_2, 0)]$  such that  $2a_1 \geq a_2 \geq a_1 \geq 0$ . The regular dominant weights at twisting level  $\tau$  in  $G_2$  are given by representative triples  $[(a_1, a_2, 0)]$  such that  $2a_1 > a_2 > a_1 > 0$  and  $\tau > a_2$ . The normalized Killing form is

$$\langle \mathbf{a}, \mathbf{b} \rangle = \left( \sum_{i=1}^3 a_i b_i \right) - \frac{1}{3} \left( \sum_{i=1}^3 a_i \right) \left( \sum_{i=1}^3 b_i \right).$$

First, notice that,

$$\frac{\langle \mathbf{a}, L_i - L_3 \rangle}{\tau} = \frac{a_i}{\tau},$$

so, in order for the denominator to be a power of  $p$ ,  $\tau' | a_i$  for  $i = 1, 2$ , and we can define integers  $b_i = a_i / \tau'$ . Also notice that

$$\frac{\langle \mathbf{a}, -L_3 \rangle}{\tau} = \frac{a_1 + a_2}{3\tau} = \frac{b_1 + b_2}{3p^k},$$

so, when  $p \neq 3$ , it must also be the case that  $3|b_1 + b_2$ . This shows the given conditions are necessary. To show they are sufficient, observe that

$$\frac{\langle \mathbf{a}, L_i \rangle}{\tau} = \frac{a_i - \frac{a_1 + a_2}{3}}{\tau} = \frac{b_i}{p^k} - \frac{b_1 + b_2}{3p^k} = \frac{b_i}{p^k} - \frac{c}{p^k}$$

where  $c = (b_1 + b_2)/3 \in \mathbb{Z}$ . □

$F_4$ : For  $p = 2$ , the number  $N(F_4, \tau, 2)$  is the number of tuples

$$(b_1, b_2, b_3, b_4) \in \mathbb{Z}^4 \cup (\mathbb{Z} + \frac{1}{2})^4$$

such that

$$(2.45) \quad b_2 > b_3 > b_4 > 0,$$

$$(2.46) \quad b_1 > b_2 + b_3 + b_4,$$

$$(2.47) \quad p^k > b_1 + b_2.$$

For  $p > 2$ ,  $N(F_4, \tau, p)$  is the number of tuples

$$(b_1, b_2, b_3, b_4) \in \mathbb{Z}^4$$

which satisfy (2.45), (2.46), (2.47), and

$$(2.48) \quad 2|(b_1 + b_2 + b_3 + b_4).$$

*Proof.* The weight lattice  $\Pi^*$  for  $F_4$  is  $\mathbb{Z}^4 \cup (\mathbb{Z} + \frac{1}{2})^4$ . In other words this weight lattice is the same as for  $B_4$ . A typical weight in the weight lattice for  $F_4$  is a 4-tuple  $(a_1, a_2, a_3, a_4)$  where the  $a_i$  are all integers or half-integers. The roots of  $F_4$  are  $(\pm L_1 \pm L_2 \pm L_3 \pm L_4)/2$  along with the roots of  $B_4$  (the pairwise sums  $\pm L_i \pm L_j$  when  $i \neq j$ , and  $\pm L_i$ ). The dominant weights are given by 4-tuples  $(a_1, a_2, a_3, a_4)$  such that  $a_1 \geq a_2 \geq a_3 \geq a_4 \geq 0$  and  $a_1 \geq a_2 + a_3 + a_4$ . The regular dominant weights at twisting level  $\tau$  in  $F_4$  are given by 4-tuples  $(a_1, a_2, a_3, a_4)$  such that  $a_1 > a_2 > a_3 > a_4 > 0$ ,  $a_1 > a_2 + a_3 + a_4$ , and  $\tau > a_1 + a_2$ . The normalized Killing form is

$$\langle \mathbf{a}, \mathbf{b} \rangle = \sum_{i=1}^4 a_i b_i.$$

As generators of the weight lattice, we may take  $L_1, \dots, L_4, (\sum L_i)/2$ . Let  $\mathbf{a} \in A_\tau$  be a regular dominant weight, then

$$\frac{\langle \mathbf{a}, L_i \rangle}{\tau} = \frac{a_i}{\tau}.$$

If we write  $a_i = A_i/2$ , where  $A_i \in \mathbb{Z}$ , then

$$\frac{a_i}{\tau} = \frac{A_i}{2p^k \tau'}.$$

If  $p = 2$ , then  $\tau' | A_i$ , and  $b_i = a_i/\tau'$  are either all integers or half integers, satisfying the given conditions. If  $p > 2$ , then  $2\tau' | A_i$ , so all the  $a_i$  must be integers, likewise for the  $b_i$ . Finally, examining

$$\frac{\langle \mathbf{a}, (\sum L_i)/2 \rangle}{\tau} = \frac{\sum b_i}{2p^k},$$

we see we get no new conditions if  $p = 2$ , but if  $p > 2$ , we must include the condition that  $2 | \sum b_i$ . □

$E_8$ : The number  $N(E_8, \tau, p)$  is the number of tuples

$$(b_1, \dots, b_8) \in \mathbb{Z}^8 \cup (\mathbb{Z} + \frac{1}{2})^8$$

such that

$$(2.49) \quad b_1 > \dots > b_7 > |b_8|,$$

$$(2.50) \quad b_1 > b_2 + \dots + b_7 - b_8,$$

$$(2.51) \quad p^k > b_1 + b_2.$$

and (2.36) for  $n = 8$ .

*Proof.* The weight lattice  $\Pi^*$  for  $E_8$  can be given as

$$\begin{aligned} \Pi^*(E_8) &\cong \{v \in \mathbb{Z}^8 \cup (\mathbb{Z} + 1/2)^8 \mid \sum v_i \equiv 0 \pmod{2}\} \\ &= \text{rowspan} \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 1/2 & 1/2 & 1/2 & 1/2 & 1/2 & 1/2 & 1/2 & 1/2 \end{bmatrix} \end{aligned}$$

A typical weight in the weight lattice for  $E_8$  is  $\mathbf{a} = (a_1, \dots, a_8)$ , where the  $a_i$  are either all integers or all half integers and the sum of the  $a_i$  is even. The roots are the pairwise sums  $\pm L_i \pm L_j$  when  $i \neq j$ , along with the  $(\pm L_1 \pm \dots \pm L_8)/2$  such that the number of minuses is even. The dominant weights are given by 8-tuples  $(a_1, \dots, a_8)$  such that  $a_2 \geq \dots \geq a_7 \geq |a_8|$  and  $a_1 \geq a_2 + \dots + a_7 - a_8$ . The regular dominant weights at twisting level  $\tau$  are given by 8-tuples  $(a_1, \dots, a_8)$  such that  $a_2 > \dots > a_7 > |a_8|$ ,  $a_1 > a_2 + \dots + a_7 - a_8$ , and  $\tau > a_1 + a_2$ . The normalized Killing form is

$$\langle \mathbf{a}, \mathbf{b} \rangle = \sum_{i=1}^8 a_i b_i.$$

As generators of the lattice, we may take the rows of the matrix displayed above.

We will denote the  $i$ th row by  $M_i$ . Let  $\mathbf{a} \in A_\tau$  be a regular dominant weight, then

$$\begin{aligned} \frac{\langle \mathbf{a}, M_1 \rangle}{\tau} &= \frac{2a_1}{\tau}, \\ \frac{\langle \mathbf{a}, 2M_i \rangle}{\tau} &= \frac{2a_i - 2a_{i-1}}{\tau}, \end{aligned}$$



for  $i = 2, \dots, 7$ , and

$$\frac{\langle \mathbf{a}, 4M_8 \rangle}{\tau} = \frac{\sum 2a_i}{\tau}.$$

Write  $a_i = A_i/2$ , where  $A_i \in \mathbb{Z}$ . From pairing with  $M_1$ ,

$$\frac{2a_1}{\tau} = \frac{A_1}{p^k \tau'},$$

we see that  $\tau' | A_1$ . Pairing with  $2M_i$ , we inductively conclude that  $\tau' | A_i$ , for  $i = 2, \dots, 7$ . Finally, pairing with  $4M_8$ , we get that  $\tau' | A_8$ , so we may define  $b_i = a_i/\tau'$ , where all  $b_i$  are integers or all are half integers. Thus the conditions we give above are necessary. To see they are sufficient, take any  $\mathbf{b}$  satisfying the above conditions, and set  $\mathbf{a} = \tau' \mathbf{b}$ , then the denominators in

$$\begin{aligned} \frac{\langle \mathbf{a}, M_1 \rangle}{\tau} &= \frac{2a_1}{\tau} = \frac{2b_1}{p^k}, \\ \frac{\langle \mathbf{a}, M_i \rangle}{\tau} &= \frac{a_i - a_{i-1}}{\tau} = \frac{b_i - b_{i-1}}{p^k}, \end{aligned}$$

for  $i = 2, \dots, 7$ , and

$$\frac{\langle \mathbf{a}, M_8 \rangle}{\tau} = \frac{\sum a_i}{2\tau} = \frac{\sum b_i}{2p^k},$$

are all powers of  $p$ .

