On the instructional triangle and sources of justification for actions in mathematics teaching

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Abstract
We elaborate on the notion of the instructional triangle, to address the question of how the nature of instructional activity can help justify actions in mathematics teaching. We propose a practical rationality of mathematics teaching composed of norms for the relationships between elements of the instructional system and obligations that a person in the position of the mathematics teacher needs to satisfy. We propose such constructs as articulations of a rationality that can help explain the instructional actions a teacher takes in promoting and recognizing learning, supporting work, and making decisions.

1. Introduction

Concerns about the students' learning from mathematics instruction have brought researchers' attention to the quality of teaching. Efforts to assess the quality of teaching have usually been conceived in terms of assessing the quality of individual teachers. In particular much research in mathematics education has examined teachers' personal beliefs or their knowledge of various kinds. That line of work has continued to conceive of actions in teaching as individual expressions (Skott, 2009). In parallel, international comparisons of mathematics teaching have used evidence to argue that teaching is a cultural activity (Stigler& Hiebert, 1999), and in particular, that there are cultural scripts that call teachers to make moves that fit into those scripts. Efforts to evaluate and improve teaching within a particular culture, however, need to be framed within considerations that include the notion of individual choice as well as the need for individual teachers to adapt to the workings of a complex system of interrelated agents. Thus, value distinctions in teaching will be possible to make but in ways that acknowledge systemic, culturally grounded, definitions of what is appropriate. In an attempt to complement those individual-centered perspectives, we present our attempt to build a theoretical perspective that considers teaching as an activity system involving positions, roles, and relationships, where individual choice is possible but not cost-free. Buchmann (1987, p. 529) speaks about the need for justification in teaching so as to ensure that teaching will pass muster, since "what teachers do is neither natural nor necessary but based on choice." In describing the kind of justification needed, Buchmann adds that, "personal reasons can be appropriate when explaining a given action to others, but they carry less weight in considering the wisdom of an action or decision." And "when one wants to understand why someone did something, one wants to know what actually motivated him or her; but if one wants to know whether what was done was right, one wants to hear and assess justifications. Here it is important that the reasons be

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good reasons, and it becomes less important whether they were operating at the time" (p. 529).

We identify systemic sources of justification for actions by teachers. Our approach is descriptive, not prescriptive: We attempt to capture systemic sources of justification that are compelling for practicing professionals, not necessarily argue for sources that ought to be so. Yet we do so with a progressive goal in mind, consonant with Buchmann's approach: Teaching needs to pass muster rather than "harden into custom or dissipate into whim" (p. 529). It may be useful to clarify, however, that the justifications that we envision are not necessarily ones that a teacher would naturally use when teaching: The practice of teaching rarely contains occasions in which a teacher must justify him or herself. Yet actions by teachers could be explained rationally by creating accounts that capture, from the researcher's perspective (Simon & Tzur, 1999) the givens and possibilities available to them as attested, among other things, by what they perceive and value in narratives about their action. An approach to develop justifications based on positions, roles, and relationships in activity systems could eventually lead to the development of assessments of the quality of specific actions; assessments that appraise and seek to improve teach*ing* actions in context rather than judge or sanction the individual teach*er* as a whole. The relational conception of teaching offered by the so-called "instructional triangle" has been foundational in this perspective.

2. The instructional triangle in earlier scholarship

The expression "instructional triangle" has become popular to name a diagram that Cohen, Raudenbush, and Ball (2003) included with their definition of instruction. Yet, triadic conceptions of the teaching and learning of subject matter have been proposed before. For example, in his review of research on the teaching of secondary mathematics, Henderson (1963) provided a triadic conception of teaching that enabled him to distinguish people from the roles they play, and to conceive of relationships between teacher and students and between students and content to describe the kind of research that was being done at the time. The need for a triad that put the subject-matter in relation with the teacher and the student is also found in Hawkins (1967/1974), who argued for the importance of subject matter in the classroom as the key in enabling the teacher-student relationship.

2.1. The instructional triangle in recent American scholarship on instruction and policy

Writing for an audience of educational policy researchers, Cohen et al. (2003) proposed a view of instruction that takes distance from a preexisting conception that, in earlier policy research, had been used to account for students' learning: Instruction had then been seen as the set of resources (including materials, teacher quality, student characteristics, and so forth) of the environment within which students would learn. Cohen et al. (2003) propose an alternative conception of instruction: "interactions among teachers and students around content, in environments" (p. 122). This definition allowed them to conjecture that what accounts for differences in learning is the *use* of resources (through the practice of instruction so defined), rather than just the *having* of resources.

Cohen et al. (2003) assert not only that there are three elements at the vertices of the instructional triangle, but also that there are relationships among the three elements and that those elements and relationships are situated in environments. The idea is actually foreshadowed in Lampert's (2001) analysis of teaching through problems. Lampert characterizes practices that realize those interactions and describes the complexities attached to the elements at the vertices of the triangle (p. 445).

