Performance Scalability in Communication Networks

by

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To my parents and sister

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TABLE OF CONTENTS

LIST OF FIGURES

Figure

LIST OF TABLES

Table

LIST OF APPENDICES

Appendix

CHAPTER I

Introduction

Performance scalability is one of the central problems that needs to be addressed when designing large-scale communication networks. An often-overlooked issue in large-scale networks is that limited local resources such as memory and power can be critical factors in determining the overall network performance, as the size of the network increases. The finiteness of local resources can create performance bottlenecks. Such performance bottlenecks are usually not prominent in small-size networks. However, minor deficiencies that can be tolerated in small-size networks can accumulate and become dominant factors that limit the performance in large-scale networks. We emphasize that this issue cannot be solved simply by over-provisioning local resources of each network device, since required upgrades in a large legacy network are likely to have a prohibitive cost. Hence, there is a need for investigating the fundamental performance limits due to the finiteness of local resources in large-scale networks.

In this dissertation, we aim to understand fundamental properties of various performance characteristics of large-scale communication networks with limited local resources. We focus on theoretical investigation, since conducting experimental studies is impractical due to the considerable cost of building large prototypes, and even simulating such networks is sometimes not feasible because of computational limitations. This dissertation addresses the following three topics on performance scalability: (i) effectiveness of application-layer coding, (ii) existence of fixed-point processes in linear loss networks, and (iii) throughput scalability for linear finite-buffer networks. We develop mathematical frameworks for studying these topics, and investigate asymptotic characteristics of the networks, as the number of users or the size of the network increases. Furthermore, we illustrate by simulation studies that these asymptotic results are reasonably accurate and applicable for networks of finite size. Our analytical results provide good guidelines for the development and optimization of scalable network architectures and protocols.

We note that we consider networks that do not utilize feedback. Feedback mechanisms for large-scale networks are usually difficult to design and not straightforward to implement. Moreover, the value of feedback information decreases to the point where it is sometimes detrimental to use due to large delay, as the size of the network increases. Thus, it is reasonable to employ simple control protocols without feedback in large-scale networks, and we focus on such networks.

1.1 Effectiveness of application-layer coding

In this section, we summarize the results on effectiveness of application-layer coding provided in Chapter II. Application-layer coding is one of the mechanisms to achieve reliable communication in packet networks. If application-layer coding is employed, sources encode their data packets into coded packets and transmit them instead. Then, even though some of the encoded packets are lost from the network, receivers can reconstruct the original data packets by decoding a received subset of coded packets. In this research, we study the effectiveness of application-layer coding in a wired network with a large number of users. In wired networks, transmission errors due to channel noise are rare, and, thus, it is reasonable to assume that losses are due to buffer overflows only (i.e., transmissions are error-free). We note that employing coding has two conflicting effects on the network performance: (i) additional packets increase the overall offered load, which results in a higher drop probability, and (ii) some of the dropped packets can be recovered at the end users after decoding, which decreases the loss rate. The effectiveness of coding depends on which of the above-mentioned effects is dominant, and it is unknown a priori when application-layer coding is advantageous.

Formally, we consider a sequence of networks, indexed by the number of users (sources), consisting of a single link with a finite buffer shared by all users. This finite-buffer node with a single transmission link, in which some packets are dropped when the buffer is full, can be thought of as an erasure channel. For analytical simplicity, we consider systematic linear block codes as our coding scheme. If a systematic code is employed, a fixed number of additional coded packets are generated per each coding block consisting of a fixed number of data packets. The additional coded packets are transmitted along with the data packets. All data and coded packets are assumed to have the same priority in the buffer. If data packets are given priority, coded packets do not affect drops of data packets, and, thus, coding does not degrade the network performance. Nevertheless, we consider a system without priority, since such a system is straightforward to implement and users have no incentive to mislabel their packets intentionally.

In order to investigate the effectiveness of coding, we first establish a relevant scaling regime (i.e., the critical-loading scaling) of the network parameters (i.e., arrival rate, link capacity, buffer size and coding block length), in terms of the number of users. Under this regime, we compare the asymptotic loss probabilities (due to buffer overflows) in networks with and without coding, as the number of users increases. These asymptotic results indicate that the network parameter space is partitioned into two regions where coding is beneficial and detrimental, respectively. In particular, the critical regime that we establish contains the boundary between these two regions. On the boundary, networks with and without coding have the same performance. Informally, it is argued that coding can be advantageous in under-loaded networks only; in over-loaded networks, the overhead of coding exceeds its benefit, and coding only worsens the network performance. Finally, we demonstrate on examples that our asymptotic results render reasonable approximations for networks with a finite number of users.

1.2 Existence of fixed-point processes in linear loss networks

This section outlines the research contributions on the asymptotic characteristics of the departure (output) processes of linear loss networks presented in Chapter III. A linear network is a tandem network consisting of a series of identical nodes, in which customers (packets) enter the network at a fixed source node and are relayed from one node to another in a fixed order until they exit the network at a fixed destination node. This tandem network is a representative model of large-scale communication networks with limited or no cross-traffic interference along the paths from the sources to destinations. An example of such a network is a sensor network where concurrent traffic volume is small relative to the size of the network.

In a linear network, the output process from a queue (node) is in turn the input process to the next queue (node). If the input and output processes are equal in distribution, then such processes are called fixed-point processes. If the number of buffers at each node is limited, the input is always denser than the output. Thus, in order to investigate their asymptotic characteristics, it is necessary to scale the departure processes properly, in terms of the network size. We focus on linear loss networks consisting of bufferless nodes each of which operates as a $\cdot/M/1/0$ queue. Under a proper scaling, as the size of the network increases, scaled inter-departure times of the customers from the destination node tend to consecutive distances between coalescing Brownian motions in a one-dimensional space, in which any two Brownian motions coalesce into one whenever they hit each other. The coalescing procedure of the Brownian motions is called one-species two-body diffusion-limited reaction and was studied in [5–7, 22]. By exploiting connections between the two areas, we provide a complete characterization of the asymptotic departure process (i.e., the joint probability density function of any finite number of consecutive inter-departure times). This asymptotic property of the departure process is completely attributed to the characteristics of the network itself (i.e., the distributions of service times and the number of buffers) and is not impacted by the input as long as the input does not vary with the size of the network.

1.3 Throughput scalability for linear finite-buffer networks

This section provides a summary of the results on the critical regime for linear finite-buffer networks presented in Chapter IV. A critical loading regime is a scaling regime of the input arrival rate, in terms of the size of the network, under which the input and output (throughput) rates are proportional, i.e., the asymptotic loss probability due to limited buffer space is strictly within $(0, 1)$, as the size of the network increases. Such a regime is of interest since it delivers a relatively high throughput at low network (energy) cost. If the offered load is (order-wise) higher than the critical load, then the loss probability tends to 1 asymptotically. In that case, only a negligible fraction of packets is delivered, and, thus, operating a network in such a way is not (energy) efficient. On the other hand, the asymptotic loss probability of 0 occurs when the offered load is (order-wise) lower than the critical load. Then, higher throughputs can be achieved with small increments of the network cost. Hence, in this case, the network capacity is not efficiently utilized. Under the critical regime, one balances between two conflicting goals: (i) achieving high throughput, and (ii) maintaining low loss probability (low network cost).

In this research, we identify a critical loading regime for linear networks consisting of finite-buffer nodes each of which operates as a $\cdot/M/1/b$ queue for fixed $b \geq 1$. To this end, we first develop a multi-dimensional random walk within a wedge that approximately describes the evolution of the relative distances between a finite number of consecutive packets in a linear finite-buffer network. In particular, an event of the random walk visiting the origin corresponds to a loss of a packet in the linear network. Thus, the loss probability of a (stationary) packet can be evaluated approximately by analyzing hitting probabilities of random walks within a wedge. Our analytical results indicate that the input rate under the critical loading regime is $\Omega(1/\sqrt{k})$ and $O(\sqrt{\log k/k})$ for $b = 1$, and $\Theta(k^{-1/(b+1)})$ for $b \geq 2$, as $k \to \infty$, where k denotes the number of nodes in the network.[1] It was shown in [35] that the critical loading regime for linear networks with bufferless nodes (i.e., $b = 0$) occurs when the input rate is $\Theta(1/\sqrt{k})$, as $k \to \infty$. From these results, we conclude that the qualitative behavior of the critical loading regime for linear networks depends on whether the buffer size is greater than 1. Finally, we illustrate with simulation studies that these asymptotic approximations are reasonably accurate for finite-size

[1]Throughout the dissertation, we use the standard asymptotic notation; e.g., see [20, Section I.3.1].

networks.

1.4 Organization of the dissertation

The dissertation is organized as follows. In Chapter II, we discuss the effectiveness of application-layer coding. Linear networks are studied in Chapters III and IV. In particular, Chapter III investigates the departure process of the linear loss network, and Chapter IV establishes a critical loading regime for linear finite-buffer networks. Chapter V contains conclusions and the future work of the dissertation. Technical proofs omitted in the main text and can be found in the Appendices.

CHAPTER II

Effectiveness of application-layer coding

2.1 Introduction

The primary reason for losses in packet networks is buffer overflow – each link in a network has finite capacity, and intermediate routers have limited memories to store packets. In general, there are two basic approaches to overcome this kind of packet losses:

- *Retransmission mechanism.* The source transmits its data packets to the receiver. Packets that have not been acknowledged (explicitly or implicitly) are retransmitted.

- *Application-layer coding.* The source encodes its data packets into coded packets and transmits them instead. The receiver reconstructs the original data packets by decoding received coded packets.

In wired networks, transmission errors due to channel noise are rare, and, thus, it is reasonable to assume that losses are due to buffer overflows only (i.e., transmissions are error-free). This study focuses on evaluating the effectiveness of application-layer coding in such networks. In wireless networks, however, packet losses attributed to transmission errors (due to unreliable channels) can be considerable and must be accounted for. We note that application-layer coding (i.e., fountain coding) for such networks has been studied substantially as one of the possible methods to achieve reliable communication (e.g., see [17, 41]).

Application-layer coding allows end users to recover the original data packets from the received subset of coded packets by decoding. However, employing such coding results in a higher offered load, and, therefore, increases drop probability. From this perspective, it is unclear when application-layer coding is advantageous; application-layer coding was shown to be advantageous in certain cases [10]. Hence, it is of interest to investigate the effectiveness of such coding. As a first step, we study the effectiveness of coding in the baseline model consisting of a single link with a finite buffer. In particular, a sequence of systems indexed by the number of users N is considered. We first discuss an appropriate scaling of the system parameters for investigating the effectiveness of coding, and establish that the critical-load scaling is the relevant one. Under the critical-load scaling, system utilization and drop probability behave as $1-\Theta(1/\sqrt{N})$ and $\Theta(1/\sqrt{N})$, respectively, when the number of users N is large. We then examine the loss probabilities in systems with and without coding. Our asymptotic analysis indicates that application-layer coding can be advantageous in under-loaded systems; in over-loaded systems, however, the overhead of coding exceeds its benefit, and coding only worsens the system performance. In addition, we demonstrate on examples that our asymptotic results render reasonable approximations for systems with a finite number of users.

The rest of the chapter is organized as follows. In the next section, we describe a system model and assumptions that are used throughout the chapter. We discuss a relevant scaling for investigating the effectiveness of coding in Section 2.3. In Section 2.4 we review erasure codes and their performance. Section 2.5 contains the analysis for the *loss* probability without coding. Then, we analyze the *drop* proba-

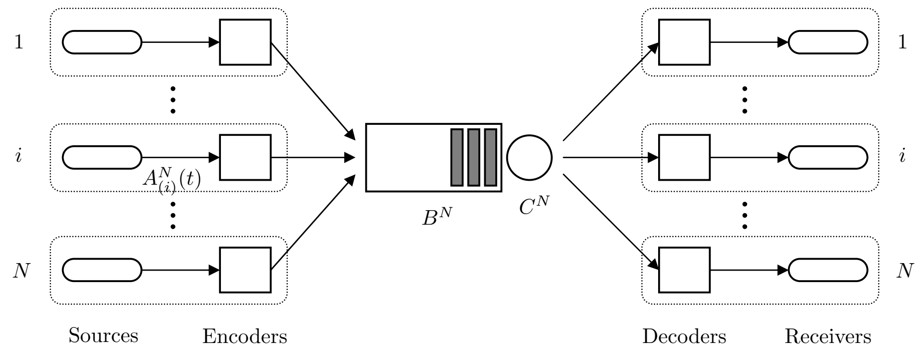


Figure 2.1: The source i, $1 \leq i \leq N$, generates $A_{(i)}^N(t) \in \{0,1\}$ packets in the time slot $t \geq 1$. When application-layer coding is employed, source-generated packets are encoded into coded packets by individual encoders. All packets from the encoders of N sources are transmitted to a single link of capacity C^N with a buffer of size B^N. The packets that are not dropped from the buffer are first delivered to the decoder of each source and decoded into the original packets. End users receive the output packets of their own decoders.

bility with coding and discuss the coding overhead due to the increased offered load in the following section. In Section 2.7 we explore the *loss* probability with coding and establish the boundary where systems with and without coding have the same performance. A discussion on the system performance for a systematic minimum-distance-separable (MDS) code is presented in Section 2.8. Concluding remarks and technical proofs can be found in Section 2.9 and Appendix A, respectively.

2.2 System model

2.2.1 Model

We consider a sequence of systems indexed by N, where N is the number of sources that transmit packets to a link with a finite buffer. Let C^N and B^N denote the link capacity and the buffer size, respectively. Time slotted operations are assumed. In addition, let $A_{(i)}^N(t)$, $1 \leq i \leq N$, $t \geq 1$, denote the number of packets generated by the source i in the time slot t. The processes $\{A_{(i)}^N(t), t \geq 1\}$, $i = 1, 2, \cdots, N$, are assumed to be independent Bernoulli random processes with parameter λ, i.e., $A_{(i)}^N(t) \in \{0,1\}$ and $\mathbb{E}A_{(i)}^N(t) = \lambda$ for $1 \leq i \leq N$ and $t \geq 1$. If present, application-layer coding is performed at each source (see Figure 2.1). All packets from the encoders are transmitted to the queue consisting of a single link with a finite buffer. The packets that are not dropped from the queue are first delivered to the decoder of each source and decoded into the original packets. End users receive the output packets of their own decoders.

We examine loss probability as a measure of the system performance. The loss probability is defined as the long-term ratio of the number of lost packets to the total number of source-generated packets. A dropped packet is a packet that is discarded from the queue when the buffer is full, and a lost packet is a packet that is not delivered to the end users. In a system without coding, every dropped packet is also a lost packet since no dropped packets can be recovered. If a system utilizes coding, however, some of dropped packets can be recovered at the end users, and, therefore, we differentiate a lost packet (loss probability) from a dropped packet (drop probability) in this case. Even though the drop probability increases due to the additional offered load attributed to coding, the loss probability can decrease by means of coding if enough dropped packets are recovered.

2.2.2 Coding scheme

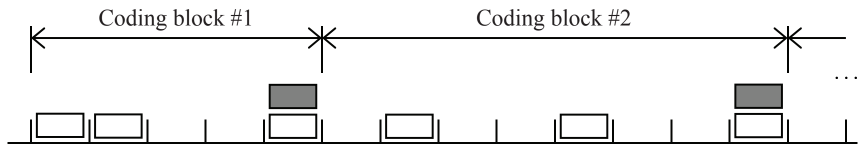
The queue, in which some packets are dropped when the buffer is full, can be thought of as an erasure channel, e.g., see [21, 40]. Two main features of our model are as follows:

- *Systematic linear block code.* We assume that each encoder uses a linear block code for producing additional α packets per each coding block consisting of M^N data packets generated by its source. For this operation, each encoder is assumed to have a memory space for storing copies of M^N most recent data packets from its source. A data packet generated by a source is transmitted in the same time slot (without any delay due to the encoder). The encoder produces α coded packets when the source generates the last data packet of the block. These additional α packets are transmitted in the same time slot as the last data packet of the block[1]; α does not vary with N. See Figure 2.2 for an example. Note that under this scheme, the decoding delay is positive only in the presence of packet drops.

- *Non-priority queue.* It is assumed that all packets have the same priority in the queue and that they are served on the first-come, first-serve basis. If a system gives priority to data packets over coded packets in the queue, then coded packets do not affect drops of data packets, and, thus, coding does not degrade the system performance. Nevertheless, we consider a system without priority since such a system is straightforward to implement and users have no incentive to mislabel their packets intentionally (cheat). Moreover, when the loss probability is very low (to be made precise later in the chapter; regime studied in [10]), coding is beneficial in both systems with ([10]) and without (our model) priority, i.e., in that regime, the priority does not impact results in a qualitative way.

Let $H^N_{(i)}(t) \in \{0, 1, 1+\alpha\}$, $1 \leq i \leq N$, $t \geq 1$, denote the total number of packet arrivals (both data and coded packets) from the encoder of the source i in the time slot t. We say that a coding block "ends" at the time slot t if the M^Nth packet of the block is generated by the source at time t; the following block "starts" at the

[1]This assumption is not crucial and does not impact the nature of our main results. Other schemes are possible, e.g., additional packets can be transmitted in consecutive time slots.



Time t	1	2	3	4	5	6	7	8	9	10	11	12	13	14
$A^N_{(i)}(t)$	1	1	0	0	1	0	1	0	0	1	0	0	1	0
$X^N_{(i)}(t)$	0	1	2	2	2	0	0	1	1	1	2	2	2	0
$H^N_{(i)}(t)$	1	1	0	0	2	0	1	0	0	1	0	0	2	0

Figure 2.2: The figure illustrates an example of our coding scheme for $M^N = 3$ and $\alpha = 1$. In this example, data packets (white) are generated (and transmitted) in the time slots with indices 1, 2, 5, 7, 10 and 13; additional coded packets (gray) are generated (and transmitted) in the time slots 5 and 13. The first coding block starts at $t = 1$ and ends at $t = 5$; the second coding block starts at $t = 6$ and ends at $t = 13$. We also indicate the values of $A^N_{(i)}(t)$, $X^N_{(i)}(t)$ and $H^N_{(i)}(t)$ for each t.

subsequent $(t+1)$ time slot (the first block starts at time $t = 1$). Then, we have

$$H^N_{(i)}(t) = A^N_{(i)}(t) + \alpha 1_{\{A^N_{(i)}(t)=1,\, X^N_{(i)}(t)=M^N-1\}},$$

where $X^N_{(i)}(t) \in \{0, 1, \dots, M^N - 1\}$ is the number of data packets generated by the source i from the beginning of the current coding block up to (and including) the time slot $t - 1$; by definition, $X^N_{(i)}(t) = 0$ if coding block starts at time t. See Figure 2.2 for an example. Now observe that the arrival process $\{H^N_{(i)}(t), t \geq 1\}$ is completely determined by a Markov chain $\{(A^N_{(i)}(t), X^N_{(i)}(t)), t \geq 1\}$ with the state space $S = \{(a, x) : a \in \{0, 1\}, x \in \{0, 1, \dots, M^N - 1\}\}$ and transition probabilities $P_{\mathbf{ij}}$, $\mathbf{i}, \mathbf{j} \in S$, given by

$$P_{\mathbf{ij}} = \begin{cases} \lambda, & a_j = 1, x_j = (x_i + 1_{\{a_i=1\}}) \mod M^N, \\ 1 - \lambda, & a_j = 0, x_j = (x_i + 1_{\{a_i=1\}}) \mod M^N, \end{cases}$$

where $\mathbf{i} = (a_i, x_i)$ and $\mathbf{j} = (a_j, x_j)$. Since this Markov chain is finite, aperiodic and irreducible, it has an unique stationary distribution $\pi(a, x)$, $a \in \{0, 1\}$, $x \in$

$\{0, 1, \ldots, M^N - 1\}$, given by

$$\pi(a, x) = \begin{cases} (1-\lambda)/M^N, & a = 0, \\ \lambda/M^N, & a = 1. \end{cases} \tag{2.1}$$

2.3 Scaling

In this section, we discuss an appropriate scaling (as the number of users increases, $N \to \infty$) for investigating the effectiveness of application-layer coding. Recall that we assume that α additional packets are generated per each block of length M^N. The additional offered load due to coding is then equal to $\alpha\lambda N/M^N$ while the spare capacity of the link is $C^N - \lambda N$. Thus, we consider the block length M^N such that $\alpha\lambda N/M^N = \Theta(C^N - \lambda N)$, as $N \to \infty$, since this scaling allows one to examine both under- and over-loaded systems by adjusting appropriate constants (system parameters). Next we review three possible scalings for C^N and M^N[2]. Let p and p_D denote the loss probability without coding and the drop probability with coding, respectively.

- *Under-load scaling*: for $\beta > 0$ and $m > \alpha/\beta$,

  $$C^N = \lfloor \lambda N + \beta N \rfloor, \quad M^N = \lfloor m\lambda \rfloor.$$

  In this regime, p and p_D are asymptotically $O(e^{-\theta N})$ and $O(e^{-\theta' N})$, respectively, as $N \to \infty$, for some positive constants θ and θ' such that $\theta' < \theta$ (e.g., see [42, Ch. 12]); θ' approaches θ when m increases. Informally, despite the fact that drop probability increases due to coding, the expected number of dropped packets in a block is close to 0 for large N. If at least one coded packet is added per block ($\alpha \geq 1$), we can recover most of the dropped packets as long as the

[2]Although other scalings are possible, these three cover the main tradeoffs between efficiency and quality.

coding block is large enough so that $(\alpha+1)\theta' > \theta$. Therefore, coding improves the system performance in this case as suggested in [10].

- *Over-load scaling*: for $\beta > 0$ and $m > \alpha/\beta$,

$$C^N = \lfloor \lambda N + \beta \rfloor, \quad M^N = \lfloor m\lambda N \rfloor.$$

Due to the central limit theorem (CLT), both p and p_D are asymptotically given by $\Theta(1/\sqrt{N})$, as $N \to \infty$. In this case, the expected number of dropped packets in a block is $\Theta(\sqrt{N})$ since the block length is $\Theta(N)$. However, the maximum number of dropped packets that can be recovered in a block is only $\alpha = \Theta(1)$. For large N, hence, the possibility of recovering dropped packets is very small. Hence, in this scaling, coding worsens the system performance.

- *Critical-load scaling*:

$$C^N = \lfloor \lambda N + \beta\sqrt{N} \rfloor, \quad M^N = \lfloor m\lambda\sqrt{N} \rfloor. \tag{2.2}$$

Under this scaling, both p and p_D behave as $\Theta(1/\sqrt{N})$ in the limit as $N \to \infty$ (e.g., see [45, Ch. 10]). Since the block length is $\Theta(\sqrt{N})$, the expected number of dropped packets in a block is $\Theta(1)$, i.e., the numbers of dropped and additional packets are of the same order. Therefore, in this case, the effectiveness of coding depends on α and m for given system parameters such as β (capacity) and b (buffer size), and it is feasible to find the critical points where systems with and without coding have the same performance.

The scaling for the buffer size B^N stems from the fact that if $B^N = o(\sigma^N)$, as $N \to \infty$, where σ^N denotes the standard deviation of the total arrival process, then the performance of the system is asymptotically equal to the one with $B^N = 0$ (as $N \to \infty$); on the other hand, if $B^N = \omega(\sigma^N)$, as $N \to \infty$, then the system behaves

Table 2.1: Summary of key notations (under critical-load scaling)

System	
N	Number of users (sources)
λ	Arrival rate per source
C^N	Link capacity
B^N	Buffer size
β	Scaled spare capacity ($C^N = \lfloor \lambda N + \beta\sqrt{N} \rfloor$)
b	Scaled buffer size ($B^N = \lfloor b\sqrt{N} \rfloor$)
User	
α	Number of additional packets per block
M^N	Coding block length (in packets)
m	Scaled coding block length ($M^N = \lfloor m\lambda\sqrt{N} \rfloor$)

asymptotically as the one with $B^N = \infty$ (as $N \to \infty$). Hence, for evaluating the effect of the buffer size on the system performance, the relevant buffer size should satisfy $B^N = \Theta(\sigma^N)$, as $N \to \infty$. For the considered model, we have $\sigma^N = \Theta(\sqrt{N})$, as $N \to \infty$, and, thus, we let, for $b \geq 0$,

$$B^N = \lfloor b\sqrt{N} \rfloor. \tag{2.3}$$

In the following sections, we demonstrate that the *critical-load* scaling is the relevant scaling as far as the effectiveness of coding is concerned. Under the *critical-load* scaling, the following scaled variables are useful in obtaining the drop and loss probabilities:

$$\begin{aligned} \hat{C}^N &= (C^N - \lambda N)/\sqrt{N} \to \beta, \\ \hat{B}^N &= B^N/\sqrt{N} \to b, \end{aligned} \tag{2.4}$$

as $N \to \infty$. Moreover, Table 2.1 summarizes the key notations (under the *critical-load* scaling) that will be used throughout the chapter; note that β, b and m are the scaled parameters that determine spare capacity, buffer size and coding block length, respectively (see (2.2) and (2.3)).

We note that in [10], application-layer coding was studied in the context of a system with priority (data packets are given priority). In particular, the authors considered the *under-load* scaling only (linear scaling between the link capacity and the number of users), and concluded that application-layer coding improves performance (in the limit as $N \to \infty$). In contrast, we focus on the *critical-load* scaling since under this scaling, application-layer coding can be either beneficial or detrimental depending on the exact parameters of the system. Our result indicates that application-layer coding is beneficial in the *under-load* regime (as $N \to \infty$) even if data packets are not prioritized over coded packets. As mentioned in Subsection 2.2.2, utilizing application-layer coding does not degrade the performance of a system with priority. However, as will be discussed in Section 2.9, the *critical-load* scaling also plays a role in the model with priority. Namely, in this regime, the ratio of loss probabilities in two corresponding systems with and without coding tends to a constant strictly within $(0, 1)$, as $N \to \infty$. On the other hand, this ratio tends to 0 (exponentially fast in N) in the *under-load* regime [10], and to 1 in the *over-load* regime.

2.4 Erasure codes

In this section, we review erasure codes and their performance. The relevance of such codes is due to the fact that the finite-buffer queue can be thought of as an erasure channel, e.g., see [21,40]. We consider $(M+\alpha, M)$ linear block codes – M data packets are used to generate $M+\alpha$ packets to be transmitted. Let $\mathbf{v} = [v_1\ v_2\ \cdots\ v_M]$ be the data packets in a single coding block, and let $\mathbf{u} = [u_1\ u_2\ \cdots\ u_{M+\alpha}]$ be the output packets encoded from these data packets. The output packets are generated from the data packets according to the following rule:

$$\mathbf{u} = \mathbf{v}\mathbf{G}, \tag{2.5}$$

where $\mathbf{G}$ is a generator matrix that depends on a specific code. All arithmetic is over $GF(q)$ for some positive integer q (e.g., see [28, Ch. 5]). Next we examine various erasure codes.

2.4.1 Ideal block code

Let D denote the number of dropped packets among the $M + \alpha$ output packets from a single block, and let L denote the number of lost packets in the same block, i.e., L original data packets can not be reconstructed after decoding. We define the *ideal block code* as a code that satisfies the following property:

$$L = (D - \alpha)^+ = D - (D \wedge \alpha). \tag{2.6}$$

Note that if D output packets are dropped, then a decoder can recover only $M+\alpha-D$ linear equations in (2.5) from the remaining output packets. From $M + \alpha - D$ linear equations, at most $M + (\alpha - D)^-$ data packets can be decoded correctly. Therefore, the ideal block code, if it exists, achieves the best performance among all linear block codes.

2.4.2 Systematic MDS code

A linear block code with minimum distance d can recover all of the original data packets in a block when the number of dropped packets in the block is less than d. If a $(M + \alpha, M)$ linear block code has minimum distance $d = \alpha + 1$, we call such codes as MDS codes; these MDS codes achieve equality in the Singleton bound (e.g., see [28, Ch. 15]). Reed-Solomon codes belong to the class of MDS codes. When a code is systematic, the output packets from a block contain M original data packets and additional α coded packets, i.e., $u_i = v_i$ for $i = 1, 2, \ldots, M$. Given a block, let D_d and D_c denote the numbers of dropped packets among the M data packets and the additional α coded packets, respectively. If $D_d + D_c \leq \alpha$, then all M data packets can be reconstructed from (2.5). On the other hand, if $D_d + D_c > \alpha$, then no dropped data packets can be recovered from (2.5) and only $M - D_d$ data packets are obtained. Therefore, letting L be the number of lost packets after decoding leads to

$$L = D_d \cdot 1_{\{D_d + D_c > \alpha\}}. \tag{2.7}$$

2.4.3 Partial coding

Suppose that a systematic MDS code is applied to only ρ fraction of data packets in a block. That is, $\lfloor \rho M \rfloor$ data packets are used to generate $\lfloor \rho M \rfloor + \alpha$ output packets, and remaining $M - \lfloor \rho M \rfloor$ data packets are transmitted without any encoding. In this case, only the dropped packets from the ρ fraction of the block can potentially be recovered. Let $\tilde{D}_d$ and $\bar{D}_d$ denote the numbers of dropped packets among $\lfloor \rho M \rfloor$ data packets in the coding part and $M - \lfloor \rho M \rfloor$ data packets in the non-coding part, respectively. Moreover, let D_c denote the number of dropped packets among additional α coded packets in the coding part. Setting L to be the number of lost packets after decoding yields

$$L = \bar{D}_d + \tilde{D}_d \cdot 1_{\{\tilde{D}_d + D_c > \alpha\}} = D_d - \tilde{D}_d \cdot 1_{\{\tilde{D}_d + D_c \leq \alpha\}}, \tag{2.8}$$

where $D_d = \tilde{D}_d + \bar{D}_d$.

2.4.4 Comparison

In Figure 2.3, we illustrate the difference between these three coding schemes on an example. In particular, we compare the conditional expectation of the number of lost packets given the value of the number of dropped packets in a block for the ideal

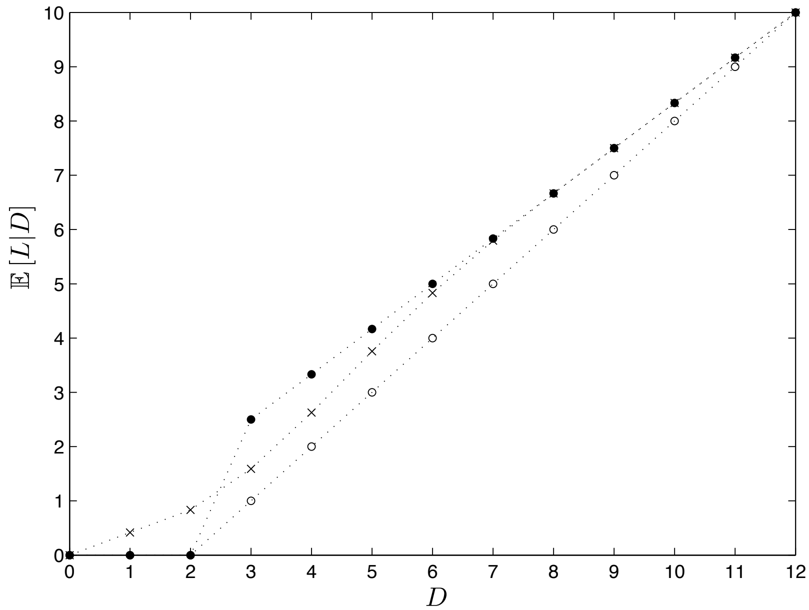


Figure 2.3: The conditional expectation of the number of lost packets $\mathbb{E}[L|D]$ given the value of the number of dropped packets D in a block for the ideal block code ($\circ$), a systematic MDS code ($\bullet$) and partial coding with $\rho = 0.5$ ($\times$) when $M = 10$ and $\alpha = 2$. In this example, it is assumed that all packet drops are independent with the same drop probability.

block code, a systematic MDS code and partial coding with $\rho = 0.5$. The block length M and the number of additional coded packets α are set to be 10 and 2, respectively. Just for this example, all packet drops are assumed to be independent with the same drop probability. As expected, the ideal block code has the smallest expected value of the number of lost packets for a given value of the number of dropped packets. When a systematic MDS code is employed, all dropped data packets can be recovered if the number of dropped packets is at most α ($\alpha = 2$ in this example); otherwise, no dropped data packets can be recovered. When partial coding is used, even though the number of dropped packets is greater than α, the dropped data packets that belong to the coding part can be recovered if the number of dropped packets in the coding part is at most α; in this case, thus, partial coding has better performance than pure block coding. On the other hand, if the number of dropped packets is not greater than α, then partial coding underperforms pure block coding since the dropped data packets in the non-coding part can not be recovered.

2.4.5 Coding with overlapping blocks

In this subsection, we examine one particular scheme that utilizes overlapping blocks. Suppose that each half of a block overlaps with either one of its adjacent blocks and that $\alpha/2$ additional packets are generated from each block of length M; M and α are assumed to be even for simplicity. Note that the number of additional packets per block is halved for fair comparison to the scheme with non-overlapping blocks since the number of blocks is doubled. Let $\{v_i, i \in \mathbb{Z}\}$ be the sequence of data packets from a source. The nth coding block $\mathbf{v}_n$, $n \in \mathbb{Z}$, is given by $\mathbf{v}_n = [\dot{\mathbf{v}}_n \ \dot{\mathbf{v}}_{n+1}]$, where $\dot{\mathbf{v}}_n = [v_{(n-1)M/2+1} \ v_{(n-1)M/2+2} \ \ldots \ v_{nM/2}]$. The output packets $\mathbf{u}_n$, which are generated from $\mathbf{v}_n$, include the data packets in $\dot{\mathbf{v}}_n$ and additional $\alpha/2$ coded packets. Observe that the data packets in $\dot{\mathbf{v}}_n$ are used to generate two sets of $\alpha/2$ coded packets. It is assumed that a systematic MDS code is used to encode each block, and each block is decoded independently, i.e., no dropped data packets are recovered if the number of dropped packets in a block is greater than $\alpha/2$.

Let $\dot{D}_d^n$ denote the number of dropped packets in $\dot{\mathbf{v}}_n$, and let $\dot{D}_c^n$ denote the number of dropped packets among the additional $\alpha/2$ coded packets that are generated from $\mathbf{v}_n$. In addition, let $\dot{L}^n$ denote the number of lost packets in $\dot{\mathbf{v}}_n$ after decoding. The following lemma characterizes the number of lost packets in one half of a block when the scheme with overlapping blocks is employed.

Lemma 2.1. *Suppose that the system is in stationarity. If $\{\dot{D}_d^n, n \in \mathbb{Z}\}$ and $\{\dot{D}_c^n, n \in \mathbb{Z}\}$ are two independent i.i.d. sequences, then*

$$\mathbb{E}[\dot{L}^n | \dot{D}_d^n = k] = k(1 - \xi(k))^2,$$

where $\xi(0) = 1$, $\xi(k) = 0$, $k > \alpha/2$, and $\xi(k)$, $1 \leq k \leq \alpha/2$, satisfies the following *equation:*

$$\xi(k) = \mathbb{P}[\dot{D}_d^n + \dot{D}_c^n \leq \alpha/2 - k] + \sum_{i=1}^{\alpha/2} \xi(i)\mathbb{P}[\alpha/2 - k - i < \dot{D}_c^n \leq \alpha/2 - k]\mathbb{P}[\dot{D}_d^n = i].$$

Proof. See Appendix A.1.

2.5 Loss probability without coding

This section discusses the loss probability due to buffer overflow in a system without coding. Since the link capacity is finite, if the number of packets generated by users exceeds the capacity, some packets should either be stored in the buffer, if possible, or be dropped from the queue. In a system without coding, every dropped packet is also a lost packet; thus, in this case, the loss probability is equal to the drop probability. We first study queue occupancy, i.e., the number of packets stored in the buffer, and, then, use it to analyze the loss probability in the following subsection.

2.5.1 Queue occupancy

Recall that $A_{(i)}^N(t)$, $1 \leq i \leq N$, $t \geq 1$, is the number of packet arrivals in the time slot t from the source i. Let $A^N(t)$, $t \geq 1$, denote the number of packets generated from all N sources in the time slot t:

$$A^N(t) = \sum_{i=1}^{N} A_{(i)}^N(t).$$

The queue occupancy $Q^N(t)$, $t \geq 0$, is defined to be the number of packets that remain in the buffer at the end of the time slot t. The packets that are transmitted in the time slot t include the packets that were in the buffer at the end of the previous time slot as well as newly arrived packets in the time slot t. Recall that the link is capable of transmitting C^N packets in one time slot and that at most B^N packets can be stored in the buffer. Therefore, the queue occupancy satisfies the following well-known equation:

$$Q^N(t) = (Q^N(t-1) + A^N(t) - C^N)^+ \wedge B^N. \tag{2.9}$$

The random variable $Q^N(t)$, $t \geq 1$, depends on $Q^N(t-1)$ and $A^N(t)$. As stated in Section 2.3, the buffer size under the critical-load scaling satisfies $B^N = \Theta(\sqrt{N})$; this implies that the queue occupancy also behaves as $\Theta(\sqrt{N})$. Hence, we consider the scaled queue occupancy $\hat{Q}^N(t)$, $t \geq 0$, defined as

$$\hat{Q}^N(t) = Q^N(t)/\sqrt{N}.$$

Note that (2.9) can be rewritten in the following form:

$$\hat{Q}^N(t) = (\hat{Q}^N(t-1) + \hat{A}^N(t) - \hat{C}^N)^+ \wedge \hat{B}^N,$$

where $\hat{A}^N(t) = (A^N(t) - \lambda N)/\sqrt{N}$. For fixed t, the distribution of $\hat{A}^N(t)$ tends to the normal distribution with zero mean and variance $\lambda(1-\lambda)$, as $N \to \infty$ (due to the CLT). Moreover, $\hat{C}^N \to \beta$ and $\hat{B}^N \to b$, as $N \to \infty$ (see (2.4)). Assuming that all processes are in their stationary regimes, it can be shown that for fixed t (e.g., see [45, Sec. 2.3 and Ch. 5])

$$\hat{Q}^N(t) \Rightarrow \hat{Q},$$

as $N \to \infty$, for a random variable $\hat{Q}$, whose distribution function $F_{\hat{Q}}$ satisfies the following integral equation:

$$F_{\hat{Q}}(x) = \begin{cases} 0, & x < 0, \\ \int_{[0,b]} \Phi_{-\beta,\sigma^2}(x-y)\,dF_{\hat{Q}}(y), & 0 \leq x < b, \\ 1, & x = b, \end{cases} \tag{2.10}$$

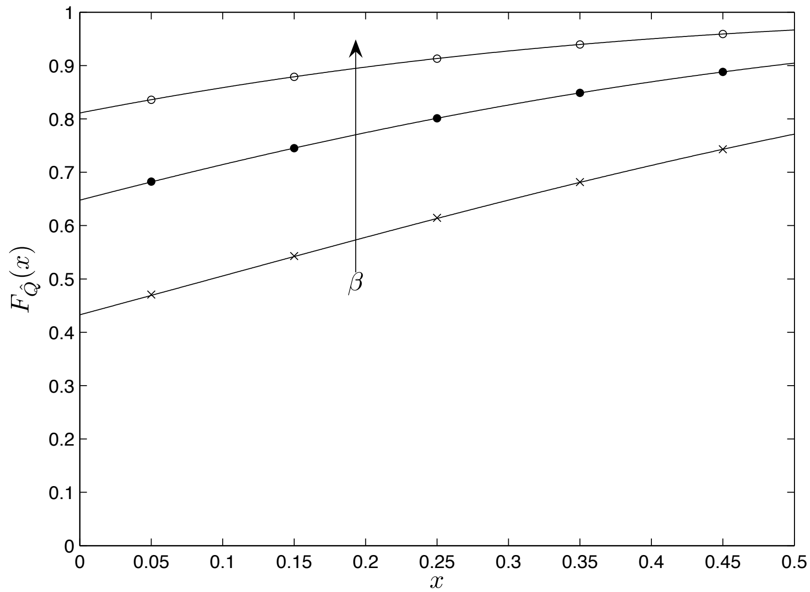


Figure 2.4: The distribution function $F_{\hat{Q}}$ of the stationary scaled queue occupancy $\hat{Q}^N(t) = Q^N(t)/\sqrt{N}$ in the limit as $N \to \infty$ (see (2.10)) for $\lambda = 0.5$, $b = 0.5$ and $\beta \in \{0.1, 0.3, 0.5\}$. Simulation results for $N = 100$ and $\beta = 0.1$ ($\times$), $\beta = 0.3$ ($\bullet$) and $\beta = 0.5$ ($\circ$) are also shown.

where $\Phi_{-\beta,\sigma^2}$ denotes the normal distribution function with mean $-\beta$ and variance $\sigma^2 = \lambda(1-\lambda)$. Note that $F_{\hat{Q}}$ has discontinuities at $x = 0$ and $x = b$.