□

$E_7$ : For  $p = 2$ , the number  $N(E_7, \tau, 2)$  is the number of tuples

$$(b_1, \dots, b_7) \in \frac{1}{\sqrt{2}}\mathbb{Z} \times (\mathbb{Z}^6 \cup (\mathbb{Z} + \frac{1}{2})^6)$$

such that

$$(2.52) \quad b_2 > \dots > b_6 > |b_7|,$$

$$(2.53) \quad \sqrt{2}b_1 > b_2 + \dots + b_6 - b_7,$$

$$(2.54) \quad p^k > \sqrt{2}b_1,$$

$$(2.55) \quad 2|(\sqrt{2}b_1 + \cdots + b_6 - b_7).$$

For  $p > 2$ ,  $N(E_7, \tau, p)$  is the number of tuples

$$(b_1, \dots, b_7) \in \frac{1}{\sqrt{2}}\mathbb{Z} \times (\mathbb{Z}^6 \cup (\mathbb{Z} + \frac{1}{2})^6)$$

with

$$(2.56) \quad 2b_i \equiv \sqrt{2}b_1 \pmod{2} \text{ for } i = 2, \dots, 7$$

such that (2.52), (2.53), (2.54), and

$$(2.57) \quad 2|(b_2 + \cdots + b_7).$$

*Proof.* The weight lattice  $\Pi^*$  for  $E_7$  can be described as a subset of

$$M = \text{frac}1\sqrt{2}\mathbb{Z} \times (\mathbb{Z}^6 \cup (\mathbb{Z} + \frac{1}{2})^6)$$

by

$$\begin{aligned} \Pi^*(E_7) &\cong \{v \in M | \sqrt{2}a_1 \equiv (a_2 - a_7) + \cdots + (a_6 - a_7)(2)\} \\ &= \{v \in M | \sqrt{2}a_1 + a_2 + \cdots + a_6 - a_7 \equiv 0(2)\} \\ &= \text{rowspan} \begin{bmatrix} 2/\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 1/\sqrt{2} & 1 & 0 & 0 & 0 & 0 & 0 \\ 1/\sqrt{2} & 0 & 1 & 0 & 0 & 0 & 0 \\ 1/\sqrt{2} & 0 & 0 & 1 & 0 & 0 & 0 \\ 1/\sqrt{2} & 0 & 0 & 0 & 1 & 0 & 0 \\ 1/\sqrt{2} & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1/2 & 1/2 & 1/2 & 1/2 & 1/2 & 1/2 \end{bmatrix} \end{aligned}$$

The second expression for  $\Pi^*(E_7)$  is just the projection of  $\Pi^*(E_8)$  to the hyperplane orthogonal to the vector  $??$ . The roots are the pairwise sums  $\pm L_i \pm L_j$  for  $i \neq j \geq 2$ ,  $\pm\sqrt{2}L_1$ , and

$$\pm\{\sqrt{2}L_1 \pm L_2 \pm \cdots \pm L_7\}/2 \mid \text{the number of minus signs is odd}\}.$$

The dominant weights are given by 7-tuples  $(a_1, \dots, a_7)$  such that  $a_2 \geq \cdots \geq a_6 \geq |a_7|$  and  $\sqrt{2}a_1 \geq a_2 + \cdots + a_6 - a_7$ . The regular dominant weights at twisting level  $\tau$  are given by 7-tuples  $(a_1, \dots, a_7)$  such that  $a_2 > \cdots > a_6 > |a_7|$  and  $\tau > \sqrt{2}a_1 > a_2 + \cdots + a_6 - a_7$ . The normalized Killing form is the same as for  $E_8$ .

Write  $a_1 = A_1/\sqrt{2}$ ,  $a_i = A_i/2$  for  $1 < i \leq 7$  where  $A_j \in \mathbb{Z}$ . Like for the  $E_8$  case, label the rows of the above matrix  $M_i$ , then

$$\begin{aligned} \frac{\langle \mathbf{a}, M_1 \rangle}{\tau} &= \frac{\sqrt{2}a_1}{\tau} = \frac{A_1}{p^k \tau'}, \\ \frac{\langle \mathbf{a}, 2M_i \rangle}{\tau} &= \frac{\sqrt{2}a_1 + 2a_i}{\tau} = \frac{A_1 + A_i}{p^k \tau'} \end{aligned}$$

for  $1 < i < 7$ , and

$$\frac{\langle \mathbf{a}, 4M_7 \rangle}{\tau} = 4 \frac{a_2 + \cdots + a_7}{2\tau} = \frac{A_2 + \cdots + A_7}{p^k \tau'}$$

From the first line, we conclude that  $\tau' \mid A_1$ . From the second line,  $\tau' \mid A_i$  for  $1 < i < 7$ . From the third line,  $\tau' \mid A_7$ . Thus we may define  $b_i = a_i/\tau'$ , showing that our conditions for the case  $p = 2$  are necessary. If  $p > 2$ , we see the additional two conditions are necessary by examining

$$\frac{\langle \mathbf{a}, M_i \rangle}{\tau} = \frac{\sqrt{2}a_1 + 2a_i}{2\tau} = \frac{\sqrt{2}b_1 + 2b_i}{2p^k},$$

and

$$\frac{\langle \mathbf{a}, M_7 \rangle}{\tau} = \frac{a_2 + \cdots + a_7}{2\tau} = \frac{b_2 + \cdots + b_7}{2p^k}.$$

□

$E_6$ : For  $p = 3$ , the number  $N(E_6, \tau, 3)$  is the number of tuples

$$(b_1, \dots, b_6) \in \left(\frac{1}{\sqrt{3}}\mathbb{Z} \times \mathbb{Z}^5\right) \cup \left(\frac{1}{\sqrt{3}}\left(\mathbb{Z} + \frac{1}{2}\right) \times \left(\mathbb{Z} + \frac{1}{2}\right)^5\right)$$

such that

$$(2.58) \quad b_2 > \dots > b_5 > |b_6|,$$

$$(2.59) \quad \sqrt{3}b_1 > b_2 + \dots + b_5 - b_6,$$

$$(2.60) \quad p^k > (\sqrt{3}b_1 + b_2 + \dots + b_6)/2$$

$$(2.61) \quad 2|(\sqrt{3}b_1 + b_2 + \dots + b_6)|.$$

For  $p \neq 3$ ,  $N(E_6, \tau, p)$  is the number of tuples

$$(b_1, \dots, b_6) \in (\sqrt{3}\mathbb{Z} \times \mathbb{Z}^5) \cup (\sqrt{3}(\mathbb{Z} + \frac{1}{2}) \times (\mathbb{Z} + \frac{1}{2})^5)$$

such that (2.58), (2.59), (2.60), (2.61).

*Proof.* The weight lattice  $\Pi^*$  for  $E_6$  can be described as a subset of

$$M = \left(\frac{1}{\sqrt{3}}\mathbb{Z} \times \mathbb{Z}^5\right) \cup \left(\frac{1}{\sqrt{3}}\left(\mathbb{Z} + \frac{1}{2}\right) \times \left(\mathbb{Z} + \frac{1}{2}\right)^5\right)$$

by

$$\begin{aligned} \Pi^*(E_6) &\cong \{v \in M \mid \sqrt{3}a_1 - 3a_6 \equiv (a_2 - a_6) + \dots + (a_5 - a_6)(2)\} \\ &= \{v \in M \mid \sqrt{3}a_1 + a_2 + \dots + a_6 \equiv 0(2)\} \\ &= \text{rowspan} \begin{bmatrix} 2/\sqrt{3} & 0 & 0 & 0 & 0 & 0 \\ 1/\sqrt{3} & 1 & 0 & 0 & 0 & 0 \\ 1/\sqrt{3} & 0 & 1 & 0 & 0 & 0 \\ 1/\sqrt{3} & 0 & 0 & 1 & 0 & 0 \\ 1/\sqrt{3} & 0 & 0 & 0 & 1 & 0 \\ \sqrt{3}/2 & 1/2 & 1/2 & 1/2 & 1/2 & 1/2 \end{bmatrix} \end{aligned}$$

The second expression for  $\Pi^*(E_6)$  is just the projection of  $\Pi^*(E_8)$  to the 6-plane orthogonal to the vector ?? and ??. The roots are the pairwise sums  $\pm L_i \pm L_j$  for  $i \neq j \geq 2$ ,  $\pm\sqrt{3}L_1$ , and the  $(\pm\sqrt{3}L_1 \pm \dots \pm L_5 \pm L_6)/2$  such that the number of minuses is even. The dominant weights are given by 6-tuples  $(a_1, \dots, a_6)$  such that  $a_2 \geq \dots \geq a_5 \geq |a_6|$  and  $\sqrt{3}a_1 \geq a_2 + \dots + a_5 - a_6$ . The regular dominant weights at twisting level  $\tau$  are given by 6-tuples  $(a_1, \dots, a_6)$  such that  $a_2 > \dots > a_5 > |a_6|$ ,  $\sqrt{3}a_1 > a_2 + \dots + a_5 - a_6$ , and  $\tau > (\sqrt{3}a_1 + a_2 + \dots + a_6)/2$ . The normalized Killing form is again as for  $E_8$ .

Again label the rows of the above matrix  $M_i$ . Write  $a_1 = A_1/2\sqrt{3}$ ,  $a_i = A_i/2$  for  $1 < i \leq 6$  where  $A_j \in \mathbb{Z}$ . First observe that

$$\frac{\langle \mathbf{a}, M_1 \rangle}{\tau} = \frac{2\sqrt{3}a_1}{3\tau} = \frac{A_1}{3p^k\tau'}$$

so if  $p \neq 3$ , in addition to  $\tau'|A_1$ , we must have that  $3|A_1$ , which is the same as saying that  $b_1 = a_1/\tau' \in \sqrt{3}\mathbb{Z}$ . From

$$\frac{\langle \mathbf{a}, 2M_i \rangle}{\tau} = \frac{2\sqrt{3}a_1 + 6a_i}{3\tau} = \frac{A_1 + 3A_i}{3p^k\tau'}$$

for  $1 < i < 6$ , we see that  $\tau'|A_i$ . From

$$\frac{\langle \mathbf{a}, 2M_6 \rangle}{\tau} = 2\frac{\sqrt{3}a_1 + \dots + a_6}{2\tau} = \frac{A_1 + A_2 + \dots + A_6}{2\tau}$$

we see that  $\tau'|A_6$ , so we may define  $b_i = a_i/\tau'$ . Finally, examining

$$\frac{\langle \mathbf{a}, M_6 \rangle}{\tau} = \frac{\sqrt{3}a_1 + \dots + a_6}{2\tau} = \frac{\sqrt{3}b_1 + \dots + b_6}{2p^k}$$

shows that we must further demand that  $2|\sqrt{3}b_1 + \dots + b_6$ .  $\square$

### 2.3 Proof of Theorem II.2

We begin with a lemma which shows that there exists a number  $N$  such that  $(R^\tau(LG))_{LG}^\wedge$  is a finitely generated module over the  $N$ -adic numbers  $\mathbb{Z}_N = \varprojlim \mathbb{Z}/N^n\mathbb{Z}$ .

Observe that by the Chinese remainder theorem,

$$\mathbb{Z}_N \cong \prod_{p|N} \mathbb{Z}_p,$$

so the completion is a finite sum of finitely generated  $\mathbb{Z}_p$  modules for finitely many primes  $p$ .