2.2. The didactic system in French *didactique* of mathematics

Writing against a different backdrop, Chevallard (1991/1980) had proposed quite congenial ideas even if not using the word *instruction*. In an attempt to specify the object of study of *didactique* of mathematics as different from the object of study of *pedagogie*, Chevallard (1991, p. 14, our translation) noted, "The didacticien of mathematics is interested in the play that unfolds… among a teacher, some students, and an element of mathematical knowledge. Three places then: that's the didactical system." Chevallard's goal in presenting that definition was to lay the foundation for the study of didactical transposition—the transformations that happen to mathematical knowledge in the process of being taught. To establish a reference between the knowledge taught and other versions of this knowledge, Chevallard brings in a notion comparable to that of *environments* in Cohen et al.'s (2003): "The environment of a didactical system is first constituted by the educational system that gathers the set of didactical systems and includes a diverse set of structural devices … for example, official and unofficial means of regulation of the flow of students between didactical systems" (Chevallard, 1991, p. 23, 24) among which Chevallard includes, at the minimum, parents, mathematicians, and politicians involved in educational debates and decisions.

To the notion of a didactical system, situated in an educational system, Brousseau (1997) has ascribed the property of having a didactical contract—a tacit set of responsibilities that bind teacher, student, and content to each other and to the background environment. As Brousseau (1997) describes the didactical contract: "The teacher is supposed to create sufficient conditions for the appropriation of knowledge and must 'recognize' this appropriation when it occurs. The student is supposed to be able to satisfy these conditions. The didactical relationship must 'continue' at all costs. The teacher therefore assumes that earlier learning and the new conditions provide the student with the possibility of new learning" (p. 32). Thus being a teacher of a particular domain of mathematics (e.g., geometry) to high school students is a role that ties the teacher contractually to teach geometry and to teach their students. Those roles are played by agents that are expected to satisfy environmental conditions: The "geometry" that enters those relationships is realized by elements of knowledge that are expected to bear some resemblance with concepts in the mathematical domain of Geometry; likewise "students" are realized by adolescents that satisfy conditions of the background environment (e.g., they must meet institutional requisites to take that class).

In this paper, we elaborate on how we have built on those earlier contributions to the instructional triangle to develop an approach to the study of mathematics teaching and of the justifications for teaching actions.

3. Instructional Systems

We are interested in articulating a theory of the rationality behind the actions of mathematics teachers. This rationality is expected to lay the grounds for justification of teaching actions as teachers manage interactions with students and knowledge in classrooms. Our intent is to complement individual-centered accounts based on teacher knowledge or beliefs. Could the justifications of what teachers do with content and students be grounded on characteristics of their activity and the environments that warrant and make possible that activity? The notion of instructional system (or didactical system) has been useful for us to build a basis for that examination of the grounds on which teachers' actions can be justified, which we refer to with the expression *practical rationality* (Herbst & Chazan, 2003, 2011). In the following we describe how we conceive of an instructional system and the rationality that can be drawn upon to justify or critique how teachers play their role in an instructional system.

3.1. The Roles Involved in an Instructional System

The notions of *role*, *relationships*, and *environments* are central to our conceptualization of instructional systems: An instructional system is a system of relationships between three roles and those roles and relationships are warranted by environmental constraints.[2] A course of studies in a given mathematics domain, for a particular class of students, deployed over a period of time instantiates an instructional system.

The relationships that constitute an instructional system are established by a rule that one could think of as the most basic element of the didactical contract binding the agents that will eventually play those roles: This rule can be expressed by saying that the teacher helps the student study the content. That simple formulation calls for a number of specifications that concern how those relationships will be enacted. It calls attention to the work that the student will do with knowledge—studying—and to what the teacher will do with the former relationship—aid in this study. The particular manner in which actual people will enact those basic relationships can vary, but their roles need to be constructed in such a way that teacher, students, and content can realize that basic rule. The following statements propose a set of minimal descriptors of those roles that can enable teacher, student, and content so conceptualized to enter into the relationship described by that basic rule.

The content or knowledge[3] at stake needs to be amenable to study—needs to be something that can be deployed in a scope of work that the student can be asked to do and can be represented as a cultural item, for example by way of a label (e.g., Pythagorean theorem). Also, the student, has to be someone who can have different kinds of relationships to the knowledge at stake at different moments in time (e.g., not to know it

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[2] We use *constraint* in the sense of bind or dependence, and without assuming that constraints are necessarily impediments. They can impede some things and facilitate others, just like the rules of a game do.

[3] The content of studies is not the same as disciplinary knowledge, but the result of the process of didactic transposition (Chevallard, 1991) or alchemy of school subjects (Popkewitz, 2004). The two characteristics listed here are minimal in order to develop the instructional triangle so as to account for the work of the teacher. In an analysis of the knowledge itself, it would be desirable to make finer distinctions among these various versions of knowledge.

once, to know it later), and can engage in a scope of work in which his or her relationship to the knowledge at stake might account for performance differences. Finally, the teacher needs to be someone who can organize and manage the engagement of the student with a scope of work such as noted above and assess the correspondence between the work (to be) done and itemized representations of knowledge. Those statements define the roles in terms of the relationships they need to be able to entertain with other roles in the system. However, the notion that instructional systems exist inside environments places more requirements on who (or what) can take up those roles.

3.2. How Environments Shape the Various Roles at Play in Instruction

Cohen et al. (2003) provide examples of what those environments can be: incentive policies or pedagogical practices subscribed by the school administration, parents' views of the curriculum, and so on. Chevallard (1991) and Lampert (2001) also articulate how instructional systems are subject to various kinds of environmental determinations. In the following we provide a way to think about how environments matter in the filling of roles in the instructional triangle.

3.2.1. Knowledge

In addition to the stipulations about its amenability to being studied described above, the determination of the knowledge at stake in instruction is also under the scrutiny of the larger socio-cultural environment to which schools contribute. The academic disciplines that lay claim on mathematics (as cultural artifact or cultural practice) provide means for that scrutiny. The disciplines that create or use mathematics have a vested interested in the mathematics studied in school, as dissemination of their knowledge or practices or as instrument for the development or selection of human resources. They thus play an important role in sanctioning what gets to be the knowledge at stake in instruction.