Figure 2.4 shows the distribution functions of $\hat{Q}$ for $\lambda = 0.5$, $b = 0.5$ and $\beta \in \{0.1, 0.3, 0.5\}$, which are numerically computed from (2.10). For a fixed value of x, $0 \leq x < b$, the value of $F_{\hat{Q}}(x)$ increases as β increases since larger β implies a larger capacity. In addition, this figure includes the estimated values of $F_{\hat{Q}^N}(x)$ (by simulation) for $N = 100$, $B^N = 5$ ($b = 0.5$) and $C^N \in \{51, 53, 55\}$ ($\beta \in \{0.1, 0.3, 0.5\}$).

2.5.2 Loss probability

Let $L^N(t)$, $t \geq 1$, denote the number of lost packets in the time slot t. Without coding, a dropped packet is also a lost packet since no dropped packets can be recovered. Therefore, we have

$$L^N(t) = (Q^N(t-1) + A^N(t) - C^N - B^N)^+. \tag{2.11}$$

The loss probability p^N is defined to be the long-term ratio of the number of lost packets to the total number of arrivals from N sources. Given that the system is in stationarity and it is ergodic, p^N can equivalently be represented by

$$p^N = \mathbb{E}[L^N(t)]/\mathbb{E}[A^N(t)]. \tag{2.12}$$

The preceding equality and (2.11) yield

$$p^N = \mathbb{E}\,(Q^N + A^N - C^N - B^N)^+/\lambda N, \tag{2.13}$$

where A^N is equal in distribution to $A^N(t)$, and Q^N has the stationary distribution of $Q^N(t)$; the random variables Q^N and A^N are independent. As discussed in Section 2.3, the loss probability under the critical-load scaling behaves as $\Theta(1/\sqrt{N})$, as $N \to \infty$, and, thus, we define the scaled loss probability $\hat{p}^N$ by

$$\hat{p}^N = \lambda\sqrt{N}p^N = \mathbb{E}\,(\hat{Q}^N + \hat{A}^N - \hat{C}^N - \hat{B}^N)^+,$$

where $\hat{A}^N = (A^N - \lambda N)/\sqrt{N}$ and $\hat{Q}^N = Q^N/\sqrt{N}$. Further-more, the limiting (as $N \to \infty$) scaled loss probability $\hat{p}$ is defined by

$$\hat{p} = \lim_{N\to\infty} \hat{p}^N = \mathbb{E}\,(\hat{Q} + \hat{A} - \beta - b)^+, \tag{2.14}$$

where $\hat{A}^N \Rightarrow \hat{A}$, $\hat{Q}^N \Rightarrow \hat{Q}$, as $N \to \infty$, and the random variables $\hat{A}$ and $\hat{Q}$ are independent.

Figure 2.5 shows $\hat{p}$ as a function of β for $\lambda = 0.5$ and $b \in \{0, 0.5, 1\}$. As expected, the loss probability decreases when β (capacity) or b (buffer size) increase. Moreover, this figure includes estimated values of $\hat{p}^N$ (by simulation) for $N = 100$ and $B^N \in \{0, 5, 10\}$ ($b \in \{0, 0.5, 1\}$). This example illustrates the applicability of our asymptotic analysis to systems with a finite number of users.

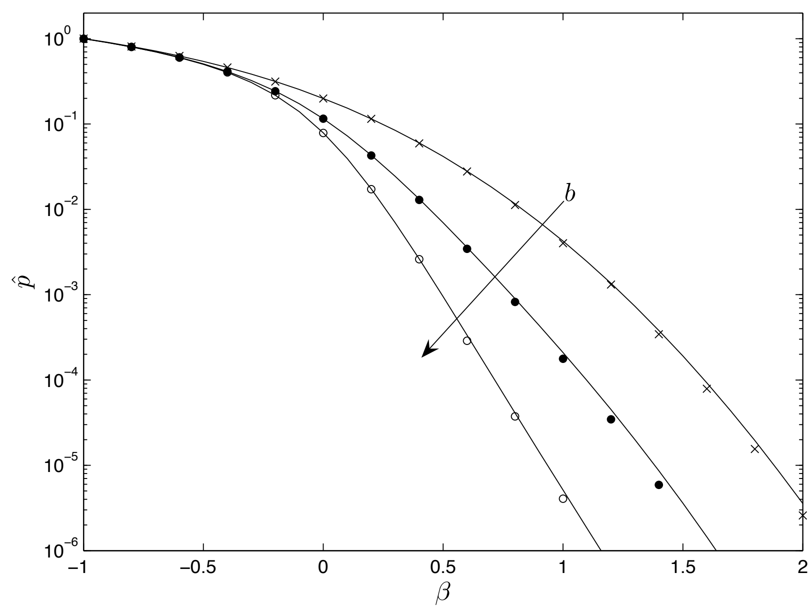


Figure 2.5: The scaled loss probability without coding $\hat{p}^N = \lambda\sqrt{N}p^N$ in the limit as $N \to \infty$ (see (2.14)) for $\lambda = 0.5$ and $b \in \{0, 0.5, 1\}$. Simulation results for $N = 100$ and $b = 0$ ($\times$), $b = 0.5$ ($\bullet$) and $b = 1$ ($\circ$) are also shown.

The loss probability can be approximated for large values of $|\beta|$. To this end, we have $\hat{p} \approx \mathbb{E}\,(\hat{A} - \beta - b)^+$ for $\beta \gg 0$ since the buffer is likely to be empty when the link capacity is larger than the offered load. In this case, it follows that

$$\hat{p} \approx \int_{\beta+b}^{\infty} (x - \beta - b)\varphi_{0,\sigma^2}(x)dx \approx \frac{\sigma^4}{(\beta+b)^2}\varphi_{0,\sigma^2}(\beta+b), \tag{2.15}$$

where φ_{0,σ^2} is the probability density function of the normal distribution with zero mean and variance $\sigma^2 = \lambda(1-\lambda)$; the approximation follows from $(x^{-1} - x^{-3})\varphi_{0,1}(x) < 1 - \Phi_{0,1}(x) < x^{-1}\varphi_{0,1}(x)$ (e.g., see [24, p.175]). On the other hand, if $\beta \ll 0$, then the buffer is likely to be full since the offered load is greater than the link capacity. Thus, in that case, we obtain

$$\hat{p} \approx \mathbb{E}\,(\hat{A} - \beta)^+ \approx -\beta. \tag{2.16}$$

In such an over-loaded system, all extra arrivals, which exceeds the capacity, are likely to be dropped from the queue since the buffer is full with high probability.

2.6 Drop probability with coding

In this section, we examine the *drop* probability when coding is employed. When a system utilizes coding, the offered load is increased by additional coded packets, and, consequently, more packets are likely to be dropped from the buffer, compared to a system without coding. Note that in this case, the *drop* probability should be differentiated from the *loss* probability since some of the dropped packets can be recovered from the received subset of packets by decoding. We discuss the *loss* probability under coding in the next section.

Recall that $H^N_{(i)}(t)$, $1 \le i \le N$, $t \ge 1$, denotes the number of packet arrivals from the encoder of the source i to the buffer in the time slot t. Assuming that the system is in stationarity, (2.1) implies

$$\mathbb{P}\left[H^N_{(i)}(t) = h\right] = \begin{cases} 1 - \lambda, & h = 0, \\ \lambda - \lambda/M^N, & h = 1, \\ \lambda/M^N, & h = 1 + \alpha, \end{cases} \tag{2.17}$$

for $1 \le i \le N$, $t \ge 1$. The mean λ^N_* and the variance $(\sigma^N_*)^2$ of $H^N_{(i)}(t)$ are respectively given by

$$\begin{aligned} \lambda^N_* &= \lambda + \alpha\lambda/M^N, \\ (\sigma^N_*)^2 &= \lambda^N_*(1 - \lambda^N_*) + \alpha(1+\alpha)\lambda/M^N; \end{aligned} \tag{2.18}$$

note that $\lambda^N_* \to \lambda$ and $(\sigma^N_*)^2 \to \lambda(1-\lambda)$, as $N \to \infty$. Let $H^N(t)$, $t \ge 1$, denote the total number of packets sent from the encoders of N sources to the buffer in the time slot t:

$$H^N(t) = \sum_{i=1}^{N} H^N_{(i)}(t).$$

The drop probability p^N_D is defined to be the long-term ratio of the number of dropped packets to the total number of arrivals from the encoders. Then, analogously to (2.13), we have

$$p_D^N = \mathbb{E}\,(Q_*^N + H^N - C^N - B^N)^+/\lambda_*^N N,$$

where H^N is equal in distribution to $H^N(t)$, and Q_*^N has the stationary distribution of the queue occupancy when coding is used. When coding is employed, original arrival processes are altered by additional coded packets as stated in Section 2.2. Thus, the queue occupancy is also affected by the coding scheme. Under the critical-load scaling, the drop probability is $\Theta(1/\sqrt{N})$, as $N \to \infty$. Therefore, we consider the scaled drop probability $\hat{p}_D^N$ defined by

$$\begin{aligned}\hat{p}_D^N &= \lambda\sqrt{N}p_D^N \\ &= \frac{\lambda}{\lambda_*^N}\mathbb{E}\,(\hat{Q}_*^N + \hat{H}^N + (\lambda_*^N - \lambda)\sqrt{N} - \hat{C}^N - \hat{B}^N)^+,\end{aligned}$$

where $\hat{H}^N = (H^N - \lambda_*^N N)/\sqrt{N}$ and $\hat{Q}_*^N = Q_*^N/\sqrt{N}$; note that $(\lambda_*^N - \lambda)\sqrt{N} \to \alpha/m$, as $N \to \infty$. Next we define $\hat{p}_D$ as the limiting (as $N \to \infty$) scaled drop probability:

$$\hat{p}_D = \lim_{N\to\infty} \hat{p}_D^N = \mathbb{E}\,(\hat{Q}_* + \hat{H} + \alpha/m - \beta - b)^+, \tag{2.19}$$

where $\hat{H}^N \Rightarrow \hat{H}$, $\hat{Q}_*^N \Rightarrow \hat{Q}_*$, as $N \to \infty$, and the random variables $\hat{H}$ and $\hat{Q}_*$ are independent. It can be shown that $\hat{H}$ has the normal distribution with zero mean and variance $\lambda(1-\lambda)$ (due to the CLT) and that the distribution of $\hat{Q}_*$ satisfies (2.10) with β replaced by $\beta - \alpha/m$ (e.g., see [12, Sec. 25]).

We define the coding overhead ζ as a function of α/m:

$$\zeta(\alpha/m) = \frac{\hat{p}_D}{\hat{p}} = \frac{\mathbb{E}\,(\hat{Q}_* + \hat{H} + \alpha/m - \beta - b)^+}{\mathbb{E}\,(\hat{Q} + \hat{A} - \beta - b)^+}. \tag{2.20}$$

In Figure 2.6, the solid lines show $\zeta(\alpha/m)$ for $\lambda = 0.5$, $b = 0.5$ and $\beta \in \{0, 0.5, 1\}$. The dashed lines are for $\lambda = 0.5$, $\beta = 0.5$ and $b \in \{0, 1\}$. Since the additional offered load due to coding increases as α/m increases, ζ is an increasing function of α/m.

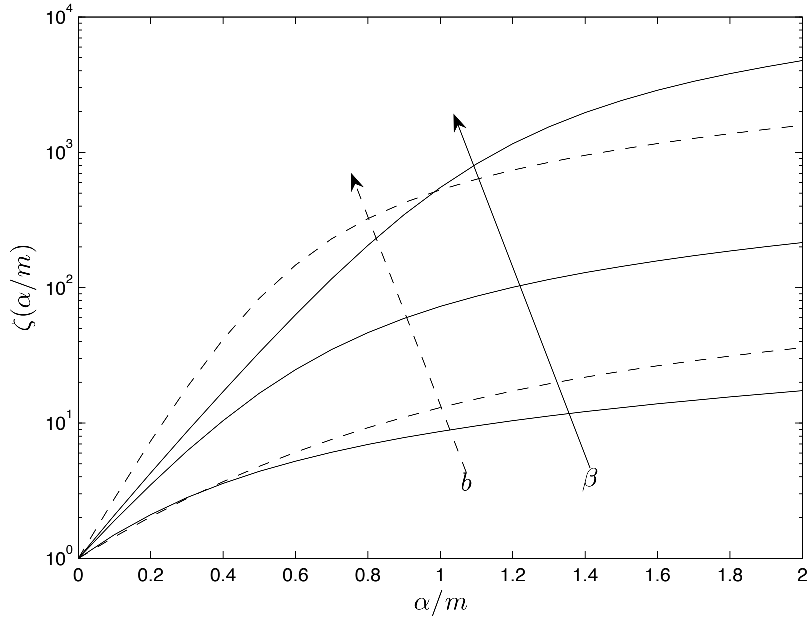


Figure 2.6: The coding overhead $\zeta\,(\alpha/m) = \hat{p}_D/\hat{p}$ (see (2.20)) for $\lambda = 0.5$: solid lines for $b = 0.5$ and $\beta \in \{0, 0.5, 1\}$, and dashed lines for $\beta = 0.5$ and $b \in \{0, 1\}$.

The figure also illustrates that, for a fixed value of α/m, the value of ζ increases when β (capacity) or b (buffer size) increase. Note that $\hat{p}_D$ is exactly equal to $\hat{p}$ when β is replaced by $\beta - \alpha/m$ in (2.14). As seen in Figure 2.5, the larger the β is, the faster the $\hat{p}$ decreases as β increases. Thus, $\hat{p}$ decreases faster than $\hat{p}_D$ when β increases. Approximations given in (2.15) and (2.16) also support this observation. Namely, for $\alpha/m \gg \beta$, $\hat{p}_D$ decreases linearly when β increases while $\hat{p}$ decreases exponentially; when $\alpha/m \ll \beta$, both $\hat{p}$ and $\hat{p}_D$ decrease exponentially, but $\hat{p}$ decreases faster than $\hat{p}_D$ due to α/m term. Similar reasoning can be applied to the case of b.

Since decoding is performed on a per-block basis, the loss probability depends not only on the drop probabilities of individual packets but also on the distribution of the number of dropped packets in a block. Thus, in order to evaluate the loss probability, one needs to consider the behavior of the packet drops in a block. The following theorem characterizes the number of dropped packets in a block in the limit as $N \to \infty$.

Theorem 2.1. *Suppose that the system is in stationarity, and consider the critical-load scaling. Let D_d^N be the number of dropped packets among M^N data packets in a block. Then, in the limit as $N \to \infty$, D_d^N is Poisson:*

$$\mathbb{P}\left[D_d^N = k\right] \to \frac{(m\hat{p}_D)^k}{k!} e^{-m\hat{p}_D},$$

as $N \to \infty$, where $\hat{p}_D$ is the limiting scaled drop probability that satisfies (2.19)*. Furthermore, if D_c^N is the number of dropped packets among additional α coded packets in a block, then, as $N \to \infty$,*

$$\mathbb{P}\left[D_c^N = 0\right] \to 1.$$

Proof. See Appendix A.2.

Informally, the theorem can be interpreted as follows. Consider a single block, and suppose that the packets in this block are dropped independently with drop probability equal to p_D^N. Then, the number of dropped packets in the block of length $M^N = \lfloor m\lambda\sqrt{N} \rfloor$ follows the binomial distribution:

$$\mathbb{P}\left[D_d^N = k\right] = \binom{\lfloor m\lambda\sqrt{N} \rfloor}{k} \left(p_D^N\right)^k \left(1 - p_D^N\right)^{\lfloor m\lambda\sqrt{N} \rfloor - k}.$$

It is straightforward to verify that this binomial distribution tends to the Poisson distribution with mean $m\hat{p}_D$ in the limit as $N \to \infty$. However, packet drops are not independent in a system with finite N. The drop probability of a packet in a fixed time slot t depends on the total number of arrivals from the encoders of N sources in the time slot t and the queue occupancy at the end of the time slot $t-1$. Since both the total arrival process and the queue occupancy have the Markov property, as discussed in Section 2.2, packet drops have dependency across time. However, Theorem 2.1 shows that the effect of this time dependency becomes negligibly small

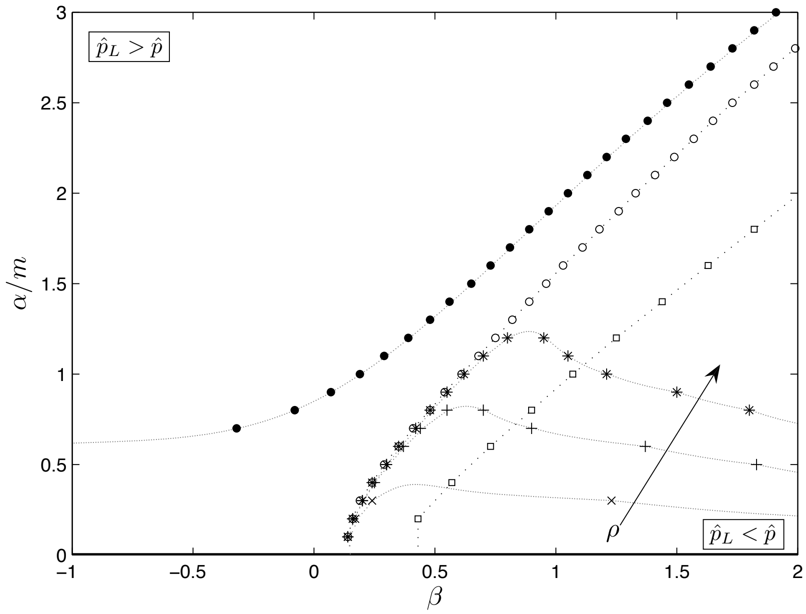


Figure 2.7: The boundary where $\hat{p}_L^N = \hat{p}^N$, as $N \to \infty$, for the ideal block code ($\bullet$), a systematic MDS code ($\circ$), coding with overlapping blocks ($\square$) and partial coding with $\rho = 0.9$ ($\times$), $\rho = 0.99$ ($+$) and $\rho = 0.999$ ($*$) when $\lambda = 0.5$, $b = 0.5$ and $m = 10$. Note that $\alpha/m \in \{0.1, 0.2, \dots\}$ since $\alpha \in \mathbb{N}$. For each boundary, the lower-right and upper-left areas are the regions where application-layer coding is beneficial ($\hat{p}_L < \hat{p}$) and detrimental ($\hat{p}_L > \hat{p}$), respectively.

in the limit as $N \to \infty$. Given that the drop probability in a fixed time slot t is $\Theta(1/\sqrt{N})$, the possibility that a packet is dropped shortly after another packet is dropped from the same block diminishes as $N \to \infty$. That is, when a packet is dropped, we can assure, with high probability, that enough time has elapsed for the system to enter its stationary regime.

Finally, Theorem 2.1 also indicates that the number of dropped packets in a block is $\Theta(1)$. Since the number of additional coded packets for each block is also $\Theta(1)$, the dropped packets can be recovered in some cases, as intended by means of coding. This result verifies the relevance of the considered critical-load scaling to the study of the effectiveness of application-layer coding.

2.7 Loss probability with coding

Here, we consider the loss probability with coding for erasure codes discussed in Section 2.4. The loss probability p_L^N is defined as the long-term ratio of the number of lost packets after decoding to the total number of data packets. Let L^N denote the number of lost packets among $M^N = \lfloor m\lambda\sqrt{N}\rfloor$ data packets in a block. Then, analogously to (2.12), we have

$$p_L^N = \mathbb{E}\, L^N / M^N.$$

The scaled loss probability $\hat{p}_L^N$ is given by

$$\hat{p}_L^N = \lambda\sqrt{N} p_L^N = \mathbb{E}\, L^N / m^N, \tag{2.21}$$

where $m^N = \lfloor m\lambda\sqrt{N}\rfloor / \lambda\sqrt{N}$, and the limiting (as $N \to \infty$) scaled loss probability $\hat{p}_L$ is defined by

$$\hat{p}_L = \lim_{N\to\infty} \hat{p}_L^N = \lim_{N\to\infty} \mathbb{E}\, L^N / m. \tag{2.22}$$

The following theorem specifies the limiting scaled loss probability for the coding schemes discussed in Section 2.4.

Theorem 2.2. *Suppose that the system is in stationarity, and consider the critical-load scaling. Let $D_{(x)}$, $x > 0$, denote a Poisson random variable with mean $xm\hat{p}_D$, where $\hat{p}_D$ is the limiting scaled drop probability in* (2.19)*. Then*

(i) for the ideal block code:

$$\hat{p}_L = \mathbb{E}\,(D_{(1)} - \alpha)^+ / m, \tag{2.23}$$

(ii) for the systematic MDS code:

$$\hat{p}_L = \mathbb{E}\big[D_{(1)} \cdot 1_{\{D_{(1)} > \alpha\}}\big] / m, \tag{2.24}$$

(iii) for the partial coding:

$$\hat{p}_L = \hat{p}_D - \mathbb{E}\big[D_{(\rho)} \cdot 1_{\{D_{(\rho)} \le \alpha\}}\big]/m,$$

(iv) for the coding with overlapping blocks:

$$\hat{p}_L = 2\mathbb{E}\big[D_{(1/2)}(1 - \xi(D_{(1/2)}))^2\big]/m,$$

where $\xi(0) = 1$*,* $\xi(k) = 0$*,* $k > \alpha/2$*, and* $\xi(k)$*,* $1 \le k \le \alpha/2$*, satisfies the following equation:*

$$\xi(k) = \mathbb{P}[D_{(1/2)} \le \alpha/2 - k] + \sum_{i=\alpha/2-k+1}^{\alpha/2} \xi(i)\mathbb{P}[D_{(1/2)} = i].$$

Proof. See Appendix A.6. □

2.7.1 Ideal block code

By using (2.14), (2.19) and (2.23), one can compare the loss probabilities with and without coding for a given set of parameters (β, b, α, m). In Figure 2.7, we show the boundary where $\hat{p}_L = \hat{p}$ for the ideal block code when $\lambda = 0.5$, $b = 0.5$ and $m = 10$. Note that the boundary partitions the parameter space $(\beta, \alpha/m)$ into two regions: the upper left region where $\hat{p}_L > \hat{p}$ (no coding is preferable) and the lower right region where $\hat{p}_L < \hat{p}$ (coding is preferable). The ideal block code has the largest region where coding is advantageous among all linear block codes since it achieves the best performance among such codes. It is interesting to observe that employing even the ideal block code can be counter-productive for some set of system parameters. In particular, consider an over-loaded system ($\lambda N > C^N$ or, equivalently, $\beta < 0$). In this case, the expected number of dropped packets in a single block *due to coding overhead* is $(p_D - p) \cdot M^N \approx \alpha$. However, the number of dropped packets in a block is not a constant; this leads to a situation where more than α packets are dropped

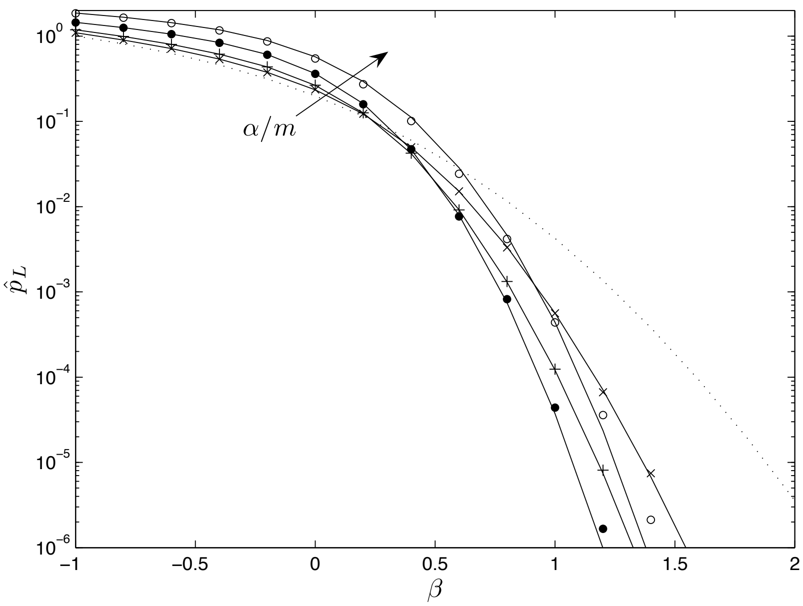


Figure 2.8: The scaled loss probability with coding $\hat{p}_L^N = \lambda\sqrt{N}p_L^N$ in the limit as $N \to \infty$ (see (2.24)) for a systematic MDS code when $\lambda = 0.5$, $b = 0$, $m = 10$ and $\alpha \in \{1, 2, 5, 10\}$. The variances of arrival processes are adjusted to be $(\sigma_*^N)^2$ for $N = 900$ (see (2.18)) instead of the limiting value $\lambda(1-\lambda)$. Simulation results for $N = 900$ and $\alpha = 1$ ($\times$), $\alpha = 2$ ($+$), $\alpha = 5$ ($\bullet$) and $\alpha = 10$ ($\circ$) are also shown. The dotted line is for the scaled loss probability without coding.

in some blocks and fewer than α packets are dropped in the others. In such a case, coding is not efficient in recovering dropped packets.

2.7.2 Systematic MDS code

Figure 2.8 shows $\hat{p}_L$ as a function of β for a systematic MDS code when $\lambda = 0.5$, $b = 0$, $m = 10$ and $\alpha \in \{1, 2, 5, 10\}$. Note that, just for this example, we use the variance $(\sigma_*^N)^2$ for $N = 900$ (see (2.18)) instead of the limiting value $\lambda(1-\lambda)$ when we compute $\hat{p}_D$, which determines $\hat{p}_L$. Since the loss probability is sensitive to the variances of arrival processes, this adjustment is needed to make our asymptotic result to be applicable for finite N. The figure also shows the estimated values of $\hat{p}_L^N$ (by simulation) for $N = 900$, $M^N = 150$ ($m = 10$) and $\alpha \in \{1, 2, 5, 10\}$. One can observe that simulation results agree with analytical results well in this example. For an over-loaded system ($\beta \ll 0$), $\hat{p}_L$ increases as α increases because the number of dropped packets in a block is likely to be beyond the number that can be recovered. Therefore, in this case, additional packets behave just as overhead. On the other hand, if a system is under-loaded ($\beta \gg 0$), then $\hat{p}_L$ decreases at first as α increases since the benefit of coding exceeds its overhead in this case. For some value of α, $\hat{p}_L$ is minimized, i.e., the coding benefit is maximized. If α is increased further, however, $\hat{p}_L$ starts increasing, and coding is not beneficial anymore. The dotted line represents the limiting scaled loss probability without coding $\hat{p}$ (see (2.14)). One can find a point where $\hat{p}_L = \hat{p}$ for each value of α/m. These points correspond to the boundary where schemes with and without coding have the same performance (shown in Figure 2.7). We refer the reader to Section 2.8 for a further discussion on application-layer coding with a systematic MDS code.

2.7.3 Partial coding

Figure 2.7 includes the boundary where $\hat{p}_L = \hat{p}$ for partial coding with $\rho \in \{0.9, 0.99, 0.999\}$ when $\lambda = 0.5$, $b = 0.5$ and $m = 10$. Note that the partial coding scheme with $\rho = 1$ is identical to the scheme with pure block coding. As seen in the figure, the region where coding is advantageous expands as ρ increases. Recall that the partial coding scheme might be beneficial only when the number of dropped packets in a block is greater than the number of additional packets in the block (see Section 2.4). In the region where coding is advantageous, however, the drop probability is so small that the number of dropped packets is not likely to be greater than the number of additional packets in a block. One can observe that partial coding is getting worse as β (capacity) increases. This result is consistent with the previous observation since larger β results in smaller drop probability.

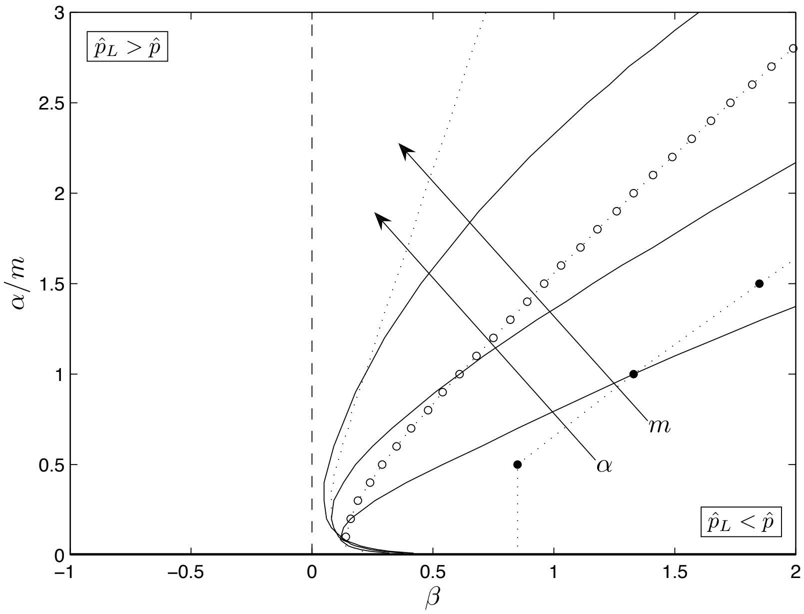


Figure 2.9: The boundary where $\hat{p}_L^N = \hat{p}^N$, as $N \to \infty$, for a systematic MDS code when $\lambda = 0.5$ and $b = 0.5$: solid lines for $\alpha \in \{2, 10, 50\}$, and dotted lines for $m = 2$ (•), $m = 10$ (∘) and $m = 50$ (·). For each boundary, the lower-right and upper-left areas are the regions where application-layer coding is beneficial ($\hat{p}_L < \hat{p}$) and detrimental ($\hat{p}_L > \hat{p}$), respectively.

2.7.4 Coding with overlapping blocks

In Figure 2.7, we plot the boundary where $\hat{p}_L = \hat{p}$ for coding with overlapping blocks. As seen in the figure, the described coding scheme with overlapping blocks underperforms compared to the one with non-overlapping blocks as far as probability of loss is concerned. This stems from the fact that the non-overlapping scheme can recover up to α dropped packets per block, while the overlapping version is capable of recovering only $\alpha/2$ dropped packets per half block. It should be noted, however, that the overlapping scheme might result in shorter decoding delays.

2.8 Discussion

In this section, we discuss the performance of application-layer coding with a systematic MDS code. Figure 2.9 shows the boundary where $\hat{p}_L = \hat{p}$ for a systematic MDS code when $\lambda = 0.5$, $b = 0.5$, $m \in \{2, 10, 50\}$ and $\alpha \in \{2, 10, 50\}$. If we increase

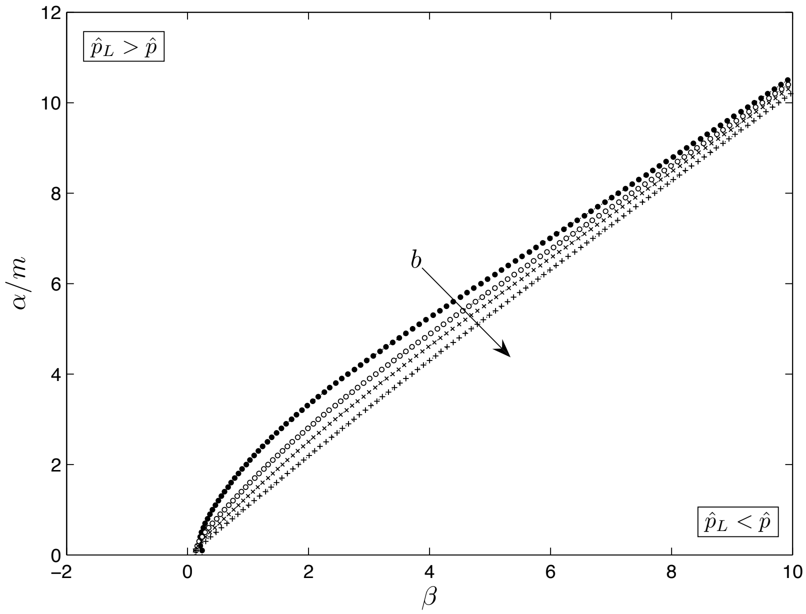


Figure 2.10: The boundary where $\hat{p}_L^N = \hat{p}^N$, as $N \to \infty$, for a systematic MDS code when $\lambda = 0.5$, $m = 10$ and $b \in \{0, 0.5, 1, 2\}$. For each boundary, the lower-right and upper-left areas are the regions where application-layer coding is beneficial ($\hat{p}_L < \hat{p}$) and detrimental ($\hat{p}_L > \hat{p}$), respectively.

the length of a block while increasing the number of additional packets as well, then the asymptotic drop probability does not change, but the number of possible packet drop patterns that can be recovered in a block increases. For example, suppose that $\alpha = 1$ and 1 dropped packet can be recovered in a block. If we double the length of a block and generate 2 coded packets per double-length block, then 2 dropped packets can be recovered in one half of a block provided that no packets are dropped in the other half. Note that larger m (and larger α for fixed α/m) implies a longer block. Hence, the region where coding is advantageous to no coding increases as m (and α for fixed α/m) increases. Note that this reasoning applies to a large class of block codes.

Figure 2.10 shows the boundary where $\hat{p}_L = \hat{p}$ for a systematic MDS code when $\lambda = 0.5$, $m = 10$ and $b \in \{0, 0.5, 1, 2\}$. Recall that Figure 2.6 indicates that the coding overhead $\zeta = \hat{p}_D/\hat{p}$ increases as b (buffer size) increases. Thus, larger b results in a

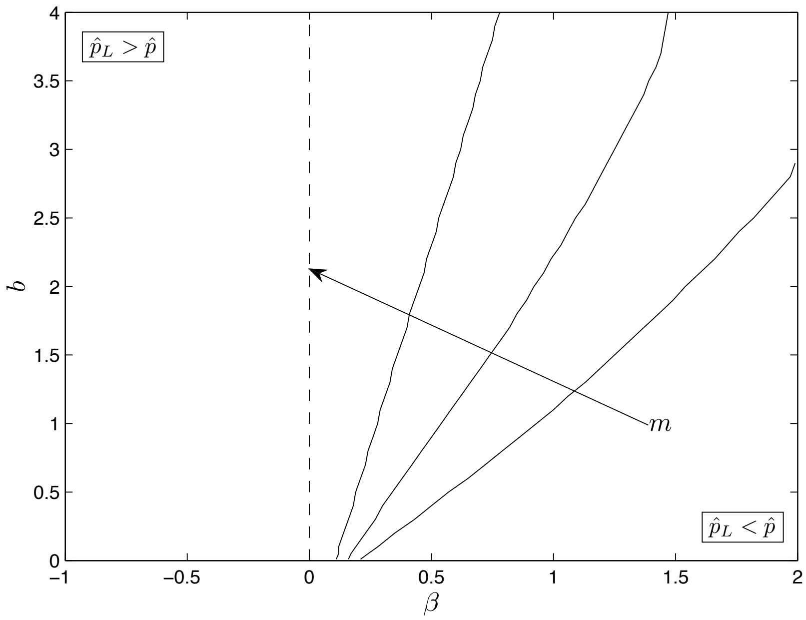


Figure 2.11: The boundary where a buffer-less system with coding (using a systematic MDS code) and a system with a buffer but no coding have the same performance, i.e., $\min_\alpha \hat{p}_L^N$ for $b = 0$ and $\hat{p}^N$ for $b > 0$ are equal, as $N \to \infty$, for $\lambda = 0.5$ and $m \in \{10, 20, 50\}$. For each boundary, the lower-right and upper-left areas are the regions where the buffer-less system with coding has better performance than the system with a buffer but no coding ($\hat{p}_L < \hat{p}$) and vice versa ($\hat{p}_L > \hat{p}$), respectively. The figure also shows that the boundary tends to $\beta = 0$ (the dashed line) in the limit as $m \to \infty$.

smaller region where coding outperforms no coding. It is interesting to observe that the critical values of α/m for different values of b converge as β (capacity) increases. In particular, the boundary tends to $\alpha/m = \beta$ as β increases. As long as the system is under-loaded ($\alpha/(m\beta) < 1$), for large β (and, hence, large $\beta - \alpha/m$ for a fixed ratio of $\alpha/(m\beta)$), the buffer is likely to be empty with high probability; when the system is over-loaded ($\alpha/(m\beta) > 1$), however, the buffer is likely to be full. This behavior is not significantly impacted by the buffer size b. Thus, (for $\beta \gg 0$) the value of b only has a secondary effect on the loss probability, and, therefore, does not perform a significant role in determining the boundary.

In Figure 2.11, we plot the boundary where a buffer-less system with coding (using a systematic MDS code) and a system with a buffer but no coding have the same performance, i.e., $\min_\alpha \hat{p}_L$ for $b = 0$ and $\hat{p}$ for $b > 0$ are equal, for $\lambda = 0.5$ and

$m \in \{10, 20, 50\}$. Similarly to Figure 2.9, this figure also illustrates that larger m (block length) results in a larger region where coding is advantageous. Moreover, the figure indicates that the boundary tends to $\beta = 0$ (the dashed line) in the limit as $m \to \infty$. Informally, from (2.24), we can derive

$$\hat{p}_L = \hat{p}_D \sum_{k=\alpha}^{\infty} \frac{(m\hat{p}_D)^k}{k!} e^{-m\hat{p}_D}.$$

In the limit as $m \to \infty$, the Poisson distribution tends to the normal distribution with mean $m\hat{p}_D$ and variance $m\hat{p}_D$:

$$\hat{p}_L \approx \hat{p}_D \left(1 - \Phi_{0,1}\left(\frac{\alpha - m\hat{p}_D}{\sqrt{m\hat{p}_D}}\right)\right);$$

this, in turn, implies (for large values of m)

$$\hat{p}_L \approx \begin{cases} 0, & \alpha/m > \hat{p}_D, \\ \hat{p}_D/2, & \alpha/m = \hat{p}_D, \\ \hat{p}_D, & \alpha/m < \hat{p}_D. \end{cases}$$

Now, for an under-loaded system ($\beta > 0$), there exists some α that satisfies $\alpha/m > \hat{p}_D$ for large m. Thus, it is possible to reduce $\hat{p}_L$ to an arbitrary small value by increasing m (block length). On the other hand, for an over-loaded system ($\beta < 0$), we have $\hat{p}_D \approx \alpha/m - \beta > \alpha/m$ from (2.16). In this case, $\hat{p}_L \approx \hat{p}_D \approx \alpha/m - \beta \geq -\beta > 0$ can not be made arbitrarily close to zero even if m is increased indefinitely.

Table 2.2 shows estimated (by simulation) drop and loss probabilities (i.e., p_D^N and p_L^N, respectively) for different values of λ, M^N and α in a finite system with fixed $N = 100$, $C^N = 50$ and $B^N = 0$; note that not only λ, but also M^N and α impact offered load and, thus, utilization. For the considered values of λ, the results show that drop and loss probabilities are of the same order. Furthermore, we can see that employing coding is either beneficial or detrimental depending on the specific values

Table 2.2: Estimated (by simulation) drop and loss probabilities (i.e., p_D^N and p_L^N, respectively) for different values of λ, α and M^N in a finite system with fixed $N = 100$, $C^N = 50$ and $B^N = 0$. The table also shows approximated loss probabilities ($\tilde{p}_L^N$) based on Theorem 2.2(ii).