**Lemma II.3.** *There is a number  $N$  such that  $N \cdot 1 \in I_G/J_\tau \cap I_G$ .*

*Proof.* If the augmentation of  $w \in J_\tau$  is  $N \neq 0$ , then  $N \in \bar{I}_G = I_G/J_\tau \cap I_G$ , otherwise we would have a trivial element in  $R^\tau(LG)$  with a nonzero augmentation. We shall show that  $J_\tau \not\subseteq I_G$ . Suppose, to the contrary, that  $J_\tau \subseteq I_G$ . In other words, suppose that every  $w \in J_\tau \subset R(G)$  augments to 0. Then the augmentation  $\epsilon : R(G) \rightarrow \mathbb{Z}$  factors through  $R^\tau(LG)$ . Also notice that  $R^\tau(LG) \xrightarrow{\prod_a \varphi_a} \prod_a \mathbb{C}$  becomes an isomorphism after tensoring with  $\mathbb{C}$ , so the complexified augmentation must factor through a map of rings  $\prod_a \mathbb{C} \rightarrow \mathbb{C}$ . Since such a map preserves idempotents, it must be a projection. Thus the complexified augmentation is equal to  $\varphi_a$  for some regular weight  $a$ . However, averaging over complex conjugates of roots of unity, we see that the augmentation must be the map which is 1 for any weight  $w$  such that  $\tau|\langle w, a \rangle$  and 0 otherwise. This is clearly not true for the original augmentation  $\epsilon$ , since this would forbid weights which augment to 0 from appearing in representations, whereas any weight can appear in a representation.  $\square$

Notice that the inclusion  $R^\tau(LG) \xrightarrow{\prod_a \varphi_a} \prod_a \mathbb{C}$  factors through a finite integral extension  $\mathbb{Z}'$  of  $\mathbb{Z}$  (adjoin the necessary roots of unity), so we obtain a map of  $\mathbb{Z}'$  algebras

$$(2.62) \quad R^\tau(LG) \otimes \mathbb{Z}' \xrightarrow{f} \prod_a \mathbb{Z}'.$$

If we let  $R = R^\tau(LG) \otimes \mathbb{Z}'$  and  $R' = \prod_a \mathbb{Z}'$  in 2.62, then we have a short exact sequence of  $R$  modules

$$0 \rightarrow R \xrightarrow{f} R' \rightarrow C \rightarrow 0$$

where the cokernel  $C$  is finite since it is finitely generated and vanishes after tensoring with  $\mathbb{C}$ . We proceed to compute the completion of  $R'$ , guided by the idea that the finite cokernel will not make a difference after passing to completions.

### 2.3.1 Completing the Verlinde Algebra

Let  $I$  denote the augmentation ideal of  $R$ . Let  $R'$  have the topology defined by the sequence of ideals  $\{I^n R'\}$  and give  $R$  and  $C$  the induced topologies, i.e., the topologies defined by  $\{(I^n R') \cap R\}$  and  $\{(I^n R')/((I^n R') \cap R)\}$  respectively. Then, by Corollary 10.3 in [2], the induced sequence

$$0 \rightarrow \widehat{R} \rightarrow \widehat{R}' \rightarrow \widehat{C} \rightarrow 0$$

is also exact. Here  $\widehat{R} = \varprojlim R/((I^n R') \cap R)$ , etc.

Now we show

**Proposition II.4.** *There is an isomorphism*

$$\widehat{R} = \varprojlim R/((I^n R') \cap R) \cong \varprojlim R/(I^n R) = R_I^\wedge.$$

*Proof.* First note that there is a nonzero integer  $\lambda$  such that  $\lambda R' \subset R$  (this follows from applying the functor  $(-\otimes \mathbb{C})$ ). Now note that for any finitely generated abelian group  $A$ ,  $[A : \lambda A] \leq \lambda^N$ , where  $N$  is the number of summands of  $A$ . Consequently, in the exact sequence

$$0 \rightarrow ((I^n R') \cap R)/(I^n R) \rightarrow R/(I^n R) \rightarrow R/((I^n R') \cap R) \rightarrow 0,$$

the kernel is finite for each  $n$ , and its order is uniformly bounded independent of  $n$ , thus by passing to limits we obtain an exact sequence

$$0 \rightarrow (\text{finite}) \rightarrow \varprojlim R/(I^n R) \rightarrow \varprojlim R/((I^n R') \cap R) \rightarrow 0.$$

Consequently, the two completions differ at most by torsion. The following lemma completes the argument.  $\square$

**Lemma II.5.**  $\varprojlim R/(I^n R)$  is torsion free.

*Proof.* Order the elements of  $A_\tau = \{a_1, \dots, a_N\}$ . Let  $J_i = \ker(p_i) \subset R$ , where  $p_i = \pi_i \circ \varphi$  and  $\pi_i$  is the projection  $R' = \bigoplus_{j=1}^N \mathbb{Z}' \rightarrow \bigoplus_{j=1}^i \mathbb{Z}'$ . Thus we have a finite filtration of  $R$  by ideals

$$R \supseteq J_1 \supseteq \dots \supseteq J_N \supseteq 0$$

such that  $J_i/J_{i+1}$  is isomorphic to an ideal of  $\mathbb{Z}'$ . Again applying Corollary 10.3 [2] we see that  $\varprojlim R/(I^n R)$  is filtered by  $\{\widehat{J}_i\}$ , where  $\widehat{J}_i$  is the completion of  $J_i$  with respect to the filtration  $F(i)_n = J_i \cap I^n$ . Since the graded object associated to the filtration  $F(i)_n$  is a sum of completions of  $J_i/J_{i+1}$  with respect to the filtration  $F'(i)_n := F(i)_n/F(i+1)_n$ , it suffices to prove that  $(\widehat{J_i/J_{i+1}})_{F'(i)}$  is torsion free.

Since  $I^n J_i \subseteq F(i)_n$ , there is an onto map

$$(\widehat{J_i/J_{i+1}})_I \xrightarrow{q} (\widehat{J_i/J_{i+1}})_{F'(i)},$$

therefore the target of  $q$  can be nonzero torsion only when it is nonzero. But  $\mathbb{Z}'$  is a Dedekind domain, so a completion of any of its ideals by another nonzero ideal  $I$  is isomorphic to the completion of  $\mathbb{Z}'$  itself by  $I$ . Our augmentation ideal  $I$  is nonzero since the least common multiple of the dimensions of the representations generating the Verlinde ideal is not zero. Note that  $\widehat{\mathbb{Z}'}_I$  is isomorphic to a product of completions of  $\mathbb{Z}'$  at various primes.



In other words, summing over  $i$ , we see that the domain of  $q$ , namely  $\bigoplus_i (\widehat{J_i/J_{i+1}})_I$ , becomes the  $I$ -completion of  $R' = \bigoplus_{j=1}^N \mathbb{Z}'$ . Therefore, if for some  $i$  we had that  $(\widehat{J_i/J_{i+1}})_{F'(i)}$  contained torsion at a prime  $\pi$  of  $Z'$ , then the  $\mathbb{Z}'_\pi$ -rank of the completion of  $R'$  would be greater than the  $\mathbb{Z}'_\pi$ -rank of the completion of  $R$ , but this would contradict the fact from the proof of the preceding proposition that the completions of  $R$  and  $R'$  differ at most by torsion, hence they must have the same  $\mathbb{Z}'_\pi$ -rank.  $\square$

## CHAPTER III

### A General Completion Theorem

The purpose of this section is to prove the following theorem.

**Theorem.** *We have*

$$K^*(B(LG^{\tau^\vee})) \cong R(S^1 \times LG^\tau)_I^\wedge$$

where  $I$  is the augmentation ideal of  $R(S^1 \times G)$ . Here  $LG^\tau$  is a central extension of  $LG$  by the positive cocycle  $\tau$ .

#### 3.1 Bundle Gerbes

The proof will use the language of (bundle) gerbes [11], [3] and their associated groupoids, so we begin by recalling that language.

##### 3.1.1 Non-equivariant Gerbes

With the reader in mind, we will first review the language in the simpler non-equivariant case. Let  $X$  be a paracompact topological space. Just as we may interpret an element  $c \in H^2(X, \mathbb{Z})$  as an  $S^1$ -bundle on  $X$ , we may interpret an element  $\gamma \in H^3(X, \mathbb{Z})$  as an “ $S^1$ -gerbe.”

**Definition III.1.** An  $S^1$ -gerbe  $\gamma = ((U_i)_{i \in I}, c_\bullet)$  on a paracompact space  $X$  consists of an open cover  $(U_i)_{i \in I}$  of  $X$ , and, for  $i, j, k \in I$ , a map

$$c_{ijk} : U_i \cap U_j \cap U_k \rightarrow S^1$$

such that for all  $x \in U_i \cap U_j \cap U_k \cap U_\ell$ ,

$$c_{ijk}(x)c_{ij\ell}(x)^{-1}c_{ik\ell}(x)c_{jkl}(x)^{-1} = 1.$$

In other words, for our purposes, an  $S^1$ -gerbe is a Čech 2-cocycle with values in the constant sheaf  $\underline{S^1}$ .

We must now say when we take two gerbes  $\gamma$  and  $\delta$  to be equivalent. Our notion of equivalence has two components, depending on whether the open covers defining  $\gamma$  and  $\delta$  are the same, or one is a refinement of the other.

**Definition III.2** (Equivalence of gerbes). Let  $\gamma = ((U_i)_{i \in I}, c_\bullet)$  and  $\delta = ((V_j)_{j \in J}, d_\bullet)$  be  $S^1$ -gerbes.

**Same cover** Suppose that  $\gamma$  and  $\delta$  are defined on the same open cover,  $(V_j) = (U_i)$ .

Then we say  $\gamma$  and  $\delta$  are equivalent if there exists a function (coboundary)

$$\phi : U_i \cap U_j \rightarrow S^1$$

such that on  $U_i \cap U_j \cap U_k$ ,

$$c_{ijk}d_{ijk}^{-1} = \phi_{ij}\phi_{ik}^{-1}\phi_{jk}.$$

**Refinement** Suppose that  $(V_j)$  is a refinement of  $(U_i)$ . In other words, we have a map  $\iota : J \rightarrow I$  such that  $V_j \subseteq U_{\iota(j)}$ . Then  $\gamma$  is equivalent to  $\delta$  if

$$d_{ijk}(x) = c_{\iota(i)\iota(j)\iota(k)}(x)$$

for all  $i, j, k \in J$ .

For general  $\gamma$  and  $\delta$ , we say they are equivalent if we may find a path from one to the other consisting of these two elementary equivalences.

Note that this definition of equivalence is simply a concrete expression of the two kinds of identifications that come into play in the definition of classes in  $\check{H}^2(X; S^1) := \varinjlim \check{H}^2((U_i); S^1)$ , where the limit is over the poset of open covers of  $X$  ordered by refinement. As a consequence of the above discussion, we obtain the following lemma.