The school institutions where instructional systems are realized also play a role in determining what knowledge is at stake. This can be perceived mostly at the program level, as the knowledge at stake is deployed and packaged into programs of study. Finally, the techno-literacy demands of social life and the workplace are independent instances for identifying the knowledge at stake.

3.2.2. Students

There are environmental constraints on who can take the role of the student in an instructional system. It matters whether the given instructional system exists in a voluntary, customizable transactional environment (e.g., tutoring an individual student in calculus), whether it belongs in a loosely organized system of vocational offerings (e.g., a mathematics club in a manifold of afterschool offerings), or is part of an educational system (e.g., a high school geometry class taught within compulsory schooling). In probably all of those cases, there are biological and psychological considerations that mediate who can be a student: Are the individuals involved able to function independently or is it otherwise possible to support the accommodations that they require In the case of instructional systems within school programs, individuals who take the role of students are also subject to institutional determinations such as prerequisite experiences (that often entitle the teacher of a course to make some assumptions about students' prior knowledge that mediate how he or she interprets individual students' performance).

3.2.3. Teacher

There are environmental constraints on who can take the role of teacher in an instructional system. These constraints may vary depending on which environment calls for the instructional system. A customizable system like one-on-one tutoring may only call for somebody knowledgeable in the subject of studies. Instructional systems in school programs make more demands of individual teachers. These demands can be grouped into four categories. Mathematics teachers are expected to know and uphold the intellectual values of the mathematics discipline they teach. Teachers are expected to model and uphold social skills such as fairness and respect. Teachers are expected to care for their students as individuals. And teachers are expected to meet professional requirements and support policies and practices issued from a number of institutions including department, school, district, and professional association. We come back to these environmental demands below when we describe these as professional obligations.

In developing a theory of the practical rationality of mathematics teaching we seek to identify various sources of justification for instructional actions from the perspective of the mathematics teacher. The foregoing notion of instruction as interaction among roles subject to environmental demands suggests two sources of regulation. The first of these sources derives from the *relationships* that a person playing the teacher *role* needs to entertain with the knowledge at stake and the student in a given instructional system. The second of those sources derives from the *position* from which a person can take the *role* of teacher in a given instructional system and the obligations this person has to the various stakeholders of the environment where that instructional system exists. Below, we propose that those two sources of regulation, considered across instructional systems, articulate the practical rationality that can justify or rebuke instructional actions.

4. The Nature of Instructional Activity as a Source of Justifications for Teaching Actions

This section elaborates on the idea that the norms of the didactical contract are a source of justification for instructional actions. We decompose the responsibilities of the teacher in the didactical contract in terms of smaller realms of teaching practice and identify specific kinds of justification elements in each of those realms.

4.1. The Didactical Contract

The hypothesis that a didactical contract exists stresses the notion that the relationships that constitute an instructional system do not just emerge or develop out of the spontaneous gathering of individuals exercising their free will. The teacher, the student, and the mathematics to be studied are kept together by virtue of an implicit contract that identifies the content as something that exists outside the student and that the student must come to know with the assistance of the teacher. The contract makes the teacher responsible for the students' acquisition of the knowledge at stake, and makes the students responsible for engaging in the activities that the teacher organizes for their learning of that knowledge, including activities in which students demonstrate that they have learned that knowledge.

Different contracts may exist that purport to meet the responsibilities listed above in different ways. Each of those contracts could be described by listing the norms[4] that teacher and students are expected to follow as they play out their roles. For example, many contracts for the teaching and learning of algebra contain as a norm that the teacher can expect students to always provide a traceable written record of the method by which they solved each exercise. It is conceivable that a contract for algebra might have a different norm instead (e.g., the teacher can expect students to describe how they solved each exercise if asked). We conjecture that while those particular contracts may be very different from each other in terms of the specific norms on which they depend, they preserve the basic role-relationships described by the hypothesis that a contract exists.

One basic way to describe the practice of teaching is to say that when individuals play the role of teacher in an instructional system, they engage in actions that satisfy or breach the norms of the didactical contract in place for that system. Thus, the norms of a contract can be a source of justification for actions in teaching. The 'show your work' norm exemplified above would justify why a teacher might not award full credit for a correct solution to a problem if the student's written response omitted some steps.

The practice of teaching is not determined by the norms of a contract. Rather, as a person plays the teacher role, some of their actions may be justifiable by reference to contractual norms but other actions may be unaccounted for by norms of the contract. When people take on the role of the teacher, they comply with or breach those rules, but they do so in a particular manner, expressing their individual skill or personality. They also may carry out actions that are unaccounted for by the norms of the contract: actions that are not called for, and are yet unremarkable, from a contractual stance; actions more likely to be justified by recourse to the teacher's individuality.