λ	M^N	α	Utilization[a]	p_D^N	p_L^N	$\tilde{p}_L^N$[b]
0.45	20	0	0.900	$9.14 \cdot 10^{-3}$	$9.14 \cdot 10^{-3}$	$9.14 \cdot 10^{-3}$
		1	0.945	$2.26 \cdot 10^{-2}$	$8.18 \cdot 10^{-3}$	$8.22 \cdot 10^{-3}$
		2	0.990	$4.51 \cdot 10^{-2}$	$1.07 \cdot 10^{-2}$	$1.03 \cdot 10^{-2}$
		3	1.035	$7.34 \cdot 10^{-2}$	$1.53 \cdot 10^{-2}$	$1.34 \cdot 10^{-2}$
	30	0	0.900	$9.14 \cdot 10^{-3}$	$9.14 \cdot 10^{-3}$	$9.14 \cdot 10^{-3}$
		1	0.930	$1.73 \cdot 10^{-2}$	$7.02 \cdot 10^{-3}$	$7.00 \cdot 10^{-3}$
		2	0.960	$3.06 \cdot 10^{-2}$	$7.30 \cdot 10^{-3}$	$7.16 \cdot 10^{-3}$
		3	0.990	$4.79 \cdot 10^{-2}$	$9.04 \cdot 10^{-3}$	$8.41 \cdot 10^{-3}$
0.49	100	0	0.9800	$3.12 \cdot 10^{-2}$	$3.12 \cdot 10^{-2}$	$3.12 \cdot 10^{-2}$
		1	0.9898	$3.61 \cdot 10^{-2}$	$3.50 \cdot 10^{-2}$	$3.51 \cdot 10^{-2}$
		2	0.9996	$4.19 \cdot 10^{-2}$	$3.85 \cdot 10^{-2}$	$3.86 \cdot 10^{-2}$
		3	1.0094	$4.85 \cdot 10^{-2}$	$4.17 \cdot 10^{-2}$	$4.18 \cdot 10^{-2}$
	200	0	0.9800	$3.12 \cdot 10^{-2}$	$3.12 \cdot 10^{-2}$	$3.12 \cdot 10^{-2}$
		1	0.9849	$3.36 \cdot 10^{-2}$	$3.35 \cdot 10^{-2}$	$3.36 \cdot 10^{-2}$
		2	0.9898	$3.64 \cdot 10^{-2}$	$3.61 \cdot 10^{-2}$	$3.62 \cdot 10^{-2}$
		3	0.9947	$3.96 \cdot 10^{-2}$	$3.87 \cdot 10^{-2}$	$3.90 \cdot 10^{-2}$

[a] Utilization = $(1 + \alpha/M^N)\lambda N/C^N$.
[b] Loss probability approximation based on Theorem 2.2(ii):

$$\tilde{p}_L^N = p_D^N \left(1 - \sum_{k=0}^{\alpha-1} \frac{(M^N p_D^N)^k}{k!} e^{-M^N p_D^N}\right);$$

note that p_D^N is evaluated by simulation.

of α and M^N. In particular, one can determine the boundary where systems with and without coding have the same performance (e.g., see $\alpha = 1$ and 2 for $\lambda = 0.45$ and $M^N = 20$). The results indicate that the derived approximations based on Theorem 2.2(ii) are fairly accurate when $N = 100$ and drop/loss probabilities are on the order of $10^{-3} \div 10^{-2}$. The applicability of the approximation is due to the fact that it is based on the CLT. The case $\lambda = 0.49$ can be viewed as an instance of a system in the over-load regime. However, the approximation from Theorem 2.2 provides a reasonable estimate as can be seen in the table. The analysis of the under-load regime is based on the large-deviation theory (e.g., see [10]), and, thus, the asymptotic approximation is valid only for extremely small drop/loss probabilities (e.g., on the order of $10^{-9} \div 10^{-7}$, and $N \gg 100$).

2.9 Concluding remarks

In this chapter, we investigated the effectiveness of application-layer coding for systems with a large number of users. The system consists of a single link with a finite buffer, and the loss probability was considered as the measure of the system performance. We first showed that the critical-load scaling is the relevant scaling to explore the effectiveness of coding. Next we examined the asymptotic behavior of the loss probabilities with and without coding, and established the boundary that partitions the system parameter space into two regions where coding is beneficial and detrimental. The asymptotic results showed that coding is advantageous for under-loaded systems with a certain set of system parameters; in over-loaded systems, however, coding is detrimental since the coding overhead exceeds its benefit. That is, application-layer coding enhances the performance in systems with low drop probabilities, but employing such coding in systems with high drop probabilities only worsens the performance. In addition, we illustrated in some simulation examples that our asymptotic results provide reasonable approximations for systems with a finite number of users.

Finally, we conclude this chapter with a comment on systems with priority. As stated in Subsection 2.2.2, a system can employ priority in order to improve its performance (implementing priority requires an extra level of system complexity). In systems with priority, data packets have priority over additional coded packets in the queue, and, thus, coded packets do not impact the drops of data packets.

Therefore, in this case, coding does not harm the system performance, and one can generate, ideally, as many coded packets as needed to enhance the performance. Let $W^N(t)$, $t \geq 1$, denote the number of *received* coded packets in the time slot t by all N users. In the ideal setup (assuming that every user generates a large enough number of coded packets to fill up the spare capacity in every time slot), we have $W^N(t) = (C^N - A^N(t) - Q^N(t-1))^+$; for simplicity, it is assumed that coded packets are not buffered. Since there are N receivers in the system, the *receiving rate* of coded packets per receiver p_W^N is then given by $p_W^N = \mathbb{E}\,(C^N - A^N - Q^N)^+/N$. From this expression, we can derive the *critical-load scaling* for systems with priority that is similar to the one for systems with non-priority (see Section 2.3). Namely, the link capacity and the buffer size are given by $C^N = \lfloor \lambda N + \beta\sqrt{N} \rfloor$ and $B^N = \lfloor b\sqrt{N} \rfloor$, respectively. Under this scaling, the drop probability of data packets and the receiving rate of coded packets are both $\Theta(1/\sqrt{N})$, as $N \to \infty$. Given a block of length $M^N = \lfloor m\lambda N \rfloor$, the expected numbers of dropped data packets and received coded packets are of the same order (i.e., $\Theta(1)$), and, therefore, it is feasible to find the boundary that partitions the system parameter space into two regions where employing coding is significantly and marginally beneficial.

CHAPTER III

Existence of fixed-point processes in linear loss networks

3.1 Introduction

A linear network is a tandem network consisting of a series of identical nodes in which customers (packets) enter the network at a fixed source node and are relayed from one node to another in a fixed order until they exit the network at a fixed destination node. This stochastic tandem network is motivated by various large-scale communication networks. One specific example is large-scale wireless networks with limited or no cross-traffic interference along the paths from sources to destinations [26, 29]; a sensor network where concurrent traffic volume is small relative to the size of the network belongs to such an example. Another motivating example is next-generation optical networks [39]. Even though the model we consider is basic, it is interesting to understand how the fundamental performance properties, such as throughput, behave as the size of the network increases, while the local resources at each node remain fixed. In this chapter, we particularly investigate asymptotic characteristics of the departure process of a linear network consisting of bufferless nodes, as the size of the network increases. The departure process is of interest since various performance properties of the network such as throughput and traffic burstiness can be obtained from it. Currently, little is known about the departure processes of linear networks with finite-buffer nodes, and, here, we focus on the simplest case where each node has no buffer. We note that as the size of the network grows large, the value of the feedback information from the destination decreases to the point where it is not useful due to large delays. Thus, it is reasonable to employ a simple control protocol and to assume that the input of the network is a fixed process determined by the information available at the source only, without any feedback from the network.

In a linear network, the output process from a node is in turn the input process to the next node. If the input and output processes at each node are equal in distribution, then such processes are called fixed-point processes. For infinite-buffer queues, the uniqueness and existence of fixed-point processes have been studied extensively [1, 18, 34, 38]. In particular, it is well known that Poisson processes are fixed-point processes for $\cdot/M/1/\infty$ queues [16]. The uniqueness of fixed-point processes for $\cdot/M/1/\infty$ queues was established in [1]. This result was extended in [18] for $\cdot/GI/1/\infty$ queues having service time distributions with finite means and unbounded supports. A similar result requiring only a finite mean can be found in [38]. In [34], the existence of fixed point processes was studied for $\cdot/GI/1/\infty$ queues with service time S satisfying $\int \mathbb{P}[S \geq u]^{1/2} du < \infty$. The linear network with $\cdot/M/1/\infty$ queues was considered in [36], and it was shown that for any ergodic input process with a finite rate, the departure process converges in distribution to the fixed-point process (i.e., a Poisson process) with the same rate, if the input rate is less than the service rate, as the size of the network increases. A similar asymptotic property was established in [38] for linear networks with $\cdot/GI/1/\infty$ queues.

In a linear network with finite-buffer queues, input and output processes are not comparable directly due to losses; in general, arrivals are denser than departures.

Under a proper scaling (to be specified in Section 3.3), however, it is feasible to characterize the asymptotic behavior of the (scaled) departure process, as the size of the network grows. This chapter focuses on a linear loss network consisting of $\cdot/M/1/0$ queues and provides an asymptotic characterization of the scaled departure process (i.e., the joint probability density function of any finite number of consecutive inter-departure times) as the size of the network increases. The asymptotic form of the departure process in such a network is not impacted by the input process as long as the input does not vary with the size of the network.

3.1.1 Notation

Throughout the chapter, we use the following notation:

(i) For $a \le b$, $a, b \in \mathbb{N}$, let $x_{a:b}$ be the $(b-a+1)$-tuple vector consisting of x_i, $a \le i \le b$, i.e., $x_{a:b} \equiv (x_a, \ldots, x_b)$; $x_{b:a} \equiv \emptyset$, $x_{a:a} \equiv x_a$, $(x_{a:c})_b \equiv x_b$ and $(x_{a:c})_{a:b} \equiv x_{a:b}$ for $a \le b \le c$. Moreover, let $x_{a:b}+c \equiv (x_a+c, x_{a+1}+c, \ldots, x_b+c)$.

(ii) Boldface symbols are used for denoting processes, e.g., $\mathbf{x} \equiv \{x(t), t \ge 0\}$; $\mathbf{x}_{a:b} \equiv \{x_{a:b}(t), t \ge 0\}$ for $a \le b$, $a, b \in \mathbb{N}$; $\mathbf{x}_{b:a} \equiv \emptyset$, $\mathbf{x}_{a:a} \equiv \mathbf{x}_a$, $(\mathbf{x}_{a:c})_b \equiv \mathbf{x}_b$ and $(\mathbf{x}_{a:c})_{a:b} \equiv \mathbf{x}_{a:b}$ for $a \le b \le c$. Let $\mathbf{x}_{a:b} + c_{a:b} \equiv \{x_{a:b}(t) + c_{a:b}, t \ge 0\}$.

(iii) $\mathbb{R}^n$, $n \in \mathbb{N}$, is the n-dimensional Euclidean space; $\mathbb{R} \equiv \mathbb{R}^1$. Let $\mathbb{R}^n_{\leqslant}$ be a subset of $\mathbb{R}^n$ such that

$$\mathbb{R}^n_{\leqslant} \equiv \{x_{1:n} \in \mathbb{R}^n : x_1 \le x_2 \le \cdots \le x_n\}.$$

(iv) For a right continuous function $f : \mathbb{R} \to \mathbb{R}$ with left limits, the left limit $f(t-)$ is defined as $f(t-) \equiv \lim_{s \uparrow t} f(s)$.

(v) For $f : (S, m) \to (S', m')$, let $\mathrm{Disc}(f)$ be the set of discontinuity points of f (see [45, p. 86]).

(vi) For $n \in \mathbb{N}$, let $\mathcal{D}^n[a,b] \equiv \mathcal{D}([a,b],\mathbb{R}^n)$ be the space of all $\mathbb{R}^n$-valued functions on $[a,b]$, which are right continuous at all $t \in [a,b)$ and have left limits at all $t \in (a,b]$, endowed with usual Skorohod J_1 topology; let $d_{[a,b]}(\cdot,\cdot)$ denote the standard J_1 metric on the space $\mathcal{D}^n[a,b]$ (see [45, Ch. 12]). As an extension, we set $\mathcal{D}^n \equiv \mathcal{D}([0,\infty),\mathbb{R}^n)$ as the space of all $\mathbb{R}^n$-valued functions, which are right continuous and have left limits everywhere in $[0,\infty)$ and $(0,\infty)$, respectively; we use $d(\cdot,\cdot)$ for denoting the standard J_1 metric on the space $\mathcal{D}^n$ (e.g., see [13, Section 16]). For notational simplicity, for $\mathbf{x}_{1:n}, \mathbf{y}_{1:n} \in \mathcal{D}^n$, let

$$d_{[a,b]}(\mathbf{x}_{1:n}, \mathbf{y}_{1:n}) \equiv d_{[a,b]}(\{x_{1:n}(t),\, a \le t \le b\}, \{y_{1:n}(t),\, a \le t \le b\}).$$

(vii) Let $\mathcal{D}^n_{\leqslant}$, $n \in \mathbb{N}$, be a subset of $\mathcal{D}^n$ such that

$$\mathcal{D}^n_{\leqslant} \equiv \{\mathbf{x}_{1:n} \in \mathcal{D}^n : x_{1:n}(t) \in \mathbb{R}^n_{\leqslant},\ \forall t \ge 0\}.$$

(viii) Let $\|\cdot\|_{[a,b]}$ be the uniform norm on $\mathcal{D}^n[a,b]$ (e.g., see [45, p. 393]), i.e.,

$$\|\mathbf{x}_{1:n}\|_{[a,b]} \equiv \sup_{a \le t \le b} \max_{1 \le i \le n} |x_i(t)|. \tag{3.1}$$

As an extension, we use $\|\cdot\|$ for the uniform norm on $\mathcal{D}^n$. Moreover, $\|x_{1:n}\| \equiv \max_{1 \le i \le n} |x_i|$ for $x_{1:n} \in \mathbb{R}^n$.

(ix) For $n \in \mathbb{N}$, let $\mathcal{C}^n[a,b] \equiv \mathcal{C}([a,b],\mathbb{R}^n)$ and $\mathcal{C}^n \equiv \mathcal{C}([0,\infty),\mathbb{R}^n)$ be the space of all continuous $\mathbb{R}^n$-valued functions on $[a,b]$ and $[0,\infty)$, respectively.

(x) Let $\stackrel{d}{=}$ denote equality in distribution. Symbols $\vee$, $\wedge$ and $\circ$ denote the maximum, minimum and composition operators, respectively.

(xi) Let $\mathbf{B}_i \equiv \{B_i(t),\, t \ge 0\}$, $i \in \mathbb{Z}$, be independent standard one-dimensional Brownian motions with $B_i(0) = 0$ a.s.. For $a \le b$, $a, b \in \mathbb{Z}$, $\mathbf{B}_{a:b} \equiv \{B_{a:b}(t),\, t \ge 0\}$ denotes a standard $(b-a+1)$-dimensional Brownian motion.

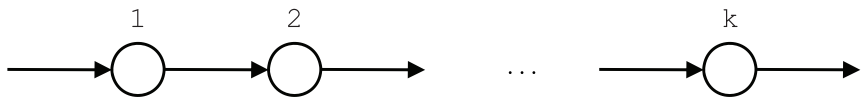


Figure 3.1: A linear loss network consists of a series of k bufferless nodes.

3.1.2 Organization

The rest of the chapter is organized as follows. The next section introduces the k-node linear loss network. In Section 3.3, we present the main result of the chapter in the following section. Section 3.4 contains some preliminary results for proving the main result. Concluding remarks and technical proofs can be found in Section 3.5 and Appendix B, respectively.

3.2 Model

A linear loss network consists of a series of k $\cdot/M/1/0$ nodes indexed by $1, 2, \ldots, k$ (see Figure 3.1). Each node has no waiting room (buffer), and service times at each node are exponentially distributed with unit mean; service times are assumed to be independent both across customers and across nodes. In the linear network, incoming customers enter the network at node 1, and, then, are relayed between adjacent nodes until they are either serviced at node k or dropped at some node due to the lack of buffer space. The nodes employ a work-conserving scheduling policy (i.e., they service customers whenever feasible). Since there are no waiting rooms at each node, in general, a customer dropping policy has to be specified. When the service times are exponentially distributed, the remaining service time of a customer being serviced is equal in distribution to the service time of a newly arrived customer (due to the memoryless property of the exponential distribution). Thus, departure times are insensitive to the dropping policy as long as the policy does not consider realizations of service times. We will define a policy that is convenient for analysis.

The input arrival process to the network is assumed to be stationary and ergodic. Given this assumption, the departure process is stationary and ergodic as well; this is due to the fact that the queue occupancy process is stationary and ergodic, which follows from the finite-buffer version of the Lindley's recursion and Loynes' construction (e.g., see [3, Section 2.6] or [33]). Finally, the input arrival rate is positive and the input process does not vary with the size of the network k; the input process and service times are independent.

3.3 Main result

This section provides the main result of the chapter, Theorem 3.1, that characterizes the departure process of the k-node linear loss network.

Definition 3.1 ([5])**.** The sequence of functions $H_n(x_{1:2n}) : \mathbb{R}^{2n}_{\leqslant} \to [0, \infty)$, $n \in \mathbb{N}$, is such that

$$H_1(x_{1:2}) = \frac{1}{\sqrt{\pi}} \int_{x_2 - x_1}^{\infty} e^{-t^2/4} dt,$$

and, for $n \geq 2$, it satisfies the following recursion:

$$H_n(x_{1:2n}) = \sum_{i=1}^{n} H_1(x_1, x_{2i}) H_{n-1}(x_{2:2i-1}, x_{2i+1:2n}) - \sum_{i=2}^{n} H_1(x_1, x_{2i-1}) H_{n-1}(x_{2:2i-2}, x_{2i:2n}).$$

Theorem 3.1. *Consider the k-node linear network with each node operating as a $\cdot/M/1/0$ queue. Suppose that the input arrival process to the network is a stationary and ergodic process with fixed rate, which does not vary with k. Let $X_{1:n}(k)$, $n, k \in \mathbb{N}$, be the n-tuple vector consisting of n consecutive stationary inter-departure times at node k. Then, as $k \to \infty$,*

$$\hat{X}_{1:n}(k) \equiv \frac{1}{\sqrt{k}} X_{1:n}(k) \Rightarrow Z_{1:n},$$

where $Z_{1:n}$ *is an n-tuple random vector with the joint probability density function given by*

$$f_{Z_{1:n}}(z_{1:n}) = -\sqrt{\pi}\frac{\partial^{n+1}}{\partial x_1 \partial x_3 \cdots \partial x_{2n-1} \partial x_{2n}} H_n(x_{1:2n})\Big|_{x_i=\sum_{j=1}^{\lfloor i/2 \rfloor} z_j,\, 1\le i\le 2n},$$

$$z_i \geq 0,\ 1 \leq i \leq n.$$

Proof. See Appendix B.1. □

Corollary 3.1. *We have*

$$f_{Z_1}(z) = \frac{z}{2} e^{-z^2/4}, \quad z \geq 0,$$

$$\text{and} \quad f_{Z_{1:2}}(z_{1:2}) = \frac{z_1 + z_2}{2\sqrt{\pi}} \left(e^{-(z_1^2+z_2^2)/4} - e^{-(z_1+z_2)^2/4} \right), \quad z_1, z_2 \geq 0,$$

implying that the correlation coefficient ρ_{Z_1,Z_2} *satisfies*

$$\rho_{Z_1,Z_2} = \frac{3-\pi}{4-\pi} \approx -0.16.$$

Remark 3.1. The limiting scaled departure process is not impacted by the input process as long as the input process is fixed. Thus, the asymptotic form of the output process is completely attributed to the characteristics of the network itself (i.e., the exponential distribution of service times and the lack of buffer space).

The following example illustrates Theorem 3.1. Given that the input arrival process is Poisson with rate 0.2, Figure 3.2 shows estimated (by simulation) probability density functions of the scaled stationary inter-departure times in the k-node linear loss network (i.e., $\hat{X}_1(k)$) for $k = 100$, 500 and 1000. Observe that as k increases, the density function $f_{\hat{X}_1(k)}$ approaches f_{Z_1} (solid line) described in Corollary 3.1, which is Rayleigh with mean $\sqrt{\pi}$.

For the purpose of illustrating the dependency between inter-departure times in the considered linear loss network, in Figure 3.3, we plotted estimated (by simulation)

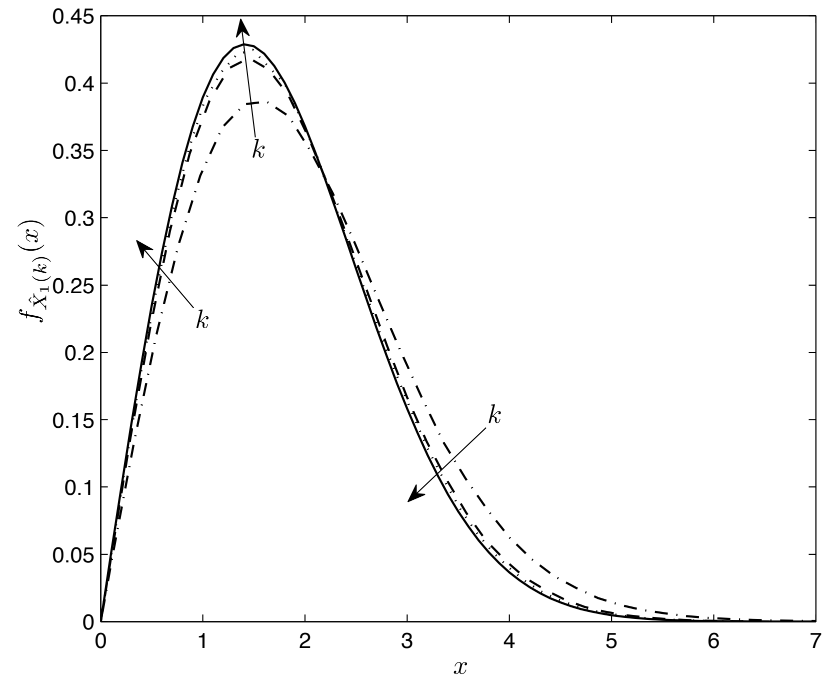


Figure 3.2: Estimated (by simulation; the bin size is 0.2 and the number of samples is $2.5 \cdot 10^6$) density functions of the scaled stationary inter-departure times at node k, i.e., $\hat{X}_1(k)$, for $k = 100$ (dash-dot line), $k = 500$ (dashed line) and $k = 1000$ (dotted line); the input arrival process is Poisson with rate 0.2. The asymptotic density function f_{Z_1} given in Corollary 3.1 is also plotted (solid line).

probability density functions of the sums of two scaled consecutive stationary inter-departure times at node k (i.e., $\hat{X}_1(k) + \hat{X}_2(k)$) for $k = 100$, 500 and 1000; as in the previous example, the input is Poisson with rate 0.2. From Corollary 3.1, we have

$$f_{Z_1+Z_2}(z) = ze^{-z^2/8}\left(\sqrt{2}\Phi(-z/2) - \frac{ze^{-z^2/8}}{2\sqrt{\pi}}\right),$$

for $z \geq 0$, where Φ denotes the standard normal distribution function. The figure shows that the density function $f_{\hat{X}_1(k)+\hat{X}_2(k)}$ approaches $f_{Z_1+Z_2}$ (solid line) as k increases. For comparison, we also plotted the density function of the sum of two independent Rayleigh random variables with the same mean $\sqrt{\pi}$ (dotted line with circles), which would be obtained if the inter-departure times were independent. As seen in the figure, assuming independence yields a more "dispersed" density function. This result is consistent with Corollary 3.1, which states that two consecutive

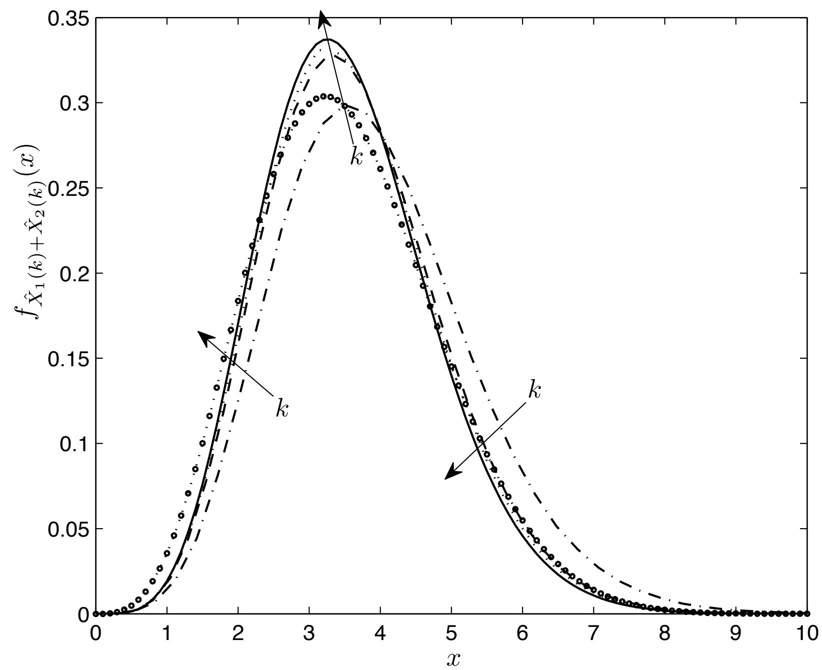


Figure 3.3: Estimated (by simulation; the bin size is 0.2 and the number of samples is $2.5 \cdot 10^6$) density functions of the sums of two consecutive scaled stationary inter-departure times at node k, i.e., $\hat{X}_1(k) + \hat{X}_2(k)$, for $k = 100$ (dash-dot line), $k = 500$ (dashed line) and $k = 1000$ (dotted line); the input arrival process is Poisson with rate 0.2. The asymptotic density function $f_{Z_1+Z_2}$ obtained from Corollary 3.1 (solid line) and the density function of a sum of two independent Rayleigh random variables with the same mean $\sqrt{\pi}$ (dotted line with circles) are plotted as well.

inter-departure times are negatively correlated.

Finally, the correlation coefficients ρ_{Z_1,Z_n} for $2 \leq n \leq 6$ computed from Theorem 3.1 by means of numerical integration are as follows: $\rho_{Z_1,Z_2} \approx -1.6 \cdot 10^{-1}$, $\rho_{Z_1,Z_3} \approx -1.8 \cdot 10^{-2}$, $\rho_{Z_1,Z_4} \approx -2.3 \cdot 10^{-3}$, $\rho_{Z_1,Z_5} \approx -3.0 \cdot 10^{-4}$ and $\rho_{Z_1,Z_6} \approx -4.5 \cdot 10^{-5}$.

3.4 Preliminary results

In this section, we provide some preliminary results for proving our main result, Theorem 3.1.

3.4.1 Convergence

Let $\{A_i\}_{i \in \mathbb{Z}}$ be the increasing sequence of (stationary) arrival times of customers at node 1; without loss of generality, we set $A_1 = 0$. Customers arriving at node 1 are labeled by integers in the increasing order according to their arrival times. The fixed arrival rate is denoted by $\lambda > 0$. Let $\{d_i(j)\}_{i\in\mathbb{Z}}$, $1 \le j \le k$, be the increasing sequence of (stationary) departure times of customers at node j. Then, we have

$$(3.2) \qquad X_i(k) = d_{i+1}(k) - d_i(k) \quad \text{and} \quad d_{i+1}(k) - d_1(k) = \sum_{j=1}^{i} X_j(k), \quad i \ge 1.$$

Recall that characteristics of the departure process are insensitive to the dropping policy due to the memoryless property of exponential service times (see Section 3.2). Here, for analytical and notational simplicity, we assume that customer 1 has the highest priority over all other customers. Under this policy, $d_1(j)$, $1 \le j \le k$, denotes the departure time of customer 1 from node j. Due to potential losses of customers at intermediate nodes, some customers (except customer 1) may not reach and be serviced at some node j. Hence, $d_i(j)$ for some $i \ne 1$ does not necessarily correspond to the departure time of customer i from node j.

We next consider an altered k-node linear loss network in which the input arrival process is produced by deterministically thinning the original input process by a factor of $\alpha > 1$, $\alpha \in \mathbb{R}$. In particular, we let $A_i^\alpha = A_{\lfloor \alpha(i-1) \rfloor + 1}$ for $i \in \mathbb{Z}$, where $\{A_i^\alpha\}_{i\in\mathbb{Z}}$ is the increasing sequence of arrival times of customers at node 1 in the altered system; note that $A_1^\alpha = A_1 = 0$. Customer i, $i \in \mathbb{Z}$, in the altered system corresponds to customer $\lfloor \alpha(i-1) \rfloor + 1$ in the original system. Suppose that service completion times at the nodes with the same indices of the original and altered systems are coupled. That is, whenever a customer departs from a node in one system, a customer (if present) also departs from the node with the same index in the other system. Due to the memoryless property of the exponential distribution, service times in the two networks are still exponentially distributed even though they are coupled as described above. In this altered system, we let $\{d_i^\alpha(j)\}_{i\in\mathbb{Z}}$, $1 \le j \le k$, be the increasing sequence of (stationary) departure times of customers at node j.

Similarly to (3.2), we have

$$X_i^\alpha(k) = d_{i+1}^\alpha(k) - d_i^\alpha(k) \quad \text{and} \quad d_{i+1}^\alpha(k) - d_1^\alpha(k) = \sum_{j=1}^{i} X_j^\alpha(k), \quad i \geq 1, \tag{3.3}$$

where $X_{1:n}^\alpha(k)$, $n, k \in \mathbb{N}$, denotes the n-tuple random vector consisting of n consecutive stationary inter-departure times at node k in the altered system. As in the original system, customer 1 is given priority over all other customers, and $d_1^\alpha(j)$, $1 \leq j \leq k$, denotes the departure time of customer 1 from node j. Furthermore, note that $d_1^\alpha(j) = d_1(j)$, $1 \leq j \leq k$, when the two systems are coupled.

The following Proposition 3.1 shows that departure processes from node $\lfloor \alpha^2 t \rfloor$ in the original and altered systems converge in probability as $\alpha \to \infty$ and $t \to \infty$, when the two systems are coupled as stated above. From this result, it is feasible to characterize asymptotic properties of the departure process of the original system by analyzing the altered system.

Proposition 3.1. *For any $\epsilon > 0$, we have*

$$\lim_{t\to\infty} \overline{\lim_{\alpha\to\infty}} \mathbb{P}[\|\hat{X}_{1:n}^\alpha(\lfloor \alpha^2 t \rfloor) - \hat{X}_{1:n}(\lfloor \alpha^2 t \rfloor)\| > \epsilon] = 0,$$

where

$$\hat{X}_{1:n}^\alpha(k) \equiv \frac{1}{\sqrt{k}} X_{1:n}^\alpha(k).$$

Proof. See Appendix B.2. □

3.4.2 Thinned system

Consider a linear network consisting of an infinite number of nodes. The input process is the thinned process, as described in Section 3.4.1. The departure process from node k in the infinite-node network is the same as the departure process from node k in the k-node linear network. Instead of examining a sequence of finite-node networks increasing in size, we investigate a sequence of departure processes in the infinite-node linear network. The latter approach is of interest since departure processes from different nodes in the linear loss network are dependent. The basic idea behind our analysis is to relate departure processes from different nodes in the network to the arrival process at the first node; recall that the arrival process is the altered process produced by thinning the original process by a factor of $\alpha > 1$. This is achieved in several steps. First, we define potential departure times at downstream nodes (Section 3.4.2). These potential departure times do not take into account packet drops. Hence, as an intermediate step, coalesced departure times are defined (Section 3.4.2) in terms of potential departure times. Finally, inter-departure times at any node are derived from coalesced departure times (Section 3.4.2).

Potential departure times

As stated earlier, we obtain the vector of inter-departure times $X^{\alpha}_{1:n}(k)$ at node k in terms of the sequence $\{A^{\alpha}_i\}_{i\in\mathbb{Z}}$ in several steps due to an intricate dependency between the two quantities. We start by introducing *potential* departure processes $\mathbf{D}^{\alpha}_i \equiv \{D^{\alpha}_i(t),\, t \geq 0\}$, $i \in \mathbb{Z}$, given by

$$D^{\alpha}_i(t) = A^{\alpha}_i + \sum_{j=1}^{\lfloor t \rfloor} S^{\alpha}_i(j), \tag{3.4}$$

where $\{S^{\alpha}_i(j)\}_{i\in\mathbb{Z},\, j\in\mathbb{N}}$ is a sequence of i.i.d. exponential random variables with unit mean, independent of $\{A^{\alpha}_i\}_{i\in\mathbb{Z}}$. It is appropriate to think of $S^{\alpha}_i(j)$ as the service time of customer i at node j if no other customers were present in the network. In that case, it follows that $D^{\alpha}_i(t)$ is the departure time of customer i from node $\lfloor t \rfloor$ if the customer had priority over all other customers. Note that if customer i in fact departs from node $\lfloor t \rfloor$, i.e., it is not lost at nodes $1, 2, \ldots, \lfloor t \rfloor$, then $D^{\alpha}_i(t)$ is equal to the actual departure time of customer i at node $\lfloor t \rfloor$; hence, we use the term potential.

Next, we let $\hat{\mathbf{D}}_i^\alpha \equiv \{\hat{D}_i^\alpha(t),\, t \geq 0\}$ for $i \in \mathbb{Z}$, where

$$\hat{D}_i^\alpha(t) \equiv \frac{1}{\alpha}(D_i^\alpha(\alpha^2 t) - \alpha^2 t); \tag{3.5}$$

observe that the values of $\hat{\mathbf{D}}_i^\alpha$ at different values of the argument t provide scaled potential departure times of customer i in the altered system. From (3.4) and (3.5), it follows that

$$\hat{D}_i^\alpha(t) = \frac{1}{\alpha}\left(A_i^\alpha + \sum_{j=1}^{\lfloor \alpha^2 t \rfloor} S_i^\alpha(j) - \alpha^2 t\right). \tag{3.6}$$

Coalesced departure times

Recall that we are interested in the vector of inter-departure times $X_{1:n}^\alpha(k)$ at node k, and aim to derive it from potential departure times defined in the previous section. Here, as an intermediate step, we define *coalesced* departure times. We first introduce two operators: τ and ψ_n.

Definition 3.2. The operator $\tau : \mathcal{D}^2 \to [0, \infty]$ is such that

$$\tau(\mathbf{x}_{1:2}) = \inf\{t \geq 0 : x_1(t) \geq x_2(t-)\},$$

where $x_2(0-) \equiv x_2(0)$.

Definition 3.3. The operator $\psi_n : \mathcal{D}^n \to \mathcal{D}^n_{\leqslant}$ for $n \in \mathbb{N}$ is such that $\psi_n(\mathbf{x}_{1:n}) = \mathbf{y}_{1:n}$, where $\mathbf{y}_1 = \mathbf{x}_1$ and $\mathbf{y}_i \equiv \{y_i(t),\, t \geq 0\}$ for $2 \leq i \leq n$ satisfies

$$y_i(t) = \begin{cases} x_i(t), & 0 \leq t < \tau(\mathbf{y}_{i-1}, \mathbf{x}_i), \\ y_{i-1}(t), & t \geq \tau(\mathbf{y}_{i-1}, \mathbf{x}_i). \end{cases}$$

In particular, ψ_1 is the identity operator; and if $\psi_2(\mathbf{x}_{1:2}) = \mathbf{y}_{1:2}$, then $\mathbf{y}_1 = \mathbf{x}_1$ and $\mathbf{y}_2$ is given by $y_2(t) = x_2(t)1_{\{t < \tau(\mathbf{x}_{1:2})\}} + x_1(t)1_{\{t \geq \tau(\mathbf{x}_{1:2})\}}$ for $t \geq 0$.

Remark 3.2. Note that $\mathbf{y}_{1:n} = \psi_n(\mathbf{x}_{1:n})$ can be obtained recursively. Initially, we have $\mathbf{y}_1 = \psi_1(\mathbf{x}_1) = \mathbf{x}_1$, since ψ_1 is the identity operator. Next, in an increasing order of i for $2 \le i \le n$, one can compute $\mathbf{y}_i$ from $(\mathbf{y}_{i-1}, \mathbf{y}_i) = \psi_2(\mathbf{y}_{i-1}, \mathbf{x}_i)$.

Remark 3.3. For $a, b > 0$ and $c \in \mathbb{R}$, if $\hat{x}_i(t) = ax_i(bt) + ct$, $1 \le i \le n$, $t \ge 0$, then $(\psi_n(\hat{\mathbf{x}}_{1:n}))_i(t) = a(\psi_n(\mathbf{x}_{1:n}))_i(bt) + ct$ for $1 \le i \le n$ and $t \ge 0$.

In view of Remark 3.2, the following lemma is straightforward since $\mathbf{x}_{a+1:b}$ does not play a role in determining $(\psi_b(\mathbf{x}_{1:b}))_{1:a}$.

Lemma 3.1. *For any $a, b \in \mathbb{N}$ such that $1 \le a \le b$, we have $\psi_a(\mathbf{x}_{1:a}) = (\psi_b(\mathbf{x}_{1:b}))_{1:a}$.*

Coalesced departure times of a customer are equal to actual departure times (and, consequently, potential departure times) before the customer is lost. However, if a customer is displaced from the network by another customer with a higher priority at some node, we set the coalesced departure times of the lost customer equal to the coalesced departure times of the displacing customer, from that node on. Recall that the characteristics of the departure process are insensitive to the dropping policy due to the memoryless property of exponential service times (see Section 3.2). Nevertheless, it is necessary to specify the dropping policy in order to formulate coalesced departure times. For analytical and notational simplicity, we let customer 1 have the highest priority over all other customers, and the rest of the customers follow the earlier arriving priority rule. Under this policy, customers with indices $i < 1$ play no role on determining the departure times of customers with indices $i \ge 1$. Recall that customers are labeled according to their arrival times at node 1.