**Lemma III.3.**  *$S^1$ -gerbes on  $X$  are classified by elements of  $\check{H}^2(X; S^1) \cong H^3(X; \mathbb{Z})$ .*

*Remark III.4.* The isomorphism in the lemma follows from considering the long exact sequence arising from the coefficient short exact sequence

$$0 \rightarrow \mathbb{Z} \rightarrow \mathbb{R} \rightarrow S^1 \rightarrow 0.$$

**Definition III.5** (Groupoid associated to a gerbe). To an  $S^1$ -gerbe  $\gamma$ , we can associate the following groupoid  $\Gamma(\gamma)$ :

$$\text{Obj}(\Gamma(\gamma)) = \coprod_{i \in I} U_i, \quad \text{Mor}(\Gamma(\gamma)) = \coprod_{i, j \in I} (U_i \cap U_j) \times S^1.$$

For  $x \in U_i \cap U_j$ , let  $x_i$  denote  $x$  viewed as an element of  $U_i$ ; let  $x_{ij}$  denote  $x$  viewed as an element of  $U_i \cap U_j$ . Then the space of morphisms from  $x_i$  to  $x_j$

$$\text{Mor}(x_i, x_j) = \{(x_{ij}, \alpha) \mid \alpha \in S^1\}$$

is homeomorphic to  $S^1$ . With this notation, for  $x \in U_i \cap U_j \cap U_k$ , composition is given by

$$(x_{jk}, \beta) \circ (x_{ij}, \alpha) = (x_{ik}, \beta \cdot \alpha \cdot c_{ijk}(x)).$$

It is easy to see that equivalent gerbes  $\gamma$  give rise to equivalent groupoids  $\Gamma(\gamma)$  in the sense of [6].

*Remark III.6.* As motivation for the above definition, recall that the primary definition of a twist of a groupoid  $\Gamma$  is a pair  $(\Gamma', L)$  consisting of (1) a groupoid  $\Gamma'$  locally

equivalent to  $\Gamma$  and (2) a central extension  $L$  of  $\Gamma'$ . Now note that for an open cover  $(U_i)$  of  $X$ , the groupoid

$$(3.1) \quad \coprod_{i,j \in I} (U_i \cap U_j) \rightrightarrows \coprod_{i \in I} U_i$$

is *locally equivalent* [6] to the constant groupoid

$$X \rightrightarrows X,$$

and our  $\Gamma(\gamma)$  is an  $S^1$ -central extension of (3.1).

### 3.1.2 Equivariant Gerbes

Now let us consider the equivariant case. Let  $G$  be a topological group,  $X$  a paracompact  $G$ -space.

**Definition III.7.** A  $G$ -equivariant  $S^1$ -gerbe  $\gamma = ((U_i)_{i \in I}, c_\bullet)$  consists of an open cover  $(U_i)_{i \in I}$  of  $X$  by  $G$ -equivariant sets (meaning for all  $g \in G, i \in I, gU_i = U_i$ ), and for  $i, j, k \in I$ , a continuous map

$$c_{ijk} : U_i \cap U_j \cap U_k \times G \times G \rightarrow S^1$$

such that the following cocycle condition is satisfied for all  $x \in U_i \cap U_j \cap U_k \cap U_\ell$ , and all triples of elements  $f, g, h \in G$ :

$$c_{ijk}(x, g, f)c_{ij\ell}(x, hg, f)^{-1}c_{ik\ell}(x, h, gf)c_{j\ell}(x, h, g)^{-1} = 1.$$

**Definition III.8** (Equivalence of equivariant gerbes). Let  $\gamma = ((U_i)_{i \in I}, c_\bullet)$  and  $\delta = ((V_j)_{j \in J}, d_\bullet)$  be  $G$ -equivariant  $S^1$ -gerbes. As before, equivalence is generated by two relations:

**Same cover** Suppose that  $\gamma$  and  $\delta$  are defined on the same open cover,  $(V_j) = (U_i)$ .

Then  $\gamma$  and  $\delta$  are equivalent if there exists a function (coboundary)

$$\phi : U_i \cap U_j \times G \rightarrow S^1$$

such that on  $U_i \cap U_j \cap U_k$ ,

$$c_{ijk}(x, g, f)d_{ijk}(x, g, f)^{-1} = \phi_{ij}(x, f)\phi_{ik}(x, gf)^{-1}\phi_{jk}(x, g).$$

**Refinement** Suppose that  $(V_j)$  is a refinement of  $(U_i)$ . In other words, we have a map  $\iota : J \rightarrow I$  such that  $V_j \subseteq U_{\iota(j)}$ . Then  $\gamma$  is equivalent to  $\delta$  if

$$d_{ijk}(x) = c_{\iota(i)\iota(j)\iota(k)}(x)$$

for all  $i, j, k \in J$ .

For general  $\gamma$  and  $\delta$ , we say they are equivalent if we may find a path from one to the other consisting of these two elementary equivalences.

**Lemma III.9.** *Equivalence classes of  $G$ -equivariant  $S^1$ -gerbes are classified by the Borel cohomology group  $H_G^3(X, \mathbb{Z})$ .*

**Definition III.10** (Groupoid associated to equivariant gerbe). As above, let  $X$  be a paracompact  $G$ -space. Let  $p : G \times X \rightarrow X$  be the second projection. Let  $m : G \times X \rightarrow X$  be the action map. For an open cover  $(U_i)$  of  $X$ , let

$$W_{ij} = p^{-1}(U_i) \cap m^{-1}(U_j) = \{(g, u_i) \in G \times U_i \mid g \cdot u_i \in U_j\}$$

To a  $G$ -equivariant  $S^1$ -gerbe  $\gamma$ , we associate the following groupoid  $\Gamma(\gamma)$ :

$$\text{Obj}(\Gamma(\gamma)) = \coprod_{i \in I} U_i, \quad \text{Mor}(\Gamma(\gamma)) = \coprod_{i, j \in I} W_{ij} \times S^1.$$

The source and target maps of the groupoid are given on components by

$$\begin{aligned} s : W_{ij} \times S^1 &\rightarrow U_i, \quad ((g, u_i), \alpha) \mapsto u_i \\ t : W_{ij} \times S^1 &\rightarrow U_j, \quad ((g, u_i), \alpha) \mapsto g \cdot u_i. \end{aligned}$$

Two morphisms  $((g, x), \alpha)$  and  $((h, y), \beta)$  are composable if  $y = g \cdot x$ , in which case composition is given by

$$((h, g \cdot x), \beta) \circ ((g, x), \alpha) = ((hg, x), \beta \cdot \alpha \cdot c_{ijk}(x, h, g)).$$

It is easy to see that equivalent gerbes  $\gamma$  give rise to equivalent groupoids  $\Gamma(\gamma)$  in the sense of [6].

### 3.2 The Theorem and Proof

Our main theorem, the goal of this section, will follow from the following proposition and theorem.

**Proposition III.11.** *Let  $G$  be a simply-connected simple compact Lie group and let  $\tau$  be a  $G$ -equivariant  $S^1$ -gerbe on  $G$  acting on itself by conjugation. Then*

$$(3.2) \quad B(\Gamma(\tau)) \simeq B(LG^\tau),$$

where  $LG^\tau$  is the central extension of the loop group  $LG$  of level  $\tau$ .

Proof of this proposition is postponed to the end of this section. In the theorem below,  $K(\Gamma(\tau^\vee))$  is the  $K$ -theory of the groupoid  $\Gamma(\tau^\vee)$ , obtained by applying the Grothendieck construction to the monoid of Hilbert bundles on  $\Gamma(\tau^\vee)$ . More precisely, a Hilbert bundle on a groupoid  $\Gamma = (\Gamma_0, \Gamma_1)$  is a groupoid  $H = (H_0, H_1)$  together with a functor  $p : H \rightarrow \Gamma$  such that  $p_i : H_i \rightarrow \Gamma_i$  is a Hilbert bundle and all groupoid structure maps are maps of Hilbert bundles [6].

**Theorem III.12.** *When  $G$  is a simply connected simple compact Lie group and  $\gamma$  is a non-degenerate twisting, then*

$$K(\Gamma(\tau^\vee)) \cong R(S^1 \times LG^\tau) = R(LG^\tau)[q, q^{-1}],$$

where by  $R(S^1 \times LG^\tau)$  we mean the free abelian group on the set of lowest weight irreducible Hilbert representations of  $S^1 \times LG^\tau$  with  $S^1$  acting by bodily rotation [12].

*Proof.* We proceed as in Sections III.5 and III.6 of [7]. Let  $H$  be a compact Lie group,  $X$  a compact  $H$ -CW complex, and  $\gamma(= \tau^\vee)$  an  $H$ -equivariant  $S^1$ -gerbe on  $X$ . Furthermore, suppose  $X$  is fixed by a closed subgroup  $M \subseteq H$ . Write  $W = H/M$ . Then Construction (5.7) of [7] assigns to this situation an  $W$ -equivariant covering space  $Y \rightarrow X$  whose fibers label isomorphism classes of irreducible  $\gamma$ -projective representations of  $M$ . In addition, we get a  $W$ -equivariant  $S^1$ -gerbe  $\gamma'$  on  $Y$  which, as an  $H$ -equivariant  $S^1$ -gerbe, satisfies  $\gamma' \cong p^*\gamma - \gamma_R$  where  $R$  is the tautological bundle on  $Y$  of projective representations of  $M$ , and  $\gamma_R$  is the corresponding  $S^1$ -gerbe. In this language, Lemma 5.8 of [7] says

**Lemma III.13.** *We have*

$$(3.3) \quad K^*(\Gamma(\gamma)) \cong K_c^*(\Gamma(\gamma'))$$

where  $K_c^*$  denotes  $K$ -theory with compact supports.

We begin by considering the case where the identity component of our Lie group  $N$  is a torus  $T$ . Observe that while this situation is not specifically covered by the statement of our theorem –  $N$  is neither simple nor simply connected – it will nevertheless give us precisely the foothold we need on the problem when we induce up from the normalizer  $N$  of a maximal torus  $T$  to the whole group  $G$ .



For  $f \in W$ , let  $N(f)$  denote the stabilizer (under conjugation) of  $fT$ . We then have an induced  $N(f)$ -equivariant  $S^1$ -gerbe  $\gamma_f$  on  $fT$  where  $N(f)$  is the stabilizer of  $fT$ , and we may decompose  $K^*(\Gamma(\gamma))$  as

$$(3.4) \quad K^*(\Gamma(\gamma)) = \bigoplus_{\langle f \rangle} K^*(\Gamma(\gamma_f))$$

where the sum is over conjugacy classes  $\langle f \rangle$  in  $W$ .