At the same time, a contractual norm may provide justification for actions that, at a finer grain of analysis, might be deemed substantially different. That is, contractual norms are fairly coarse in grain size—in particular, while they may be adapted to regulate the teaching and studying of a domain of mathematics with norms such as "in algebra you have to show your work," these norms don't differentiate actions that concern the teaching and studying of different items of knowledge at stake. As an example, and apropos of the contractual norm that entitles the teacher to assign problems to students, consider the possibility that an algebra teacher assigned an exercise such as (1) solve for x, $x^2 = 2^x$ or (2) solve for x, $5x - 1 = 2x + 5$ (Chazan, Yerushalmy, & Leikin, 2008). Each of those assignments would be contractually legitimate in that the teacher would be using their prerogative to assign work to students and both assignments concern objects of study in algebra. Yet, we predict that a teacher of algebra would be much more likely to ask students to do a problem like (2) than a problem like (1); indeed we surmise that experienced teachers of algebra would judge problem (1) as inappropriate. What would

[4] By *norm* we mean statements of behaviors that are unmarked or unremarked upon when participants do them but that call for elaborations or repairs by participants when those behaviors are missing. The word norm has thus an objective sense, as most frequent behavior in a recurrent social encounter, and a subjective sense, as behavior expected by actors of a recurrent social encounter. Herbst, Nachlieli, and Chazan (2011) show how norms can be empirically confirmed using an adaptation of the ethnomethodological practice of breaching experiments (Garfinkel & Sacks, 1970).

justify a teacher's resistance to assign a problem like (1)? The notion of instructional situation as a system of norms adapted to regulate the exchange of work on particular kinds of tasks for particular objects of knowledge (Herbst, 2006) is our proposal for addressing such differences. The next two sections build the ideas needed to introduce instructional situations and their norms.

4.2. Instructional Exchanges

To account for the proposition that giving one 'solve for *x*' problem might seem more justifiable for an algebra teacher than giving another, even though both acts comply with the same general norm of the didactical contract (e.g., the teacher has the right to assign problems), we further complicate the notion of norm. In the way instructional system was defined above, a teacher has the responsibility to organize and sustain actitivies for students in which the work that students do can be described by appealing to a culturally current representation of the knowledge at stake. For example, the students' work operating algebraically on both sides of the equation on a problem such as "solve for x, $5x - 1 = 2x + 5$" could be described as *solving equations in one variable*. We use the expression *instructional exchanges* to refer to those interpretive acts that the teacher needs to do--labeling a sequence of actions with the name of an item of knowledge or, conversely, deploying an item of knowledge as a scope of work. In the manner we've defined an instructional system, the teacher is (contractually) responsible to manage those exchanges—it is the teacher who has to account for students mathematical work in terms of items of knowledge at stake.

Instructional exchanges involve unequal terms, unequal representations of knowledge. One of them is the mathematical work done on a particular task or series of tasks while the other is an object of study that might ordinarily be represented with special names or statements for concepts, procedures, or theorems involved. An instructional exchange is, therefore, not a simple thing for a teacher to manage. It requires that a teacher engages in: (1) Deploying mathematical objects of study in the form of work for students to do and (2) Interpreting work (being) done by students in light of a mathematical object of study (we refer to this as *cashing*). The terms of this exchange exist in different timescales (Lemke, 2000): While the work that students do exists and can change at the scale of the fraction of second (every utterance may constitute a move in that work), the elements of the knowledge at stake exist in a larger timescale, say that of the year or semester in which the course is offered. Their differences are not only temporal but semiotic as well. The work that students do is multimodal: it involves strokes on paper, utterances, silences, gestures, physical movements, etc. Many of those moves do not have conventional names, let alone mathematical names—some of them may not even be noticed if done outside of the sequence of actions where they ordinarily appear. On the other hand the knowledge at stake in an instructional exchange (e.g., solving equations in one variable) is, in general, stated through a combination of elements of the mathematical register (using language, symbols, and sometimes diagrams) all of which have some degree of stability and status in adult mathematical discourse. All knowledge at stake can be stated explicitly in some way, yet its transaction, in exchange for work done, requires the teacher to engage in serious interpretation of much more opaque signs in the realm of the students' work.

Building on Brousseau (1997), we have proposed (Herbst, 2003, 2006) two fundamental ways in which teachers make that exchange of unequal terms. One of those is *negotiation of task* (this is short for negotiation of how the didactical contract applies to the task) and it describes what a teacher needs to do to handle "novel" tasks—tasks that are new to students. Apropos of these tasks the teacher needs to engage students in identifying, perhaps deciding upon, how the didactical contract will apply to the task at hand. In particular, the negotiation may initially include identifying what the students might need to do and what they might expect the teacher to do. The negotiation may later include figuring out what aspects of the work point to the knowledge at stake.

The other way in which that exchange is facilitated is by *default to an instructional situation*, namely by framing the exchange according to norms that have framed other exchanges (possibly set up before through negotiation). This case requires the teacher to identify, or cue students into, the situation by engaging in actions that comply with the norms that constitute the situation. A situation frames that exchange by saving the effort of having to negotiate what needs to be done and what its exchange value will be.[5]

Negotiation of task and default to an instructional situation are two ideal types. In actual classroom interaction one should expect to see activity that combines default to an instructional situation and negotiation of how to handle whatever breaches the task at hand creates in the situation to which that task might default. This theorization helps describe how tasks that involve exchange of new kinds of work as representing the learning of new content might be deployed with the support of existing, specific systems of norms, and enable the establishment of slightly different norms that are specific to the study of the new content. Thus, novelty in classroom interaction is constructed by negotiating how to handle a breach in an existing instructional situation.[6] A new instructional situation may be installed as a result. In the paragraphs below we unpack the job description of the teacher in making instructional exchanges possible.