From now on, we consider customers with positive indices only. Let $\mathbf{C}_i^\alpha \equiv \{C_i^\alpha(t),\, t \ge 0\}$, $i \in \mathbb{N}$, be the coalesced departure process of customer i. Namely,

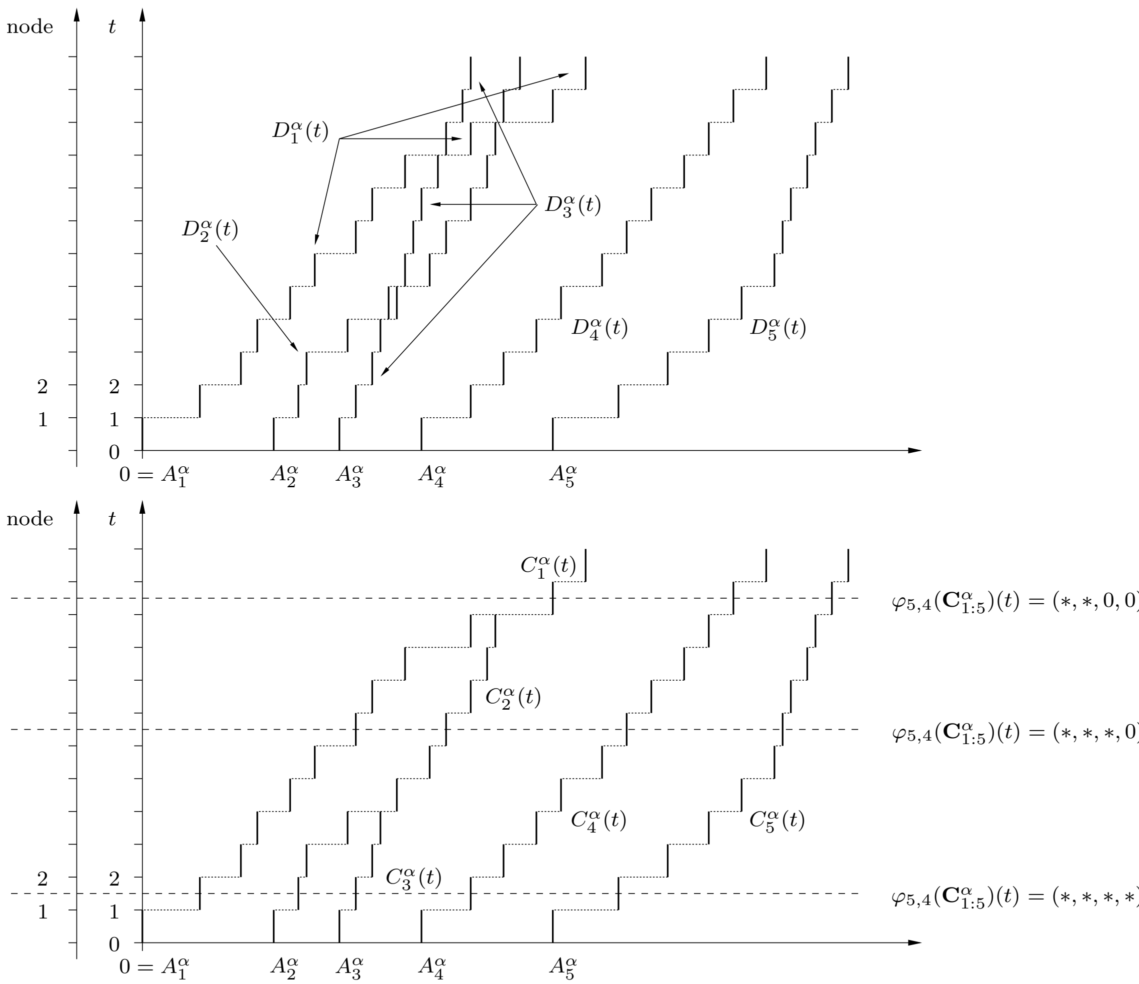


Figure 3.4: An example of sample paths of $\mathbf{D}_{1:5}^\alpha$ and $\mathbf{C}_{1:5}^\alpha$.

$C_i^\alpha(t)$ is the coalesced departure time of customer i from node $\lfloor t \rfloor$. By definition, if customer i is displaced from the network by customer $j < i$ at node l, then $\mathbf{C}_i^\alpha = \{C_i^\alpha(t), t \geq 0\}$ is given by

$$C_i^\alpha(t) = \begin{cases} D_i^\alpha(t), & 0 \leq t < l, \\ C_j^\alpha(t), & t \geq l. \end{cases} \tag{3.7}$$

An example of sample paths of $\mathbf{D}_{1:5}^\alpha$ and $\mathbf{C}_{1:5}^\alpha$ is shown in Figure 3.4. Under the considered dropping policy, if $C_c^\alpha(t) = C_a^\alpha(t)$ for $1 \leq a \leq b \leq c$, then $C_b^\alpha(t) = C_a^\alpha(t)$. Hence, in (3.7), it holds that

$$C_i^\alpha(t) = C_j^\alpha(t) = C_{i-1}^\alpha(t), \quad t \geq l. \tag{3.8}$$

Moreover, if customer i is lost due to customer j at node l, then $D_i^\alpha(l-1) \leq D_j^\alpha(l) = C_j^\alpha(l)$. Thus, by using (3.8), it can be shown that l in (3.7) satisfies

$$l = \inf\{n \in \mathbb{N} : C_{i-1}^\alpha(n) \geq D_i^\alpha(n-1)\} = \tau(\mathbf{C}_{i-1}^\alpha, \mathbf{D}_i^\alpha); \tag{3.9}$$

the second equality stems from the fact that $\lfloor t- \rfloor = t-1$ for $t \in \mathbb{N}$. Finally, from (3.7)–(3.9), it follows that for any $N \in \mathbb{N}$,

$$\mathbf{C}_{1:N}^\alpha = \psi_N(\mathbf{D}_{1:N}^\alpha).$$

Inter-departure times

In order to derive inter-departure times from coalesced departure times, we define the following operators ν_N, $\phi_{N,n}$ and $\varphi_{N,n}$. The operator ν_N counts the number of (strictly) positive differences between adjacent elements of the input vector. By utilizing this counting operator, the operator $\phi_{N,n}$ yields an n-tuple vector filled (from the left) with positive adjacent element differences of the input vector as much as possible, and padded with zeros elsewhere. The operator $\varphi_{N,n}$ is similar to $\phi_{N,n}$, but can be applied to processes rather than vectors.

Definition 3.4. The operator $\nu_N : \mathbb{R}_{\leqslant}^N \to \{0\} \cup \mathbb{N}$ for $N \in \mathbb{N}$ is such that

$$\nu_N(x_{1:N}) = \sum_{i=1}^{N-1} 1_{\{x_{i+1} - x_i > 0\}}.$$

Definition 3.5. The operator $\phi_{N,n} : \mathbb{R}_{\leqslant}^N \to \mathbb{R}^n$ for $N, n \in \mathbb{N}$ is such that $\phi_{N,n}(x_{1:N}) = y_{1:n}$ satisfies

$$y_i = \begin{cases} x_{a_i} - x_{a_i - 1}, & 1 \leq i \leq n \wedge \nu_N(x_{1:N}), \\ 0, & n \wedge \nu_N(x_{1:N}) < i \leq n, \end{cases}$$

where $a_0 = 1$ and $a_i = \min\{j > a_{i-1} : x_j - x_{j-1} > 0\}$ for $1 \leq i \leq n \wedge \nu_N(x_{1:N})$.

Definition 3.6. The operator $\varphi_{N,n} : \mathcal{D}^N_{\leqslant} \to \mathcal{D}^n$ for $N, n \in \mathbb{N}$ is such that $\varphi_{N,n}(\mathbf{x}_{1:N}) = \mathbf{y}_{1:n}$, where

$$y_{1:n}(t) = \phi_{N,n}(x_{1:N}(t)).$$

Remark 3.4. For $a > 0$ and $c \in \mathbb{R}$, if $\hat{x}_i = ax_i + c$, $1 \leq i \leq N$, then $\nu_N(\hat{x}_{1:N}) = \nu_N(x_{1:N})$ and $\phi_{N,n}(\hat{x}_{1:N}) = a\phi_{N,n}(x_{1:N})$. Hence, letting $\hat{x}_i(t) = ax_i(bt) + ct$, $1 \leq i \leq N$, $t \geq 0$, for $a, b > 0$ and $c \in \mathbb{R}$, yields $\varphi_{N,n}(\hat{\mathbf{x}}_{1:N})(t) = a\varphi_{N,n}(\mathbf{x}_{1:N})(bt)$.

Note that the operator $\varphi_{N,n}$ produces a finite number of consecutive inter-departure times from the coalesced departure processes $\mathbf{C}^{\alpha}_{1:N} = \psi_N(\mathbf{D}^{\alpha}_{1:N})$ since the operator $\varphi_{N,n}$ returns only positive differences between adjacent elements of the input; recall that if a customer is displaced from the network, we set the coalesced departure time of the lost customer equal to the coalesced departure time of the displacing customer, from that node on. In particular, for a finite, fixed $t \geq 0$, if N is large enough, the random vector $\varphi_{N,n} \circ \psi_N(\mathbf{D}^{\alpha}_{1:N})(t) = \varphi_{N,n}(\mathbf{C}^{\alpha}_{1:N})(t)$ consists of n inter-departure times at node $\lfloor t \rfloor$. However, if N is not large enough, it contains fewer than n inter-departure times, and the rest of the elements are zeros. Let $\upsilon^{\alpha}_n(t)$, $n \in \mathbb{N}$, $t \geq 0$, be the minimum number of customers (after and including customer 1) that need to be taken into account in order to obtain at least n inter-departure times at node $\lfloor t \rfloor$. Formally, $\upsilon^{\alpha}_n(t)$ is defined as

$$\upsilon^{\alpha}_n(t) = \inf\{N \in \mathbb{N} : \nu_N(\psi_N(\mathbf{D}^{\alpha}_{1:N})(t)) \geq n\}. \tag{3.10}$$

The following lemma shows that the stopping time $\upsilon^{\alpha}_n(t)$ is finite for fixed $n \in \mathbb{N}$ and $t \geq 0$.

Lemma 3.2. *For $\alpha > 1$, $n \in \mathbb{N}$ and $t \geq 0$, it holds that $\mathbb{P}[\upsilon^{\alpha}_n(t) > N] \to 0$, as $N \to \infty$.*

Proof. See Appendix B.3. ◻

For fixed $n \in \mathbb{N}$ and $t \geq 0$, as long as we consider more than $\upsilon_n^\alpha(t)$ customers, the number of considered customers N does not impact the resulting first n inter-departure times at node $\lfloor t \rfloor$ (i.e., $\varphi_{N,n} \circ \psi_N(\mathbf{D}_{1:N}^\alpha)(t)$) since the customers from $\upsilon_n^\alpha(t)+1$ play no role on determining these first n inter-departure times under our customer dropping policy (see Lemma 3.1). Hence, it follows that for all $N \geq \upsilon_n^\alpha(t)$,

$$\varphi_{N,n} \circ \psi_N(\mathbf{D}_{1:N}^\alpha)(t) = \varphi_{\upsilon_n^\alpha(t),n} \circ \psi_{\upsilon_n^\alpha(t)}(\mathbf{D}_{1:\upsilon_n^\alpha(t)}^\alpha)(t). \tag{3.11}$$

The next lemma indicates that the joint distribution of any finite number of consecutive stationary inter-departure times at node k can be obtained by considering a sufficiently large number of customers. Intuitively, one can obtain $n+1$ departures at node $\lfloor t \rfloor$ by considering a sufficiently large number of arrivals at node 1.

Lemma 3.3. *For $\alpha > 1$, $n \in \mathbb{N}$ and $t \geq 0$, we have, as $N \to \infty$,*

$$\varphi_{N,n} \circ \psi_N(\mathbf{D}_{1:N}^\alpha)(t) \Rightarrow X_{1:n}^\alpha(\lfloor t \rfloor).$$

Proof. See Appendix B.4. □

Now consider the scaled processes $\hat{\mathbf{D}}_i^\alpha = \{\hat{D}_i^\alpha(t),\, t \geq 0\}$, $i \in \mathbb{Z}$, defined in (3.5). Analogously to (3.10), we let

$$\hat{\upsilon}_n^\alpha(t) = \inf\{N \in \mathbb{N} : \nu_N(\psi_N(\hat{\mathbf{D}}_{1:N}^\alpha)(t)) \geq n\}, \tag{3.12}$$

and, then, similarly to (3.11), for all $N \geq \hat{\upsilon}_n^\alpha(t)$, we have

$$\varphi_{N,n} \circ \psi_N(\hat{\mathbf{D}}_{1:N}^\alpha)(t) = \varphi_{\hat{\upsilon}_n^\alpha(t),n} \circ \psi_{\hat{\upsilon}_n^\alpha(t)}(\hat{\mathbf{D}}_{1:\hat{\upsilon}_n^\alpha(t)}^\alpha)(t). \tag{3.13}$$

The following corollary restates Lemma 3.2 and 3.3 for scaled quantities.

Corollary 3.2. *For $\alpha > 1$, $n \in \mathbb{N}$ and $t \geq 0$, we have, as $N \to \infty$,*

$$\mathbb{P}[\hat{\upsilon}_n^\alpha(t) > N] \to 0,$$

and

$$\varphi_{N,n} \circ \psi_N(\hat{\mathbf{D}}^\alpha_{1:N})(t) \Rightarrow \frac{1}{\alpha} X^\alpha_{1:n}(\lfloor \alpha^2 t \rfloor).$$

Proof. Due to (3.5) and Remark 3.3, we have

$$(\psi_N(\hat{\mathbf{D}}^\alpha_{1:N}))_i(t) = \frac{1}{\alpha}((\psi_N(\mathbf{D}^\alpha_{1:N}))_i(\alpha^2 t) - \alpha^2 t), \tag{3.14}$$

for $1 \le i \le N$ and $t \ge 0$. This, together with Remark 3.4, further results in

$$\nu_N(\psi_N(\hat{\mathbf{D}}^\alpha_{1:N})(t)) = \nu_N(\psi_N(\mathbf{D}^\alpha_{1:N})(\alpha^2 t)).$$

Hence, it follows from (3.10), (3.12) and the preceding equality that

$$\hat{\upsilon}^\alpha_n(t) = \upsilon^\alpha_n(\alpha^2 t).$$

Lemma 3.2 and the preceding equality render the first statement of the corollary. Furthermore, from (3.14) and Remark 3.4, we have

$$\varphi_{N,n} \circ \psi_N(\hat{\mathbf{D}}^\alpha_{1:N})(t) = \frac{1}{\alpha} \varphi_{N,n} \circ \psi_N(\mathbf{D}^\alpha_{1:N})(\alpha^2 t),$$

and the second statement of the corollary follows from Lemma 3.3 and the preceding equality. □

The following lemma is the main result of this section.

Lemma 3.4. *For $N, n \in \mathbb{N}$, we have*

$$\varphi_{N,n} \circ \psi_N(\hat{\mathbf{D}}^\alpha_{1:N}) \Rightarrow \varphi_{N,n} \circ \psi_N(\mathbf{B}_{1:N} + \beta_{1:N}),$$

in $(\mathcal{D}^n, J_1)$, as $\alpha \to \infty$, where $\beta_{1:N} = \lambda^{-1}(0, 1, \ldots, N-1)$.

Proof. See Appendix B.5. □

Corollary 3.3. *For $N, n \in \mathbb{N}$ and any fixed $t \ge 0$, we have*

$$\varphi_{N,n} \circ \psi_N(\hat{\mathbf{D}}^\alpha_{1:N})(t) \Rightarrow \varphi_{N,n} \circ \psi_N(\mathbf{B}_{1:N} + \beta_{1:N})(t),$$

as $\alpha \to \infty$, where $\beta_{1:N} = \lambda^{-1}(0, 1, \ldots, N-1)$.

Proof. We have (see (B.44) and Lemma B.6)

$$\mathrm{Disc}(\varphi_{N,n} \circ \psi_N(\mathbf{B}_{1:N} + \beta_{1:N})) \subseteq \{\tau(\mathbf{B}_i + \beta_i, \mathbf{B}_j + \beta_j), 1 \leq i < j \leq N\},$$

and, thus, it follows that for any fixed $t \geq 0$,

$$\mathbb{P}[t \in \mathrm{Disc}(\varphi_{N,n} \circ \psi_N(\mathbf{B}_{1:N} + \beta_{1:N}))] \leq \sum_{1 \leq i < j \leq N} \mathbb{P}[\tau(\mathbf{B}_i + \beta_i, \mathbf{B}_j + \beta_j) = t] = 0.$$

Hence, Lemma 3.4 implies the statement of the corollary (e.g., see [13, p. 138-139]).

□

3.4.3 Brownian system

Consider a collection of one-dimensional Brownian motions indexed by integers in the increasing order according to their initial values (at $t = 0$). Let $\{\mathbf{B}_i + \tilde{\beta}_i\}_{i \in \mathbb{Z}}$ denote such a collection of Brownian motions; note that $\tilde{\beta}_i$, $i \in \mathbb{Z}$, is the initial position of the ith Brownian motion and we have $\tilde{\beta}_{a:b} \in \mathbb{R}_{\leqslant}^{b-a+1}$ for any $a, b \in \mathbb{Z}$ such that $a \leq b$. This collection of Brownian motions serves as a basis for a system of coalescing Brownian motions [2]. In this new system, two Brownian motions coalesce whenever they hit each other. After the coalescing time, the merged process is a one-dimensional Brownian motion. The described coalescing procedure is called one-species diffusion-limited coalescence and was studied in [5–7, 22]. Now let

$$\tilde{\beta}_{a:b} = \lambda^{-1}(a-1, \ldots, b-1) + U, \tag{3.15}$$

for any $a, b \in \mathbb{Z}$ such that $a \leq b$, where U is a random variable uniformly distributed on $[0,1]$ and independent of $\{\mathbf{B}_i + \beta_i\}_{i \in \mathbb{Z}}$. Let $W_{1:n}(t)$, $t \geq 0$, $n \in \mathbb{N}$, be n-tuple random vector consisting of n consecutive distances between neighboring coalesced Brownian motions at time t when the initial positions of them are given as in (3.15). Then, it was shown that $W_{1:n}(t)$ satisfies the following limit as $t \to \infty$ [5, 6]:

$$\frac{1}{\sqrt{t}} W_{1:n}(t) \Rightarrow Z_{1:n}, \tag{3.16}$$

where $Z_{1:n}$ is as in the statement of Theorem 3.1.

Remark 3.5. The distribution of $W_{1:n}(t)$ does not change if one replaces U with 0 in (3.15), since $W_{1:n}(t)$ only depends on the relative distances between coalesced Brownian motions. Hence, from now on, we let $\tilde{\beta}_{a:b} = \beta_{a:b} = \lambda^{-1}(a-1,\ldots,b-1)$ for any $a, b \in \mathbb{Z}$ such that $a \leq b$.

Similarly to (3.10), we define $\omega_n(t)$ as follows:

$$\omega_n(t) = \inf\{N \in \mathbb{N} : \nu_N(\psi_N(\mathbf{B}_{1:N} + \beta_{1:N})(t)) \geq n\};$$

for fixed $n \in \mathbb{N}$ and $t \geq 0$, $\omega_n(t)$ is the minimum number of original Brownian motions that need to be considered in order to obtain at least $n+1$ coalesced Brownian motions at time t. Analogously to (3.11), for all $N \geq \omega_n(t)$, we have

$$\varphi_{N,n} \circ \psi_N(\mathbf{B}_{1:N} + \beta_{1:N})(t) = \varphi_{\omega_n(t),n} \circ \psi_{\omega_n(t)}(\mathbf{B}_{1:\omega_n(t)} + \beta_{1:\omega_n(t)})(t). \tag{3.17}$$

The following proposition provides equivalent results to Lemma 3.2 and 3.3 for the Brownian system.

Proposition 3.2. *If $\beta_{1:N} = \lambda^{-1}(0,1,\ldots,N-1)$, then, for $n \in \mathbb{N}$ and $t \geq 0$, we have*

$$\lim_{N\to\infty} \mathbb{P}[\omega_n(t) > N] = \lim_{N\to\infty} \lim_{\alpha\to\infty} \mathbb{P}[\hat{v}_n^{\alpha}(t) > N] = 0;$$

furthermore, as $N \to \infty$,

$$\varphi_{N,n} \circ \psi_N(\mathbf{B}_{1:N} + \beta_{1:N})(t) \Rightarrow W_{1:n}(t).$$

Proof. See Appendix B.6. □

3.5 Concluding remarks

In this chapter, we studied the asymptotic characteristics of the departure process of a linear network consisting of bufferless nodes with exponential service times. We first established a relevant scaling regime for the departure process that becomes sparser (due to losses) as the size of the network increases. Under the relevant scaling regime, we characterized the limiting behavior of the scaled departure process by determining the joint probability density function of any finite number of consecutive inter-departure times. We discovered that this asymptotic behavior of the departure process is completely attributed to the characteristics of the network itself (i.e., the distribution of service times and lack of buffers) and is not impacted by the input as long as the input does not vary with the size of the network.

The complete characterization of the departure process we provide in this chapter contains information on the output traffic of the linear network, and many performance properties of the network can be obtained from it. For example, the negative correlation coefficients between the inter-departure times indicate that the output traffic is not bursty, since bursty departures have positive correlation between the inter-departure times. Moreover, we observed that the correlation coefficient between the inter-departure times converges to zero very fast as the lag between the inter-departure times increases. This indicates that the dependency between the departure times vanishes very quickly. Finally, we note that the departure process from a network is of value since it often constitutes an input to another system. The departing customers (packets) from a network can be fed into another system for further processing, and, then, the knowledge of a network's departure process can be utilized in designing such subsequent systems.

CHAPTER IV

Throughput scalability for linear finite-buffer networks

4.1 Introduction

This chapter considers large linear networks consisting of a series of identical nodes with finite-buffers. Packets (customers) enter the network at the first node and are relayed from one node to another in a fixed order until they exit the network at the last node. Such a tandem network is a representative model of large-scale networks with no or limited cross-traffic interference along the paths from sources to destinations, e.g., a sensor network where concurrent traffic volume is small relative to the size of the network. As the size of the network increases, feedback information received at the source is delayed to an extent that it is of marginal value. Thus, it is reasonable to employ simple control protocols that insert packets into the network based only on the information available at the source, without any feedback from the network.

Performance scalability is one of the critical issues in designing modern communication systems that continue to expand in terms of traffic volume, the numbers of users and nodes, as well as the range of applications. Here, we focus on understanding the fundamental performance properties of linear finite-buffer networks, as the size of the network grows while the local resources (i.e., the buffer space) at each node remain fixed. In particular, we aim to identify a critical loading regime under which the loss probability defined as the long-term fraction of packets lost due to limited buffer space is strictly within $(0,1)$ asymptotically as the size of the network increases. Such a regime is of interest given that it delivers a relatively high throughput at low network cost. The loss probability tends to 1 asymptotically when the offered load is (order-wise) higher than the critical load. In that case, only a negligible fraction of packets is delivered, i.e., throughputs close to the maximal come with high network costs, an undesirable feature in certain systems such as sensor networks. On the other hand, the asymptotic loss probability of 0 occurs when the offered load is (order-wise) lower than the critical load. Then, higher throughputs can be achieved with small increments of the network cost. Under the critical regime, one balances between the throughput and network cost.

Linear networks with bufferless nodes were studied in [19, 35]. It was shown that the maximum throughput of the linear bufferless network with exponential service times is $\Theta(1/\sqrt{k})$, as $k \to \infty$, where k denotes the size of the network; moreover, the critical loading regime occurs when the input rate is $\Theta(1/\sqrt{k})$, as $k \to \infty$. The specific relation between the throughput and network cost was also identified in [35]. The main result in [19] provided an asymptotic characterization of a properly scaled limiting departure process, i.e., the joint probability density function of any finite number of consecutive inter-departure times. In this study, we consider a more general model of linear networks, where each node has a finite buffer. Our results indicate that under the critical loading regime, the input rate is $\Omega(1/\sqrt{k})$ and $O(\sqrt{\log k/k})$ for $b = 1$, and $\Theta(k^{-1/(b+1)})$ for $b \geq 2$, as $k \to \infty$, where b denotes the size of buffer space at each node. Finally, we argue that these asymptotic approximations are reasonably accurate for finite-size networks.

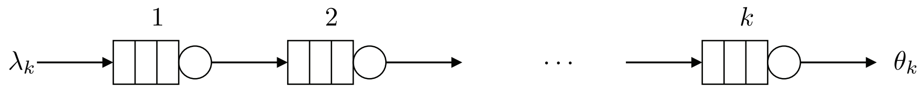


Figure 4.1: A linear finite-buffer network consisting of a series of k identical nodes with finite buffers. In this example, the size of buffer space at each node is equal to 3 (i.e., $b = 3$).

4.1.1 Notation

Throughout the chapter, we use the following notation:

(1) For $a \leq b$, $a, b \in \mathbb{Z}$, let $x_{a:b}$ be the $(b-a+1)$-tuple vector consisting of x_i, $a \leq i \leq b$, i.e., $x_{a:b} \equiv (x_a, \ldots, x_b)$.

(2) Let $e^i_{1:n}$, $1 \leq i \leq n$, $n \in \mathbb{N}$, be the n-tuple vector with an 1 in the ith element and 0s elsewhere. Let $1_{1:n}$, $n \in \mathbb{N}$, be the n-tuple vector with all 1s.

(3) For two real-valued functions $f(x)$ and $g(x)$, $f(x) \sim g(x)$, as $x \downarrow 0$, denotes $f(x)/g(x) \to 1$, as $x \downarrow 0$.

(4) For a right continuous function $f : \mathbb{R} \to \mathbb{R}$ with left limits, the left limit $f(t-)$ is defined as $f(t-) \equiv \lim_{s \uparrow t} f(s)$.

4.1.2 Organization

The rest of the chapter is organized as follows. We introduce a k-node linear finite-buffer network in the next section. Section 4.3 contains some preliminary results, while the main results are presented in the section that follows. Concluding remarks and technical proofs can be found in Section 4.5 and Appendix C.

4.2 Model

We consider a k-node linear network consisting of a series of $\cdot/M/1/b$ queues, indexed by $1, 2, \ldots, k$ – each node has a finite buffer of size b (see Figure 4.1). Service times are exponentially distributed with unit mean; service times are independent both across packets and across nodes. In this tandem network, incoming packets enter the network at node 1, and are relayed between adjacent nodes until they are either serviced at node k or lost at some intermediate node due to limited buffer space. The input arrival process to the network is assumed to be Poisson; the input process and service times are independent.

For a finite-buffer system, in general, a buffer management scheme needs to be specified. A buffer management scheme includes (i) service discipline and (ii) dropping policy. The service discipline determines the order of packets to be serviced (transmitted) at a node; in this chapter, nodes are assumed to employ a work-conserving scheduling policy. The dropping policy prioritizes packets to be dropped upon new arrivals at a node with the full buffer. Since service times are i.i.d. with the exponential distribution, the remaining service time of a packet is equal in distribution to the service time of a newly arrived packet. Thus, in that case, the choice of the buffer management scheme does not impact throughput results as long as the policy does not consider realizations of service times. Throughout the chapter, the first-come, first-serve (FCFS) discipline is assumed to be employed. A suitable dropping policy that is convenient for analysis will be defined later (see Section 4.3.3).

In the linear network model, a node can accommodate $b+1$ packets (including any one in service), and a packet is lost if a new packet arrives at a node with a full buffer. For $b = 0$, it was shown in [19] that, under a proper scaling, locations of packets in the linear network can be viewed as coalescing Brownian motions, where two Brownian motions coalesce whenever they hit each other. This coalescing procedure is called as one-species two-body diffusion-limited reaction and was studied in [5–7, 22]. More general one-species many-body diffusion-limited reactions, where Brownian motions coalesce when more than two of them collide, were investigated in [4, 8, 32, 37, 44].

However, we point out that these results are not directly applicable in analyzing the critical regime for linear networks with $b \geq 1$.

4.3 Preliminary results

This section provides some preliminary results.

4.3.1 Multidimensional random walk within a wedge

In this subsection we consider an n-dimensional random walk relevant in analyzing the interaction of $(n+1)$ consecutive packets in the original network. Let $\mathcal{W}^n$ be a wedge-shaped n-dimensional Euclidean subspace (we simple call it a wedge) such that

$$\mathcal{W}^n \equiv \{x_{1:n} \in \mathbb{Z}^n : 0 \leq x_1 \leq x_2 \leq \cdots \leq x_n\}.$$

Given a parameter $\rho \in (0,1)$, an n-dimensional discrete-time random walk $\{X_{1:n}(i),\ i \geq 0\}$ in the wedge $\mathcal{W}^n$ is defined by the transition probabilities $P_{x_{1:n},y_{1:n}}$, $x_{1:n}, y_{1:n} \in \mathcal{W}^n$, given by

$$P_{x_{1:n},y_{1:n}} = \begin{cases} \frac{I_i}{n+1-\rho}, & y_{1:n} = x_{1:n} - e^i_{1:n},\ 1 \leq i \leq n, \\ \frac{J}{n+1-\rho}, & y_{1:n} = x_{1:n}, \\ \frac{1-\rho}{n+1-\rho}, & y_{1:n} = x_{1:n} + 1_{1:n}, \\ 0, & \text{otherwise}, \end{cases} \tag{4.1}$$

where $I_i \equiv 1_{\{x_i > x_{i-1}\}}$, $J \equiv \sum_{i=1}^n 1_{\{x_i = x_{i-1}\}}$ and $x_0 \equiv 0$. It can be verified that the stationary distribution $\pi_{x_{1:n}}$, $x_{1:n} \in \mathcal{W}^n$, satisfies

$$\pi_{x_{1:n}} = \rho^n (1-\rho)^{x_n}. \tag{4.2}$$

Remark 4.1. After a proper scaling ($\rho \downarrow 0$), the random walk converges weakly to an n-dimensional reflected Brownian motion with a constant drift, which has an exponential stationary distribution. Hence, the preceding result is in agreement with the one in [27].

Let $\tau_{0_{1:n}}$ be a first hitting time:

$$\tau_{0_{1:n}} = \inf\{i \geq 0 : X_{1:n}(i) = 0_{1:n}\}, \tag{4.3}$$

i.e., the first time the random walk hits the origin. The following lemma provides an estimate of $\tau_{0_{1:n}}$.

Lemma 4.1. *For $m, n \in \mathbb{N}$, we have*

$$\frac{1}{|\mathcal{V}_m^n|} \sum_{x_{1:n} \in \mathcal{V}_m^n} \mathbb{E}[\tau_{0_{1:n}} | \, X_n(0) = x_{1:n}] \leq (n + 1 - \rho) \sum_{i=1}^{m} \sum_{j=1}^{n} \binom{n-1}{j-1} \frac{(j-1)!}{i^{j-1}} \frac{1}{\rho^j},$$

where $\mathcal{V}_m^n \equiv \{x_{1:n} \in \mathbb{Z}^n : 1 \leq x_1 \leq x_2 \leq \cdots \leq x_n = m\}$.

Proof. See Appendix C.1.

Remark 4.2. The reason for considering the set $\mathcal{V}_m^n$ in Lemma 4.1 is related to the (initial) distribution of the distances between $(n+1)$ consecutive packets in the linear network (see Section 4.3.4 – Remark 4.3 in particular).

Let $\xi(s)$, $s \in \mathbb{N}$, be the number of times the increment of the random walk is equal to $1_{1:n}$ during the first s steps:

$$\xi(s) = \sum_{i=1}^{s} 1_{\{X_{1:n}(i) - X_{1:n}(i-1) = 1_{1:n}\}}. \tag{4.4}$$

Note that Wald's equality yields (e.g., see [25, p. 64-67])

$$\mathbb{E}[\xi(\tau_{0_{1:n}}) | X_{1:n}(0) = x_{1:n}] = \frac{1-\rho}{n+1-\rho} \mathbb{E}[\tau_{0_{1:n}} | X_{1:n}(0) = x_{1:n}],$$

for all $x_{1:n} \in \mathcal{W}^n$, and, therefore,

$$\mathbb{E}\xi(\tau_{0_{1:n}}) = \frac{1-\rho}{n+1-\rho} \mathbb{E}\tau_{0_{1:n}}; \tag{4.5}$$

the expectations above implicitly depend on the initial condition $X_{1:n}(0)$. We conclude this subsection with an asymptotic characterization of $\mathbb{E}\xi(\tau_{0_{1:n}})$.

Lemma 4.2. *Suppose that the initial distribution of the random walk satisfies the following three properties:*

(A1) $\sum_{m=1}^{\infty} \mathbb{P}[X_{1:n}(0) \in \mathcal{V}_m^n] = 1.$

(A2) For $m \in \mathbb{N}$ *and* $x_{1:n} \in \mathcal{V}_m^n$,

$$\mathbb{P}[X_{1:n}(0) = x_{1:n} | \, X_n(0) = m] = |\mathcal{V}_m^n|^{-1}.$$

(A3) $\mathbb{E}X_n(0) \sim c\rho^{-1}$, *as* $\rho \downarrow 0$, *for finite, fixed* $c > 0$.

Then, for $n = 2$,

$$\overline{\lim_{\rho \downarrow 0}} \frac{\rho^2}{\log \rho^{-1}} \mathbb{E}\xi(\tau_{0_{1:2}}) \leq 1,$$

and, for $n \geq 3$,

$$\overline{\lim_{\rho \downarrow 0}} \rho^n \mathbb{E}\xi(\tau_{0_{1:n}}) \leq (n-1)^2(n-3)!.$$

Proof. See Appendix C.2.

4.3.2 Extended linear network

In this subsection, we consider an extended linear network constructed by adding m infinite-buffer nodes in tandem in front of the first node of the original k-node network; each infinite-buffer node operates as a $\cdot/M/1/\infty$ queue with the unit service rate (all service times are independent). In this new network, nodes are labeled by integers from $-m+1$ to k in increasing order according to their positions in the network. Incoming packets enter the network at node $-m+1$ and pass through the infinite-buffer nodes first. Within this part of the network, no packets are lost. Packets departing from node 0 are then relayed between the finite-buffer nodes until they are either serviced at node k or dropped at some node due to limited buffer space, like in the original network.

Let the arrival process at node $-m+1$ be Poisson with rate $\lambda_k < 1$. Then, the output process at node 0 in the extended network described above is also Poisson with rate λ_k [16]. Since no packets are lost from the infinite-buffer nodes, this further results in

$$p_k^m = p_k^0, \tag{4.6}$$

for all $m \in \mathbb{N}$, where p_k^m is the loss probability defined as the long-term fraction of packets that are lost before they reach node k and complete service successfully in the extended $(m+k)$-node network described above. In other words, augmenting the original network with infinite-buffer nodes does not alter the loss probability.

4.3.3 Dropping policy

Recall that throughput results are insensitive to the dropping policy (see Section 4.2). In this subsection, we describe a suitable dropping policy convenient for analyzing the loss probability in (4.6). To this end, we consider two identical linear networks – they have the same numbers of packets at the nodes with the same indices initially, and input processes at node $-m+1$ are also same in both networks. Packets are labeled by integers in $(-\infty, \infty)$ in increasing order according to their arrival times at node $-m+1$, and when some arbitrary packet arrives at node $-m+1$, both networks are already in stationarity. Suppose that these two identical networks adopt the following two different dropping policies (P1) and (P2), respectively:

(P1) All packets follow the earlier arriving priority rule.

(P2) Packets from 0 to b are given priority over all other packets, and the rest of the packets follow the earlier arriving priority rule.

Moreover, it is assumed that service completion times at the nodes with the same indices of the two linear networks are coupled. That is, whenever a packet departs from a node in one network, a packet (if present) also departs from the node with the same index in the other network (regardless of the indices of packets). Due to the memoryless property of the exponential distribution, service times in the two networks are still exponentially distributed even though they are coupled as described above. Thus, throughputs of the networks are not altered by this coupling.

Recall that both networks are already in stationarity when some packet arrives at node $-m+1$. Under the dropping policy (P1), given that the network is in stationarity, loss probabilities of individual packets are equal due to symmetry. Hence, the loss probability p_k^m in (4.6), i.e., the long-term fraction of lost packets, is equal to the loss probability of an arbitrary packet. However, under the dropping policy (P2), the loss probabilities of individual packets can be different. Note that packets with indices from 0 to b are never lost, since they have priority over all other packets and each node can accommodate $b+1$ packets (including any one in service). On the other hand, packet $b+1$ is displaced from the network when it enters a node with full buffer that holds all packets from 0 to b. Now we compare the loss probabilities of packet $b+1$ in both networks. Observe that in both policies, all packets arriving before packet $b+1$ have priority over packet $b+1$. Moreover, recall that the FCFS discipline is assumed to be employed (see Section 4.2). Then, since two linear networks are identical and service completion times at the nodes with the same indices are coupled as described above, the numbers of packets present in the nodes where packet $b+1$ enters are always same in both networks even though their (packet) indices can be different. That is, the loss probabilities of packet $b+1$ are same for both networks. Thus, the loss probability p_k^m in (4.6) can be obtained by evaluating the probability that packet $b+1$ is lost under the dropping policy (P2).

4.3.4 Initial distribution

From now on suppose that the network employs the dropping policy (P2). Recall that under (P2), packet $b+1$ is lost if and only if packets from 0 to $b+1$ happen to be at the same node before packet 0 departs from node k. Packets arriving after packet $b+1$ do not impact on the behavior of packets from 0 to $b+1$, i.e., the loss of packet $b+1$ is determined by packets $0, 1, \ldots, b+1$ only. In this subsection, we characterize the positions of packets from 0 to $b+1$ when packet 0 arrives at node 1; this results will be utilized in the following subsection to identify the loss probability of packet $b+1$.

Let $\{A_i\}_{i\in\mathbb{Z}}$ be the increasing sequence of arrival times of packets at node $-m+1$ (in the extended network). Recall that packets are labeled by integers in increasing order according to their arrival times (see Section 4.3.3). Let L_i^m, $i \geq 1$, be the index of the node where packet i is located when packet 0 arrives at node 1; in order to properly define L_i^m for all $i \geq 1$, we set $L_i^m = -m$, if packet i has not entered the network by the time packet 0 departs from node 0. The following lemma characterize the positions of packets from 1 to $b+1$ at the time when packet 0 enters node 1.

Lemma 4.3. *For $x_{1:b+1} \in \mathcal{W}^{b+1}$ such that $x_{b+1} \leq m$, we have*

$$\mathbb{P}[L_{1:b+1}^m = -x_{1:b+1}] = \lambda_k^{\sum_{i=1}^{b+1} 1\{x_i \,\leq\, m+1\}}(1-\lambda_k)^{x_{b+1}}.$$

Proof. See Appendix C.3.

Now consider a random vector $L_{1:b+1}^\infty$ such that

$$\mathbb{P}[L_{1:b+1}^\infty = -x_{1:b+1}] = \lambda_k^{b+1}(1-\lambda_k)^{x_{b+1}}, \tag{4.7}$$

for $x_{1:b+1} \in \mathcal{W}^{b+1}$. Note that setting $m = \infty$ in Lemma 4.3, even though it is informal, simply yields (4.7). A formal relation between $L_{1:b+1}^m$ and $L_{1:b+1}^\infty$ is given in the following corollary.

Corollary 4.1. *Given* $m \in \mathbb{N}$*, we have*

$$\mathbb{P}[L^m_{1:b+1} = -x_{1:b+1}] = \mathbb{P}[L^\infty_{1:b+1} = -x_{1:b+1}],$$

for all $x_{1:b+1} \in \mathcal{W}^{b+1}$ *such that* $x_{b+1} \le m - 1$*.*

Remark 4.3. $L^\infty_{1:b+1}$ can be thought of the random vector consisting of the positions of packets from 1 to $b+1$ when packet 0 arrives at node 1 in the extended linear network with $m = \infty$. Now suppose that the initial distribution of the random walk is given by

$$X_{1:b+1}(0) = 1_{1:b+1} - L^\infty_{1:b+1}.$$

When packet 0 arrives at node 1, packets from 1 to $b+1$ are located at some nodes with indices less than or equal to 0, i.e., $X_i(0) = 1 - L^\infty_i \ge 1$ for all $1 \le i \le b+1$. Then, it is straightforward that (A1) in Lemma 4.2 holds. Moreover, it can be shown from (4.7) that (A2) in Lemma 4.2 is also satisfied. This explains why we consider the set $\mathcal{V}^n_m$ in Lemma 4.1 (see Remark 4.2). Finally, if $\rho \sim r\lambda_k$, as $\lambda_k \downarrow 0$, for some finite, fixed $r > 0$, then

$$\begin{aligned} \mathbb{E}X_{b+1}(0) &= 1 - \mathbb{E}L^\infty_{b+1} \\ &= 1 + (b+1)\lambda_k^{-1} \sim (b+1)r\rho^{-1}, \end{aligned}$$

as $\rho \downarrow 0$, and this implies (A3) in Lemma 4.2.

4.3.5 Loss probability

In this subsection we focus on the behavior of packets $0, 1, \ldots, b+1$ in the extended linear network after packet 0 departs from node 0. Results in here, together with those in the previous subsection, are utilized to obtain the loss probability p^m_k in (4.6). In particular, we consider an infinite-node linear network consisting of m infinite-buffer nodes and an infinite number of finite-buffer nodes, i.e., the linear network described in Section 4.3.2 with $k = \infty$. Recall that the loss probability p_k^m is equal to the loss probability of packet $b+1$ in the $(m+k)$-node linear network under the dropping policy (P2) (see Section 4.3.3). This probability can be obtained by evaluating the probability that packet $b+1$ is lost before it reaches node $k+1$ in the infinite-node linear network.