For simplicity, let us first discuss the summand in (3.4) corresponding to the identity component  $T$ . This is the situation  $H = N$ ,  $M = X = T$ . By lemma (III.13), we have a  $W$ -equivariant gerbe  $\gamma'_1$  on  $Y$  and an isomorphism

$$K^*(\Gamma(\gamma_1)) \cong K_c^*(\Gamma(\gamma'_1)).$$

Note that the underlying space of  $\gamma'_1$ , namely  $Y = \mathfrak{t} \times_{\Pi} {}^{\gamma_1}P$ , is a disjoint union of copies of  $\mathfrak{t}$ , the Lie algebra of  $T$ . Here,  $\Pi = \pi_1(T)$ ,  ${}^{\gamma_1}P$  is the set of  $\gamma_1$ -affine weights of  $T$ , and  $\Pi$  acts on  ${}^{\gamma_1}P$  by means of a map

$${}^{\gamma_1}\kappa : H_1(T, \mathbb{Z}) \rightarrow H^2(BT, \mathbb{Z}) \cong H^1(T, \mathbb{Z})$$

distilled from  $\gamma_1 \in H^3(X, \mathbb{Z})$  (see [7], Section III.6). Following [7], we assume  ${}^{\gamma_1}\kappa$  is injective, so we may further simplify  $\mathfrak{t} \times_{\Pi} {}^{\gamma_1}P$  to  $\mathfrak{t} \times ({}^{\gamma_1}P/{}^{\gamma_1}\kappa(\Pi))$ . Letting  $\sigma(\mathfrak{t})$  denote the  $S^1$ -gerbe corresponding to the  $W$ -equivariant Thom class of  $\mathfrak{t}$ , we have

$$(3.5) \quad K^*(\Gamma(\gamma)) \cong K^{*-dim(T)}(\Gamma(\gamma'_1 - \sigma(\mathfrak{t}))).$$

Observe that the above Thom isomorphism (sometimes loosely referred to as “integration along  $\mathfrak{t}$ ”), reduces the underlying space on the right hand side to the finite collection of points  ${}^{\gamma_1}P/{}^{\gamma_1}\kappa(\Pi)$ . Also note that, in general, the  $\lambda$ -twisted  $G$ -equivariant

$K$ -theory of a point is isomorphic to the representation ring of the corresponding central extension  $1 \rightarrow S^1 \rightarrow G^\lambda \rightarrow G \rightarrow 1$ , i.e., we have  $K^*(\Gamma(\lambda)) \cong R(G^\lambda)$ . In particular, in unraveling the right hand side of (3.5), we need to account for the stabilizer groups  $W_x$  of the individual affine weights  $x \in {}^\gamma P / {}^\gamma \kappa(\Pi)$ . For *regular* weights, the stabilizer group (by definition) is trivial, so  $R(W_x^{\gamma_1}) \cong R(S^1)$ . As we shall see below, we only need be concerned with regular weights, so this observation is sufficient for our purposes.

We now apply the above discussion to  $G$ , acting on itself by conjugation. Let  $T$  be a maximal torus in  $G$ ,  $N$  its normalizer. Consider the ‘‘Weyl map’’

$$\omega : G \times_N T \rightarrow G, (g, t) \mapsto gtg^{-1}.$$

(In defining  $G \times_N T$ , we take  $N$  to be acting on  $G$  by right translation, and on  $T$  by conjugation.) Let  $\gamma$  be a  $G$ -equivariant  $S^1$ -gerbe on  $G$ , and form its restriction  $\gamma_N$  to an  $N$ -equivariant  $S^1$ -gerbe on  $T$ . This, in turn, induces a  $G$ -equivariant  $S^1$ -gerbe  $\gamma'_N$  on  $G \times_N T$ . We then get the restriction map

$$\omega^* : K^*(\Gamma(\gamma_N)) = K^*(\Gamma(\gamma'_N)) \rightarrow K^*(\Gamma(\gamma))$$

and an induction map

$$\omega_\# : K^*(\Gamma(\gamma)) \rightarrow K^*(\Gamma(\gamma_N)) = K^*(\Gamma(\gamma'_N)).$$

Parallel to Theorem 7.9 of [7], it can be shown that

$$\omega_\# \omega^* = Id.$$

Examining the effect of this idempotent on weights, similarly to [7], one concludes that the image consists precisely of summands (3.5) corresponding to regular weights.

As remarked above, we have an  $R(S^1)$  summand in (3.5) corresponding to each of these regular weights. Thus, we get a free abelian group on pairs  $(w, n)$ , where  $w$  is a

regular weight and  $n \in \mathbb{Z}$ . This is the free abelian group on irreducible lowest weight Hilbert representations of  $\widetilde{LG}$  together with an integer, which is the same thing as irreducible lowest weight Hilbert representations of  $S^1 \times \widetilde{LG}$ .  $\square$

By the completion theorem for groupoid K-theory, Proposition III.11 and Theorem III.12 then give the following

**Theorem III.14.** *We have*

$$(3.6) \quad K^*(B(LG^{\tau^\vee})) \cong R(S^1 \times LG^\tau)_I^\wedge$$

where  $I$  is the augmentation ideal of  $R(S^1 \times G)$  (note that  $R(S^1 \times LG^\tau)$  is canonically an  $R(S^1 \times G)$ -module). Here  $LG^{\tau^\vee}$  is a central extension of  $LG$  by the positive cocycle  $\tau^\vee = \tau + h^\vee$ .

$\square$

To prove Proposition III.11, recall from Deligne-Mumford [5] the definition of a fibration of groupoids.

**Definition III.15.** A functor of groupoids  $F : \widetilde{\Gamma} \rightarrow \Gamma$  is called a *fibration* if the following two conditions hold:

1. For every morphism  $f : X \rightarrow Y$  in  $\Gamma$  and every object  $y$  above  $Y$  there exists a morphism  $F : x \rightarrow y$  above  $f$
2. For every diagram in  $\widetilde{\Gamma}$

$$\begin{array}{ccc} & x & \\ & \downarrow f & \\ y & \xrightarrow{g} & z \end{array}$$

and every morphism  $\chi : F(x) \rightarrow F(y)$  with  $F(g)\chi = F(f)$  there is a unique morphism  $h : x \rightarrow y$  above  $\chi$  such that  $gh = f$ .

The full subcategory of  $\tilde{\Gamma}$  over an object  $X$  is called a *fiber* of  $U$ .

**Proposition III.16.** *Consider a fibration of groupoids*

$$\Delta \rightarrow \tilde{\Gamma} \rightarrow \Gamma.$$

*Then applying the bar construction, one obtains a fibration*

$$B\Delta \rightarrow B\tilde{\Gamma} \rightarrow B\Gamma.$$

□

**Proof of Proposition III.11:** Let  $\Gamma = G//G$  be the action groupoid of  $G$  acting on  $G$  by conjugation. Let  $\gamma \in H_G^3(G, \mathbb{Z})$  be a  $G$ -equivariant gerbe on  $G$ . Then, by definition, we have a fibration of groupoids

$$(3.7) \quad S^1 \rightarrow \Gamma(\gamma) \rightarrow \Gamma.$$

(Note that the fibers over different objects are equivalent.) Applying the bar construction to (3.7), we get

$$(3.8) \quad BS^1 \rightarrow B\Gamma(\gamma) \rightarrow BLG.$$

Given our assumptions,  $BLG$  is 2-connected, so (3.8) is a principal fibration, and it is well known that these all come by applying the bar construction to  $S^1$ -central extensions of  $LG$ . Moreover, both the data (3.7) and (3.8) are classified by  $\mathbb{Z}$ , and an isomorphism between the two groups of data is given by transgression. □

## CHAPTER IV

### Verlinde $K$ -theory of Representation Spheres

#### 4.1 Verlinde $K$ -theory of Adjoint Representation Spheres

Forgetting extra structure which we will not need, the building  $\Delta(G)$  associated to a semi-simple complex Lie group  $G$  is a simplicial complex carrying a simplicial  $G$ -action. In particular, the collection of  $k$ -simplices  $\Delta(G)_k \subset \Delta(G)$  is

$$\bigsqcup_{i \in I_k} G/P_i,$$

where  $\{P_i\}_{i \in I_k}$  is a set of representatives of the conjugacy classes of “height  $k$ ” parabolic subgroups of  $G$ . [Note: the notion of height here may feel a bit backwards, because the height is (one greater than) the dimension of the simplices represented. A height  $k$  subgroup is generated by the Borel subgroup and  $\text{rank}(G) - k$  one-parameter subgroups.  $G$  is height 0 (the empty (-1)-simplex); maximal proper parabolics are height 1 (vertices, 0-simplices); the Borel subgroup  $B$  is height  $\text{rank}(G)$  (chambers, maximal simplices).] Since any parabolic subgroup  $P$  is self-normalizing, the  $G$ -space  $G/P$ , with  $G$  acting by conjugation, may (and will) be viewed as representing the space of parabolic subgroups which are conjugate to  $P$ . The face relation in  $\Delta(G)$  is by reverse inclusion:  $g \cdot P = gPg^{-1} \in G/P \subset \Delta_k$  is a face of  $h \cdot Q = hQh^{-1} \in G/Q \subset \Delta_{k+1}$  if  $gPg^{-1} > hQh^{-1}$ . This gives us a diagram:

$$\Delta_{\text{rank}(G)} \rightarrow \cdots \rightarrow \Delta_2 \rightarrow \Delta_1 \rightarrow \Delta_0 \rightarrow \Delta_{-1},$$

more explicitly

$$G/B \rightarrow \cdots \rightarrow \bigsqcup_{i \in I_1} G/Q_i \rightarrow \bigsqcup_{i \in I_0} G/P_i \rightarrow G/G.$$

*Remark IV.1.* In the context of *algebraic groups*, it is common to treat the sets of simplices of a building as discrete sets (for example, when one is treating finite groups of Lie type like  $SL_3(F_2)$ ). In such cases, (the geometric realization of) the building associated to a semi-simple algebraic group is homotopic to a wedge of spheres of dimension  $\text{rank}(G) - 1$ . In contrast, in our chosen context where our semi-simple group is *complex Lie*, the collection of vertices  $\Delta_0 = \bigsqcup_{P \text{ maximal parabolic}} G/P$  has a topology on it induced from the topology of  $G$ , and we shall take that topology into account in forming the geometric realization of the building. We then topologize the rest of the  $\Delta_k$  by taking the topology induced by the map  $\Delta_k \rightarrow \Delta_0^{k+1}$ . [Claim: This topology on  $\Delta_k$  coincides with the more naive choice of topology given by the fact that  $\Delta_k = \bigsqcup G/P$  is a disjoint union of quotients of the topological group  $G$ .] Henceforth,  $\Delta(G)$  will refer to this *topological building*.