4.2.1 Mathematical Tasks and their Norms for the Teacher

Mathematical task and instructional situation are constructs that help us dig deeper into the notion of didactical contract so as to bring more detail to the problem of justifying actions in teaching. One of the manifestations of mathematics in the classroom is through the work that students do. *Task* is a way of parsing that work into smaller units of meaning. This notion of task[7] as a way to account for the curriculum draws from Doyle (1988) and contains, as a particular case, what Brousseau (1997) has called "adidactical situation." Under the aegis of the didactical contract the teacher has the right to engage students in tasks—the contract allocates time and space for this sort of engagement to happen. Brousseau (1997) has contributed the notion of the *milieu*—a counterpart environment that provides feedback on the actions of students. Thus one can define a mathematical task as the engagement of students in actions with and against a

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[5] This use of situation and framing comes from interpretive sociology, dating back to the work of Goffmann (1964/1997).

[6] Examples of situations include "solving for x" in algebra I, "doing proofs" in high school Geometry; also calculating a measure and exploring a figure in middle and high school geometry, to mention a few.

[7] Conceptions of task have also been proposed to examine teacher education (e.g., Zaslavsky & Sullivan, 2011) and teacher learning from teaching (Leikin, 2010) suggesting that task might be a more central notion than what we provide here. An examination of that issue is presently beyond the scope of this paper.

milieu.[8] In Doyle's terms, this milieu could be described as including the *goal* students are working toward and the *resources* with which students are operating. The interactions between student and milieu—both students' actions on the milieu and their interpretation of the feedback from the milieu are among the *operations* Doyle's task model includes.

As students work on mathematical tasks, the practice of the teacher includes supporting that work by ensuring that the milieu functions as expected. A type of norm, *task norms*, can be relied on to provide grounds for the justification of those teacher actions. If a task involves students' reliance on the feedback of a device such as a calculator, the teacher may need to ensure the calculator has batteries and a functional keypad. If a task requires the teacher to provide such feedback (e.g., in a number game of in which students ask questions to guess a number), the teacher may need to calculate correctly and respond truthfully. In general, there is a range of actions that the teacher needs to do to support the function of the milieu. These actions can be justified or critiqued on the basis of the norms of a task.

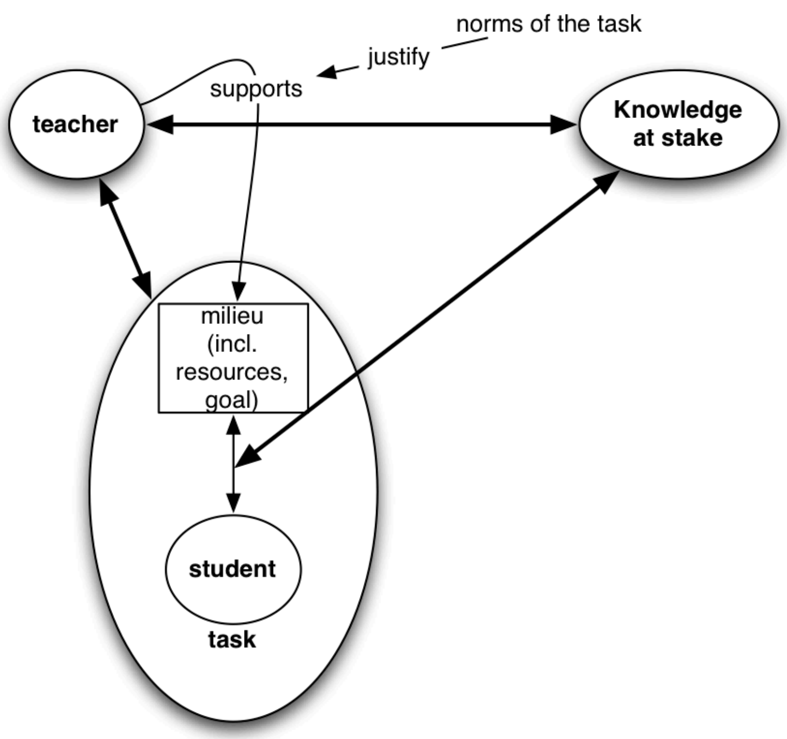

Figure 1. Mathematical task and how its norms may justify a teacher's action.

Figure 1 represents this first approximation to the sources of justification for teacher action. The diagram in Figure 1 complicates the image of the instructional triangle by representing the "students" corner of the triangle as a subsystem—students'

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[8] When Brousseau introduced the subject-milieu system, his main interest was in showing how adidactical situations could embody mathematical ideas. Thus, Brousseau's emphasis was on the adidacticity of the milieu—the possibility that the student perceive the milieu as devoid of didactical (instructional) intention. In our account of "task" we continue to find helpful to speak about a milieu, though the milieu may or may not be adidactical. Clearly not all tasks are adidactical situations.

mathematical work on tasks—including the interaction between the students and their milieu. The figure identifies as norms of the task those that regulate, for a particular task, what the teacher needs to do to install and support the function of the students' milieu related to that task. Those norms are, by definition, task-specific, and can also be diverse in nature—some of them logistic, while others mathematical. They hang together in the sense that they enable or sustain students' work on a chosen task.

4.2.2. Instructional Exchanges and Instructional Situations

In a second approximation to the practice of the teacher we consider the management of exchanges between students' mathematical work and the knowledge at stake. To the extent that students are doing mathematical work in an instructional system the teacher's practice includes more than sustaining the students' work. It includes (1) designing or otherwise choosing that mathematical work on account that it purportedly allows students to study and eventually lay claim to learning a given item of knowledge. The teacher's practice also includes (2) observing the unfolding of that mathematical work, gathering evidence that might warrant the claim that students have studied and, eventually, learned a given item of knowledge.