The loss of packet $b+1$ occurs if and only if packets from 0 to $b+1$ happen to be at the same node in the network. We first consider the behavior of packet 0. Without loss of generality, let time 0 be the time instance when packet 0 arrives at node 1, and define $W_0(i)$, $i \in \mathbb{N}$, to be the sojourn time of packet 0 at node i. The location (node index) of packet 0 at time $t \geq 0$ can be represented by a right-continuous function $\{L_0^m(t),\, t \geq 0\}$ that satisfies

$$L_0^m(t) = n + 1, \tag{4.8}$$

for $T_0(n) \leq t < T_0(n+1)$, $n \geq 0$, where

$$T_0(n) \equiv \sum_{j=1}^{n} W_0(j). \tag{4.9}$$

Note that $T_0(n)$, $n \geq 0$, corresponds to the departure time of packet 0 from node i. The behavior of packet 0 is impacted by packets with negative indices (i.e., the packets that enter the network before packet 0). The characteristics of sojourn times of packet 0 reflect all the effects of the packets with negative indices. Due to the intrinsic dependency in the behavior of packets that interact along the network, the sojourn times at different nodes are dependent, and it is not easy to characterize them exactly. In the next section, we approximate these sojourn times to simplify the analysis and compute the loss probability approximately (see Approximation 4.1).

Next we consider packets $1, 2, \ldots, b+1$. For now, assume that all packets from 1 to $b+1$ have entered the network at time 0, i.e., $L_{b+1}^m \geq -m+1$. The behavior of packets from 1 to $b+1$ is determined by service completion times of them at each node and their relative distances to packet 0. Imagine that the service completion times are controlled by timers from 1 to $b+1$; timer i is associated with packet i. All timers are set at time 0, and each timer expires and is reset every random amount of time exponentially distributed with unit mean; these random times are all independent. If packet i is in service when timer i expires, then packet i completes service and proceed to the next node; otherwise, if packet i is waiting in the buffer when timer i expires, then nothing changes. Due to the memoryless property of the exponential distribution, the resulting service times of packets are still independent and exponentially distributed with unit mean. This procedure renders right-continuous functions $\{L_i^m(t),\, t \geq 0\}$, $1 \leq i \leq b+1$, satisfying

$$L_i^m(t) = L_i^m + \sum_{j=1}^{n} 1_{\{L_i^m(T_i(j)-) \,<\, L_{i-1}^m(T_i(j)-)\}}, \tag{4.10}$$

for $T_i(n) \leq t < T_i(n+1)$, $n \geq 0$, where

$$T_i(n) \equiv \sum_{j=1}^{n} S_i(j), \tag{4.11}$$

and $\{S_i(j)\}_{1 \leq i \leq b+1,\, j \in \mathbb{N}}$ is a sequence of i.i.d. exponential random variables with unit mean. Under the FCFS discipline, it is appropriate to think of $L_i^m(t)$ for $1 \leq i \leq b+1$ as the index of the node where packet i is located at time t (before packet $b+1$ is lost).

Finally we construct a discrete-time random process $\{Y_{1:b+1}^m(n),\, n \geq 0\}$ such that

$$Y_{1:b+1}^m(n) \equiv L_0^m(T(n))1_{1:b+1} - L_{1:b+1}^m(T(n)), \tag{4.12}$$

where $\{T(n)\}_{n \geq 0}$ is the (strictly) increasing sequence generated from all elements in $\{T_i(n)\}_{n \geq 0}$ for $0 \leq i \leq b+1$. This discrete-time process describes the evolution of the relative distances between packets in the network. We note that this process is not Markovian since the increasing sequence $\{T_0(n)\}_{n\geq 0}$ has dependent increments, which follows from the fact that the sojourn times of packet 0 at different nodes, i.e., $W_0(i)$, $i \in \mathbb{N}$, are dependent (see (4.9)). Observe that the packet $b+1$ is lost if the process hits the origin. Similarly to (4.3), we define $\tau^m_{0_{1:b+1}}$ as the first hitting time for this discrete-time process:

$$\tau^m_{0_{1:b+1}} = \inf\{i \geq 0 : Y^m_{1:b+1}(i) = 0_{1:n}\}.$$

The packet $b+1$ is lost before it reaches node $k+1$ if and only if the location of packet 0 at time $T(\tau^m_{0_{1:n}})$ is less than or equal to k, i.e., $L^m_0(T(\tau^m_{0_{1:n}})) \leq k$. Note that $L^m_0(T(s))$ for any fixed $s \in \mathbb{N}$ can be represented by

$$L^m_0(T(s)) = \xi^m(s) + 1,$$

where $\xi^m(s)$, $s \in \mathbb{N}$, is defined, analogously to (4.4), as

$$\xi^m(s) = \sum_{i=1}^{s} 1_{\{Y^m_{1:n}(i) - Y^m_{1:n}(i-1) = 1_{1:n}\}};$$

this follows from the fact that $Y^m_{1:b+1}(i)$ increases by $1_{1:b+1}$ only when packet 0 proceeds to the next node. Therefore, the loss of packet $b+1$ occurs if and only if $\xi^m(\tau^m_{0_{1:n}}) \leq k-1$. Lemma 4.3 implies

$$\mathbb{P}[L^m_{b+1} \leq -m] = \mathbb{P}[Y^m_{b+1}(0) \geq m+1] \to 0,$$

as $m \to \infty$, i.e., asymptotically, all packets from 1 to $b+1$ enter the (extended) network before packet 0 departs from node 0. Then, it is not difficult to see from (4.6) and the preceding argument that

$$\begin{aligned} p^0_k &= \lim_{m\to\infty} p^m_k \\ &= \lim_{m\to\infty} \mathbb{P}[\xi^m(\tau^m_{0_{1:b+1}}) \leq k-1,\, Y^m_{b+1}(0) \leq m]. \end{aligned} \tag{4.13}$$

4.4 Main results

This section provides the main results of the chapter that characterize the critical loading regime for linear finite-buffer networks. We first present a technical lemma. Let $\theta_k(\lambda)$, $k \in \mathbb{N}$, $\lambda \geq 0$, be the throughput of a k-node linear network when the input arrival process at node 1 is Poisson with rate λ.

Lemma 4.4. *If* $0 \leq \lambda' \leq \lambda$, *then* $\theta_k(\lambda') \leq \theta_k(\lambda)$.

Proof. See Appendix C.4.

Now we consider a sequence of linear finite-buffer networks indexed by the size of the network k. The input arrival process at node 1 is Poisson with rate λ_k that varies with k. A critical loading regime is defined as a scaling regime of the input arrival rate λ_k in terms of the size of the network k in which the loss probability defined as the long-term fraction of lost packets are strictly within $(0, 1)$ asymptotically as the size of the network increases. In this regime, the input and output (throughput) rates are proportional. Such a regime is of interest since it delivers a relatively high throughput at low network cost per packet delivered. Asymptotic loss probabilities of 0 and 1 correspond to under-loaded and over-loaded regimes, respectively. Under the over-loaded regime, only a small fraction of packets are delivered, and, thus, operating a network in this regime is not (energy) efficient. On the other hand, under the under-loaded regime, higher throughputs are feasible to achieve with small increments of the network cost; hence, in this regime, the network capacity is not efficiently utilized. Under the critical loading regime, one balances between two conflicting goals: (i) achieving high throughput and (ii) maintaining low loss probability (low network cost).

The critical loading regime for $b = 0$ is already established in [35], which is given by $\lambda_k = \Theta(1/\sqrt{k})$, as $k \to \infty$. Thus, we focus on the case that $b \geq 1$. Let r_k be the ratio of the throughput to the input arrival rate, i.e.,

$$r_k \equiv \frac{\theta_k(\lambda_k)}{\lambda_k}.$$

The following subsection provides an asymptotic lower-bound on this ratio.

4.4.1 Lower-bound

The throughput $\theta_k(\lambda_k)$ is bounded below by (see [29, (7)])

$$\theta_k(\lambda_k) \geq \lambda_k - kc_b\lambda_k^{b+2}, \tag{4.14}$$

where $c_b^{-1} = \sum_{i=0}^{b+1} \lambda_k^i$. This yields the following asymptotic properties on the ratio r_k.

Proposition 4.1. *If $\lambda_k = o(k^{-1/(b+1)})$, as $k \to \infty$, then*

$$\lim_{k\to\infty} r_k = 1.$$

Furthermore, if $\lambda_k = \Theta(k^{-1/(b+1)})$, as $k \to \infty$, then

$$\varliminf_{k\to\infty} r_k > 0. \tag{4.15}$$

Proof. The first statement of the proposition is straightforward from (4.14). For the second statement of the proposition, it suffices to show that the statement holds if $\lambda_k \sim ak^{-1/(b+1)}$, as $k \to \infty$, for any finite, fixed $a > 0$. If $a \in (0,1)$, then (4.15) follows from (4.14) since $c_b \leq 1$. If $a \geq 1$, then there exists some $a' \in (0,1)$ such that

$$\lambda_k \geq \lambda_k' = a'k^{-1/(b+1)},$$

for all $k \in \mathbb{N}$, and Lemma 4.4 yields

$$\begin{aligned} \varliminf_{k\to\infty} \frac{\theta_k(\lambda_k)}{\lambda_k} &\geq \varliminf_{k\to\infty} \frac{\theta_k(\lambda_k')}{\lambda_k'}\frac{\lambda_k'}{\lambda_k} \\ &= \frac{a'}{a} \varliminf_{k\to\infty} \frac{\theta_k(\lambda_k')}{\lambda_k'} > 0. \end{aligned} \tag{4.16}$$

Hence, the second statement of the proposition holds. Proposition 4.1 implies that the critical loading regime should satisfy, as $k \to \infty$,

$$\lambda_k = \Omega(k^{-1/(b+1)}). \tag{4.17}$$

4.4.2 Critical loading regime

In this subsection, we first identify an (approximate) upper-bound on the input rate under the critical loading regime. This result along with (4.17) is then used to characterize the critical loading regime. By definition (see Section 4.3.2), it is straightforward that

$$r_k = 1 - p_k^0. \tag{4.18}$$

In order to evaluate p_k^0 from (4.13), it is necessary to characterize the sojourn times $W_0(i)$, $i \in \mathbb{N}$, explicitly (see Section 4.3.5). In this chapter, in order to simplify the analysis, we utilize the following approximation.

Approximation 4.1. Sojourn times $W_0(i)$, $i \in \mathbb{N}$, are independent and exponentially distributed with mean $(1-\rho)^{-1}$, where $\rho \sim r\lambda_k$, as $\lambda_k \downarrow 0$, for some fixed $r \in (0, 1)$.

Remark 4.4. Under a critical-loading regime, by definition, the loss probability is strictly within $(0, 1)$ asymptotically as the size of the network increases. This implies that the output rate (throughput) at each node of the network is proportional to the input arrival rate. In that case, the density of packets in the network (i.e., the expected number of packets per node) is asymptotically proportional to the input arrival rate; this stems from the fact that the expected number of packets in a node is asymptotically equal to the output rate. Now, observe that the service of packet 0 at a node can be blocked (delayed) by its preceding packets – packet 0 has to wait in the buffer if the server is busy. This results in non-zero waiting times. It can be argued that the probability that packet 0 is blocked from being serviced due to its preceding packets is proportional to the density of packets. Under the critical regime, this blocking probability, denoted by ρ, is asymptotically proportional to the input arrival rate. Finally, Approximation 4.1 follows by simplifying that such blocking events occur independently within and across nodes (with probability ρ).

Let $\tilde{r}_k$ be the approximation of r_k under Approximation 4.1. The following lemma provides an upper-bound on $\tilde{r}_k$.

Lemma 4.5. *We have*

$$\tilde{r}_k \leq \frac{\mathbb{E}[\xi(\tau_{0_{1:b+1}}) |\, X_{1:b+1}(0) = 1_{1:b+1} - L^{\infty}_{1:b+1}]}{k},$$

where $L^{\infty}_{1:b+1}$ satisfies (4.7)*.*

Proof. See Appendix C.5.

Lemma 4.5 yields the following asymptotic results. For notational simplicity, we use

$$u_b(k) \equiv \begin{cases} \sqrt{\log k/k}, & b = 1, \\ k^{-1/(b+1)}, & b \geq 2. \end{cases} \tag{4.19}$$

Proposition 4.2. *For $b \geq 1$, if $\lambda_k = \omega(u_b(k))$, as $k \to \infty$, then*

$$\lim_{k\to\infty} \tilde{r}_k = 0.$$

Furthermore, if $\lambda_k = \Theta(u_b(k))$, as $k \to \infty$, then

$$\overline{\lim_{k\to\infty}}\, \tilde{r}_k < 1. \tag{4.20}$$

Proof. In view of Remark 4.3, if $\rho \sim r\lambda_k$, as $\lambda_k \downarrow 0$, for some fixed $r \in (0,1)$, which is given in Approximation 4.1, then $X_{1:b+1}(0) = 1_{1:b+1} - L^{\infty}_{1:b+1}$ satisfies (A1)–(A3) in Lemma 4.2. Therefore, Lemmas 4.2 and 4.5 yield, if $b = 1$,

$$\varlimsup_{k\to\infty} \tilde{r}_k \le \frac{1}{r^2} \varlimsup_{k\to\infty} \frac{\log \lambda_k^{-1}}{k\lambda_k^2}, \tag{4.21}$$

and, if $b \ge 2$,

$$\varlimsup_{k\to\infty} \tilde{r}_k \le \frac{b^2(b-2)!}{r^{b+1}} \varlimsup_{k\to\infty} \frac{1}{k\lambda_k^{b+1}}. \tag{4.22}$$

The first statement of the proposition is straightforward from the preceding inequalities. For the second statement of the proposition, as in the proof of Proposition 4.1, it suffices to show that the statement is true if $\lambda_k \sim au_b(k)$, as $k \to \infty$, for any finite, fixed $a > 0$. For notational simplicity, let

$$C_b \equiv \begin{cases} r^{-1}, & b = 1, \\ r^{-1}(b^2(b-2)!)^{1/(b+1)}, & b \ge 2. \end{cases}$$

If $a > C_b$, then (4.20) follows from (4.21) and (4.22). If $a \in (0, C_b]$, then there exists some $a' > C_b(C_b/a)^{1/b} \ge C_b$ such that

$$\lambda_k \le \lambda'_k = a'u_b(k),$$

for all $k \in \mathbb{N}$. Similarly to (4.16), it can be shown that

$$\varlimsup_{k\to\infty} \frac{\theta_k(\lambda_k)}{\lambda_k} \le \varlimsup_{k\to\infty} \frac{\theta_k(\lambda'_k)}{\lambda'_k}\frac{\lambda'_k}{\lambda_k} = \frac{a'}{a}\left(\frac{C_b}{a'}\right)^{b+1} < 1,$$

and the second statement of the proposition follows.

Recall that Approximation 4.1 is applicable for networks operating in the critical loading regime only (see Remark 4.4). We can identify a critical loading regime by first assuming that the network operates in a critical loading regime and evaluating the loss probability using Approximation 4.1; if the resulting loss probability is strictly within $(0,1)$, as $k \to \infty$, it can be argued that the assumed critical loading regime and Approximation 4.1 were appropriate. In this case, the constant r in Approximation 4.1 corresponds to the limiting value of r_k, as $k \to \infty$, approximately, which is strictly within $(0,1)$ in the critical loading regime. From this point of view, Proposition 4.2 implies that the critical loading regime for $b \geq 1$ satisfies (approximately)

$$\lambda_k = O(u_b(k)), \tag{4.23}$$

as $k \to \infty$, and this provides an upper-bound on the critical-loading regime.

From (4.17) and (4.23), the critical loading regime for $b \geq 2$ is reduced to $\lambda_k = \Theta(k^{-1/(b+1)})$, as $k \to \infty$ (see (4.19)). For $b = 1$, the upper- and lower- bounds of the critical loading regime, given in (4.17) and (4.23), respectively, do not coincide. We conjecture that the upper-bound in (4.23) is tight despite it stems from an approximation, and that the critical loading regime for $b = 1$ is given by $\lambda_k = \Theta(\sqrt{\log k/k})$, as $k \to \infty$. It is argued that the critical loading is not impacted substantially if we increase the size of the buffer from 0 to 1. For $b \geq 2$, the critical loading regime is considerably impacted by the buffer size.

4.4.3 Simulation results

This subsection provides some simulation results that illustrate our analytical results presented in the previous subsection. First, in order to see the behavior of the input arrival rate λ_k under the critical loading regime, in Figure 4.2, we plotted estimated (by simulation) input arrival rates that deliver loss probability of 0.5 (i.e., $r_k = \theta_k(\lambda_k)/\lambda_k = 0.5$) for different values of $k \in [10^2, 10^5]$ and $b \in \{0, 1, 2, 3\}$. We note that 0.5 is just an arbitrary value within $(0,1)$, and that for different choices of the loss probability, the qualitative behavior of the critical input rate remains

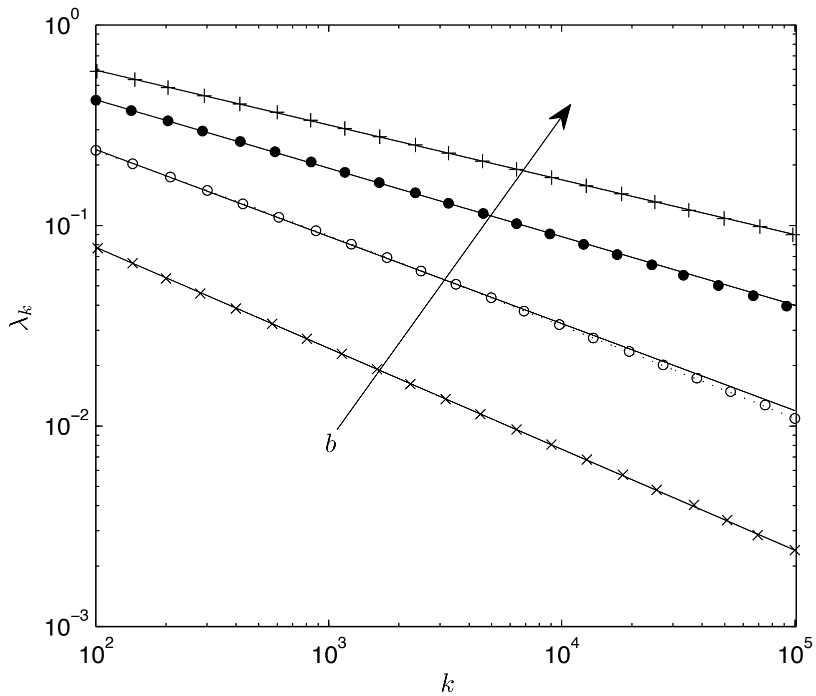


Figure 4.2: Estimated (by simulation) input arrival rate λ_k at node 1 that delivers $\theta_k(\lambda_k)/\lambda_k = 0.5$ for $b = 0$ ($\times$), $b = 1$ ($\circ$), $b = 2$ ($\bullet$), and $b = 3$ ($+$). The solid lines are obtained by fitting the simulated data to the equation $\lambda_k = ak^{-c}$; the estimated constants are $a = 0.79, 1.762, 2.039, 2.092$ and $c = 0.5034, 0.4338, 0.3414, 0.2732$ for $b = 0, 1, 2, 3$, respectively. For $b = 1$, we also plot the dotted line obtained by fitting the simulated data to the equation $\lambda_k = a\sqrt{\log k}k^{-c}$; the estimated constants are $a = 1.154$ and $c = 0.511$.

the same. Along with the simulated data, we also plotted fitted curves (solid lines) obtained by fitting the simulated data to the equation $\lambda_k = ak^{-c}$ with parameters a and c; we used a robust (using the bisquare weights method) non-linear least-squares estimation in data fitting. For $b = 0, 1, 2, 3$, the estimated parameters are $a = 0.79, 1.762, 2.039, 2.092$ and $c = 0.5034, 0.4338, 0.3414, 0.2732$, respectively. Observe that the estimated values of c are reasonably consistent with our analytical results for $b \in \{0, 2, 3\}$. As b increases, however, the discrepancy between the estimated and theoretical values of c becomes larger. This is due to the fact that the analytical results are asymptotically true, as $k \to \infty$, in particular, as $\lambda_k \to 0$; observe that the estimated input rate λ_k for given k goes away from 0, as b increases. For $b = 1$, we also plotted a fitted curve (dotted line) that obtained by fitting the simulated data

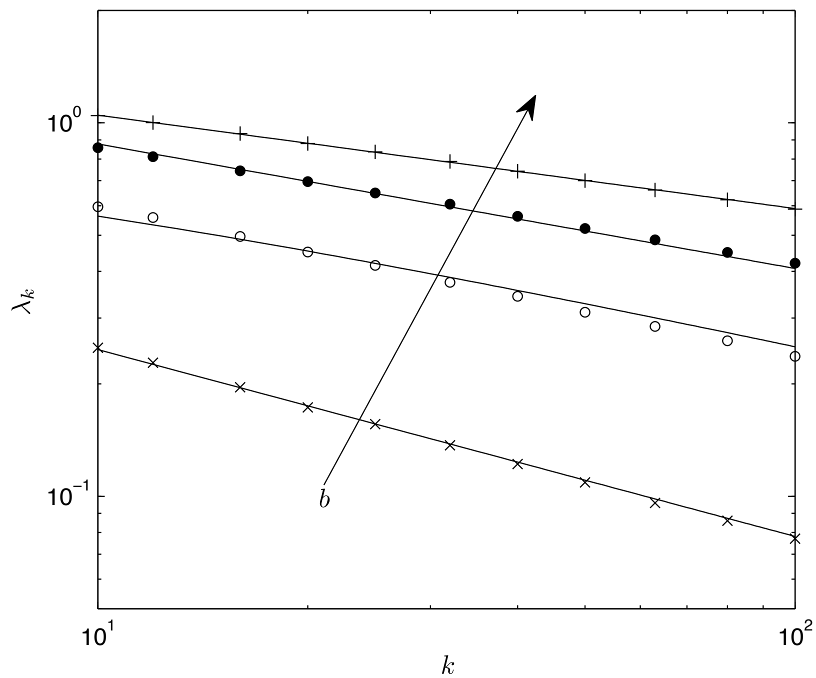


Figure 4.3: Estimated (by simulation) input arrival rate λ_k at node 1 that delivers $\theta_k(\lambda_k)/\lambda_k = 0.5$ for $b = 0$ ($\times$), $b = 1$ ($\circ$), $b = 2$ ($\bullet$), and $b = 3$ ($+$). The solid lines are obtained by fitting the data to the equation $\lambda_k = a/\sqrt{k}$ for $b = 0$ and $\lambda_k = au_b(k)$ for $b \geq 1$ (see (4.19)); the estimated constants are $a = 0.7812, 1.172, 1.891, 1.865$ for $b = 0, 1, 2, 3$, respectively.

to the equation $\lambda_k = a\sqrt{\log k}k^{-c}$ with parameters a and c; the estimated parameters are $a = 1.154$ and $c = 0.511$. As k increases, the simulated data fit better to the equation $\lambda_k = a\sqrt{\log k}k^{-c}$, which supports that our analytical results become accurate asymptotically as $k \to \infty$.

In Figure 4.2, we plotted the estimated (by simulation) input arrival rates λ_k that result in loss probability of 0.5 for different values of $k \in [10, 100]$ and $b \in \{0, 1, 2, 3\}$. The solid lines are obtained by fitting the simulated data to the equation $\lambda_k = a/\sqrt{k}$ for $b = 0$ and $\lambda_k = au_b(k)$ for $b \geq 1$ (see (4.19)); the estimated parameters are $a = 0.7812, 1.172, 1.891, 1.865$ for $b = 0, 1, 2, 3$, respectively. This figure illustrates that our asymptotic results are reasonably applicable to finite-size networks as well.

4.5 Concluding remarks

A critical loading regime is a scaling regime of the input arrival rate in terms of the size of the network in which the asymptotic loss probability is strictly within $(0, 1)$, as the size of the network increases. Such a regime is of interest since it delivers a relatively high throughput at low network cost. The critical loading regime was previously established only for linear networks with bufferless nodes. In this chapter, we considered a more general model of the linear networks, where each node has a finite buffer. Our analytical results indicated that the input rate under the critical loading regime is $\Omega(1/\sqrt{k})$ and $O(\sqrt{\log k/k})$ for $b = 1$, and $\Theta(k^{-1/(b+1)})$ for $b \geq 2$, as $k \to \infty$, where k denotes the number of nodes in the network. It was shown in [35] that the critical loading regime for linear networks with bufferless nodes (i.e., $b = 0$) occurs when the input rate is $\Theta(1/\sqrt{k})$, as $k \to \infty$. These results indicate that the qualitative behavior of the critical loading regime for linear networks depends on whether the buffer size is greater than 1.

This chapter established a qualitative relation between the achievable throughput and the required buffer space at each node of the network under the critical loading regime. The throughput and the input rate are proportional under the critical loading regime, and, thus, given the size of available buffers at each node of the network, this qualitative relation provides some guideline in determining an appropriate input rate at the source of the network for achieving a relatively high throughput with low (energy) cost. Conversely, this can be utilized in determining a necessary amount of buffers for achieving a certain throughput with low (energy) cost, and, moreover, is applicable to some of the resource allocation problems in large-scale networks with multiple users. When the buffer space at each node is shared by multiple users in the network, our results on the critical loading regime provide a qualitative criterion in determining an appropriate amount of buffers that need to be allocated to each user based on the requirements on its input rate, throughput and/or network cost.

CHAPTER V

Conclusions

Due to rapid expansion of modern communication networks, performance scalability is a central problem in designing next-generation network protocols and architectures. In large-scale networks, performance bottlenecks attributed to the finiteness of local resources can be critical factors in determining the overall network performance. While this problem was considered to some extent by practitioners and system engineers, there has been a limited amount of work in establishing its mathematical foundations. The need for theoretical investigation is even more apparent in view of the fact that conducting experimental studies on large-scale networks is prohibitively expensive. In this dissertation, we focused on three models of large-scale communication networks with limited local resources, and investigated their asymptotic performance characteristics, as the size of the network or the number of users increase.

First, we considered a packet network with a large number of users and investigated the effectiveness of application-layer coding for recovering packet losses due to buffer overflows. Application-layer coding is one of mechanisms for achieving reliable communication in packet networks. Coding has two conflicting effect on the network performance: (i) on the positive side, coding can enable end users to recover some of the dropped packets, thus, it reduces the packet loss rate. (ii) on the negative side, coding introduces additional packets, increases the overall offered load, and results in higher packet drop probability. The effectiveness of coding depends on which of the above-mentioned effects is dominant. Our analytical results indicate that the network parameter space can be partitioned into two regions where coding is beneficial and detrimental, respectively. The results provide good guidelines to network designers who consider application-layer coding as one of the methodologies to improve the network performance. Informally, we concluded that coding can be advantageous in under-loaded networks only.

Next, we studied the asymptotic characteristics of the departure processes of linear loss networks, as the size of the network increases. A linear network is a tandem network consisting of a series of identical nodes, which is a representative model of large-scale communication networks with limited or no cross-traffic interference. The departure process is of interest since various performance properties of the network such as throughput and traffic burstiness can be obtained from it. In this research, we characterized the asymptotic behavior of the departure process of the linear network consisting of bufferless nodes with exponential service times by determining the joint probability density function of any finite number of consecutive inter-departure times. This asymptotic behavior of the departure process is completely attributed to the characteristics of the network itself (i.e., the distributions of service times and the number of buffers) and is not impacted by the input as long as the input does not vary with the size of the network.

Finally, we focused on linear networks consisting of finite-buffer nodes, and identified a critical loading regime of the input under which the loss probability due to buffer overflows is strictly within $(0, 1)$, asymptotically as the size of the network increases. Such a regime is of interest given that it delivers a relatively high throughput at low network (energy) cost. Our analytical results using an approximation indicate that the qualitative behavior of the achievable throughput under the critical loading regime depends on whether the buffer size is greater than 1.

In large-scale communication networks with limited local resources, the achievable performance limits are usually given by functions of the network parameters, including the amount of local resources available at each node of the network. Such relations between the performance and network resources can be utilized in various ways. If the available resources at each node are fixed, then these results provide some guidelines to system designers or network operators in developing efficient network algorithms or in tuning network parameters for achieving some performance requirements. For example, using the results on application-layer coding in Chapter II, one can determine whether employing coding is advantageous in a network with specific network parameters such as the link capacity and buffer size. From the results on the critical loading regime of linear finite-buffer networks in Chapter IV, one can determine an appropriate input rate which delivers a relatively high throughput with low network cost. On the other hand, the relations between the performance and network resources can be utilized in determining a necessary amount of local resources that guarantee a certain performance limit. Furthermore, they are applicable in addressing some of the resource allocation problems in large-scale networks with multiple users. Namely, one can decide appropriate amounts of resources that need to be allocated to individual users with different performance and/or quality requirements in a network with multiple users. For example, recall that the linear network is a representative model for large-scale multi-hop wireless networks where each node has limited buffers. If there are multiple source-destination pairs (users) in the network and the local resources at each node are allocated to individual users disjointly, then the individual paths can be thought of independent linear networks. Then, by using the results in Chapter IV, one can determine the necessary amounts of local resources (i.e., buffer space and transmission capacity) for individual users based on their input rates and/or throughput requirements.

We conclude this dissertation with a discussion of the future work. First, it is of interest to establish a complete relationship between the throughput and network (energy) cost in linear finite-buffer networks. Although we identified the critical loading regime in which a relatively high throughput can be achieved with low network cost, an explicit relation between these two properties is still unknown for general linear finite-buffer networks. In order to identify this relation, one needs to obtain an exact expression of the throughput for any given input process using an approach different from the one we employed in this dissertation. Next, performance scalability under specific QoS (Quality-of-Service) requirements (e.g., delay and loss rate) is also an interesting problem to investigate. In order to address this problem, one needs to identify the relation (or tradeoff) among various performance characteristics of the network such as throughput, loss probability and delay. The results on this problem can provide some guidelines in determining feasible QoS requirements for various network services. So far, we have mainly focused on understanding the fundamental performance properties of large-scale communication networks, but the development of scalable network algorithms is also a problem of great importance. An algorithm can be said to be scalable in the sense that it requires only a limited amount of resources (memory, power and computing power) at each node while keeping some performance guarantee even though the size of the network increases. Designing scalable network algorithms is a challenging problem since it requires not only mathematical insights but also creative ideas. In general, simple algorithms are desirable since complex algorithm are difficult to implement. Moreover, if an algorithm is complex, it may even not be feasible to verify its correctness and evaluate its performance either theoretically or experimentally.

APPENDICES

APPENDIX A

Appendix for Chapter II

A.1 Proof of Lemma 2.1

Utilizing a systematic MDS code, we have either $\{\dot{L}^n = 0\}$ or $\{\dot{L}^n = k\}$ on the event $\{\dot{D}_d^n = k\}$. Therefore, it follows that

$$\mathbb{E}[\dot{L}^n|\dot{D}_d^n = k] = k(1 - \mathbb{P}[\dot{L}^n = 0|\dot{D}_d^n = k]). \tag{A.1}$$

Recall that the data packets in $\dot{\mathbf{v}}_n$ are included in both $\mathbf{v}_n$ and $\mathbf{v}_{n-1}$. Assuming that at least one packet from $\dot{\mathbf{v}}_n$ is dropped, define two events $\mathcal{E}_1^n = \{\dot{\mathbf{v}}_n \text{ is recovered by decoding } \mathbf{v}_n\}$ and $\mathcal{E}_2^n = \{\dot{\mathbf{v}}_n \text{ is recovered by decoding } \mathbf{v}_{n-1}\}$. For $k > 0$, we have

$$\begin{aligned}\mathbb{P}[\dot{L}^n = 0|\dot{D}_d^n = k] &= \mathbb{P}[\mathcal{E}_1^n \cup \mathcal{E}_2^n|\dot{D}_d^n = k] \\ &= 1 - (1 - \mathbb{P}[\mathcal{E}_1^n|\dot{D}_d^n = k])(1 - \mathbb{P}[\mathcal{E}_2^n|\dot{D}_d^n = k]),\end{aligned}$$

where the second equality follows from the assumptions of the lemma. By combining the preceding equality with (A.1), one can obtain

$$\mathbb{E}[\dot{L}^n|\dot{D}_d^n = k] = k(1 - \xi_1^n(k))(1 - \xi_2^n(k)), \tag{A.2}$$

where $\xi_1^n(k) = \mathbb{P}[\mathcal{E}_1^n|\dot{D}_d^n = k]$ and $\xi_2^n(k) = \mathbb{P}[\mathcal{E}_2^n|\dot{D}_d^n = k]$; for notational simplicity, we extended the definition of $\xi_1^n(k)$ and $\xi_2^n(k)$ for $k = 0$. Observe that if $\dot{D}_d^n = 0$

then $\dot{L}^n = 0$ since there are no dropped packets. Besides, if $\dot{D}_d^n > \alpha/2$ then $\dot{L}^n = \dot{D}_d^n$ since no dropped packets can be recovered in this case. Formally, we have

$$\xi_1^n(k) = \xi_2^n(k) = \begin{cases} 1, & k = 0, \\ 0, & k > \alpha/2. \end{cases} \tag{A.3}$$

Let D^n denote the number of dropped packets among the data packets in $\mathbf{v}_n$ and the additional $\alpha/2$ coded packets generated from $\mathbf{v}_n$, i.e., $D^n = \dot{D}_d^n + \dot{D}_d^{n+1} + \dot{D}_c^n$. If $D^n \leq \alpha/2$, then all dropped packets in $\mathbf{v}_n$ can be recovered. Otherwise, the dropped packets in $\mathbf{v}_n$ can be recovered only when the dropped packets in $\dot{\mathbf{v}}_{n+1}$ are recovered by decoding the next block $\mathbf{v}_{n+1}$ and the number of (remaining) unrecovered dropped packets is at most $\alpha/2$. This argument leads to

$$\xi_1^n(k) = \mathbb{P}[D^n \leq \alpha/2 | \dot{D}_d^n = k] + \mathbb{P}[D^n > \alpha/2,\, D^n - \dot{D}_d^{n+1} \leq \alpha/2,\, \mathcal{E}_1^{n+1} | \dot{D}_d^n = k],$$

for $1 \leq k \leq \alpha/2$. The assumptions of the lemma imply that the event $\mathcal{E}_1^{n+1}$ is independent of both $\dot{D}_d^n$ and $\dot{D}_c^n$. Thus, the second term on the right-hand side of the preceding equality can be expressed as

$$\begin{aligned} &\mathbb{P}[D^n > \alpha/2,\, D^n - \dot{D}_d^{n+1} \leq \alpha/2,\, \mathcal{E}_1^{n+1} | \dot{D}_d^n = k] \\ &\qquad = \mathbb{P}[\alpha/2 - \dot{D}_d^{n+1} < \dot{D}_c^n + k \leq \alpha/2,\, \mathcal{E}_1^{n+1}], \end{aligned}$$

and, if we represent this as the sum of conditional probabilities (conditioned on the event $\{\dot{D}_d^{n+1} = i\}$ for $1 \leq i \leq \alpha/2$), then we have

$$\begin{aligned} &\mathbb{P}[\alpha/2 - \dot{D}_d^{n+1} < \dot{D}_c^n + k \leq \alpha/2,\, \mathcal{E}_1^{n+1}] \\ &\qquad = \sum_{i=1}^{\alpha/2} \mathbb{P}[\alpha/2 - i < \dot{D}_c^n + k \leq \alpha/2,\, \mathcal{E}_1^{n+1},\, \dot{D}_d^{n+1} = i]. \end{aligned}$$

For notational simplicity, let $\dot{\alpha} = \alpha/2$. Then, one can derive the following equation from the preceding argument:

$$\text{(A.4)} \quad \xi_1^n(k) = \mathbb{P}[\dot{D}_d^{n+1} + \dot{D}_c^n + k \le \dot{\alpha}] + \sum_{i=1}^{\dot{\alpha}} \xi_1^{n+1}(i)\mathbb{P}[\dot{\alpha} - i < \dot{D}_c^n + k \le \dot{\alpha}]\mathbb{P}[\dot{D}_d^{n+1} = i],$$

for $1 \le k \le \dot{\alpha}$. Likewise, it can be shown that

$$\text{(A.5)} \quad \xi_2^n(k) = \mathbb{P}[\dot{D}_d^{n-1} + \dot{D}_c^{n-1} + k \le \dot{\alpha}] + \sum_{i=1}^{\dot{\alpha}} \xi_2^{n-1}(i)\mathbb{P}[\dot{\alpha} - i < \dot{D}_c^{n-1} + k \le \dot{\alpha}]\mathbb{P}[\dot{D}_d^{n-1} = i],$$

for $1 \le k \le \dot{\alpha}$. Under steady-state, both (A.4) and (A.5) have the same solution $\xi(k) = \xi_1^n(k) = \xi_2^n(k)$. In this case, (A.2)–(A.5) yield

$$\mathbb{E}[\dot{L}^n | \dot{D}_d^n = k] = k(1 - \xi(k))^2,$$

where $\xi(k)$, $k \ge 0$, is given as in the statement of the lemma. This concludes the proof of Lemma 2.1.

A.2 Proof of Theorem 2.1

Let $D^N(t)$, $t \ge 1$, denote the number of dropped packets in the time slot t:

$$D^N(t) = (Q_*^N(t-1) + H^N(t) - C^N - B^N)^+,$$

where $Q_*^N(t)$ denotes the queue occupancy at the end of the time slot t. Since all packets are assumed to have the same priority, the drop probability of a packet at the time slot t is given by

$$\text{(A.6)} \qquad p_D^N(t) = D^N(t)/H^N(t).$$

Without loss of generality, consider the arrival process of the source 1. Let $\{\tau_s, 1 \leq s \leq M^N\}$ be the sequence of arrival times of data packets in a block of the source 1. Then, the probability that k data packets are dropped in a block is given by

$$\mathbb{P}[D_d^N = k] = \sum_{S \in \mathcal{S}^k} \mathbb{E}\left[\prod_{s \in S} p_D^N(\tau_s) \prod_{s \notin S} (1 - p_D^N(\tau_s))\right], \tag{A.7}$$

where $\mathcal{S}^k$ is the collection of all k-subsets of $\{1, 2, \ldots, M^N\}$.

Now define, for $l \in \mathbb{N}$,

$$\mathcal{S}_l^k = \{S : |i - j| > l, \forall i, j \in S \in \mathcal{S}^k\}. \tag{A.8}$$

The following lemma states that under the critical-load scaling, the intervals between packet drops in a block are asymptotically $\Omega(\log N)$, as $N \to \infty$.

Lemma A.1. *Consider the critical-load scaling. Suppose that $l = \lfloor a \log N \rfloor$ for fixed $a > 0$. Then, as $N \to \infty$,*

$$\sum_{S \in \mathcal{S}^k \setminus \mathcal{S}_l^k} \mathbb{E}\left[\prod_{s \in S} p_D^N(\tau_s) \prod_{s \notin S} (1 - p_D^N(\tau_s))\right] \to 0.$$

Proof. See Appendix A.3.

Lemma A.2. *Consider the critical-load scaling. Suppose that $l = \lfloor a \log N \rfloor$ for fixed $a > 0$. Then, as $N \to \infty$,*

$$\sum_{S \in \mathcal{S}_l^k} \mathbb{E} \prod_{s \in S} \frac{p_D^N(\tau_s)}{1 - p_D^N(\tau_s)} \to \frac{(m\hat{p}_D)^k}{k!},$$

where $\hat{p}_D$ is the limiting scaled drop probability that satisfies (2.19)*. Moreover, we have*

$$\limsup_{N \to \infty} \sum_{S \in \mathcal{S}_l^k} \left(\mathbb{E} \prod_{s \in S} \left(\frac{p_D^N(\tau_s)}{1 - p_D^N(\tau_s)}\right)^2\right)^{1/2} < \infty.$$

Proof. See Appendix A.4.

Lemma A.3. *Consider the critical-load scaling. Then*

$$\prod_{s=1}^{M^N}(1 - p_D^N(\tau_s)) \xrightarrow{\mathbb{P}} e^{-m\hat{p}_D},$$

as $N \to \infty$, where $\hat{p}_D$ is the limiting scaled drop probability that satisfies (2.19).

Proof. See Appendix A.5.