The point of all of this is that, by the work of Burns-Spatzier [4], we know that the geometric realization of  $\Delta(G)$  as a topological building is a sphere of dimension  $\dim(G)$ . More precisely, it is  $G$ -homeomorphic to  $S^{\mathfrak{g}}$ , the one-point compactification of the adjoint representation of  $G$  on its Lie algebra  $\mathfrak{g}$ .

**Proposition IV.2.**  $K_G^{\tau+*}(\Delta(G)) \cong K_G^{\tau+*-\dim(G)}(*)$

*Proof.* Applying the functor  $K_G^\tau$  to the above diagram yields the chain complex  $C$

$$R^\tau(G) \rightarrow \bigoplus_{i \in I_0} R^\tau(P_i) \rightarrow \bigoplus_{i \in I_1} R^\tau(Q_i) \rightarrow \cdots \rightarrow R^\tau(B)$$

where the differentials are alternating sums using the same sign conventions as for an augmented simplicial complex.

Let  $M_1, \dots, M_n$  be the maximal (= height  $n - 1$ ) parabolics containing  $B$ ,  $n = \text{rank}(G)$ .<sup>1</sup> Each minimal (= height 0) parabolic  $\bigcap_{j \neq i} M_j = m_i$  corresponds to a Weyl plane  $W_i$ . Then each  $R(P)$  where  $P$  is the intersection of some  $M_i$ 's is a subring of  $\mathbb{Z}[\Pi^*]$ , where  $\Pi^*$  is the weight lattice, namely it is the subring of elements fixed under each  $W_j$  where  $j$  is different from all the  $i$ 's involved in the intersection. In other words, if  $P_J = \bigcap_{i \in I_1 \setminus J} M_i = \langle m_j | j \in J \rangle$ , then  $R(P_J) = \mathbb{Z}[\Pi^*]^{W_J}$  where  $W_J = \langle W_j | j \in J \rangle$ .

Now filter  $R(B) = \mathbb{Z}[\Pi^*]$  by an increasing filtration, where  $F_k R(B)$  is the subgroup spanned by all elements of  $\Pi^*$  which lie on the non-positive side of at most  $k$  of the Weyl planes  $W_i$ . This induces a filtration on our chain complex  $C$ :  $F_k R(P) = R(P) \cap F_k R(B)$ . Then the associated graded chain complex is simply the free abelian group on the Venn diagram of the sets  $W_i^-$  of non-positive element with respect to the  $i$ 'th Weyl plane.

By a cohomological version of the inclusion-exclusion principle, the cohomology of this  $E_0(C)$  is isomorphic to the free abelian group of  $\Pi^* \setminus \bigcup_{i=1}^n W_i^-$ , which is the set of regular weights in the fundamental Weyl chamber. (Since all the cohomology is in a single filtration 0, the spectral sequence collapses.)  $\square$

## 4.2 Verlinde $K$ -theory of General Representation Spheres

By applying results of Freed-Hopkins-Teleman, we can extend this result to apply to more general representation spheres. To accomplish this, we need a lemma from [7] and a remark.

**Lemma IV.3.** *Let  $G$  be a compact Lie group. The central extensions*

$$U(1) \rightarrow G^\tau \rightarrow G$$

---

<sup>1</sup>These maximal parabolics correspond to the walls of the *fundamental chamber* in  $\Delta(G)$ , the chamber which corresponds to  $B$ .

are classified by  $H_G^3(*; \mathbb{Z}) = H^3(BG; \mathbb{Z})$ . The Grothendieck group  $R^\tau(G)$ , of representations of  $\tilde{G}$  on which  $U(1)$  acts according to its defining representation, can be thought of as a twisted form of  $R(G)$ . In this case, the definition of equivariant twisted  $K$ -theory gives

$$K_G^{\tau+k}(\ast) = \begin{cases} R^\tau(G) & \text{if } k = 0 \\ 0 & \text{if } k = 1. \end{cases}$$

*Remark IV.4.* Let  $X := (X_1 \rightrightarrows X_0)$  be a groupoid, where

$$s, t : X_1 \rightarrow X_0$$

are, respectively, the source and target maps, and

$$p_1, p_2, m : X_1 \times_{s,t} X_1 \rightarrow X_1$$

are, respectively, the first projection, second projection, and composition maps. Let  $V$  be a (real) vector bundle over  $X$  – which is, more precisely, a vector bundle  $V \rightarrow X_0$  together with an isomorphism  $\varphi : s^*V \rightarrow t^*V$  over  $X_1$  such that  $m^*\varphi = p_2^*\varphi \circ p_1^*\varphi$ . Let

$$(\tau) \in H^3(X; \mathbb{Z})$$

be a twist of  $K$ -theory on  $X$ , and let

$$\tau_V = (\dim V, W_3(V)) \in H^0(X; \mathbb{Z}/2) \times H^3(X; \mathbb{Z})$$

be the twist induced from  $V$  on  $X$ . Note that  $W_3 = \beta \circ w_2$  is the third integral Stiefel-Whitney class of  $V$ , where  $\beta$  is the Bockstein for the short exact sequence of coefficients  $0 \rightarrow \mathbb{Z} \rightarrow \mathbb{Z} \rightarrow \mathbb{Z}/2 \rightarrow 0$ . Then there is a tautological Thom isomorphism

$$K^{n+\tau}(B(V), S(V)) \cong K^{n+\tau-\tau_V}(X).$$



**Proposition IV.5.** *Let  $V$  be a real  $G$ -representation (equivalently, a real vector bundle on  $*//G$ ) equipped with a  $G$ -invariant inner-product,  $S^V$  the one point compactification of  $V$ , and  $\tau \in H^3(*//G; \mathbb{Z}) \cong H^3(BG; \mathbb{Z})$  a twist on  $*//G$ , then*

$$\begin{aligned} \tilde{K}_G^{n+\tau}(S^V) &\cong K_G^{n+\tau-\tau_V}(\ast) \\ &\cong \begin{cases} R^{\tau-(w_1(V)+W_3(V))}(G) & \text{if } n - \dim_{\mathbb{R}} V \equiv 0(2) \\ 0 & \text{if } n - \dim_{\mathbb{R}} V \equiv 1(2). \end{cases} \end{aligned}$$

*Proof.* First notice that The  $V$ -sphere  $S^V$  is the Thom space of  $V$  over  $*//G$ . Furthermore,  $S^V$  is equivariantly homeomorphic to  $B(V)/S(V)$ , so we obtain the isomorphisms

$$\tilde{K}_G^{n+\tau}(S^V) \cong K_G^{n+\tau}(B(V), S(V)) \cong K_G^{n+\tau-\tau_V}(\ast),$$

where the second isomorphism follows by the remark above. Now apply the lemma to see that the  $K$ -groups are trivial in every other dimension.  $\square$

*Remark IV.6.* Notice that, as pointed out in [6] in the proof of Lemma 4.1, when  $\pi_1(G)$  is torsion free,  $H^3(BG; \mathbb{Z}) = 0$ , so in this case there aren't any nontrivial twists on  $*//G$ .

As a particular application of this isomorphism, we may use the cofibration sequence

$$G \setminus \{e\} \xrightarrow{i} G \xrightarrow{f} S^{\mathfrak{g}} \simeq Ci$$

to obtain a map

$$R(G) \cong R^{\tau-\tau_{\mathfrak{g}}}(G) \cong \tilde{K}_G^{*+\tau}(S^{\mathfrak{g}}) \xrightarrow{f^*} \tilde{K}_G^{*+\tau}(G) \cong V^{\tau}(G).$$

Claim: This map is reduction modulo the Verlinde ideal.

We would also like to consider a similar situation for possibly disconnected compact Lie groups. In the context of disconnected compact Lie groups, the statement

of the Freed-Hopkins-Teleman isomorphism requires refinement. First, let us state the most general statement of Theorem 2 of [7].

**Theorem IV.7.** *For regular  $\tau$ , there is a natural isomorphism of  $R(G)$ -modules  $R^\tau(LG^s) \cong K_G^{\tau+*}(G)$ , where  $K$ -classes arise by coupling the “Dirac operator family” to admissible  $LG^s$ -modules.*

This is the most canonical statement, involving admissible  $\tau$ -projective representations of the “graded super-group”  $LG^s = LG \ltimes \text{Cliff}(L\mathfrak{g}^*)$ , and in particular clarifies the reason for the adjoint shift  $-\tau_{\mathfrak{g}}$  associated to the spin adjoint representation of  $G$ . More precisely, there is a Morita isomorphism which removes the spinor part of representations of  $LG^s$ .

Second, when  $G$  is a disconnected group, one can define certain “twisted loop groups” which must be taken into account ([7], Section 1.4).

**Definition IV.8.** For any  $f \in G$ , the twisted loop group  $L_f G$  of smooth maps  $\gamma : \mathbb{R} \rightarrow G$  satisfying  $\gamma(t + 2\pi) = f\gamma(t)f^{-1}$  depends, up to isomorphism, only on the conjugacy class in  $\pi_0(G)$  of the component  $fG_1$  of  $f$ . Let  $[fG_1] \subset G$  denote the union of conjugates of  $fG_1$ . Let  $\sigma$  denote the projective cocycle of the intertwining action of  $L_f G^s$  on the lowest weight  $\text{Cliff}(L\mathfrak{g})^*$ -module  $S$ . Let  $d$  be the dimension of the centralizer  $G^f$  of  $f$  in  $G$ .

In this context, there is a Morita isomorphism

$$R^{\tau+*}(L_f G^s) \cong R^{\tau-\sigma+*-d}(L_f G).$$

As a result of this, we have

**Theorem IV.9** (Theorem 3, [7]). *For regular  $\tau$ , there is a natural isomorphism*

$$K_G^{\tau+*}([fG_1]) \cong R^{\tau-\sigma+*-d}(L_f G).$$

## APPENDIX

## APPENDIX A

### Representation Rings of Simple Lie Groups

This appendix reviews the standard characterizations of the representation rings of the various simple Lie groups. Most of this material may be found in [8].

In each of the following four subsections  $L_i$  refers to the basis dual to some natural basis in the Lie algebra of the relevant maximal torus.

#### A.1 Type $A_n$ , $SU_{n+1}$

The weight lattice  $\Pi^*$  for  $A_n$  is

$$\mathbb{Z}\{L_1, \dots, L_{n+1}\} / \left(\sum L_i\right) \subset \mathfrak{t}^*.$$

More specifically, if  $H_i$  is the matrix  $E_{i,i}$  which has a 1 in the  $i$ th row and  $i$ th column, zeros elsewhere, then  $\mathfrak{t}$  is generated as a vector space by  $H_i - H_{i+1}$  (note that  $H_i$  is not on its own an element of  $\mathfrak{t}$ ). Define  $L_i$  by  $L_i(H_j) = \delta_{ij}$ .