The actions a teacher engages in when managing instructional exchanges in the context of an instructional situation can be justified by appeal to the norms of that situation. Consider the case of an algebra teacher picking a problem that might stand for an opportunity for students to demonstrate knowledge of how to solve linear equations in one variable: Compare the possibility of asking students to solve $20x + 5 = 5x + 65$ versus asking them to solve $7x + 5 = 5x + 65$. The teaching action in question is how to choose the coefficients for the linear expressions being equated. We surmise that a teacher's choice of which coefficient to use (20 or 7) can be justified or critiqued on grounds that are based on the possibilities of exchange between (1) what students may do when they solve the particular equation and (2) the general characteristics of the solution method that they need to learn. Along those lines, Chazan and Herbst (2012) show evidence that, when focused on teaching solving equations on one variable, while both equations are mathematically equivalent, an algebra teacher would see it as more justifiable to use the second one ($7x + 5 = 5x + 65$) rather than the first one ($20x + 5 = 5x + 65$). In the latter, students could introduce a step (dividing through by 5) that, while mathematically valid and ingenious, has no exchange value in terms of the general method for solving a linear equation in one variable, which is what students are to learn through the engagement in that work. A norm of the instructional situation appears to be the warrant for one choice over the other: The expressions the teacher chooses for both sides of the equal sign must not encourage steps that are not part of the method being taught. Consider, on the other hand, the event that after assigning students the problem of solving $7x + 5 = 5x + 65$, a student wrote $7x + 7 = 5x + 67$ on the board. While this expression is mathematically equivalent to the first one and hence true for whatever value x will be, we hypothesize that a teacher of algebra would feel more warranted to call students' attention on the poor strategic value (Chazan & Sandow, 2011) of that move than to sanction the equality of the two new expressions. Thus the teacher is more likely to say "and how does that help you?" than "and why is that true?" A possible justification for such teacher choice of feedback is a combination of a contractual norm (that the teacher is responsible to ascribe value to students' work) and a norm for students in the situation of solving equations: That students must do to both sides operations that undo those operations affecting the variable.[9]

The preceding examples illustrate the practices a teacher needs to do to exchange students' mathematical work (anticipated in one case, actual in the other) for the item of knowledge that provides currency to such work. Each of the examples shows how the norms of an instructional situation might justify a teacher's preference of one action over another. In general, the need for a teacher to manage these instructional exchanges points to actions teachers take to select and deploy work for students as well as actions teachers take to register and attach value to the actions of students as they do such work. The norms of an instructional situation serve this function. These are an eclectic set of norms, indicating who has to do what, in what order things need to be done, and what is the value or currency of those actions (Herbst & Miyakawa, 2008).

The norms for instructional situations are valuable insofar as they warrant what participants ordinarily do when they enact those situations. Additionally, like the rules of any game, these norms also open a space of possibilities for students' development of strategies and tactics to do the tasks that allegedly count for the knowledge at stake. Likewise, those norms also open a space of possibilities for teachers to generate new tasks that can be guaranteed to be of value or to hone their skills of observing students' difficulties and discoveries.

Figure 2 attempts to represent this second approximation to the sources of justification for the teacher's actions by locating the role of norms of an instructional situation. The question of what justifies the actions of a teacher can be addressed in this second approximation to the instructional triangle by saying that to the extent that an existing instructional situation can accommodate the work that the class is engaged in, the norms of that instructional situation are the ones that justify the actions that a teacher takes to shape that work initially and to account for the value of that work eventually. Actions that breach those norms (e.g., including an expression that cannot be simplified by undoing as part of a 'solve for *x*' problem, as in "solve for x, $x^2 = 2^x$") would thus be unjustifiable on those grounds. If a teacher was to engage in those actions anyway, say by posing a task that breaches a norm of the instructional situation summoned, the continuation of the work would require the teacher and the students to negotiate how the didactical contract applies to that task (e.g., using graphical representations to learn about what it means for equations over the real numbers to have solutions).

The most important value of those norms of a situation is their capacity to point at the costs of learning new ideas or engaging in novel tasks. One could describe the progress of instruction over time as the socialization of students into new instructional situations, through negotiating breaches of existing instructional situations (See Marcus & Chazan, 2010 on situations involving the solving of equations in one and two variables.). At any one moment in a course of studies, students and teacher rely on a stock of available instructional situations with which they can take on new mathematical tasks. New instructional situations are developed through teacher and students' negotiation of how to accommodate the breaches that a novel task creates in an existing instructional situation. A legitimate analogy could be proposed between this account of

[9] Needless to say, whatever individual students may actually do in response to whatever task is chosen is not determined by the norms of the task chosen in a situation.

instruction as adaptation of instructional situations and the account of individual learning as adaptation of cognitive schemes proposed by radical constructivists (von Glasersfeld, 1995). The present account of instruction helps describe the change over time in the nature of the mathematical work of a class seen as a complex organism or, in other words, the organizational learning of a class. The diagram in Figure 2 complicates the instructional triangle by proposing that the teacher needs to engage in two kinds of practices, each warranted by different kinds of norms. The first one was described earlier and concerns the teacher's support of students' work on tasks. The second one identified in this section has to do with the teachers' actions setting up the students' work so that they can engage in work that exchanges for the knowledge at stake, and actually making those exchanges.