Next we present a proof of Theorem 2.1.

Proof of Theorem 2.1. First consider D_d^N, which denotes the number of dropped data packets in a block of the source 1. The set $\mathcal{S}^k$ can be partitioned into two disjoint subsets $\mathcal{S}_l^k$ and $\mathcal{S}^k \setminus \mathcal{S}_l^k$. Thus, in view of (A.7), we have

$$\mathbb{P}[D_d^N = k] = \sum_{S \in \mathcal{S}_l^k} \mathbb{E}\Pi^N(S) + \sum_{S \in \mathcal{S}^k \setminus \mathcal{S}_l^k} \mathbb{E}\Pi^N(S), \tag{A.9}$$

where $\Pi^N(S) = \prod_{s \in S} p_D^N(\tau_s) \prod_{s \notin S}(1 - p_D^N(\tau_s))$. By letting

$$\Sigma^N = \sum_{S \in \mathcal{S}_l^k} \prod_{s \in S} \frac{p_D^N(\tau_s)}{1 - p_D^N(\tau_s)},$$

$$\Gamma^N = \prod_{s=1}^{M^N}(1 - p_D^N(\tau_s)) - e^{-m\hat{p}_D},$$

and by using the triangular inequality, it is straightforward to show that

$$\left| \sum_{S \in \mathcal{S}_l^k} \mathbb{E}\Pi^N(S) - \frac{(m\hat{p}_D)^k}{k!} e^{-m\hat{p}_D} \right| \le \left|\mathbb{E}[\Sigma^N \Gamma^N]\right| + e^{-m\hat{p}_D} \left| \mathbb{E}\Sigma^N - \frac{(m\hat{p}_D)^k}{k!} \right|. \tag{A.10}$$

For some $\epsilon > 0$, define an event $\mathcal{G}_\epsilon^N = \{|\Gamma^N| < \epsilon\}$. Then, we have

$$\begin{aligned} \left|\mathbb{E}[\Sigma^N \Gamma^N]\right| &\le \mathbb{E}[|\Sigma^N \Gamma^N| \cdot 1_{\mathcal{G}_\epsilon^N}] + \mathbb{E}[|\Sigma^N \Gamma^N| \cdot 1_{\bar{\mathcal{G}}_\epsilon^N}] \\ &\le \epsilon \mathbb{E}\Sigma^N + \mathbb{E}[\Sigma^N 1_{\bar{\mathcal{G}}_\epsilon^N}]; \end{aligned} \tag{A.11}$$

the first inequality is due to the Jensen's inequality and the second inequality follows from $|\Gamma^N| \leq 1$. Furthermore, the Cauchy-Schwarz inequality renders

$$\mathbb{E}[\Sigma^N 1_{\bar{\mathcal{G}}_\epsilon^N}] \leq \sum_{S \in \mathcal{S}_l^k} \left(\mathbb{E}\left[\prod_{s \in S} \left(\frac{p_D^N(\tau_s)}{1 - p_D^N(\tau_s)} \right)^2 \right] \mathbb{P}[\bar{\mathcal{G}}_\epsilon^N] \right)^{1/2}.$$

Let $l = \lfloor a \log N \rfloor$ for fixed $a > 0$. Then, the second statement of Lemma A.2, Lemma A.3 and the preceding inequality imply $\mathbb{E}[\Sigma^N 1_{\bar{\mathcal{G}}_\epsilon^N}] \to 0$, as $N \to \infty$. Since $\lim_{N\to\infty} \mathbb{E}\Sigma^N < \infty$ (see the first statement of Lemma A.2) and (A.11) holds for any $\epsilon > 0$, it follows that $|\mathbb{E}[\Sigma^N \Gamma^N]| \to 0$, as $N \to \infty$; combining this limit, the first statement of Lemma A.2 and (A.10) yields

$$\sum_{S \in \mathcal{S}_l^k} \mathbb{E}\Pi^N(S) \to \frac{(m\hat{p}_D)^k}{k!} e^{-m\hat{p}_D},$$

as $N \to \infty$. Due to Lemma A.1 and the preceding limit, the first statement of the theorem follows from (A.9).

Second consider D_c^N, i.e., the number of dropped packets among additional α coded packets in a block of the source 1 (recall that without loss of generality, we consider the arrival process of the source 1). Note that additional α coded packets are transmitted in the same time slot as the last data packet of the block. Moreover, all packets have the same priority in the system. Thus, it follows from the union bound:

$$\mathbb{P}[D_c^N > 0] \leq \alpha \mathbb{E} p_D^N(\tau_{M^N}), \tag{A.12}$$

where τ_{M^N} is the arrival time of the last data packet in the block. The drop probability $p_D^N(t)$ is bounded by

$$p_D^N(t) \leq \frac{(H^N(t) - C^N)^+}{H^N(t)} \leq \frac{(H^N(t) - C^N)^+}{C^N}; \tag{A.13}$$

this together with (A.12) leads to

$$\mathbb{P}[D_c^N > 0] \le \alpha(\sqrt{N}/C^N)\mathbb{E}(\check{H}^N(\tau_{M^N}))^+, \tag{A.14}$$

where $\check{H}^N(t) = (H^N(t) - C^N)/\sqrt{N}$. Moreover, the relation $x^+ < e^x$ for $x \in \mathbb{R}$ implies

$$\mathbb{E}(\check{H}^N(\tau_{M^N}))^+ \le \mathbb{E}e^{\check{H}^N(\tau_{M^N})}. \tag{A.15}$$

Since additional α packets are transmitted in the same time slot as the last data packet of the block, we have $H^N(\tau_{M^N}) = (1+\alpha) + \sum_{i=2}^N H_{(i)}^N(\tau_{M^N})$. Assuming that processes are in their stationary regimes except the one corresponding to the source 1, the random variables $H_{(i)}^N(\tau_{M^N})$, $i = 2, 3, \ldots, N$, are i.i.d.. Thus, it follows that

$$\begin{aligned}\mathbb{E}e^{\check{H}^N(\tau_{M^N})} &= \mathbb{E}\prod_{i=1}^N e^{\check{H}_{(i)}^N(\tau_{M^N})} \\ &= e^{((1+\alpha)-C^N/N)/\sqrt{N}} \left(\mathbb{E}e^{\check{H}_{(2)}^N(\tau_{M^N})}\right)^{N-1},\end{aligned}$$

where $\check{H}_{(i)}^N(t) = (H_{(i)}^N(t) - C^N/N)/\sqrt{N}$. From (2.17), it can be shown that

$$\mathbb{E}e^{\check{H}_{(2)}^N(t)} = 1 + \frac{\alpha/m - \beta}{N} + \frac{\lambda(1-\lambda)}{2N} + o\left(\frac{1}{N}\right),$$

as $N \to \infty$, and this further results in

$$\lim_{N\to\infty} \mathbb{E}e^{\check{H}^N(\tau_{M^N})} < \infty. \tag{A.16}$$

Finally, putting together (A.14)–(A.16) renders the second statement of the theorem.

A.3 Proof of Lemma A.1

First we present a preliminary technical lemma.

Lemma A.4. *If $l = o(M^N)$, as $M^N \to \infty$, then*

$$|\mathcal{S}_l^k| = ((M^N)^k/k!)(1 - o(1)),$$

and

$$|\mathcal{S}^k| - |\mathcal{S}_l^k| = O(l \cdot (M^N)^{k-1}),$$

as $M^N \to \infty$*, for fixed* $k \in \mathbb{N}$*.*

Proof. Observe that $|\mathcal{S}^k|$ is bounded above by

$$|\mathcal{S}^k| = \binom{M^N}{k} \le \frac{(M^N)^k}{k!},$$

and $|\mathcal{S}_l^k|$ is bounded below by

$$|\mathcal{S}_l^k| \ge \frac{1}{k!} \prod_{i=0}^{k-1} (M^N - i(2l+1)) \ge \frac{(M^N - k(2l+1))^k}{k!}.$$

Under the assumption of the lemma, these two inequalities imply the first statement of the lemma. Furthermore, combining these two inequalities results in

$$|\mathcal{S}^k| - |\mathcal{S}_l^k| \le \frac{(M^N)^k - (M^N - k(2l+1))^k}{k!},$$

and, then, the second statement of the lemma also follows due to the assumption of the lemma.

Now we provide a proof of Lemma A.1.

Proof of Lemma A.1. The lemma holds for $k = 0$ trivially; hence, we consider $k \ge 1$. From (A.13), it follows that

$$\prod_{s \in S} p_D^N(\tau_s) \prod_{s \notin S} (1 - p_D^N(\tau_s)) \le \prod_{s \in S} p_D^N(\tau_s) \le (\sqrt{N}/C^N)^k \prod_{s \in S} (\check{H}^N(\tau_s))^+. \tag{A.17}$$

Moreover, due to the Cauchy-Schwarz inequality, we have

$$\begin{aligned} \mathbb{E} \prod_{s \in S} (\check{H}^N(\tau_s))^+ &\le \prod_{s \in S} \left(\mathbb{E}((\check{H}^N(\tau_s))^+)^k \right)^{1/k} \\ &\le \prod_{s \in S} \left(\mathbb{E} e^{k \check{H}^N(\tau_s)} \right)^{1/k}; \end{aligned} \tag{A.18}$$

the second inequality follows from the relation $x^+ < e^x$ for $x \in \mathbb{R}$. Recall that τ_s, $1 \leq s \leq M^N$, is the arrival time of the sth data packet in a block of the source 1. Thus, we have $H^N(\tau_s) = 1 + \alpha \cdot 1_{\{s=M^N\}} + H^N_{(-1)}(\tau_s)$, where $H^N_{(-1)}(t) = \sum_{i=2}^N H^N_{(i)}(t)$. Given that all processes are in their stationary regimes except the one corresponding to the source 1, the random variables $H^N_{(-1)}(\tau_s)$, $s = 1, 2, \ldots, M^N$, are equal in distribution, and, analogously to (A.16), it can be shown that

$$\lim_{N\to\infty} \mathbb{E}e^{k\check{H}^N(\tau_s)} < \infty, \tag{A.19}$$

for all τ_s, $1 \leq s \leq M^N$. Under the assumptions of the lemma, Lemma A.4 implies $|\mathcal{S}^k| - |\mathcal{S}^k_l| \leq c\lfloor a \log N \rfloor (M^N)^{k-1}$ for some finite constant c. Combining this and (A.17)–(A.19) leads to the statement of Lemma A.1.

A.4 Proof of Lemma A.2

Consider $H^N_{(-1)}(t) - A^N_{(-1)}(t)$, i.e., the number of additional coded packets generated by the encoders of $N-1$ sources (except the source 1) in the time slot t, where $H^N_{(-1)}(t)$ and $A^N_{(-1)}(t)$ are respectively given by

$$H^N_{(-1)}(t) = \sum_{i=2}^N H^N_{(i)}(t), \quad A^N_{(-1)}(t) = \sum_{i=2}^N A^N_{(i)}(t).$$

The following lemma indicates that the number of such coded packets per time slot can be approximated by $(\alpha/m)\sqrt{N}$ during an entire block.

Lemma A.5. *Consider the critical-load scaling. Then*

$$\sup_{\tau_1 \leq t \leq \tau_{M^N}} \left| \frac{H^N_{(-1)}(t) - A^N_{(-1)}(t)}{\sqrt{N}} - \frac{\alpha}{m} \right| \xrightarrow{\mathbb{P}} 0,$$

as $N \to \infty$.

Proof. For some $\epsilon > 0$, define an event $\mathcal{E}^N_\epsilon$ as

$$\mathcal{E}^N_\epsilon = \left\{ \sup_{\tau_1 \leq t \leq \tau_{M^N}} \left| \frac{H^N_{(-1)}(t) - A^N_{(-1)}(t)}{\sqrt{N}} - \frac{\alpha}{m^N} \right| \geq \epsilon \right\},$$

where $m^N = \sqrt{N}M^N/(\lambda(N-1))$. Since $m^N \to m$, as $N \to \infty$, it is sufficient to show that for any $\epsilon > 0$,

$$\mathbb{P}[\mathcal{E}_\epsilon^N] \to 0, \tag{A.20}$$

as $N \to \infty$. The event $\mathcal{E}_\epsilon^N$ satisfies $\mathcal{E}_\epsilon^N \subseteq \bigcup_{t=\tau_1}^{\tau_{M^N}} \mathcal{E}_\epsilon^N(t)$, where

$$\mathcal{E}_\epsilon^N(t) = \left\{ \left| \frac{H_{(-1)}^N(t) - A_{(-1)}^N(t)}{\sqrt{N}} - \frac{\alpha}{m^N} \right| \geq \epsilon \right\}.$$

Thus, the union bound renders

$$\mathbb{P}[\mathcal{E}_\epsilon^N] \leq \mathbb{E} \sum_{t=\tau_1}^{\tau_{M^N}} \mathbb{P}[\mathcal{E}_\epsilon^N(t)]. \tag{A.21}$$

Furthermore, the Markov's inequality results in

$$\begin{aligned} \mathbb{P}[\mathcal{E}_\epsilon^N(t)] &= \mathbb{P}\left[\left| \frac{H_{(-1)}^N(t) - A_{(-1)}^N(t)}{\sqrt{N}} - \frac{\alpha}{m^N} \right|^4 \geq \epsilon^4 \right] \\ &\leq (\epsilon\sqrt{N})^{-4} \mathbb{E} \left(\sum_{i=2}^N U_{(i)}^N(t) \right)^4, \end{aligned} \tag{A.22}$$

where $U_{(i)}^N(t) = H_{(i)}^N(t) - A_{(i)}^N(t) - \alpha\lambda/M^N$. Note that the random variables $U_{(i)}^N(t)$, $i = 2, 3, \ldots, N$, are independent since packets are generated and encoded by individual sources and their encoders independently. Furthermore, provided that the system is in stationarity, the random variables $U_{(i)}^N(t)$, $i = 2, 3, \ldots, N$, are i.i.d. with zero mean and

$$\mathbb{P}[U_{(i)}^N(t) = u] = \begin{cases} 1 - \lambda/M^N, & u = -\alpha\lambda/M^N, \\ \lambda/M^N, & u = \alpha - \alpha\lambda/M^N, \end{cases} \tag{A.23}$$

for all $t \in [\tau_1, \tau_{M^N}]$; this stems from the fact that all processes except the one corresponding to the source 1 are in steady-state at the time slot τ_1. Therefore, it follows that

$$\mathbb{E} \left(\sum_{i=2}^N U_{(i)}^N(t) \right)^4 \leq N\mathbb{E}(U_{(2)}^N(t))^4 + 3N^2(\mathbb{E}(U_{(2)}^N(t))^2)^2, \tag{A.24}$$

for all $t \in [\tau_1, \tau_{M^N}]$. From (A.23), one can obtain

$$\mathbb{E}(U^N_{(2)}(t))^k = (1 - \lambda/M^N)(-\alpha\lambda/M^N)^k + (\lambda/M^N)(\alpha - \alpha\lambda/M^N)^k,$$

for $k \geq 2$, and this together with (A.22) and (A.24) leads to

$$\mathbb{P}[\mathcal{E}^N_\epsilon(t)] = O(1/N), \tag{A.25}$$

as $N \to \infty$, for all $t \in [\tau_1, \tau_{M^N}]$. Observe that $\tau_{M^N} - \tau_1 = \sum_{i=2}^{M^N}(\tau_i - \tau_{i-1})$, where $(\tau_i - \tau_{i-1})$, $i = 2, 3, \ldots, M^N$, are i.i.d. geometric random variables with mean $1/\lambda$ (since the sources are Bernoulli). Hence, combining (A.21) and (A.25) yields (A.20), and this concludes the proof of the lemma. Next we introduce an additional technical lemma. For some $\epsilon > 0$, consider two systems that have the same link capacity C^N and buffer size B^N; however, assume that input processes are respectively given by $A^N_{-\epsilon}(t) = A^N_{(-1)}(t) + (\alpha/m - \epsilon)\sqrt{N}$ and $A^N_{+\epsilon}(t) = A^N_{(-1)}(t) + (1 + \alpha) + (\alpha/m + \epsilon)\sqrt{N}$, instead of $H^N(t)$. Formally, the queue occupancies of these systems $Q^N_{\pm\epsilon}(t)$, $t \geq 1$, satisfy the following recursion:

$$Q^N_{\pm\epsilon}(t) = (Q^N_{\pm\epsilon}(t-1) + A^N_{\pm\epsilon}(t) - C^N)^+ \wedge B^N, \tag{A.26}$$

and the numbers of dropped packets $D^N_{\pm\epsilon}(t)$, $t \geq 1$, are respectively given by

$$D^N_{\pm\epsilon}(t) = (A^N_{\pm\epsilon}(t) + Q^N_{\pm\epsilon}(t-1) - C^N - B^N)^+. \tag{A.27}$$

Observe that the processes $\{Q^N_{\pm\epsilon}(t), t \geq 0\}$ are Markov chains since $\{A^N_{\pm\epsilon}(t), t \geq 1\}$ are i.i.d. processes.

The following lemma provides an upper and lower bound on the number of dropped packets.

Lemma A.6. *Consider the critical-load scaling. For any $\epsilon > 0$, if $Q^N_{-\epsilon}(\tau_1 - 1) = Q^N_{+\epsilon}(\tau_1 - 1) = Q^N_*(\tau_1 - 1)$, then, as $N \to \infty$,*

$$\mathbb{P}[\mathcal{D}^N_\epsilon] \to 1,$$

where $\mathcal{D}_\epsilon^N = \{D_{-\epsilon}^N(t) \le D^N(t) \le D_{+\epsilon}^N(t), \forall t \in [\tau_1, \tau_{M^N}]\}$.

Proof. For some $\epsilon > 0$, define an event $\mathcal{A}_\epsilon^N$ as

$$\mathcal{A}_\epsilon^N = \left\{ \sup_{\tau_1 \le t \le \tau_{M^N}} \left| \frac{H_{(-1)}^N(t) - A_{(-1)}^N(t)}{\sqrt{N}} - \frac{\alpha}{m} \right| < \epsilon \right\}, \tag{A.28}$$

Given the event $\mathcal{A}_\epsilon^N$, the following bound holds:

$$A_{-\epsilon}^N(t) \le H^N(t) \le A_{+\epsilon}^N(t), \quad \forall t \in [\tau_1, \tau_{M^N}]; \tag{A.29}$$

this further implies (due to the monotonicity in (A.26))

$$Q_{-\epsilon}^N(t) \le Q_*^N(t) \le Q_{+\epsilon}^N(t), \quad \forall t \in [\tau_1, \tau_{M^N}]. \tag{A.30}$$

From (A.27), (A.29) and (A.30), on the event $\mathcal{A}_\epsilon^N$, we also have

$$D_{-\epsilon}^N(t) \le D^N(t) \le D_{+\epsilon}^N(t), \quad \forall t \in [\tau_1, \tau_{M^N}]. \tag{A.31}$$

Then, the statement of the lemma follows from Lemma A.5.

Next we present a proof of Lemma A.2

Proof of Lemma A.2. First consider the first statement of the lemma. Note that the statement holds for $k = 0$ trivially; hence, we consider $k \ge 1$. The proof consists of three parts.

Part I. Observe that $H^N(t) \wedge C^N \le H^N(t) - D^N(t) \le C^N + B^N$; this implies (see (A.6))

$$\frac{D^N(t)}{C^N + B^N} \le \frac{p_D^N(t)}{1 - p_D^N(t)} \le \frac{D^N(t)}{H^N(t) \wedge C^N} = \frac{D^N(t)}{C^N}; \tag{A.32}$$

the equality is due to the fact that the event $\{H^N(t) < C^N\}$ implies $\{D^N(t) = 0\}$. Then, it is sufficient to show that for all $S \in \mathcal{S}_l^k$,

$$\mathbb{E} \prod_{s \in S} \hat{D}^N(\tau_s) \to (\hat{p}_D)^k, \tag{A.33}$$

as $N \to \infty$, where $\hat{D}^N(t) = D^N(t)/\sqrt{N}$. In particular, note that (A.32) and the preceding limit imply

$$(C^N/\sqrt{N})^k \mathbb{E} \prod_{s \in S} \frac{p_D^N(\tau_s)}{1 - p_D^N(\tau_s)} \to (\hat{p}_D)^k,$$

as $N \to \infty$, for all $S \in \mathcal{S}_l^k$; then, the statement of the lemma follows from the first statement of Lemma A.4. For $k = 1$, the limit (A.33) is straightforward (see (2.19)); thus, we consider $k \geq 2$ from now on.

Part II. The proof is based on a coupling argument (e.g., see [15, Sec. 4.1.2]). Given $\epsilon > 0$, define two events $\mathcal{C}_{-\epsilon,0}^N(t_1, t_2)$ and $\mathcal{C}_{-\epsilon,b}^N(t_1, t_2)$ for time slots t_1 and t_2 ($t_1 < t_2$):

$$\mathcal{C}_{-\epsilon,0}^N(t_1, t_2) = \{\exists t \in [t_1, t_2) : A_{-\epsilon}^N(t) < C^N - B^N\},$$

$$\mathcal{C}_{-\epsilon,b}^N(t_1, t_2) = \{\exists t \in [t_1, t_2) : A_{-\epsilon}^N(t) > C^N + B^N\},$$

and consider the event $\mathcal{C}_{-\epsilon}^N(t_1, t_2)$ given by

$$\mathcal{C}_{-\epsilon}^N(t_1, t_2) = \mathcal{C}_{-\epsilon,0}^N(t_1, t_2) \cap \mathcal{C}_{-\epsilon,b}^N(t_1, t_2). \tag{A.34}$$

The events $\{A_{-\epsilon}^N(t) < C^N - B^N\}$ and $\{A_{-\epsilon}^N(t) > C^N + B^N\}$ imply $\{Q_{-\epsilon}^N(t) = 0\}$ and $\{Q_{-\epsilon}^N(t) = B^N\}$, respectively, regardless of the queue occupancy of the previous time slot. On the event $\mathcal{C}_{-\epsilon}^N(t_1, t_2)$, thus, the buffer becomes empty and full at least once during the time interval $[t_1, t_2)$. Now consider a queue occupancy process $\{\dot{Q}_{-\epsilon}^N(t), t \geq t_1 - 1\}$ with the same arrival process $\{A_{-\epsilon}^N(t), t \geq t_1\}$ but possibly different initial distribution at $t = t_1 - 1$. If there exists $t^0 \geq t_1$ such that $Q_{-\epsilon}^N(t^0) \leq \dot{Q}_{-\epsilon}^N(t^0)$, then $Q_{-\epsilon}^N(t) \leq \dot{Q}_{-\epsilon}^N(t)$ for all $t \geq t^0$. On the other hand, if there exists $t^b \geq t_1$ such that $Q_{-\epsilon}^N(t^b) \geq \dot{Q}_{-\epsilon}^N(t^b)$, then $Q_{-\epsilon}^N(t) \geq \dot{Q}_{-\epsilon}^N(t)$ for all $t \geq t^b$. Hence, the event $\mathcal{C}_{-\epsilon}^N(t_1, t_2)$ implies that these two queue occupancy processes couple before the time slot t_2, i.e., $Q_{-\epsilon}^N(t_2) = \dot{Q}_{-\epsilon}^N(t_2)$, regardless of their initial distributions at $t = t_1 - 1$.

Let $S = \{s_1, s_2, \dots, s_k\} \in \mathcal{S}_l^k$, where $s_1 < s_2 < \cdots < s_k$. The assumption of the lemma implies $|s_i - s_j| > \lfloor a \log N \rfloor$ for all $s_i, s_j \in S$ (see (A.8)); this further results in $(\tau_{s_i} - \tau_{s_{i-1}}) > \lfloor a \log N \rfloor$ for all i, $2 \le i \le k$. Therefore, we have

$$\mathbb{P}[\bar{\mathcal{C}}_{-\epsilon}^N(\tau_{s_{i-1}} + 1, \tau_{s_i})] \le (q_0^N)^{\lfloor a \log N \rfloor} + (q_b^N)^{\lfloor a \log N \rfloor},$$

for all i, $2 \le i \le k$, where $q_0^N = \mathbb{P}[A_{-\epsilon}^N(t) \ge C^N - B^N]$ and $q_b^N = \mathbb{P}[A_{-\epsilon}^N(t) \le C^N + B^N]$. Since $q_0^N \to q_0 \in (0,1)$ and $q_b^N \to q_b \in (0,1)$, as $N \to \infty$ (due to the CLT), it follows that

$$\mathbb{P}[\bar{\mathcal{C}}_{-\epsilon}^N(\tau_{s_{i-1}} + 1, \tau_{s_i})] \to 0, \tag{A.35}$$

as $N \to \infty$, for all i, $2 \le i \le k$. The union bound and the preceding limit yield

$$\mathbb{P}\left[\bigcup_{i=2}^k \bar{\mathcal{C}}_{-\epsilon}^N(\tau_{s_{i-1}} + 1, \tau_{s_i})\right] \le \sum_{i=2}^k \mathbb{P}[\bar{\mathcal{C}}_{-\epsilon}^N(\tau_{s_{i-1}} + 1, \tau_{s_i})] \to 0, \tag{A.36}$$

as $N \to \infty$. One can define a corresponding event $\mathcal{C}_{+\epsilon}^N(t_1, t_2)$ (as in (A.34)) for the case "$+\epsilon$", and it can be shown that

$$\mathbb{P}[\bar{\mathcal{C}}_{+\epsilon}^N(\tau_{s_{i-1}} + 1, \tau_{s_i})] \to 0, \tag{A.37}$$

for all i, $2 \le i \le k$, and

$$\mathbb{P}\left[\bigcup_{i=2}^k \bar{\mathcal{C}}_{+\epsilon}^N(\tau_{s_{i-1}} + 1, \tau_{s_i})\right] \to 0, \tag{A.38}$$

as $N \to \infty$.

Part III. For some $\epsilon > 0$ and $S = \{s_1, s_2, \dots, s_k\} \in \mathcal{S}_l^k$, $s_1 < s_2 < \cdots < s_k$, let

$$\mathcal{C}_{\epsilon,i}^N(S) = \mathcal{C}_{-\epsilon}^N(\tau_{s_{i-1}} + 1, \tau_{s_i}) \cap \mathcal{C}_{+\epsilon}^N(\tau_{s_{i-1}} + 1, \tau_{s_i}),$$

for $2 \le i \le k$, and consider the event $\mathcal{C}_\epsilon^N(S) = \bigcap_{i=2}^k \mathcal{C}_{\epsilon,i}^N(S)$. It is straightforward to show that

$$\mathbb{E}[\Psi_{-\epsilon}^N(S) 1_{\mathcal{C}_\epsilon^N(S)}] = \mathbb{E}\left[\hat{D}_{-\epsilon}^N(\tau_{s_1}) \prod_{i=2}^k \left(\hat{D}_{-\epsilon}^N(\tau_{s_i}) 1_{\mathcal{C}_{\epsilon,i}^N(S)}\right)\right],$$

where $\Psi^N_{-\epsilon}(S) = \prod_{s\in S} \hat{D}^N_{-\epsilon}(\tau_s)$ and $\hat{D}^N_{-\epsilon}(t) = D^N_{-\epsilon}(t)/\sqrt{N}$. For each i, $2 \le i \le k$, consider a queue occupancy process $\{Q^{*N}_{-\epsilon,i}(t), \tau_{s_{i-1}} \le t \le \tau_{s_i} - 1\}$ that is in stationarity at the time slot $t = \tau_{s_{i-1}}$ and follows the same recursion (A.26) for $t \in [\tau_{s_{i-1}} + 1, \tau_{s_i} - 1]$. In particular, we assume that the initial queue occupancies $Q^{*N}_{-\epsilon,i}(\tau_{s_{i-1}})$, $i = 2, 3, \ldots, k$, are i.i.d. and that they do not depend on the arrival process $\{A^N_{-\epsilon}(t), t \ge 1\}$. In addition, let $D^{*N}_{-\epsilon,i}(t)$, $\tau_{s_{i-1}} + 1 \le t \le \tau_{s_i}$, $2 \le i \le k$, denote the number of dropped packets that corresponds to the queue occupancy $Q^{*N}_{-\epsilon,i}(\cdot)$:

$$D^{*N}_{-\epsilon,i}(t) = (A^N_{-\epsilon}(t) + Q^{*N}_{-\epsilon,i}(t-1) - C^N - B^N)^+. \tag{A.39}$$

Due to the coupling argument given in the previous part, we have, for all i, $2 \le i \le k$,

$$D^N_{-\epsilon}(\tau_{s_i}) 1_{\mathcal{C}^N_{\epsilon,i}(S)} = D^{*N}_{-\epsilon,i}(\tau_{s_i}) 1_{\mathcal{C}^N_{\epsilon,i}(S)}. \tag{A.40}$$

Note that each random variable $D^{*N}_{-\epsilon,i}(\tau_{s_i}) 1_{\mathcal{C}^N_{\epsilon,i}(S)}$, $2 \le i \le k$, is determined by the initial queue occupancy $Q^{*N}_{-\epsilon,i}(\tau_{s_{i-1}})$ and the random variables $A^N_{-\epsilon}(t)$, $t \in [\tau_{s_{i-1}} + 1, \tau_{s_i}]$. Since the random variables $A^N_{-\epsilon}(t)$, $t = 1, 2, \ldots$, are i.i.d. and the queue occupancies $Q^{*N}_{-\epsilon,i}(\tau_{s_{i-1}})$, $i = 2, 3, \ldots, k$, are assumed to be i.i.d. with stationary distribution (and they do not depend on the arrival process), it follows that

$$\mathbb{E}[\Psi^N_{-\epsilon}(S) 1_{\mathcal{C}^N_\epsilon(S)}] = \mathbb{E}[\hat{D}^N_{-\epsilon}(\tau_{s_1})] \prod_{i=2}^{k} \mathbb{E}[\hat{D}^{*N}_{-\epsilon,i}(\tau_{s_i}) 1_{\mathcal{C}^N_{\epsilon,i}(S)}], \tag{A.41}$$

where $\hat{D}^{*N}_{-\epsilon,i}(t) = D^{*N}_{-\epsilon,i}(t)/\sqrt{N}$. From the bound $D^{*N}_{-\epsilon,i}(t) \le (A^N_{-\epsilon}(t) - C^N)^+$ and the Cauchy-Schwarz inequality, we have

$$\begin{aligned} \mathbb{E}[\hat{D}^{*N}_{-\epsilon,i}(\tau_{s_i}) 1_{\bar{\mathcal{C}}^N_{\epsilon,i}(S)}] &\le \mathbb{E}[(\check{A}^N_{-\epsilon}(\tau_{s_i}))^+ 1_{\bar{\mathcal{C}}^N_{\epsilon,i}(S)}] \\ &\le \left(\mathbb{E}[((\check{A}^N_{-\epsilon}(\tau_{s_i}))^+)^2] \mathbb{P}[\bar{\mathcal{C}}^N_{\epsilon,i}(S)]\right)^{1/2}, \end{aligned} \tag{A.42}$$

for all i, $2 \leq i \leq k$, where $\check{A}^N_{-\epsilon}(t) = (A^N_{-\epsilon}(t) - C^N)/\sqrt{N}$. By using the similar steps as in (A.15) and (A.16), one can obtain

$$\limsup_{N\to\infty} \mathbb{E}((\check{A}^N_{-\epsilon}(\tau_s))^+)^2 < \infty. \tag{A.43}$$

Furthermore, due to (A.35) and (A.37), we have $\mathbb{P}[\mathcal{C}^N_{\epsilon,i}(S)] \to 1$, as $N \to \infty$, for all i, $2 \leq i \leq k$. Then, from (A.42) and (A.43), it follows that, as $N \to \infty$,

$$\mathbb{E}[\hat{D}^{*N}_{-\epsilon,i}(\tau_{s_i}) 1_{\mathcal{C}^N_{\epsilon,i}(S)}] - \mathbb{E}\hat{D}^{*N}_{-\epsilon,i}(\tau_{s_i}) \to 0, \tag{A.44}$$

for all i, $2 \leq i \leq k$. Recall that the initial queue occupancies $Q^{*N}_{-\epsilon,i}(\tau_{s_{i-1}})$, $i = 2, 3, \ldots, k$, are assumed to have the stationary distribution; this implies $\mathbb{E}\hat{D}^{*N}_{-\epsilon,i}(\tau_{s_i}) = \mathbb{E}\hat{D}^N_{-\epsilon}(\tau_{s_1})$, for all i, $2 \leq i \leq k$. Thus, combining (A.41) and (A.44) renders

$$\mathbb{E}[\Psi^N_{-\epsilon}(S) 1_{\mathcal{C}^N_{\epsilon}(S)}] - (\mathbb{E}\hat{D}^N_{-\epsilon}(\tau_{s_1}))^k \to 0, \tag{A.45}$$

as $N \to \infty$.

The bound $D^N_{-\epsilon}(t) \leq (A^N_{-\epsilon}(t) - C^N)^+$ and the Cauchy-Schwarz inequality yield (see Lemma A.6)

$$\begin{aligned} \mathbb{E}[\Psi^N_{-\epsilon}(S) 1_{\mathcal{C}^N_{\epsilon}(S)} 1_{\bar{\mathcal{D}}^N_{\epsilon}}] &\leq \mathbb{E}[\Psi^N_{-\epsilon}(S) 1_{\bar{\mathcal{D}}^N_{\epsilon}}] \\ &\leq \left(\mathbb{E}\left[\prod_{s\in S} ((\check{A}^N_{-\epsilon}(\tau_s))^+)^2 \right] \mathbb{P}[\bar{\mathcal{D}}^N_{\epsilon}] \right)^{1/2}. \end{aligned} \tag{A.46}$$

Since random variables $\check{A}^N_{-\epsilon}(\tau_s)$, $s \in S$, are i.i.d., we have

$$\mathbb{E} \prod_{s\in S} ((\check{A}^N_{-\epsilon}(\tau_s))^+)^2 = \prod_{s\in S} \mathbb{E}((\check{A}^N_{-\epsilon}(\tau_s))^+)^2.$$

Then, by combining (A.43), (A.46) and Lemma A.6, it follows that $\mathbb{E}[\Psi^N_{-\epsilon}(S) 1_{\mathcal{C}^N_{\epsilon}(S)} 1_{\bar{\mathcal{D}}^N_{\epsilon}}] \to 0$, as $N \to \infty$; this limit and (A.45) further result in

$$\mathbb{E}[\Psi^N_{-\epsilon}(S) 1_{\mathcal{C}^N_{\epsilon}(S)} 1_{\mathcal{D}^N_{\epsilon}}] - (\mathbb{E}\hat{D}^N_{-\epsilon}(\tau_{s_1}))^k \to 0, \tag{A.47}$$

as $N \to \infty$. Likewise, one can derive a similar limit for the case "$+\epsilon$", i.e.,

$$\mathbb{E}[\Psi^N_{+\epsilon}(S)1_{\mathcal{C}^N_\epsilon(S)}1_{\mathcal{D}^N_\epsilon}] - (\mathbb{E}\hat{D}^N_{+\epsilon}(\tau_{s_1}))^k \to 0, \tag{A.48}$$

as $N \to \infty$, where $\Psi^N_{+\epsilon}(S) = \prod_{s\in S}(D^N_{+\epsilon}(\tau_s)/\sqrt{N})$. For any $\epsilon > 0$, Lemma A.6 implies

$$\Psi^N_{-\epsilon}(S)1_{\mathcal{Z}^N_\epsilon(S)} \leq \Psi^N(S)1_{\mathcal{Z}^N_\epsilon(S)} \leq \Psi^N_{+\epsilon}(S)1_{\mathcal{Z}^N_\epsilon(S)}, \tag{A.49}$$

where $\mathcal{Z}^N_\epsilon(S) = \mathcal{C}^N_\epsilon(S) \cap \mathcal{D}^N_\epsilon$ and $\Psi^N(S) = \prod_{s\in S}\hat{D}^N(\tau_s)$. In addition, due to continuity, it can be shown that

$$\lim_{\epsilon\downarrow 0}\lim_{N\to\infty}\mathbb{E}\hat{D}^N_{\pm\epsilon}(\tau_{s_1}) = \hat{p}_D.$$

Hence, from (A.47)-(A.49) and the preceding limit, one can derive

$$\lim_{\epsilon\downarrow 0}\lim_{N\to\infty}\mathbb{E}[\Psi^N(S)1_{\mathcal{Z}^N_\epsilon(S)}] = (\hat{p}_D)^k. \tag{A.50}$$

On the other hand, the bound $D^N(t) \leq (H^N(t) - C^N)^+$ and the Cauchy-Schwarz inequality lead to

$$\mathbb{E}[\Psi^N(S)1_{\bar{\mathcal{Z}}^N_\epsilon(S)}] \leq \left(\mathbb{E}\left[\prod_{s\in S}((\check{H}^N(\tau_s))^+)^2\right]\mathbb{P}[\bar{\mathcal{Z}}^N_\epsilon(S)]\right)^{1/2}. \tag{A.51}$$

Similarly to (A.18) and (A.19), one can obtain

$$\limsup_{N\to\infty}\mathbb{E}\prod_{s\in S}((\check{H}^N(\tau_s))^+)^2 < \infty. \tag{A.52}$$

Moreover, (A.36), (A.38) and Lemma A.6 imply $\mathbb{P}[\mathcal{Z}^N_\epsilon(S)] \to 1$, as $N \to \infty$. Then, from (A.51) and (A.52), we have

$$\mathbb{E}[\Psi^N(S)1_{\bar{\mathcal{Z}}^N_\epsilon(S)}] \to 0,$$

as $N \to \infty$. The preceding limit and (A.50) imply (A.33), and the first statement of the lemma follows.

Finally consider the second statement of the lemma. From (A.32) and the bound $D^N(t) \le (H^N(t) - C^N)^+$, we have

$$\mathbb{E}\prod_{s\in S}\left(\frac{p_D^N(\tau_s)}{1-p_D^N(\tau_s)}\right)^2 \le (\sqrt{N}/C^N)^{2k}\mathbb{E}\prod_{s\in S}((\check{H}^N(\tau_s))^+)^2.$$

Then, since $|\mathcal{S}_l^k| = O((M^N)^k)$, as $N\to\infty$ (see Lemma A.4), the second statement of the lemma follows from (A.52).

A.5 Proof of Lemma A.3

The proof consists of two parts.

Part I. For some $\delta > 0$, it can be shown that

$$\text{(A.53)}\quad \mathbb{P}\left[\left|\prod_{s=1}^{M^N}(1-p_D^N(\tau_s)) - e^{-m\hat{p}_D}\right| \ge \delta\right] \le \mathbb{P}\left[\left| -\sum_{s=1}^{M^N}\log(1-p_D^N(\tau_s)) - m\hat{p}_D\right| \ge \delta'\right],$$

where $\delta' = \log(1+\delta/e^{-m\hat{p}_D})$. The relation $x \le -\log(1-x) \le x/(1-x)$ for $0 \le x < 1$ renders

$$\text{(A.54)}\qquad p_D^N(t) \le -\log(1-p_D^N(t)) \le p_D^N(t)/(1-p_D^N(t)).$$

Given $\epsilon > 0$, the event $\mathcal{A}_\epsilon^N$ implies (see (A.28), (A.29) and (A.31))

$$p_D^N(t) \ge \frac{D_{-\epsilon}^N(t)}{A_{+\epsilon}^N(t)}, \qquad \frac{p_D^N(t)}{1-p_D^N(t)} \le \frac{D^N(t)}{C^N} \le \frac{D_{+\epsilon}^N(t)}{C^N},$$

for all $t \in [\tau_1, \tau_{M^N}]$; the second inequality follows from (A.32). Due to (A.54) and the preceding inequalities, it follows that on the event $\mathcal{A}_\epsilon^N$,

$$\text{(A.55)}\qquad \sum_{s=1}^{M^N} P_{-\epsilon}^N(\tau_s) \le -\sum_{s=1}^{M^N}\log(1-p_D^N(\tau_s)) \le \sum_{s=1}^{M^N} P_{+\epsilon}^N(\tau_s),$$

where $P^N_{-\epsilon}(t) = D^N_{-\epsilon}(t)/A^N_{+\epsilon}(t)$ and $P^N_{+\epsilon}(t) = D^N_{+\epsilon}(t)/C^N$. Due to Lemma A.5, we have $\mathbb{P}[\mathcal{A}^N_\epsilon] \to 1$, as $N \to \infty$, for any $\epsilon > 0$; hence, from (A.53) and (A.55), it is sufficient to show that

$$\mathbb{P}\left[\left|\sum_{s=1}^{M^N} P^N_{\pm\epsilon}(\tau_s) - m\hat{p}_D\right| \geq \delta\right] \to 0, \tag{A.56}$$

as $N \to \infty$, for any $\delta > 0$ and all sufficiently small $\epsilon > 0$.