A typical weight in the weight lattice for  $A_n$  is  $[(a_1, \dots, a_n, 0)]$ , where the  $a_i$  are integers.

The Weyl planes are  $a_i = a_j$ . There are  $\binom{n+1}{2}$  of them.

The roots are the pairwise differences  $L_i - L_j$  where  $i \neq j$ . There are  $n(n-1)$  of them.

The dominant weights (weights in the Weyl chamber) are given by the representative  $(n + 1)$ -tuples  $[(a_1, \dots, a_n, 0)]$  such that

$$a_1 \geq \dots \geq a_n \geq 0.$$

The regular dominant weights at twisting level  $m$  are given by representative  $(n + 1)$ -tuples  $[(a_1, \dots, a_n, 0)]$  such that

$$m > a_1 > \dots > a_n > 0.$$

There are  $\binom{m-1}{n}$  such weights.

Let  $V$  be the standard  $(n + 1)$ -dimensional representation of  $SU_{n+1}\mathbb{C}$ .

$$\begin{aligned} R(SU_{n+1}) &\cong R(\mathfrak{sl}_{n+1}) \\ &= \mathbb{Z}[A_1, A_2, \dots, A_n] \\ &= \mathbb{Z}[[V], [\Lambda^2 V], \dots, [\Lambda^n V]]. \end{aligned}$$

The multiplicative structure of this ring is determined by examining the character homomorphism (injection)

$$\mathbb{Z}[A_1, A_2, \dots, A_n] \hookrightarrow \mathbb{Z}[\Pi^*]$$

which sends

$$A_k \mapsto e_k(x_1, \dots, x_{n+1})$$

where  $e_k(x_1, \dots, x_{n+1})$  is the  $k$ th elementary symmetric polynomial on the  $(n + 1)$  variables  $x_1, \dots, x_{n+1}$ , and  $x_k = e(L_k)$  in the group algebra  $\mathbb{Z}[\Pi^*]$ .

## A.2 Type $B_n$ , $Spin_{2n+1}$

The weight lattice  $\Pi^*$  for  $B_n$  is

$$(\mathbb{Z}^n \cup (\mathbb{Z} + 1/2)^n) \subset \mathfrak{t}^*.$$

A typical weight in the weight lattice for  $B_n$  is  $(a_1, \dots, a_n)$ , where the  $a_i$  are either all integers or all half integers.

The Weyl planes are  $a_i = \pm a_j$  when  $i \neq j$ , as well as  $a_i = 0$ . There are  $n(n-1) + n = n^2$  of them.

The roots are the pairwise sums  $\pm L_i \pm L_j$  when  $i \neq j$ , as well as  $\pm L_i$ . There are  $2n(n-1) + 2n = 2n^2$  of them.

The dominant weights (weights in the Weyl chamber) are given by the  $n$ -tuples  $(a_1, \dots, a_n)$  such that

$$a_1 \geq \dots \geq a_n \geq 0.$$

The regular dominant weights at twisting level  $m$  are given by  $n$ -tuples  $(a_1, \dots, a_n)$  such that

$$a_1 > \dots > a_n > 0 \text{ and } m > a_1 + a_2.$$

The number of regular dominant weights at level  $m$  in  $B_n$ ,  $b(m, n)$ , is given by

$$b(m, n) = \begin{cases} \binom{\frac{m+1}{2}}{n} + 3\binom{\frac{m-1}{2}}{n} & \text{if } m \text{ odd} \\ 3\binom{\frac{m}{2}}{n} + \binom{\frac{m-2}{2}}{n} & \text{if } m \text{ even.} \end{cases}$$

This is only valid for  $n > 1$ .

Let  $V$  be the standard  $(2n+1)$ -dimensional representation of  $Spin_{2n+1}\mathbb{C}$ . Let  $S$  be the spin representation.

$$\begin{aligned} R(\text{Spin}_{2n+1}) &\cong R(\mathfrak{so}_{2n+1}) \\ &= \mathbb{Z}[B_1, B_2, \dots, B_{n-1}, B] \\ &= \mathbb{Z}[[V], [\Lambda^2 V], \dots, [\Lambda^{n-1} V], [S]]. \end{aligned}$$

The multiplicative structure of this ring is determined by examining the character homomorphism (injection)

$$\mathbb{Z}[B_1, B_2, \dots, B_{n-1}, B] \hookrightarrow \mathbb{Z}[\Pi^*]$$

which sends

$$B_k \mapsto e_k(x_1, x_1^{-1}, \dots, x_n, x_n^{-1}, 1)$$

where  $e_k(x_1, x_1^{-1}, \dots, x_n, x_n^{-1}, 1)$  is the  $k$ th elementary symmetric polynomial on the  $(2n + 1)$  variables  $x_1, x_1^{-1}, \dots, x_n, x_n^{-1}, 1$ . We also send

$$B \mapsto \sum x_1^{\pm 1/2} \dots x_n^{\pm 1/2} = \prod_{i=1}^n (x_i^{1/2} + x_i^{-1/2}).$$

### A.3 Type $C_n$ , $Sp_{2n}$

The weight lattice  $\Pi^*$  for  $C_n$  is

$$\frac{1}{\sqrt{2}}\mathbb{Z}^n \subset \mathfrak{t}^*.$$

A typical weight in the weight lattice for  $C_n$  is  $(a_1, \dots, a_n)$ , where the  $a_i$  are all integers. The extra factor of  $\frac{1}{\sqrt{2}}$  is present to account for a factor of  $\frac{1}{2}$  we need to appear in the inner product.

The Weyl planes are  $a_i = \pm a_j$  when  $i \neq j$ , as well as  $a_i = 0$ . Like for  $B_n$ , there are  $n(n - 1) + n = n^2$  of them.

The roots are the pairwise sums  $\pm L_i \pm L_j$  when  $i \neq j$ , as well as  $\pm 2L_i$ . There are  $2n(n - 1) + 2n = 2n^2$  of them.

The dominant weights (weights in the Weyl chamber) are given by  $n$ -tuples  $(a_1, \dots, a_n)$  such that

$$a_1 \geq \dots \geq a_n \geq 0.$$

The regular dominant weights at twisting level  $m$  are given by  $n$ -tuples  $(a_1, \dots, a_n)$  such that

$$m > a_1 > \dots > a_n > 0.$$

There are  $\binom{m-1}{n}$  such weights.

Let  $V$  be the standard  $2n$ -dimensional representation of  $Sp_{2n}\mathbb{C}$ . Let  $V^{(k)} = \text{Ker}(\Lambda^k V \rightarrow \Lambda^{k-2} V)$  be the kernel of the  $i$ th contraction map.

$$\begin{aligned} R(\text{Sp}_{2n}) &\cong R(\mathfrak{sp}_{2n}) \\ &= \mathbb{Z}[C_1, C_2, \dots, C_n] \\ &= \mathbb{Z}[C_1, C_2 - 1, \dots, C_n - C_{n-2}] \\ &= \mathbb{Z}[[V^{(1)}], [V^{(2)}], \dots, [V^{(n)}]]. \end{aligned}$$

The multiplicative structure of this ring is determined by examining the character homomorphism (injection)

$$\mathbb{Z}[C_1, C_2, \dots, C_n] \hookrightarrow \mathbb{Z}[\Pi^*]$$

which sends

$$C_k \mapsto e_k(x_1, x_1^{-1}, \dots, x_n, x_n^{-1})$$

where  $e_k(x_1, x_1^{-1}, \dots, x_n, x_n^{-1})$  is the  $k$ th elementary symmetric polynomial on the  $(2n)$  variables  $x_1, x_1^{-1}, \dots, x_n, x_n^{-1}$ .

#### A.4 Type $D_n$ , $Spin_{2n}$

The weight lattice  $\Pi^*$  for  $D_n$  is

$$(\mathbb{Z}^n \cup (\mathbb{Z} + 1/2)^n) \subset \mathfrak{t}^*.$$



A typical weight in the weight lattice for  $D_n$  is  $(a_1, \dots, a_n)$ , where the  $a_i$  are either all integers or all half integers.

The Weyl planes are  $a_i = \pm a_j$  when  $i \neq j$ . There are  $n(n-1)$  of them.

The roots are the pairwise sums  $\pm L_i \pm L_j$  when  $i \neq j$ . There are  $2n(n-1)$  of them.

The dominant weights (weights in the Weyl chamber) are given by  $n$ -tuples  $(a_1, \dots, a_n)$  such that

$$a_1 \geq \dots \geq a_{n-1} \geq |a_n|.$$

The regular dominant weights at twisting level  $m$  are given by  $n$ -tuples  $(a_1, \dots, a_n)$  such that

$$a_1 > \dots > a_{n-1} > a_n > -a_{n-1} \text{ and } m > a_1 + a_2.$$

The number of regular dominant weights at level  $m$  in  $D_n$ ,  $d(m, n)$ , is given by

$$d(m, n) = 2b(m, n) + b(m, n-1)$$

which we may also write as

$$d(m, n) = \begin{cases} \binom{\frac{m+3}{2}}{n} + 4\binom{\frac{m+1}{2}}{n} + 3\binom{\frac{m-1}{2}}{n} & \text{if } m \text{ odd} \\ 3\binom{\frac{m+2}{2}}{n} + 4\binom{\frac{m}{2}}{n} + \binom{\frac{m-2}{2}}{n} & \text{if } m \text{ even.} \end{cases}$$

This is only valid for  $n > 2$ .

Let  $V$  be the standard  $2n$ -dimensional representation of  $Spin_{2n}\mathbb{C}$ . Let  $S^+$  and  $S^-$  be the two half-spin representations.

$$\begin{aligned} R(Spin_{2n}) &\cong R(\mathfrak{so}_{2n}) \\ &= \mathbb{Z}[D_1, D_2, \dots, D_{n-2}, D^+, D^-] \\ &= \mathbb{Z}[[V], [\Lambda^2 V], \dots, [\Lambda^{n-2} V], [S^+], [S^-]]. \end{aligned}$$

The multiplicative structure of this ring is determined by examining the character homomorphism (injection)

$$\mathbb{Z}[D_1, D_2, \dots, D_{n-2}, D^+, D^-] \hookrightarrow \mathbb{Z}[\Pi^*]$$

which sends

$$D_k \mapsto e_k(x_1, x_1^{-1}, \dots, x_n, x_n^{-1})$$

where  $e_k(x_1, x_1^{-1}, \dots, x_n, x_n^{-1})$  is the  $k$ th elementary symmetric polynomial on the  $2n$  variables  $x_1, x_1^{-1}, \dots, x_n, x_n^{-1}$ . We furthermore send

$$D^\pm \mapsto \sum x_1^{\pm 1/2} \dots x_n^{\pm 1/2}$$

where the number of plus signs chosen is even or odd according to whether we're considering  $D^+$  or  $D^-$ .