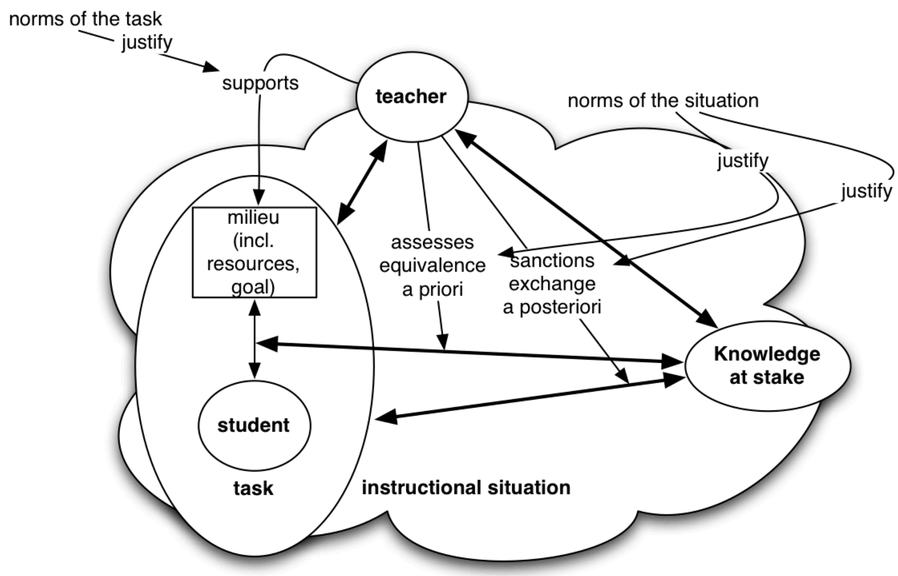

Figure 2. Instructional situations and how its norms justify teacher's actions.

This model admits being understood, on the one hand, as a set of nested activity systems (cf. Margolinas, 1995), where the norms of the contract are refined or specialized into norms of an instructional situation. Along those lines, an instructional situation includes a specification of canonical tasks and, within those, a demarcation of the students' share of labor in those canonical tasks and an identification of the milieu whose function the teacher has to support. In that way of reading the model one could see these structures as specializations of each other. On the other hand, this model admits being understood as an anticipation of tensions between competing and possibly contradictory teaching actions that are possible at any one moment in time, each of which actions might be justified by norms of different kinds. Consider the possibility, that students were involved in solving $20x + 5 = 5x + 62$ and that a student offered as the following step $4x + 1 = x + 12.4$. In figuring out what to do next a teacher might have to resolve a tension between, on the one hand, a norm of the situation which recommends appraising the step somewhat negatively because the student did not gather like terms as expected by the method being studied, and a norm of the task according to which the step does not raise any concerns since it results from a valid operation on the givens. This second reading of the model shows its usefulness in research on the practice of teaching: It provides a way of anticipating the different sources of regulation that might create the tensions or impasses that are visible when a teacher attempts to engage students in a new task. While novel tasks can quite efficiently bring those tensions about (Herbst, 2003), one could unearth these tensions at least in the form of descriptions of what has been solved or avoided, as part of the process of explaining why particular lessons appear to cohere. One could then describe instruction as an alternation of periods of stability, which encompass normed, routine instructional exchanges where the actions of the teacher appear unproblematic, and periods of crisis, which include the introduction of tasks that breach the norms of existing instructional situations, and where the teacher has to manage problems or tensions. The existence of a didactical contract requires that such critical periods exist in order for new knowledge to become what is at stake; the model helps predict that, ordinarily, such critical periods would tend to be as short as possible.

If this model described all the sources of justification for the actions of the teacher, one would be forced to legitimate the traditional contract of demonstration and practice—whereby a teacher shows how to do a problem type then assigns students many variations of it that enable them to practice and learn how to solve that problem type. What would compel a teacher to, at the same time, summon an existing instructional situation and breach it by assigning a task that does not quite fit the norms of such situation? What would justify the particular moves a teacher could make in negotiating the contract for such a task? The field has been used to answering those questions in terms of individual assets such as goals, beliefs, or knowledge (e.g., Schoenfeld, 2010). We contend that, to the extent that the demands entailed by the existence of a didactical contract make sense, some of the rationality that encourages "traditional" teaching is in fact intrinsic to the activity of teaching, not attributable to deficits of individual teachers. In the next section, we argue that the resources that justify more ambitious instruction similarly are not limited to the knowledge and will of individual teachers.

5. Professional Obligations

To describe what sources of justification are available for a teacher to do her share in the negotiation of a task, we return to the environment's constraints on the position of the teacher to introduce the notion of *professional obligations* (Herbst & Chazan, 2011). While the relationships between the three elements in the instructional triangle can be described as the enactment of instructional situations or the negotiation of tasks, both the impetus for assigning tasks that do not quite fit in a situation and the direction in which to push the negotiation of a contract for one such task may draw from the professional obligations that tie the person playing the role of mathematics teacher to the environment of the instructional system.

The people that take the roles of teacher and student, and to a large extent also the knowledge that takes the role of "content," are constrained not only by the rules that tie them to each other but also by obligations to their environments. In the case of the teacher, the ties a teacher has with the environments they belong to (and to which they must respond) provide resources with which teachers might justify deviations from the norms of instructional situations.