Part II. Consider the case "$-\epsilon$". For some $\epsilon > 0$ and $\delta > 0$, define an event $\mathcal{B}^N_{\epsilon,\delta}$ as

$$\mathcal{B}^N_{\epsilon,\delta} = \left\{\left|\sum_{s=1}^{M^N} P^N_{-\epsilon}(t) - M^N \mathbb{E} P^N_{-\epsilon}(t)\right| \geq \delta\right\}.$$

For notational simplicity, let $\bar{P}^N_{-\epsilon}(t) = P^N_{-\epsilon}(t) - \mathbb{E}P^N_{-\epsilon}(t)$. The Chebyshev's inequality yields

$$\mathbb{P}[\mathcal{B}^N_{\epsilon,\delta}] \leq \delta^{-2}\mathbb{E}\left(\sum_{s=1}^{M^N} \bar{P}^N_{-\epsilon}(\tau_s)\right)^2. \tag{A.57}$$

Recall that the set $\mathcal{S}^2$ is the collection of all 2-subsets of $\{1, 2, \ldots, M^N\}$. Thus, it follows that

$$\mathbb{E}\left(\sum_{s=1}^{M^N} \bar{P}^N_{-\epsilon}(\tau_s)\right)^2 = 2\sum_{S\in\mathcal{S}^2} \mathbb{E}\prod_{s\in S} \bar{P}^N_{-\epsilon}(\tau_s) + M^N\mathbb{E}(\bar{P}^N_{-\epsilon}(\tau_1))^2. \tag{A.58}$$

From the following bound:

$$P^N_{-\epsilon}(t) \leq \frac{(A^N_{-\epsilon}(t) - C^N)^+}{A^N_{+\epsilon}(t)} \leq \frac{(A^N_{-\epsilon}(t) - C^N)^+}{C^N}, \tag{A.59}$$

we have

$$\begin{aligned}\mathbb{E}(\bar{P}^N_{-\epsilon}(\tau_1))^2 &\leq \mathbb{E}(P^N_{-\epsilon}(\tau_1))^2 \\ &\leq (\sqrt{N}/C^N)^2\mathbb{E}((\check{A}^N_{-\epsilon}(\tau_1))^+)^2.\end{aligned} \tag{A.60}$$

Then, due to (A.43), it can be shown that, as $N \to \infty$,

$$M^N\mathbb{E}(\bar{P}^N_{-\epsilon}(\tau_1))^2 \to 0. \tag{A.61}$$

For notational simplicity, define $\Lambda^N_{-\epsilon}(S) = \prod_{s\in S} \bar{P}^N_{-\epsilon}(\tau_s)$. The set $\mathcal{S}^2$ can be partitioned into two disjoint subsets $\mathcal{S}^2_l$ and $\mathcal{S}^2 \setminus \mathcal{S}^2_l$ (see (A.8)), and, therefore, we have

$$\text{(A.62)} \qquad \sum_{S\in\mathcal{S}^2} \mathbb{E}\Lambda^N_{-\epsilon}(S) = \sum_{S\in\mathcal{S}^2_l} \mathbb{E}\Lambda^N_{-\epsilon}(S) + \sum_{S\in\mathcal{S}^2\setminus\mathcal{S}^2_l} \mathbb{E}\Lambda^N_{-\epsilon}(S).$$

The Cauchy-Schwarz inequality and the bound (A.60) render

$$\left|\mathbb{E}\Lambda^N_{-\epsilon}(S)\right| \le (\sqrt{N}/C^N)^2 \prod_{s\in S} \left(\mathbb{E}((\check{A}^N_{-\epsilon}(\tau_s))^+)^2\right)^{1/2},$$

and this together with (A.43) leads to $|\mathbb{E}[\Lambda^N_{-\epsilon}(S)]| = O(1/N)$, as $N\to\infty$. Let $l = \lfloor a\log N\rfloor$ for fixed $a>0$. In this case, Lemma A.4 implies $|\mathcal{S}^2| - |\mathcal{S}^2_l| \le c\lfloor a\log N\rfloor M^N$ for some finite constant $c>0$; hence, we have, as $N\to\infty$,

$$\text{(A.63)} \qquad \sum_{S\in\mathcal{S}^2\setminus\mathcal{S}^2_l} \mathbb{E}\Lambda^N_{-\epsilon}(S) \to 0.$$

Next consider the event $\mathcal{C}^N_{-\epsilon}(S) = \mathcal{C}^N_{-\epsilon}(\tau_{s_1}+1, \tau_{s_2})$ for some $S = \{\tau_{s_1}, \tau_{s_2}\} \in \mathcal{S}^2_l$ (see (A.34)). Recall that this event implies $D^N_{-\epsilon}(\tau_{s_2}) = D^{*N}_{-\epsilon,2}(\tau_{s_2})$, where $D^{*N}_{-\epsilon,2}(t)$, $\tau_{s_1}+1 \le t \le \tau_{s_2}$, denotes the number of dropped packets that corresponds to the queue occupancy $Q^{*N}_{-\epsilon,2}(\cdot)$ (see (A.39) and (A.40)). In a similar manner as in (A.41), it can be shown that

$$\text{(A.64)} \qquad \mathbb{E}[\Lambda^N_{-\epsilon}(S)1_{\mathcal{C}^N_{-\epsilon}(S)}] = \mathbb{E}[\bar{P}^N_{-\epsilon}(\tau_{s_1})]\mathbb{E}[\bar{P}^{*N}_{-\epsilon}(\tau_{s_2})1_{\mathcal{C}^N_{-\epsilon}(S)}] = 0,$$

where $\bar{P}^{*N}_{-\epsilon}(t) = D^{*N}_{-\epsilon,2}(t)/A^N_{+\epsilon}(t) - \mathbb{E}[D^{*N}_{-\epsilon,2}(t)/A^N_{+\epsilon}(t)]$; note that we used the fact that $\mathbb{E}\bar{P}^N_{-\epsilon}(\tau_{s_1}) = 0$. On the other hand, the Cauchy-Schwarz inequality renders

$$\text{(A.65)} \qquad \left|\mathbb{E}[\Lambda^N_{-\epsilon}(S)1_{\bar{\mathcal{C}}^N_{-\epsilon}(S)}]\right| \le \left(\mathbb{E}\left[\prod_{s\in S}(\bar{P}^N_{-\epsilon}(\tau_s))^2\right]\mathbb{P}[\bar{\mathcal{C}}^N_{-\epsilon}(S)]\right)^{1/2}.$$

From the bound (A.59), we have

$$\text{(A.66)} \qquad \begin{aligned} \mathbb{E}\prod_{s\in S}(\bar{P}^N_{-\epsilon}(\tau_s))^2 &\le (\sqrt{N}/C^N)^4 \mathbb{E}\prod_{s\in S}\left((\check{A}^N_{-\epsilon}(\tau_s))^+ + \mathbb{E}(\check{A}^N_{-\epsilon}(\tau_s))^+\right)^2 \\ &\le (\sqrt{N}/C^N)^4 \prod_{s\in S} 4\mathbb{E}((\check{A}^N_{-\epsilon}(\tau_s))^+)^2; \end{aligned}$$

the second inequality follows from the fact that the random variables $\check{A}^N_{-\epsilon}(\tau_s)$, $s \in S$, are i.i.d. and also from the Jensen's inequality. Note that Lemma A.4 implies $|\mathcal{S}^2_l| = O((M^N)^2)$, as $N \to \infty$. Thus, from (A.35), (A.43), (A.65) and (A.66), it follows that

$$\sum_{S \in \mathcal{S}^2_l} \mathbb{E}[\Lambda^N_{-\epsilon}(S) 1_{\bar{\mathcal{C}}^N_{-\epsilon}(S)}] \to 0,$$

as $N \to \infty$. Putting together (A.57), (A.58), (A.61)–(A.64) and the preceding limit yields

$$\mathbb{P}[\mathcal{B}^N_{\epsilon,\delta}] \to 0, \tag{A.67}$$

as $N \to \infty$. By using the fact that $A^N_{+\epsilon}(t)/N \to \lambda$, almost surely, as $N \to \infty$ (due to the SLLN), it can be shown that $\lim_{\epsilon \downarrow 0} \lim_{N \to \infty} M^N \mathbb{E} P^N_{-\epsilon}(t) = m\hat{p}_D$ (due to continuity), and combining this with (A.67) renders (A.56) for the case "$-\epsilon$". In a similar manner, one can derive (A.56) for the case "$+\epsilon$" as well. This concludes the proof of Lemma A.3.

A.6 Proof of Theorem 2.2

From (2.6), (2.7) and (2.21), it is straightforward to show that the scaled loss probability $\hat{p}^N_L$ satisfies

$$\hat{p}^N_L = \mathbb{E}\,(D^N_d + D^N_c - \alpha)^+ / m^N,$$

for the ideal block code, and

$$\hat{p}^N_L = \mathbb{E}\big[D^N_d \cdot 1_{\{D^N_d + D^N_c > \alpha\}}\big] / m^N,$$

for a systematic MDS code, where the limiting distributions of D^N_d and D^N_c are given in Theorem 2.1. Note that we have

$$(D^N_d, D^N_c) \Rightarrow (D_{(1)}, 0), \tag{A.68}$$

as $N \to \infty$ (e.g., see [43, Theorem 2.7]). From (2.22), the statements (i) and (ii) of the theorem follow.

For the partial coding scheme, the scaled loss probability can be computed by combining (2.8) and (2.21):

$$\hat{p}_L^N = \mathbb{E}\big[D_d^N - \tilde{D}_d^N \cdot 1_{\{\tilde{D}_d^N + D_c^N \le \alpha\}}\big]/m^N,$$

where $\tilde{D}_d^N$ is the number of dropped packets among $\lfloor \rho M^N \rfloor$ data packets in the coding part of a block. As in Theorem 2.1, it can be shown that $\tilde{D}_d^N$ tends to Poisson with mean $\rho m \hat{p}_D$, as $N \to \infty$, and, thus, we have $(\tilde{D}_d^N, D_c^N) \Rightarrow (D_{(\rho)}, 0)$, as $N \to \infty$ (see (A.68)). Then, the statement (iii) of the theorem follows from (2.22).

Finally, we consider the coding scheme with overlapping blocks. Lemma 2.1 and (2.21) yield

$$\hat{p}_L^N = 2\mathbb{E}[\dot{D}_d^N(1 - \xi^N(\dot{D}_d^N))^2]/m^N,$$

where $\xi^N(0) = 1$, $\xi^N(k) = 0$, $k > \alpha/2$, and $\xi^N(k)$, $1 \le k \le \alpha/2$, satisfies the following equation:

$$\xi^N(k) = \mathbb{P}[\dot{D}_d^N + \dot{D}_c^N \le \alpha/2 - k] + \sum_{i=1}^{\alpha/2} \xi^N(i)\mathbb{P}[\alpha/2 - k - i < \dot{D}_c^N \le \alpha/2 - k]\mathbb{P}[\dot{D}_d^N = i];$$

$\dot{D}_d^N$ and $\dot{D}_c^N$ are the numbers of dropped packets respectively among $M/2$ data packets in a half of a block and among $\alpha/2$ coded packets generated per each (overlapping) block; recall that we only consider even values of M^N and α for simplicity. Now observe that we have

$$\mathbb{E}[\dot{D}_d^N(1 - \xi^N(\dot{D}_d^N))^2] = \mathbb{E}\dot{D}_d^N - \sum_{k=0}^{\alpha/2} k(1 - (1 - \xi^N(k))^2)\mathbb{P}[\dot{D}_d^N = k]. \tag{A.69}$$

Similarly to Theorem 2.1 and (A.68), we can show that

$$(\dot{D}_d^N, \dot{D}_c^N) \Rightarrow (D_{(1/2)}, 0), \tag{A.70}$$

as $N \to \infty$. Moreover, $\xi^N(k) \to \xi(k)$, $k \geq 0$, as $N \to \infty$, where $\xi(k)$, $k \geq 0$, is given in the statement of the theorem. Then, the statement (iv) follows by letting $N \to \infty$ in (A.69) and by using (A.70). This concludes the proof of the theorem.

APPENDIX B

Appendix for Chapter III

B.1 Proof of Theorem 3.1

The proof of the theorem consists of two parts.

Part I. Here we show that

$$\frac{1}{\alpha}X^{\alpha}_{1:n}(\lfloor \alpha^2 t \rfloor) \Rightarrow W_{1:n}(t), \tag{B.1}$$

as $\alpha \to \infty$, for any fixed $t \geq 0$, where $W_{1:n}(t)$ is defined in Section 3.4.3. To this end, consider a bounded, continuous real-valued function $h : \mathbb{R}^n \to \mathbb{R}$ and a fixed $t \geq 0$. The second statements of Corollary 3.2 and Proposition 3.2 yield (e.g., see [13, p. 26])

$$\lim_{N\to\infty} |\mathbb{E}\Delta^{\alpha}_{N,n}(t)| = \left|\mathbb{E}h\left(\frac{1}{\alpha}X^{\alpha}_{1:n}(\lfloor \alpha^2 t \rfloor)\right) - \mathbb{E}h(W_{1:n}(t))\right|,$$

where $\beta_{1:N} = \lambda^{-1}(0, 1, \ldots, N-1)$ and

$$\Delta^{\alpha}_{N,n}(t) \equiv h(\varphi_{N,n} \circ \psi_N(\hat{\mathbf{D}}^{\alpha}_{1:N})(t)) - h(\varphi_{N,n} \circ \psi_N(\mathbf{B}_{1:N} + \beta_{1:N})(t)).$$

Thus, for the limit (B.1), it suffices to show that (e.g., see [13, p. 26])

$$\lim_{\alpha\to\infty} \left|\mathbb{E}h\left(\frac{1}{\alpha}X^{\alpha}_{1:n}(\lfloor \alpha^2 t \rfloor)\right) - \mathbb{E}h(W_{1:n}(t))\right| = \lim_{\alpha\to\infty}\lim_{N\to\infty} |\mathbb{E}\Delta^{\alpha}_{N,n}(t)| = 0. \tag{B.2}$$

By using the triangular inequality, one obtains

$$|\mathbb{E}\Delta^{\alpha}_{N,n}(t)| \leq |\mathbb{E}[\Delta^{\alpha}_{N,n}(t) 1_{\{\Theta^{\alpha}_n(t) > N\}}]| + |\mathbb{E}[\Delta^{\alpha}_{N,n}(t) 1_{\{\Theta^{\alpha}_n(t) \leq N\}}]|, \tag{B.3}$$

where $\Theta_n^\alpha(t) \equiv \hat{\upsilon}_n^\alpha(t) \vee \omega_n(t)$, while the Jensen's inequality yields

$$|\mathbb{E}[\Delta_{N,n}^\alpha(t) 1_{\{\Theta_n^\alpha(t) > N\}}]| \leq \mathbb{E}[|\Delta_{N,n}^\alpha(t)| 1_{\{\Theta_n^\alpha(t) > N\}}]. \tag{B.4}$$

Since the function h is bounded, there exists a finite constant $C \geq 0$ such that

$$|h(x_{1:n})| \leq C < \infty, \tag{B.5}$$

for all $x_{1:n} \in \mathbb{R}^n$, which further implies $|\Delta_{N,n}^\alpha(t)| \leq 2C < \infty$. Therefore, it is straightforward that

$$\begin{aligned} \mathbb{E}[|\Delta_{N,n}^\alpha(t)| 1_{\{\Theta_n^\alpha(t) > N\}}] &\leq 2C\mathbb{P}[\Theta_n^\alpha(t) > N] \\ &\leq 2C(\mathbb{P}[\hat{\upsilon}_n^\alpha(t) > N] + \mathbb{P}[\omega_n(t) > N]). \end{aligned} \tag{B.6}$$

From (B.4), (B.6) and the first statements of Corollary 3.2 and Proposition 3.2, it follows that

$$\lim_{N\to\infty} |\mathbb{E}[\Delta_{N,n}^\alpha(t) 1_{\{\Theta_n^\alpha(t) > N\}}]| = 0. \tag{B.7}$$

From (3.13) and (3.17), it is straightforward that $\Delta_{N,n}^\alpha(t) 1_{\{\Theta_n^\alpha(t) = i\}} = \Delta_{i,n}^\alpha(t) 1_{\{\Theta_n^\alpha(t) = i\}}$ holds for $1 \leq i \leq N$, and, therefore,

$$\mathbb{E}[\Delta_{N,n}^\alpha(t) 1_{\{\Theta_n^\alpha(t) \leq N\}}] = \sum_{i=1}^N \mathbb{E}[\Delta_{i,n}^\alpha(t) 1_{\{\Theta_n^\alpha(t) = i\}}]. \tag{B.8}$$

Due to (B.5), we have $|\Delta_{i,n}^\alpha(t)| \leq 2C$, and, consequently, $|\mathbb{E}[\Delta_{i,n}^\alpha(t) 1_{\{\Theta_n^\alpha(t) = i\}}]| \leq 2C < \infty$ for all $i \in \mathbb{N}$. Then, Fatou's lemma leads to (e.g., see [12, p. 209])

$$\begin{aligned} \underline{\lim}_{\alpha\to\infty} \lim_{N\to\infty} \sum_{i=1}^N \mathbb{E}[\Delta_{i,n}^\alpha(t) 1_{\{\Theta_n^\alpha(t) = i\}}] &\geq \lim_{N\to\infty} \sum_{i=1}^N \underline{\lim}_{\alpha\to\infty} \mathbb{E}[\Delta_{i,n}^\alpha(t) 1_{\{\Theta_n^\alpha(t) = i\}}] \\ &= \lim_{N\to\infty} \underline{\lim}_{\alpha\to\infty} \sum_{i=1}^N \mathbb{E}[\Delta_{i,n}^\alpha(t) 1_{\{\Theta_n^\alpha(t) = i\}}]. \end{aligned} \tag{B.9}$$

Combining (B.8) and (B.9) yields

$$\underline{\lim}_{\alpha\to\infty} \lim_{N\to\infty} \mathbb{E}[\Delta_{N,n}^\alpha(t) 1_{\{\Theta_n^\alpha(t) \leq N\}}] \geq \lim_{N\to\infty} \underline{\lim}_{\alpha\to\infty} \mathbb{E}[\Delta_{N,n}^\alpha(t) 1_{\{\Theta_n^\alpha(t) \leq N\}}]. \tag{B.10}$$

Similarly, it can be shown that

$$\overline{\lim_{\alpha\to\infty}}\lim_{N\to\infty}\mathbb{E}[\Delta^{\alpha}_{N,n}(t)1_{\{\Theta^{\alpha}_n(t)\,\le\,N\}}]\le\lim_{N\to\infty}\overline{\lim_{\alpha\to\infty}}\mathbb{E}[\Delta^{\alpha}_{N,n}(t)1_{\{\Theta^{\alpha}_n(t)\,\le\,N\}}].\tag{B.11}$$

Note that

$$\mathbb{E}[\Delta^{\alpha}_{N,n}(t)1_{\{\Theta^{\alpha}_n(t)\,\le\,N\}}]=\mathbb{E}\Delta^{\alpha}_{N,n}(t)-\mathbb{E}[\Delta^{\alpha}_{N,n}(t)1_{\{\Theta^{\alpha}_n(t)\,>\,N\}}].\tag{B.12}$$

Corollary 3.3 implies (e.g., see [13, p. 26])

$$\lim_{\alpha\to\infty}\mathbb{E}\Delta^{\alpha}_{N,n}(t)=0,\tag{B.13}$$

and, analogously to (B.7), the first statement of Proposition 3.2 renders

$$\lim_{N\to\infty}\lim_{\alpha\to\infty}\mathbb{E}[\Delta^{\alpha}_{N,n}(t)1_{\{\Theta^{\alpha}_n(t)\,>\,N\}}]=0.\tag{B.14}$$

Combining (B.10)–(B.14) results in

$$\lim_{\alpha\to\infty}\lim_{N\to\infty}\mathbb{E}[\Delta^{\alpha}_{N,n}(t)1_{\{\Theta^{\alpha}_n(t)\,\le\,N\}}]=\lim_{N\to\infty}\lim_{\alpha\to\infty}\mathbb{E}[\Delta^{\alpha}_{N,n}(t)1_{\{\Theta^{\alpha}_n(t)\,\le\,N\}}]=0.\tag{B.15}$$

Finally, putting together (B.3), (B.7) and (B.15) leads to (B.2). Since the limit (B.2) holds for any bounded, continuous real-valued function h and any fixed $t\ge 0$, the limit (B.1) follows.

Part II. For any fixed $t>0$, we have

$$\overline{\lim_{k\to\infty}}\mathbb{P}[\|\hat{X}^{\sqrt{k/t}}_{1:n}(k)-\hat{X}_{1:n}(k)\|>\epsilon]=\overline{\lim_{\alpha\to\infty}}\mathbb{P}[\|\hat{X}^{\alpha}_{1:n}(\lfloor\alpha^2t\rfloor)-\hat{X}_{1:n}(\lfloor\alpha^2t\rfloor)\|>\epsilon].$$

This and Proposition 3.1 yield

$$\lim_{t\to\infty}\overline{\lim_{k\to\infty}}\mathbb{P}[\|\hat{X}^{\sqrt{k/t}}_{1:n}(k)-\hat{X}_{1:n}(k)\|>\epsilon]=0.\tag{B.16}$$

The limit (B.1) derived in Part I implies, as $k\to\infty$,

$$\hat{X}^{\sqrt{k/t}}_{1:n}(k)\Rightarrow\frac{1}{\sqrt{t}}W_{1:n}(t).\tag{B.17}$$

The statement of the theorem follows from (3.16), (B.16) and (B.17) (see [13, Theorem 3.2]).

B.2 Proof of Proposition 3.1

Since the expected value of the stationary inter-departure time is equal to the reciprocal of the throughput, we have

$$\mathbb{E}\hat{X}_1(k) = \frac{1}{\sqrt{k}\theta_k}, \tag{B.18}$$

where θ_k denotes the throughput at node k in the original system. Theorem 1 in [35] states that the maximum throughput θ_k^* in the linear k-node network with unit-mean exponential service times satisfies $\sqrt{k}\theta_k^* \to 1/\sqrt{\pi}$, as $k \to \infty$. Thus, from $\theta_k \leq \theta_k^*$, it follows that

$$\varliminf_{k\to\infty} \mathbb{E}\hat{X}_1(k) \geq \sqrt{\pi}. \tag{B.19}$$

Similarly to (B.18), we have

$$\mathbb{E}\hat{X}_1^\alpha(k) = \frac{1}{\sqrt{k}\theta_k^\alpha}, \tag{B.20}$$

where θ_k^α is the throughput of the altered k-node linear network in which the input process is produced by thinning the original arrival process by a factor of $\alpha > 1$. The following lemma characterizes θ_k^α.

Lemma B.1. *We have*

$$\lim_{t\to\infty}\lim_{\alpha\to\infty} \sqrt{\lfloor \alpha^2 t\rfloor}\,\theta^\alpha_{\lfloor \alpha^2 t\rfloor} = \frac{1}{\sqrt{\pi}}.$$

Proof. See Appendix B.2.1 □

If the original and altered systems are coupled as stated in Section 3.4.1, the inter-departure times at node k of the two systems, given in (3.2) and (3.3), satisfy the following property.

Lemma B.2. *For $n, k \in \mathbb{N}$, we have*

$$\sum_{i=1}^{n} X_i(k) \leq \sum_{i=1}^{n} X_i^{\alpha}(k).$$

Proof. See Appendix B.2.2 ☐

Now we complete the proof of the proposition. Using the triangular inequality, one obtains

$$|X_i^{\alpha}(k) - X_i(k)| \leq \left|\sum_{j=1}^{i}(X_j^{\alpha}(k) - X_j(k))\right| + \left|\sum_{j=1}^{i-1}(X_j^{\alpha}(k) - X_j(k))\right|,$$

for $2 \leq i \leq n$. From Lemma B.2 and the preceding inequality, it is not difficult to see that

$$\bigcap_{i=1}^{n}\left\{\sum_{j=1}^{i}(X_j^{\alpha}(k) - X_j(k)) \leq \epsilon/2\right\} \subseteq \{\|X_{1:n}^{\alpha}(k) - X_{1:n}(k)\| \leq \epsilon\},$$

for any $\epsilon > 0$. By the union bound and Markov's inequality, the preceding relation results in

(B.21)

$$\mathbb{P}[\|\hat{X}_{1:n}^{\alpha}(\lfloor\alpha^2 t\rfloor) - \hat{X}_{1:n}(\lfloor\alpha^2 t\rfloor)\| > \epsilon] \leq 2\epsilon^{-1}\sum_{i=1}^{n}\sum_{j=1}^{i}(\mathbb{E}\hat{X}_j^{\alpha}(\lfloor\alpha^2 t\rfloor) - \mathbb{E}\hat{X}_j(\lfloor\alpha^2 t\rfloor)).$$

Lemma B.1 and (B.20) yield

$$\lim_{t\to\infty}\lim_{\alpha\to\infty}\mathbb{E}\hat{X}_1^{\alpha}(\lfloor\alpha^2 t\rfloor) = \lim_{t\to\infty}\lim_{\alpha\to\infty}\frac{1}{\sqrt{\lfloor\alpha^2 t\rfloor}\,\theta_{\lfloor\alpha^2 t\rfloor}^{\langle\alpha\rangle}} = \sqrt{\pi}. \tag{B.22}$$

The limit (B.19) implies

$$\varliminf_{t\to\infty}\varliminf_{\alpha\to\infty}\mathbb{E}\hat{X}_1(\lfloor\alpha^2 t\rfloor) \geq \sqrt{\pi}. \tag{B.23}$$

Since we have $\mathbb{E}\hat{X}_1(k) \leq \mathbb{E}\hat{X}_1^{\alpha}(k)$ for any $k \in \mathbb{N}$ from Lemma B.2, combining (B.21)–(B.23) renders the statement of the proposition.

B.2.1 Proof of Lemma B.1

Observe that the throughput θ_k^α can be obtained from the following expression:

$$\theta_k^\alpha = \frac{\lambda}{\alpha}\rho_k^\alpha, \tag{B.24}$$

where ρ_k^α, $k \in \mathbb{N}$, is the probability that a given (stationary) customer reaches node k and successfully completes service in the altered system; $\lambda > 0$ is the input arrival rate (at node 1) in the original system. In view of [35, Lemma 1] (see also [35, Remark 5]), the probability ρ_k^α can be obtained by considering only two customers in the altered system and evaluating the probability that the later arriving customer does not displace the earlier arriving customer within the k-node network. Note that as in [35], later arriving customers are given priority for now; this is feasible since the throughput results in [35] are insensitive to the customer dropping policy. Without loss of generality, we consider two customers 1 and 2 arriving at node 1 at times A_1^α and A_2^α, respectively; it is assumed that no new customers are inputted to the network. Recall that $D_i^\alpha(j)$, $i \in \mathbb{Z}$, $j \in \mathbb{N}$, denotes the potential departure time of customer i from node j (see Section 3.4.2). The probability ρ_k^α can be represented by

$$\rho_k^\alpha = \mathbb{P}\left[\inf_{1\le j\le k}\{D_2^\alpha(j-1) - D_1^\alpha(j)\} > 0\right]; \tag{B.25}$$

for notational simplicity, we set $D_2^\alpha(0) = A_2^\alpha$. From (3.4), we have

$$D_2^\alpha(j-1) - D_1^\alpha(j) = A_2^\alpha - A_1^\alpha - S_1^\alpha(1) + \sum_{l=1}^{j-1}(S_2^\alpha(l) - S_1^\alpha(l+1)). \tag{B.26}$$

The stationarity and ergodicity of the original input process imply (e.g., see [23, p. 465])

$$\alpha^{-1}(A_2^\alpha - A_1^\alpha) = \alpha^{-1}(A_{\lfloor\alpha\rfloor+1} - A_1) \to \lambda^{-1} \quad \text{a.s.},$$

as $\alpha \to \infty$; moreover, it is straightforward that

$$\alpha^{-1} S_1^{\alpha}(1) \to 0 \quad \text{a.s.},$$

as $\alpha \to \infty$. From (B.26), the preceding two limits and the functional central limit theorem (e.g., see [45, Section 4.3]) yield

$$\text{(B.27)} \qquad \{\alpha^{-1}(D_2^{\alpha}(\lfloor \alpha^2 t \rfloor - 1) - D_1^{\alpha}(\lfloor \alpha^2 t \rfloor)),\, t \geq 0\} \Rightarrow \{\lambda^{-1} + \sqrt{2} B(t),\, t \geq 0\},$$

in $(\mathcal{D}, J_1)$, as $\alpha \to \infty$, where $\{B(t),\, t \geq 0\}$ is a standard one-dimensional Brownian motion. For $\alpha > 1$ and $t > 0$, we have

$$\inf_{1 \leq j \leq \lfloor \alpha^2 t \rfloor} \{D_2^{\alpha}(j-1) - D_1^{\alpha}(j)\} = \inf_{\alpha^{-2} \leq s \leq t} \{(D_2^{\alpha}(\lfloor \alpha^2 s \rfloor - 1) - D_1^{\alpha}(\lfloor \alpha^2 s \rfloor))\}.$$

From (B.25), (B.27) and the preceding equality, one obtains (e.g., see [45, Section 14.3] and [13, p. 26])

$$\text{(B.28)} \qquad \lim_{\alpha \to \infty} \rho_{\lfloor \alpha^2 t \rfloor}^{\alpha} = \mathbb{P}\left[\inf_{0 < s \leq t} B(s) > -\frac{1}{\sqrt{2}\lambda}\right] = \sqrt{\frac{2}{\pi}} \int_0^{1/(\sqrt{2t}\lambda)} e^{-s^2/2} ds;$$

the second equality stems from the reflection principle and the strong Markov property of the Brownian motion (e.g., see [30, Section 2.6]). Since the following inequality holds

$$\lambda^{-1} e^{-1/(4t\lambda^2)} \leq \sqrt{2t} \int_0^{1/(\sqrt{2t}\lambda)} e^{-s^2/2} ds \leq \lambda^{-1},$$

it follows from (B.28) that

$$\lim_{t \to \infty} \lim_{\alpha \to \infty} \sqrt{t} \rho_{\lfloor \alpha^2 t \rfloor}^{\alpha} = \frac{1}{\sqrt{\pi}\lambda}.$$

Finally, the preceding limit, together with (B.24) and (B.25), renders the statement of the lemma.

B.2.2 Proof of Lemma B.2

Consider two single server queues (denoted by queue 1 and 2, respectively) with no waiting rooms. Let $\mathscr{A}_i \equiv \{A_{i,j}\}_{j\in\mathbb{N}}$ and $\mathscr{D}_i \equiv \{D_{i,j}\}_{j\in\mathbb{N}}$, $i = 1, 2$, be the (strictly) increasing sequences of arrival and departure times at queue i, respectively; and let $Q_i(t)$, $t \geq 0$, $i = 1, 2$, be the number of customers in queue i at time t. The input processes are assumed to start at time $t = 0$ (i.e., $A_{i,1} \geq 0$ for $i = 1, 2$), and arrival times are independent of service times. Suppose that service completion times of the queues are coupled, i.e., whenever a customer departs from one queue, a customer (if present) also departs from the other queue. The following lemma provides a relation between the departure processes of the coupled queues.

Lemma B.3. *If* $\mathscr{A}_1 \subseteq \mathscr{A}_2$ *and* $Q_1(0) = Q_2(0)$, *then* $\mathscr{D}_1 \subseteq \mathscr{D}_2$.

Proof. Since the service completion times are coupled, the statement $Q_1(t) \leq Q_2(t)$ for all $t \geq 0$ implies the statement of the lemma. Note that the value of right-continuous $\{Q_i(t), t \geq 0\}$, $i = 1, 2$, changes only when a customer arrives to or departs from queue i. Thus, it is sufficient to show that $Q_1(t) \leq Q_2(t)$ for $t \in \mathscr{A}_1 \cup \mathscr{A}_2 \cup \mathscr{D}_1 \cup \mathscr{D}_2$.

The rest of the proof is based on induction. First, we construct an increasing sequence $\{t_i\}_{i\in\mathbb{N}}$ from all the elements in $\mathscr{A}_1 \cup \mathscr{A}_2 \cup \mathscr{D}_1 \cup \mathscr{D}_2$; we set $t_0 = 0$ for the base of the induction. For each $i \geq 0$, suppose that $Q_1(t) \leq Q_2(t)$ for $0 \leq t \leq t_i$; this accordingly implies $Q_1(t) \leq Q_2(t)$ for $0 \leq t < t_{i+1}$. Next, in order to show that $Q_1(t_{i+1}) \leq Q_2(t_{i+1})$, we consider the following three (disjoint) cases: (i) $t_{i+1} \in \mathscr{A}_1 \cup \mathscr{A}_2 = \mathscr{A}_2$, (ii) $t_{i+1} \in \mathscr{D}_1$, and (iii) $t_i \in \mathscr{D}_2 \setminus \mathscr{D}_1$. No other cases are relevant because of the assumptions of the lemma. In the case (i), the assumption $\mathscr{A}_1 \subseteq \mathscr{A}_2$ yields $Q_2(t_{i+1}) = 1 \geq Q_1(t_{i+1})$. In the cases (ii) and (iii), we have $Q_1(t_{i+1}-) =$

$Q_2(t_{i+1}-) = 1$ and $Q_1(t_{i+1}-) = 0$, $Q_2(t_{i+1}-) = 1$, respectively, and, in both cases, the queues become empty at time t_{i+1} (i.e., $Q_1(t_{i+1}) = Q_2(t_{i+1}) = 0$). By combining these results, it follows that $Q_1(t) \leq Q_2(t)$ for $0 \leq t \leq t_{i+1}$. Finally, by induction, the statement of the lemma holds. □

Next, we present the proof of Lemma B.2.

Proof of Lemma B.2. Recall that the altered input process is produced by thinning the original process by a factor of $\alpha > 1$:

$$\{A_i^\alpha\}_{i\in\mathbb{N}} = \{A_{\lfloor \alpha(i-1)\rfloor+1}\}_{i\in\mathbb{N}} \subseteq \{A_i\}_{i\in\mathbb{N}}. \tag{B.29}$$

Let $Q_j(t)$ and $Q_j^\alpha(t)$, $1 \leq j \leq k$, $t \geq 0$, be the numbers of customers at node j at time t in the original and altered systems, respectively. Under the assumption that customer 1 has the highest priority over all other customers, customers with indices $i < 1$ play no role on determining the departure times of customers with indices $i \geq 1$. Hence, just for now, we suppose that no customers arrive before customer 1, and this implies $Q_j(0) = Q_j^\alpha(0)$ for all $1 \leq j \leq k$. Recall that service completion times at the nodes with the same indices of the original and altered systems are coupled. Then, (B.29) and Lemma B.3 recursively yield

$$\{d_i^\alpha(j)\}_{i\in\mathbb{N}} \subseteq \{d_i(j)\}_{i\in\mathbb{N}}, \tag{B.30}$$

for all $1 \leq j \leq k$; moreover, note that $d_1^\alpha(j) = d_1(j)$ for all $1 \leq j \leq k$. The statement of the lemma follows from (3.2), (3.3) and (B.30). □

B.3 Proof of Lemma 3.2

We first provide a technical lemma for proving Lemma 3.2.

Lemma B.4. *For $\alpha > 1$ and $c > 0$, we have $\mathbb{P}[A_N^\alpha - A_1^\alpha < c] \to 0$, as $N \to \infty$.*

Proof. Under the assumption that the arrival process at node 1 is stationary and ergodic, we have (e.g., see [23, p. 465])

$$\frac{A_N^\alpha - A_1^\alpha}{N-1} \to \alpha\lambda^{-1} \quad \text{a.s.},$$

as $N \to \infty$, which implies (e.g., see [12, p. 330])

$$\mathbb{P}\left[\left|\frac{A_N^\alpha - A_1^\alpha}{N-1} - \alpha\lambda^{-1}\right| > \epsilon\right] \to 0, \tag{B.31}$$

as $N \to \infty$, for any $\epsilon > 0$. If $0 < \epsilon < \alpha\lambda^{-1}$, then

$$\mathbb{P}[A_N^\alpha - A_1^\alpha < c] \leq \mathbb{P}\left[\left|\frac{A_N^\alpha - A_1^\alpha}{N-1} - \alpha\lambda^{-1}\right| > \epsilon\right], \tag{B.32}$$

for all $N \geq c/(\alpha\lambda^{-1} - \epsilon) + 1$. Thus, the statement of the lemma follows from (B.31) and (B.32). □

Next we present the proof of Lemma 3.2.