## A.5 $G_2$

The weight lattice  $\Pi^*$  for  $G_2$  is

$$\sqrt{2/3}(\mathbb{Z}\{L_1, L_2, L_3\}/(L_1 + L_2 + L_3)).$$

In other words this weight lattice is, up to a choice of scalar, the same as for  $A_2$ .

A typical weight in the weight lattice for  $G_2$  is ( $\sqrt{2/3}$  times)  $[(a_1, a_2, 0)]$ , where the  $a_i$  are integers.

The Weyl planes are  $a_i = a_j$  and  $a_i = 2a_j$ . There are  $2\binom{3}{2} = 6$  of them.

The roots are  $\pm\{L_1, L_2 - L_1, L_2, L_2 + L_1, L_2 + 2L_1, 2L_2 + L_1\}$ . There are 12 of them.

The dominant weights are given by representative 3-tuples  $[(a_1, a_2, 0)]$  such that  $2a_1 \geq a_2 \geq a_1 \geq 0$ .

The regular dominant weights at twisting level  $m$  in  $G_2$  are given by representative 3-tuples  $[(a_1, a_2, 0)]$  such that  $2a_1 > a_2 > a_1 > 0$  and  $m > a_2$ . The number of regular dominant weights at level  $m$  in  $G_2$ ,  $g(m, 2)$ , is given by

$$g(m, 2) = \begin{cases} \frac{(m-1)(m-3)}{4} & \text{if } m \text{ odd} \\ \frac{(m-2)^2}{4} & \text{if } m \text{ even.} \end{cases}$$

$$R(G_2) \cong R(\mathfrak{g}_2) = \mathbb{Z}[X, Y].$$

The multiplicative structure of this ring is determined by examining the character homomorphism (injection)

$$\mathbb{Z}[X, Y] \hookrightarrow \mathbb{Z}[x, x^{-1}, y, y^{-1}]$$

which sends

$$\begin{aligned} X &\mapsto 1 + (x + x^{-1}) + (yx^{-1} + xy^{-1}) + (yx^{-2} + x^2y^{-1}) \\ Y &\mapsto 2 + (x + x^{-1}) + (yx^{-1} + xy^{-1}) + (yx^{-2} + x^2y^{-1}) \\ &\quad + (y + y^{-1}) + (yx^{-3} + x^3y^{-1}) + (y^2x^{-3} + x^3y^{-2}). \end{aligned}$$

## A.6 $F_4$

The weight lattice  $\Pi^*$  for  $F_4$  is

$$\mathbb{Z}^4 \cup \left(\mathbb{Z} + \frac{1}{2}\right)^4.$$

In other words this weight lattice is the same as for  $B_4$ .

A typical weight in the weight lattice for  $F_4$  is a 4-tuple  $(a_1, a_2, a_3, a_4)$  where the  $a_i$  are all integers or half-integers.

The Weyl planes are  $a_i = 0$ ,  $a_i = \pm a_j$ , and  $a_1 \pm a_2 \pm a_3 \pm a_4 = 0$ . There are  $4 + 6 + 8 = 18$  of them.

The roots are the roots of  $B_4$  (the pairwise sums  $\pm L_i \pm L_j$  when  $i \neq j$ , as well as  $\pm L_i$ ) along with  $(\pm L_1 \pm L_2 \pm L_3 \pm L_4)/2$ . There are 48 of them.

The dominant weights are given by 4-tuples  $(a_1, a_2, a_3, a_4)$  such that  $a_1 \geq a_2 \geq a_3 \geq a_4 \geq 0$  and  $a_1 \geq a_2 + a_3 + a_4$ .

The regular dominant weights at twisting level  $m$  in  $F_4$  are given by 4-tuples  $(a_1, a_2, a_3, a_4)$  such that  $a_1 > a_2 > a_3 > a_4 > 0$ ,  $a_1 > a_2 + a_3 + a_4$ , and  $m > a_1 + a_2$ .

## A.7 The $E$ series

### A.7.1 $E_8$

The weight lattice  $\Pi^*$  for  $E_8$  can be given as

$$\begin{aligned} \Pi^*(E_8) &\cong \{v \in \mathbb{Z}^8 \cup (\mathbb{Z} + 1/2)^8 \mid \sum v_i \equiv 0 \pmod{2}\} \\ &= \text{rowspan} \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 1/2 & 1/2 & 1/2 & 1/2 & 1/2 & 1/2 & 1/2 & 1/2 \end{bmatrix} \end{aligned}$$

A typical weight in the weight lattice for  $E_8$  is  $(a_1, \dots, a_8)$ , where the  $a_i$  are either all integers or all half integers and the sum of the  $a_i$  is even.

The Weyl planes are ...

The roots are the pairwise sums  $\pm L_i \pm L_j$  when  $i \neq j$ , along with the  $(\pm L_1 \pm$

$\dots \pm L_8)/2$  such that the number of minuses is even.

The dominant weights are given by 8-tuples  $(a_1, \dots, a_8)$  such that  $a_7 \geq \dots \geq a_2 \geq |a_1|$  and  $a_8 \geq a_2 + \dots + a_7 - a_1$ .

The regular dominant weights at twisting level  $m$  are given by 8-tuples  $(a_1, \dots, a_8)$  such that  $a_7 > \dots > a_2 > |a_1|$ ,  $a_8 > a_2 + \dots + a_7 - a_1$ , and  $m > a_7 + a_8$ .

### A.7.2 $E_7$

The weight lattice  $\Pi^*$  for  $E_7$  can be described as a subset of

$$M = \text{frac}1\sqrt{2}\mathbb{Z} \times (\mathbb{Z}^6 \cup (\mathbb{Z} + \frac{1}{2})^6)$$

by

$$\begin{aligned} \Pi^*(E_7) &\cong \{v \in M \mid \sqrt{2}a_1 \equiv (a_2 - a_7) + \dots + (a_6 - a_7)(2)\} \\ &= \{v \in M \mid \sqrt{2}a_1 + a_2 + \dots + a_6 - a_7 \equiv 0(2)\} \\ &= \text{rowspan} \begin{bmatrix} 2/\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 1/\sqrt{2} & 1 & 0 & 0 & 0 & 0 & 0 \\ 1/\sqrt{2} & 0 & 1 & 0 & 0 & 0 & 0 \\ 1/\sqrt{2} & 0 & 0 & 1 & 0 & 0 & 0 \\ 1/\sqrt{2} & 0 & 0 & 0 & 1 & 0 & 0 \\ 1/\sqrt{2} & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1/2 & 1/2 & 1/2 & 1/2 & 1/2 & 1/2 \end{bmatrix} \end{aligned}$$

The second expression for  $\Pi^*(E_7)$  is just the projection of  $\Pi^*(E_8)$  to the hyperplane orthogonal to the vector  $L_7 + L_8$ .

The Weyl planes are ...

The roots are the pairwise sums  $\pm L_i \pm L_j$  for  $i \neq j \leq 6$ ,  $\pm\sqrt{2}L_7$ , and

$$\pm\{\pm L_1 \pm \dots \pm L_6 + \sqrt{2}L_7)/2 \mid \text{the number of minus signs is odd}\}.$$

The dominant weights are given by 7-tuples  $(a_1, \dots, a_7)$  such that  $a_6 \geq \dots \geq a_2 \geq |a_1|$  and  $\sqrt{2}a_7 \geq a_6 + \dots + a_2 - a_1$ .

The regular dominant weights at twisting level  $m$  are given by 7-tuples  $(a_1, \dots, a_7)$  such that  $a_6 > \dots > a_2 > |a_1|$  and  $m > \sqrt{2}a_7 > a_2 + \dots + a_6 - a_1$ .

### A.7.3 $E_6$

The weight lattice  $\Pi^*$  for  $E_6$  can be described as a subset of

$$M = \left(\frac{1}{\sqrt{3}}\mathbb{Z} \times \mathbb{Z}^5\right) \cup \left(\frac{1}{\sqrt{3}}\left(\mathbb{Z} + \frac{1}{2}\right) \times \left(\mathbb{Z} + \frac{1}{2}\right)^5\right)$$

by

$$\begin{aligned} \Pi^*(E_6) &\cong \{v \in M \mid \sqrt{3}a_1 - 3a_6 \equiv (a_2 - a_6) + \dots + (a_5 - a_6)(2)\} \\ &= \{v \in M \mid \sqrt{3}a_1 + a_2 + \dots + a_6 \equiv 0(2)\} \\ &= \text{rowspan} \begin{bmatrix} 2/\sqrt{3} & 0 & 0 & 0 & 0 & 0 \\ 1/\sqrt{3} & 1 & 0 & 0 & 0 & 0 \\ 1/\sqrt{3} & 0 & 1 & 0 & 0 & 0 \\ 1/\sqrt{3} & 0 & 0 & 1 & 0 & 0 \\ 1/\sqrt{3} & 0 & 0 & 0 & 1 & 0 \\ \sqrt{3}/2 & 1/2 & 1/2 & 1/2 & 1/2 & 1/2 \end{bmatrix} \end{aligned}$$

The second expression for  $\Pi^*(E_6)$  is just the projection of  $\Pi^*(E_8)$  to the 6-plane orthogonal to the vector  $L_7 - L_8$  and  $L_6 - L_7$ .

The Weyl planes are ...

The roots are the pairwise sums  $\pm L_i \pm L_j$  for  $i \neq j \leq 5$ ,  $\pm\sqrt{2}L_7$ , and the  $(\pm L_1 \pm \dots \pm L_6 \pm \sqrt{3}L_6)/2$  such that the number of minuses is even.

The dominant weights are given by 6-tuples  $(a_1, \dots, a_6)$  such that  $a_5 \geq \dots \geq a_2 \geq |a_1|$  and  $\sqrt{3}a_6 \geq a_5 + \dots + a_2 - a_1$ .

The regular dominant weights at twisting level  $m$  are given by 7-tuples  $(a_1, \dots, a_6)$  such that  $a_5 > \dots > a_2 > |a_1|$ ,  $\sqrt{3}a_6 > a_2 + \dots + a_6 - a_1$ , and  $m > (a_1 + \dots + a_5 + \sqrt{3}a_6)/2$ .

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