We address the environments for the position of mathematics teacher by proposing that a person in the role of mathematics teacher is in a position that obligates them to at least four stakeholders, though as an individual they may have obligations to other groups as well. Those who are in the *position* to become the teacher of a course (e.g., high school Algebra I) for a particular kind of students will come to play a *role* whose rules are described in terms of a didactical contract and its instructional situations. The *position* from which they are enabled to take on that role can be characterized as being subject to some obligations that, in the aggregate, situate the profession of mathematics teaching. These obligations bind the teacher to the environment of the instructional system.

Four sources of obligations are proposed here: the *discipline* that the teacher is meant to represent, the *individual* students the teacher is meant to serve (along with their stewards which include parents), the *socio-cultural* world of a given society and its customs and values (which are instantiated in the class of people with which the teacher is meant to share time, space, and language), and the *institution(s)* that create official time, space, and sanction for all those relationships to happen (Herbst & Chazan, 2011). The position of mathematics teacher is under those obligations[10] though individuals may vary in how and how much they recognize each obligation. We propose that these obligations that come with the position of mathematics teacher provide sources of justification for a teacher's actions, just as much as the norms that tie the role of the teacher to the other roles in the contract do.

> The ***disciplinary obligation*** says that the mathematical knowledge teachers teach needs to be a valid representation of the mathematical knowledge, practices, and applications of the discipline of mathematics.
>
> The ***individual obligation*** says that each of the persons in the position of student has the right to be treated as they individually are and feel. Students can have diverse physical and psychological traits and needs which a mathematics teacher may not ignore.
>
> The ***interpersonal obligation*** says that all of the individuals who are together in a classroom need to share resources such as time, physical space, and symbolic space in socially and culturally appropriate ways. In particular teachers need to ensure that appropriateness.
>
> The ***institutional (schooling) obligation*** says that there are regimes of coarser grain size than instruction and that condition or constrain what a teacher does in the course of studies. These include obligations to the department (e.g., textbook choices, curriculum coverage), to the school (e.g., calendar, bell schedules), to the district where the school is (e.g., assessment instruments and goals), to professional associations and unions (e.g., duration of workday).

[10] Traces of three of these obligations (disciplinary, individual, and interpersonal) can be found in Ball (1993) and in Cohen (2011). The notion that a teacher may need to manage dilemmas issued from competing commitments can be traced to Lampert (1985).

We posit that all school mathematics teachers have those obligations. In particular circumstances, an action that abides by one obligation may or may not abide by another obligation. These obligations not only constrain the teacher, they also provide resources that might help justify breaches of the norms of instructional situations.

Consider again the task of solving $x^2 = 2^x$. While the solution of equations like that one is not part of the knowledge at stake in beginning algebra, the disciplinary obligation could help a teacher justify assigning it: It creates an opportunity to learn about a different kind of equation or about the limits of algebraic manipulation (Chazan & Yerushalmy, 2003). The disciplinary obligation might actually justify offering the more general equation $x^a = a^x$, with a real and nonnegative. The individual obligation might support giving the task because of the value of challenging students intellectually, but support the choice of $a = 2$ so as not to overwhelm students' intellectual capacity. The interpersonal obligation might justify assigning the task in class (as opposed to for homework) because the class would likely have several responses to this strange task that could cross-fertilize each other. Finally, the institutional obligation could justify considering whether and how long one can afford having students work on such a problem in class, considering how much time it might take from other required work. These obligations could provide not only a basis on which to justify engaging in this novel task to begin with, but also resources to negotiate how to continue such engagement. Asking questions such as "did anybody have a different idea?", "have you seen any of these expressions before?", or "how could you prove that that is the only solution?" are moves with which the teacher could carry a negotiation of the task and that would be justified on account of some of those obligations. These negotiations might shape work that indeed exchanges for an item of mathematical knowledge: For example, students might end up seeing how graphing both sides of an equation can help visualize the solution of an equation.

6. Conclusion

The foregoing identifies sources of justification for teaching actions based on the nature of instructional activity and the position of mathematics teachers. In doing so, we have elaborated on the instructional triangle. The norms for contract, task, and instructional situations organize sources of justification for teaching actions emanating from the relationships between the roles involved in the instructional triangle. The professional obligations to the discipline, to individual students, to the class of students, and to the school institution gather sources of justification that illustrate how the environment of an instructional system can influence the interactions inside the instructional triangle through its hold on the position of the mathematics teacher. Practical rationality is the set of dispositions that result from combining norms that shape the role of the teacher inside the instructional triangle with obligations generated by the hold of the environment on the position of the teacher. We propose that the rationality of an action in teaching can be argued in terms of dispositions that combine the norms of instructional activity and the obligations of the mathematics teaching profession. This kind of justification complements attempts to explain action in terms of individual knowledge, goals, or beliefs by providing grounds on which the wisdom of an action can be determined. The approach proposed builds on the notion of the instructional triangle by elaborating on the themes of roles, relationships, and environments that scholars in the past have used to characterize instruction.

This theoretical elaboration of the instructional triangle provides elements that could support incremental and justified attempts at instructional improvement. Attempts to improve instruction can be operationalized in terms of engaging students in novel tasks by both building on existing instructional situations and breaching with some of their norms. Those breaches may need justification of their appropriateness as well as anticipation of the systemic push back that may be justified on other obligations.

The theory also suggests directions for research. Do obligations apply in the same way to instructional systems at the elementary level (where individual teachers not necessarily identify as mathematics teachers, but instead as teachers who teach children many subjects)? Do the obligations apply similarly across cultures where school may play different roles vis-à-vis the reproduction of social life? How can the existence of those obligations be confirmed empirically and compared across those boundaries?

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