Proof of Lemma 3.2. Since we have $\upsilon_{n_1}^\alpha(t) \leq \upsilon_{n_2}^\alpha(t)$ for $n_1 \leq n_2$, it is straightforward from (3.10) that

$$\mathbb{P}[\upsilon_n^\alpha(t) > N] = \mathbb{P}[\nu_N(C_{1:N}^\alpha(t)) < n], \tag{B.33}$$

where $\mathbf{C}_{1:N}^\alpha = \psi_N(\mathbf{D}_{1:N}^\alpha)$. For some $x_{1:N} \in \mathbb{R}_\leqslant^N$ and $1 \leq a < b \leq N$, if $x_a < x_b$, then there must exist at least one $a+1 \leq i \leq b$ such that $x_i - x_{i-1} > 0$. Hence, we have

$$\begin{aligned} &\{x_{1:N} \in \mathbb{R}_\leqslant^N : x_{\lfloor (i-1)N/n \rfloor + 1} < x_{\lfloor iN/n \rfloor},\ 1 \leq i \leq n\} \\ &\qquad \subseteq \{x_{1:N} \in \mathbb{R}_\leqslant^N : \nu_N(x_{1:N}) \geq n\}. \end{aligned} \tag{B.34}$$

From (B.33) and (B.34), one obtains

$$\begin{aligned} \mathbb{P}[\upsilon_n^\alpha(t) > N] &= 1 - \mathbb{P}[\nu_N(C_{1:N}^\alpha(t)) \geq n] \\ &\leq 1 - \mathbb{P}\left[C_{1:N}^\alpha(t) \in \bigcap_{1\leq i\leq n} \mathcal{S}_{N,n}^i\right] = \mathbb{P}\left[C_{1:N}^\alpha(t) \in \bigcup_{1\leq i\leq n} \bar{\mathcal{S}}_{N,n}^i\right], \end{aligned}$$

where $\mathcal{S}^i_{N,n} = \{x_{1:N} \in \mathbb{R}^N_{\leqslant} : x_{\lfloor (i-1)N/n \rfloor + 1} < x_{\lfloor iN/n \rfloor}\}$. The union bound further yields

$$\mathbb{P}[\upsilon^\alpha_n(t) > N] \leq \sum_{i=1}^n \mathbb{P}[C^\alpha_{1:N}(t) \in \bar{\mathcal{S}}^i_{N,n}] = \sum_{i=1}^n \mathbb{P}[\Gamma^{\alpha,i}_{N,n}(t) = 0], \tag{B.35}$$

where $\Gamma^{\alpha,i}_{N,n}(t) \equiv C^\alpha_{\lfloor iN/n \rfloor}(t) - C^\alpha_{\lfloor (i-1)N/n \rfloor + 1}(t) \geq 0$. For $1 \leq i \leq n$ and any $c > 0$, we have

$$\mathbb{P}[\Gamma^{\alpha,i}_{N,n}(t) = 0] \leq \mathbb{P}[\Gamma^{\alpha,i}_{N,n}(t) = 0,\, \Gamma^{\alpha,i}_{N,n}(0) \geq c] + \mathbb{P}[\Gamma^{\alpha,i}_{N,n}(0) < c]. \tag{B.36}$$

It is not difficult to derive

$$\begin{aligned} \mathbb{P}[\Gamma^{\alpha,i}_{N,n}(t) = 0,\, \Gamma^{\alpha,i}_{N,n}(0) \geq c] &\leq \mathbb{P}[\Gamma^{\alpha,i}_{N,n}(0) - \Gamma^{\alpha,i}_{N,n}(t) \geq c] \\ &\leq \mathbb{P}[C^\alpha_{\lfloor (i-1)N/n \rfloor + 1}(t) - C^\alpha_{\lfloor (i-1)N/n \rfloor + 1}(0) \geq c]; \end{aligned} \tag{B.37}$$

we used the fact that $C^\alpha_{\lfloor iN/n \rfloor}(t) - C^\alpha_{\lfloor iN/n \rfloor}(0) \geq 0$ when deriving the second inequality. Note that the Markov's inequality yields

$$\begin{aligned} &\mathbb{P}[C^\alpha_{\lfloor (i-1)N/n \rfloor + 1}(t) - C^\alpha_{\lfloor (i-1)N/n \rfloor + 1}(0) \geq c] \\ &\qquad \leq c^{-1}\mathbb{E}[C^\alpha_{\lfloor (i-1)N/n \rfloor + 1}(t) - C^\alpha_{\lfloor (i-1)N/n \rfloor + 1}(0)]. \end{aligned} \tag{B.38}$$

Recall that if some customer i is displaced from the network by some customer j at node l, then the coalesced departure time of customer i at node l is set to be equal to the departure time of customer j at node l. Since service times are i.i.d. exponential with unit mean and the exponential distribution has the memoryless property, the remaining service time of customer j at node l is also exponential with unit mean, i.e., it is equal in distribution of the service time of customer i at node l. Thus, the following holds for any $j \in \mathbb{N}$:

$$C^\alpha_{\lfloor (i-1)N/n \rfloor + 1}(j) \stackrel{d}{=} C^\alpha_{\lfloor (i-1)N/n \rfloor + 1}(j-1) + S^\alpha_{\lfloor (i-1)N/n \rfloor + 1}(j).$$

By using the preceding equality, one obtains

$$\mathbb{E}[C^\alpha_{\lfloor (i-1)N/n \rfloor + 1}(t) - C^\alpha_{\lfloor (i-1)N/n \rfloor + 1}(0)] = \lfloor t \rfloor, \tag{B.39}$$

and combining (B.37)–(B.39) renders

$$\mathbb{P}[\Gamma^{\alpha,i}_{N,n}(t) = 0,\, \Gamma^{\alpha,i}_{N,n}(0) \geq c] \leq c^{-1}t. \tag{B.40}$$

On the other hand, Lemma B.4 implies

$$\mathbb{P}[\Gamma^{\alpha,i}_{N,n}(0) < c] = \mathbb{P}[A^{\alpha}_{\lfloor iN/n\rfloor} - A^{\alpha}_{\lfloor (i-1)N/n\rfloor+1} < c] \to 0, \tag{B.41}$$

as $N \to \infty$. Putting together (B.35), (B.36), (B.40) and (B.41) yields

$$\overline{\lim_{N\to\infty}}\, \mathbb{P}[\upsilon^{\alpha}_n(T) > N] \leq nc^{-1}t,$$

and, since this holds for any $c > 0$, taking $c \to \infty$ renders the statement of the lemma. $\square$

B.4 Proof of Lemma 3.3

By the definition of $\upsilon^{\alpha}_n(t)$ (see (3.10)), all elements of $\varphi_{\upsilon^{\alpha}_n(t),n} \circ \psi_{\upsilon^{\alpha}_n(t)}(\mathbf{D}^{\alpha}_{1:\upsilon^{\alpha}_n(t)})(t)$ are strictly positive. Thus, for fixed $t \geq 0$, it consists of n consecutive stationary inter-departure times at node $\lfloor t \rfloor$; recall that the arrival process at node 1 is stationary (see Section 3.4.1), and, therefore, departure processes at the following nodes are also in stationarity. Hence, we have

$$\varphi_{\upsilon^{\alpha}_n(t),n} \circ \psi_{\upsilon^{\alpha}_n(t)}(\mathbf{D}^{\alpha}_{1:\upsilon^{\alpha}_n(t)})(t) \overset{d}{=} X^{\alpha}_{1:n}(\lfloor t \rfloor). \tag{B.42}$$

Due to (3.11), the following holds for any $\epsilon > 0$:

$$\mathbb{P}[|\varphi_{N,n} \circ \psi_N(\mathbf{D}^{\alpha}_{1:N})(t) - \varphi_{\upsilon^{\alpha}_n(t),n} \circ \psi_{\upsilon^{\alpha}_n(t)}(\mathbf{D}^{\alpha}_{1:\upsilon^{\alpha}_n(t)})(t)| > \epsilon] \leq \mathbb{P}[\upsilon^{\alpha}_n(t) > N].$$

This and Lemma 3.2 further imply

$$\varphi_{N,n} \circ \psi_N(\mathbf{D}^{\alpha}_{1:N})(t) \overset{\mathbb{P}}{\to} \varphi_{\upsilon^{\alpha}_n(t),n} \circ \psi_{\upsilon^{\alpha}_n(t)}(\mathbf{D}^{\alpha}_{1:\upsilon^{\alpha}_n(t)})(t), \tag{B.43}$$

as $N \to \infty$. From (B.42) and (B.43), the statement of the lemma follows (e.g., see [12, p. 330]).

B.5 Proof of Lemma 3.4

For $n \in \mathbb{N}$, consider a subset of $\mathcal{C}^n$ defined as

$$\mathcal{A}^n \triangleq \{\mathbf{x}_{1:n} \in \mathcal{C}^n : \mathbf{x}_{1:n} \text{ satisfies (A1), (A2) and (A3)}\},$$

where (A1), (A2) and (A3) are given by

(A1) $x_{1:n}(0) \in \mathbb{R}^n_{\leqslant}$.

(A2) $\tau(\mathbf{x}_i, \mathbf{x}_j)$, $1 \leq i < j \leq n$, have finite distinct values.

(A3) For $1 \leq i < j \leq n$, given any $\epsilon > 0$, there exist $\epsilon', \epsilon'' \in (0, \epsilon)$ such that (see [14])

$$x_i(\tau(\mathbf{x}_i,\mathbf{x}_j)+\epsilon') > x_j(\tau(\mathbf{x}_i,\mathbf{x}_j)+\epsilon') \quad \text{and} \quad x_i(\tau(\mathbf{x}_i,\mathbf{x}_j)+\epsilon'') < x_j(\tau(\mathbf{x}_i,\mathbf{x}_j)+\epsilon'').$$

Lemma B.5. *The composite operator* $\varphi_{N,n} \circ \psi_n : (\mathcal{D}^N, J_1) \to (\mathcal{D}^n, J_1)$, $N, n \in \mathbb{N}$, *is continuous at all* $\mathbf{x}_{1:N} \in \mathcal{A}^N$.

Proof. See Appendix B.5.1 □

It is straightforward that $\mathbf{B}_{1:N} + \beta_{1:N}$ satisfies (A1) since $\beta_{1:N} = \lambda^{-1}(0, 1, \ldots, N-1) \in \mathbb{R}^N_{\leqslant}$. The operator τ yields the first hitting time of the two (argument) processes. The first hitting time of two processes can be thought of the first time instance at which the difference becomes zero. If $\mathbf{B}_i$ and $\mathbf{B}_j$ are i.i.d. standard one-dimensional Brownian motions, the difference $\mathbf{B}_i - \mathbf{B}_j$ is also a one-dimensional Brownian motion. One-dimensional Brownian motions are recurrent, i.e., they keep returning to zero within arbitrary large times $t < \infty$. Thus, $\tau(\mathbf{B}_i + \beta_i, \mathbf{B}_j + \beta_j)$, $1 \leq i < j \leq N$, are all finite. Moreover, they are all distinct since the Brownian motion in two dimensions is not point recurrent, i.e., the two-dimensional Brownian motion never returns to the origin. Hence, $\mathbf{B}_{1:N} + \beta_{1:N}$ satisfies (A2) a.s. (e.g., see [31, Section 8.5]). In addition, a standard one-dimensional Brownian motion changes sign infinitely many times in any time-interval $[0,\epsilon]$, $\epsilon > 0$, with probability one (see [30, p. 94, Problem 7.18]), and, thus, $\mathbf{B}_{1:N} + \beta_{1:N}$ also satisfies (A3) a.s.. Finally, note that Brownian motion is a continuous process. To summarize, we have

$$\mathbb{P}[\mathbf{B}_{1:N} + \beta_{1:N} \in \mathcal{A}^N] = 1. \tag{B.44}$$

The stationarity and ergodicity of the original input process imply (e.g., see [23, p. 465])

$$\alpha^{-1} A_i^\alpha = \alpha^{-1}(A_{\lfloor \alpha(i-1) \rfloor + 1} - A_1) \to \lambda^{-1}(i-1) \quad \text{a.s.},$$

as $\alpha \to \infty$, for all $i \in \mathbb{N}$; recall that we set $A_1^\alpha = A_1 = 0$. Therefore, it follows that

$$\alpha^{-1} A_{1:N}^\alpha \to \lambda^{-1}(0, 1, \ldots, N-1) \quad \text{a.s.}$$

From (3.6) and the preceding limit, the functional central limit theorem yields (e.g., see [45, Section 4.3])

$$\hat{\mathbf{D}}_{1:N}^\alpha \Rightarrow \mathbf{B}_{1:N} + \beta_{1:N}, \tag{B.45}$$

as $\alpha \to \infty$, where $\beta_{1:N} = \lambda^{-1}(0, 1, \ldots, N-1)$. From (B.44), (B.45), Lemma B.5 and the continuous-mapping theorem (e.g., see [45, p. 86]), the statement of the lemma follows.

B.5.1 Proof of Lemma B.5

Lemma B.6. *For $N, n \in \mathbb{N}$ and $\mathbf{x}_{1:N} \in \mathcal{A}^N$, we have*

$$\mathrm{Disc}(\varphi_{N,n} \circ \psi_N(\mathbf{x}_{1:N})) \subseteq \{\tau(\mathbf{x}_i, \mathbf{x}_j), 1 \le i < j \le N\}.$$

Proof. The proof is based on induction. It is straightforward that

$$\varphi_{1:n} \circ \psi_1(\mathbf{x}) = \varphi_{1:n}(\mathbf{x}) = \mathbf{0}_{1:n}, \tag{B.46}$$

for any $\mathbf{x} \in \mathcal{D}^1$ and all $n \in \mathbb{N}$, where $\mathbf{0}_{1:n} \equiv \{0_{1:n}(t) = (0, 0, \ldots, 0),\, t \geq 0\}$. Thus, it is trivial that $\mathrm{Disc}(\varphi_{1:n} \circ \psi_1(\mathbf{x})) = \emptyset$, and the statement of the lemma holds for $N = 1$ and all $n \in \mathbb{N}$. As an inductive hypothesis, assume that the statement of the lemma is true for $N = l \in \mathbb{N}$ and all $n \in \mathbb{N}$. For some $\mathbf{x}_{1:l+1} \in \mathcal{A}^{l+1}$, let $\mathbf{z}_{1:n} = \varphi_{l+1,n} \circ \psi_{l+1}(\mathbf{x}_{1:l+1})$. By Definitions 3.3 and 3.6, $\mathbf{z}_{1:n} \equiv \{z_{1:n}(t),\, t \geq 0\}$ satisfies

$$z_{1:n}(t) = \begin{cases} a_{1:n}(t), & 0 \leq t < \tau(\mathbf{x}_{1:2}), \\ b_{1:n}(t), & t \geq \tau(\mathbf{x}_{1:2}), \end{cases} \tag{B.47}$$

where $\mathbf{a}_{1:n} = (\mathbf{x}_2 - \mathbf{x}_1, \varphi_{l,n-1} \circ \psi_l(\mathbf{x}_{2:l+1}))$ and $\mathbf{b}_{1:n} = \varphi_{l,n} \circ \psi_l(\mathbf{x}_1, \mathbf{x}_{3:l+1})$. Here, observe that $\tau(\mathbf{x}_{1:2})$ can be a discontinuity point of $\mathbf{z}_{1:n}$. Thus, it follows that

$$\begin{aligned} \mathrm{Disc}(\mathbf{z}_{1:n}) &\subseteq \mathrm{Disc}(\mathbf{a}_{1:n}) \cup \mathrm{Disc}(\mathbf{b}_{1:n}) \cup \{\tau(\mathbf{x}_{1:2})\} \\ &= \mathrm{Disc}(\varphi_{l,n-1} \circ \psi_l(\mathbf{x}_{2:l+1})) \cup \mathrm{Disc}(\varphi_{l,n} \circ \psi_l(\mathbf{x}_1, \mathbf{x}_{3:l+1})) \cup \{\tau(\mathbf{x}_{1:2})\} \\ &\subseteq \{\tau(\mathbf{x}_i, \mathbf{x}_j), 1 \leq i < j \leq l+1\}; \end{aligned}$$

the last relation stems from the inductive hypothesis. Hence, the statement of the lemma holds for $N = l+1$ and all $n \in \mathbb{N}$, and, by induction, this completes the proof of the lemma. □

Next we present the proof of Lemma B.5.

Proof of Lemma B.5. The proof is based on induction. It is straightforward from (B.46) that the statement of the lemma is true for $N = 1$ and all $n \in \mathbb{N}$. As an inductive hypothesis, suppose that the statement of the lemma holds for $N = l \in \mathbb{N}$ and all $n \in \mathbb{N}$. Now consider $N = l + 1$. For some $\mathbf{x}_{1:l+1} \in \mathcal{A}^{l+1}$, let $\mathbf{z}_{1:n} = \varphi_{l+1,n} \circ \psi_{l+1}(\mathbf{x}_{1:l+1})$. Note that it is sufficient to show that for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$\|\mathbf{x}_{1:l+1} - \mathbf{x}'_{1:l+1}\| < \delta \quad \text{implies} \quad d(\mathbf{z}_{1:n}, \mathbf{z}'_{1:n}) < \epsilon, \tag{B.48}$$

for any $\mathbf{x}'_{1:l+1} \in \mathcal{D}^{l+1}$, where $\mathbf{z}'_{1:n} = \varphi_{l+1,n} \circ \psi_{l+1}(\mathbf{x}'_{1:l+1})$. Recall that $\mathbf{z}_{1:n} \equiv \{z_{1:n}(t),\ t \geq 0\}$ is given by (B.47). Then, from the inductive hypothesis, for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$\text{(B.49)} \qquad \|\mathbf{x}_{1:l+1} - \mathbf{x}'_{1:l+1}\| < \delta \quad \text{implies} \quad d(\mathbf{a}_{1:n}, \mathbf{a}'_{1:n}) \vee d(\mathbf{b}_{1:n}, \mathbf{b}'_{1:n}) < \epsilon/2,$$

where $\mathbf{a}'_{1:n} = (\mathbf{x}'_2 - \mathbf{x}'_1, \varphi_{l,n-1} \circ \psi_l(\mathbf{x}'_{2:l+1}))$ and $\mathbf{b}'_{1:n} = \varphi_{l,n} \circ \psi_l(\mathbf{x}'_1, \mathbf{x}'_{3:l+1})$. Now, for notational simplicity, let $\tau_{ij} \equiv \tau(\mathbf{x}_i, \mathbf{x}_j)$ and $\tau'_{ij} \equiv \tau(\mathbf{x}'_i, \mathbf{x}'_j)$. Since $\mathbf{x}_{1:l+1} \in \mathcal{A}^{l+1}$ satisfies (A2), there exists $\eta > 0$ such that

$$\text{(B.50)} \qquad |\tau_{12} - \tau_{ij}| > \eta,$$

for all $(i, j) \neq (1, 2)$, $1 \leq i < j \leq l + 1$. Due to Lemma B.6, we have $\tau_{12} \notin \mathrm{Disc}(\mathbf{a}_{1:n}) \cup \mathrm{Disc}(\mathbf{b}_{1:n})$, and this, together with (B.50), results in

$$\text{(B.51)} \qquad [\tau_{12} - \eta, \tau_{12} + \eta] \cap (\mathrm{Disc}(\mathbf{a}_{1:n}) \cup \mathrm{Disc}(\mathbf{b}_{1:n})) = \emptyset.$$

It is not difficult to see that the operator $\tau : (\mathcal{D}^2, J_1) \to [0, \infty]$ is continuous at any $\mathbf{x}_{1:2} \in \mathcal{A}^2$; this further indicates that for any $\eta' \in (0, \eta \wedge \epsilon)$,

$$\text{(B.52)} \qquad \|\mathbf{x}_{1:l+1} - \mathbf{x}'_{1:l+1}\| < \delta \quad \text{implies} \quad |\tau_{12} - \tau'_{12}| < \eta' < \eta \wedge \epsilon,$$

for small enough $\delta > 0$. Consider a function $\lambda : [\tau_{12} - \eta, \tau_{12} + \eta] \to [\tau_{12} - \eta, \tau_{12} + \eta]$ satisfying $\lambda(\tau_{12} \pm \eta) = \tau_{12} \pm \eta$ and $\lambda(\tau'_{12}) = \tau_{12}$ and defined by linear interpolation elsewhere. Observe from (B.52) that

$$\text{(B.53)} \qquad \|\mathbf{x}_{1:l+1} - \mathbf{x}'_{1:l+1}\| < \delta \quad \text{implies} \quad \|\lambda - e\|_{[\tau_{12}-\eta, \tau_{12}+\eta]} \leq |\tau_{12} - \tau'_{12}| < \epsilon,$$

for small enough $\delta > 0$, where $e \equiv \{e(t) = t,\ t \geq 0\}$. From (B.47) and by using the triangular inequality, one obtains

(B.54)

$$\begin{aligned} \|z_{1:n} \circ \lambda - z'_{1:n}\|_{[\tau_{12}-\eta, \tau'_{12})} &= \|a_{1:n} \circ \lambda - a'_{1:n}\|_{[\tau_{12}-\eta, \tau'_{12})} \\ &\leq \|a_{1:n} \circ \lambda - a_{1:n}\|_{[\tau_{12}-\eta, \tau'_{12})} + \|a_{1:n} - a'_{1:n}\|_{[\tau_{12}-\eta, \tau'_{12})}, \end{aligned}$$

where $\|f\|_S \equiv \sup_{t\in S}\|f(t)\|$ is the uniform norm for a function f defined on a set $S \subseteq [0,\infty)$. Similarly, it can be shown that

$$\text{(B.55)}\quad \|z_{1:n}\circ\lambda - z'_{1:n}\|_{[\tau'_{12},\tau_{12}+\eta]} \le \|b_{1:n}\circ\lambda - b_{1:n}\|_{[\tau'_{12},\tau_{12}+\eta]} + \|b_{1:n} - b'_{1:n}\|_{[\tau'_{12},\tau_{12}+\eta]},$$

Note that $\mathbf{a}_{1:n}$ and $\mathbf{b}_{1:n}$ are continuous at all $t \in [\tau_{12}-\eta, \tau_{12}+\eta]$ due to (B.51). Hence, it follows from (B.53) that

$$\text{(B.56)}\quad \|\mathbf{x}_{1:l+1} - \mathbf{x}'_{1:l+1}\| < \delta \quad \text{implies}$$
$$\|a_{1:n}\circ\lambda - a_{1:n}\|_{[\tau_{12}-\eta,\tau'_{12})} \vee \|b_{1:n}\circ\lambda - b_{1:n}\|_{[\tau'_{12},\tau_{12}+\eta]} < \epsilon/2,$$

for small enough $\delta > 0$. Furthermore, from (B.49) and by using the fact that the convergence in J_1 topology to a continuous limit is equivalent to the uniform convergence (e.g., see [13, p. 157-158]), we have

$$\text{(B.57)}\quad \|\mathbf{x}_{1:l+1} - \mathbf{x}'_{1:l+1}\| < \delta \quad \text{implies}$$
$$\|a_{1:n} - a'_{1:n}\|_{[\tau_{12}-\eta,\tau'_{12})} \vee \|b_{1:n} - b'_{1:n}\|_{[\tau',\tau_{12}+\eta]} < \epsilon/2,$$

for small enough $\delta > 0$. Combining (B.54)–(B.57) results in

$$\|\mathbf{x}_{1:l+1} - \mathbf{x}'_{1:l+1}\| < \delta \quad \text{implies} \quad \|z_{1:n}\circ\lambda - z'_{1:n}\|_{[\tau_{12}-\eta,\tau_{12}+\eta]} < \epsilon,$$

for small enough $\delta > 0$, and this, together with (B.53), leads to

$$\text{(B.58)}\qquad \|\mathbf{x}_{1:l+1} - \mathbf{x}'_{1:l+1}\| < \delta \quad \text{implies} \quad d_{[\tau_{12}-\eta,\tau_{12}+\eta]}(\mathbf{z}_{1:n}, \mathbf{z}'_{1:n}) < \epsilon,$$

for small enough $\delta > 0$; note that $\tau_{12} \pm \eta \notin \mathrm{Disc}(\mathbf{z}_{1:n})$ due to (B.50) and Lemma B.6. From (B.47) and (B.49), we have

$$\text{(B.59)}\quad \|\mathbf{x}_{1:l+1} - \mathbf{x}'_{1:l+1}\| < \delta \quad \text{implies}$$
$$d_{[0,\tau_{12}-\eta]}(\mathbf{z}_{1:n}, \mathbf{z}'_{1:n}) \vee d_{[\tau_{12}+\eta,\infty)}(\mathbf{z}_{1:n}, \mathbf{z}'_{1:n}) < \epsilon/2,$$

for small enough $\delta > 0$. Finally, the statement (B.48) follows from (B.58) and (B.59) (e.g., see [13, p. 168-169]). This completes the proof of the lemma. □

B.6 Proof of Proposition 3.2

By definition (Section 3.4.2), we have

$$\mathbb{P}[\hat{\upsilon}_n^\alpha(t) > N] = \mathbb{P}[(\varphi_{N,n} \circ \psi_N(\hat{\mathbf{D}}_{1:N}^\alpha))_n(t) = 0],$$

and

$$\mathbb{P}[\omega_n(t) > N] = \mathbb{P}[(\varphi_{N,n} \circ \psi_N(\mathbf{B}_{1:N} + \beta_{1:N}))_n(t) = 0],$$

where $\beta_{1:N} = \lambda^{-1}(0, 1, \ldots, N-1)$. It follows from the preceding equalities and Corollary 3.3 that (e.g., see [13, p. 26])

$$\overline{\lim_{\alpha\to\infty}}\, \mathbb{P}[\hat{\upsilon}_n^\alpha(t) > N] \le \mathbb{P}[\omega_n(t) > N], \tag{B.60}$$

for fixed $n, N \in \mathbb{N}$ and $t \ge 0$. Let $\mathbf{C}_{1:N} = \psi_N(\mathbf{B}_{1:N} + \beta_{1:N})$. Similarly to (B.35), we have

$$\mathbb{P}[\omega_n(t) > N] \le \sum_{i=1}^{n} \mathbb{P}[\Lambda_{N,n}^i(t) = 0], \tag{B.61}$$

where $\Lambda_{N,n}^i(t) \equiv C_{\lfloor iN/n \rfloor}(t) - C_{\lfloor (i-1)N/n \rfloor + 1}(t) \ge 0$. Observe that

$$\{\Lambda_{N,n}^i(t) = 0\} = \left\{ \inf_{0\le s\le t} \{C_{\lfloor iN/n \rfloor}(s) - C_{\lfloor (i-1)N/n \rfloor + 1}(s)\} = 0 \right\}. \tag{B.62}$$

By definition, it is not difficult to see that each $\mathbf{C}_i$ for $1 \le i \le N$ is continuous at all $t \ge 0$ and peicewisely equal to one of $\mathbf{B}_i + \beta_i$, $1 \le i \le N$. For any two integers a and b such that $1 \le a < b \le N$, we have $C_a(s) = (\psi_a(\mathbf{B}_{1:a} + \beta_{1:a}))_a(s)$ and $C_b(s) = (\psi_{b-a}(\mathbf{B}_{a+1:b} + \beta_{a+1:b}))_b(s)$ for $0 \le s \le \tau(\mathbf{C}_a, \mathbf{C}_b)$, which implies that $\mathbf{C}_a$ and $\mathbf{C}_b$ are not related before they meet. Then, by the strong Markov property of Brownian motion, one can see that $\mathbf{C}_a$ and $\mathbf{C}_b$ perform independent standard one-dimensional Brownian motions before they meet. Hence, by using the reflection principle and the strong Markov property of Brownian motion, (B.62) renders (e.g., see [30, Section 2.6])

$$\mathbb{P}[\Lambda^i_{N,n}(t)=0]=\sqrt{\frac{2}{\pi}}\mathbb{E}\int_{\Lambda^i_{N,n}(0)/\sqrt{2t}}^{\infty}e^{-s^2/2}ds. \tag{B.63}$$

For any $c>0$, the following bound holds:

$$\sqrt{\frac{2}{\pi}}\mathbb{E}\int_{\Lambda^i_{N,n}(0)/\sqrt{2t}}^{\infty}e^{-s^2/2}ds\leq\sqrt{\frac{2}{\pi}}\int_{c/\sqrt{2t}}^{\infty}e^{-s^2/2}ds+\mathbb{P}[\Lambda^i_{N,n}(0)<c]. \tag{B.64}$$

Since we have $\beta_{1:N}=\lambda^{-1}(0,1,\ldots,N-1)$ for any $N\in\mathbb{N}$, it is trivial that

$$\mathbb{P}[\Lambda^i_{N,n}(0)<c]=\mathbb{P}[\beta_{\lfloor iN/n\rfloor}-\beta_{\lfloor (i-1)N/n\rfloor+1}<c]\to 0, \tag{B.65}$$

as $N\to\infty$. From (B.63)–(B.65), it follows that

$$\overline{\lim_{N\to\infty}}\,\mathbb{P}[\Lambda^i_{N,n}(0)=0]\leq\sqrt{\frac{2}{\pi}}\int_{c/\sqrt{2t}}^{\infty}e^{-s^2/2}ds.$$

The preceding inequality holds for any $c>0$, and taking $c\to\infty$ renders $\mathbb{P}[\Lambda^i_{N,n}(0)\leq 0]\to 0$, as $N\to\infty$. From (B.60), (B.61) and this limit, the first statement of the proposition follows.

In view of Remark 3.5 and similarly to (B.42), we have

$$\varphi_{\omega_n(t),n}\circ\psi_{\omega_n(t)}(\mathbf{B}_{1:\omega_n(t)}+\beta_{1:\omega_n(t)})(t)\stackrel{d}{=}W_{1:n}(t). \tag{B.66}$$

Moreover, analogously to (B.43), it can be shown that

$$\varphi_{N,n}\circ\psi_N(\mathbf{B}_{1:N}+\beta_{1:N})(t)\xrightarrow{\mathbb{P}}\varphi_{\omega_n(t),n}\circ\psi_{\omega_n(t)}(\mathbf{B}_{1:\omega_n(t)}+\beta_{1:\omega_n(t)})(t), \tag{B.67}$$

as $N\to\infty$. The second statement of the proposition follows from (B.66) and (B.67).

B.7 Summary of key notations

$X_{1:n}(k)$	Theorem 3.1	$X^{\alpha}_{1:n}(k)$	Section 3.4.1	$D^{\alpha}_i(t)$	Section 3.4.2
$\hat{X}_{1:n}(k)$	Theorem 3.1	$\hat{X}^{\alpha}_{1:n}(k)$	Proposition 3.1	$\hat{D}^{\alpha}_i(t)$	Section 3.4.2
A_i	Section 3.4.1	A^{α}_i	Section 3.4.1	$S^{\alpha}_i(j)$	Section 3.4.2
$d_i(j)$	Section 3.4.1	$d^{\alpha}_i(j)$	Section 3.4.1	$C^{\alpha}_i(t)$	Section 3.4.2
θ_k	Section B.2	θ^{α}_k	Section B.2	$\upsilon^{\alpha}_n(t)$	Section 3.4.2
λ	Section 3.4.1	$Z_{1:n}$	Theorem 3.1	$\hat{\upsilon}^{\alpha}_n(t)$	Section 3.4.2
$\beta_{1:n}$	Lemma 3.4	$W_{1:n}(t)$	Section 3.4.3	$\omega_n(t)$	Section 3.4.3
H_n	Definition 3.1	τ	Definition 3.2	ψ_n	Definition 3.3
ν_N	Definition 3.4	$\phi_{N,n}$	Definition 3.5	$\varphi_{N,n}$	Definition 3.6

APPENDIX C

Appendix for Chapter IV

C.1 Proof of Lemma 4.1

We first provide two technical lemmas. Lemma C.1 is straightforward from the definition of $\mathcal{V}_m^n$.

Lemma C.1. *For $m, n \in \mathbb{N}$, we have*

$$|\mathcal{V}_m^n| = \binom{m+n-2}{n-1}.$$

The following lemma provides a relation between the expected first hitting times for the random walks with different starting points. Let $x_{1:n} \leq y_{1:n}$ stand for $x_i \leq y_i$ for all $1 \leq i \leq n$.

Lemma C.2. *For $x_{1:n}, y_{1:n} \in \mathcal{W}^n$, if $x_{1:n} \leq y_{1:n}$, then*

$$\mathbb{E}[\tau_{0_{1:n}} \,|\, X_{1:n}(0) = x_{1:n}] \leq \mathbb{E}[\tau_{0_{1:n}} \,|\, X_{1:n}(0) = y_{1:n}].$$

Proof. It is sufficient to show that if $x_{1:n} \leq y_{1:n}$ then

$$\mathbb{P}[\tau_{0_{1:n}} \leq s |\, X_{1:n}(0) = x_{1:n}] \geq \mathbb{P}[\tau_{0_{1:n}} \leq s |\, X_{1:n}(0) = y_{1:n}], \tag{C.1}$$

for all $s \geq 0$. The following proof is based on induction. For notational simplicity, let

$$P_{x_{1:n}}(s) \equiv \mathbb{P}[\tau_{0_{1:n}} \leq s |\, X_{1:n}(0) = x_{1:n}].$$

It is trivial that

$$P_{x_{1:n}}(0) = 1_{\{x_{1:n} = 0_{1:n}\}},$$

and, thus, (C.1) is true for $s = 0$. As an inductive hypothesis, suppose that (C.1) holds for any $s \geq 0$ and all $x_{1:n}, y_{1:n} \in \mathcal{W}^n$ satisfying $x_{1:n} \leq y_{1:n}$. For $x_{1:n} \in \mathcal{W}^n$, one obtains from (4.1) that

$$\text{(C.2)} \qquad P_{x_{1:n}}(s+1) = \frac{1}{n+1-\rho} \sum_{i=1}^{n} \Pi^i_{x_{1:n}}(s) + \frac{1-\rho}{n+1-\rho} P_{x_{1:n}+1_{1:n}}(s),$$

where $x_0 \equiv 0$ and

$$\Pi^i_{x_{1:n}}(s) \equiv 1_{\{x_i > x_{i-1}\}} P_{x_{1:n}-e^i_{1:n}}(s) + 1_{\{x_i = x_{i-1}\}} P_{x_{1:n}}(s).$$

Now consider any $x_{1:n}, y_{1:n} \in \mathcal{W}^n$ such that $x_{1:n} \leq y_{1:n}$. It is straightforward that $x_{1:n} + 1_{1:n} \leq y_{1:n} + 1_{1:n}$, and, then, the inductive hypothesis yields

$$\text{(C.3)} \qquad P_{x_{1:n}+1_{1:n}}(s) \geq P_{y_{1:n}+1_{1:n}}(s).$$

Furthermore, if $x_i > x_{i-1}$ and $y_i > y_{i-1}$, then the inductive hypothesis renders

$$\text{(C.4)} \qquad \Pi^i_{x_{1:n}}(s) = P_{x_{1:n}-e^i_{1:n}}(s) \geq P_{y_{1:n}-e^i_{1:n}}(s) = \Pi^i_{y_{1:n}}(s).$$

Similarly, if $x_i = x_{i-1}$ and $y_i = y_{i-1}$, then

$$\text{(C.5)} \qquad \Pi^i_{x_{1:n}}(s) = P_{x_{1:n}}(s) \geq P_{y_{1:n}}(s) = \Pi^i_{y_{1:n}}(s),$$

and if $x_i > x_{i-1}$ and $y_i = y_{i-1}$, then

$$\text{(C.6)} \qquad \Pi^i_{x_{1:n}}(s) = P_{x_{1:n}-e^i_{1:n}}(s) \geq P_{x_{1:n}}(s) \geq P_{y_{1:n}}(s) = \Pi^i_{y_{1:n}}(s).$$

If $x_i = x_{i-1}$ and $y_i > y_{i-1}$, then $y_i > y_{i-1} \geq x_{i-1} = x_i$; this further implies $x_{1:n}+e^i_{1:n} \leq y_{1:n}$. Thus, in this case, the inductive hypothesis results in

$$\text{(C.7)} \qquad \Pi^i_{x_{1:n}}(s) = P_{x_{1:n}}(s) \geq P_{x_{1:n}+e^i_{1:n}}(s) \geq P_{y_{1:n}}(s) = \Pi^i_{y_{1:n}}(s).$$

Combining (C.2)–(C.7) yields

$$P_{x_{1:n}}(s+1) \geq P_{y_{1:n}}(s+1).$$

Finally, by induction, we show that (C.1) holds for all $s \geq 0$, and this completes the proof of the lemma.

Now we present the proof of Lemma 4.1.

Proof of Lemma 4.1. By definition (see (4.3)), it is trivial that

$$\mathbb{E}[\tau_{0_{1:n}} \,|\, X_{1:n}(0) = 0_{1:n}] = 0. \tag{C.8}$$

Let $\gamma_{0_{1:n}}$ be the first return time to the origin, i.e.,

$$\gamma_{0_{1:n}} = \inf\{i \geq 1 : X_{1:n}(i) = 0_{1:n}\};$$

note the difference between $\gamma_{0_{1:n}}$ and $\tau_{0_{1:n}}$ (see (4.3)). By using the first-step analysis (e.g., see [15, p. 65-70]), it follows from (4.1) that

$$\mathbb{E}[\gamma_{0_{1:n}} \,|\, X_{1:n}(0) = 0_{1:n}] = \frac{1-\rho}{n+1-\rho}\mathbb{E}[\tau_{0_{1:n}} \,|\, X_{1:n}(0) = 1_{1:n}] + 1;$$

Moreover, note from (4.2) that (e.g., see [15, p. 104-105])

$$\mathbb{E}[\gamma_{0_{1:n}} \,|\, X_{1:n}(0) = 0_{1:n}] = \pi_{0_{1:n}}^{-1} = \rho^{-n}.$$

The preceding two equalities yield

$$\mathbb{E}[\tau_{0_{1:n}} \,|\, X_{1:n}(0) = 1_{1:n}] = \frac{(n+1-\rho)(1-\rho^n)}{\rho^n(1-\rho)}. \tag{C.9}$$

For notational simplicity, let

$$T_{x_{1:n}} \equiv \mathbb{E}[\tau_{0_{1:n}} \,|\, X_{1:n}(0) = x_{1:n}], \tag{C.10}$$

for $x_{1:n} \in \mathcal{W}^n$; $T_{x_{1:n}} \equiv 0$ if $x_{1:n} \notin \mathcal{W}^n$. For any $x_{1:n} \in \mathcal{W}^n$, one obtains from (4.2) that

$$\text{(C.11)} \quad T_{x_{1:n}} = \frac{1}{n+1-\rho} \sum_{i=1}^{n} 1_{\{x_i > x_{i-1}\}} T_{x_{1:n} - e^i_{1:n}} + \frac{1}{n+1-\rho} \sum_{i=1}^{n} 1_{\{x_i = x_{i-1}\}} T_{x_{1:n}} + \frac{1-\rho}{n+1-\rho} T_{x_{1:n}+1_{1:n}} + 1,$$

where $x_0 \equiv 0$. For notational simplicity, let

$$\begin{aligned}
\Xi^n_m &\equiv \sum_{x_{1:n} \in \mathcal{W}^n_m} T_{x_{1:n}}, \\
\Sigma^n_m &\equiv \sum_{x_{1:n} \in \mathcal{W}^n_m} T_{x_{1:n}+1_{1:n}}, \\
A^n_m(i) &\equiv \sum_{x_{1:n} \in \mathcal{W}^n_m} 1_{\{x_i > x_{i-1}\}} T_{x_{1:n} - e^i_{1:n}}, \\
B^n_m(i) &\equiv \sum_{x_{1:n} \in \mathcal{W}^n_m} 1_{\{x_i = x_{i-1}\}} T_{x_{1:n}},
\end{aligned}$$

where

$$\mathcal{W}^n_m \equiv \{x_{1:n} \in \mathbb{Z}^n : 0 \le x_1 \le x_2 \le \cdots \le x_n = m\}.$$

For $m \ge 0$, by definition, it is straightforward that

$$\text{(C.12)} \qquad |\mathcal{W}^n_m| = |\mathcal{V}^n_{m+1}|;$$

furthermore, we have

$$\text{(C.13)} \qquad \frac{\Sigma^n_m}{|\mathcal{W}^n_m|} = \frac{1}{|\mathcal{V}^n_{m+1}|} \sum_{x_{1:n} \in \mathcal{V}^n_{m+1}} T_{x_{1:n}},$$

which is of interest to us in this lemma. For each $1 \le i \le n-1$, one obtains

$$\Xi^n_m = A^n_m(i) + B^n_m(i+1).$$

Summing $T_{x_{1:n}}$ in (C.11) for all $x_{1:n} \in \mathcal{W}^n_m$ and using the preceding equality render

$$\text{(C.14)} \qquad (2-\rho)\Xi^n_m = A^n_m(n) + B^n_m(0) + (1-\rho)\Sigma^n_m + (n+1-\rho)|\mathcal{W}^n_m|.$$

It is not difficult to see that

$$A^n_m(n) = \Xi^n_{m-1} \quad \text{and} \quad B^n_m(0) = \Xi^n_m - \Sigma^n_{m-1}. \tag{C.15}$$

Combining (C.14) and (C.15) results in

$$\Sigma^n_m - \Xi^n_m = \frac{\Sigma^n_{m-1} - \Xi^n_{m-1}}{1-\rho} - \frac{n+1-\rho}{1-\rho}|\mathcal{W}^n_m|.$$

Since the preceding equality holds for all $m \in \mathbb{N}$, it can be shown that

$$\Sigma^n_m - \Xi^n_m = \frac{T_{1_{1:n}} - T_{0_{1:n}}}{(1-\rho)^m} - \sum_{j=1}^{m} \frac{n+1-\rho}{(1-\rho)^{m+1-j}}|\mathcal{W}^n_j|;$$

note that we used the fact that $\Sigma^n_0 - \Xi^n_0 = T_{1_{1:n}} - T_{0_{1:n}}$. It follows from (C.8), (C.9) and the preceding equality that

$$\begin{aligned}
\Sigma^n_m - \Xi^n_m &= \frac{(n+1-\rho)(1-\rho^n)}{\rho^n(1-\rho)^{m+1}} - \sum_{j=1}^{m} \frac{n+1-\rho}{(1-\rho)^{m+1-j}}|\mathcal{W}^n_j| \\
&= \frac{n+1-\rho}{(1-\rho)^{m+1}}\left(\frac{1}{\rho^n} - \sum_{j=0}^{m}\binom{j+n-1}{n-1}(1-\rho)^j\right) \\
&= \sum_{j=m+1}^{\infty}\binom{j+n-1}{n-1}\frac{n+1-\rho}{(1-\rho)^{m+1-j}};
\end{aligned} \tag{C.16}$$

we used (C.12) and Lemma C.1 in deriving the second equality and the third equality stems from

$$\frac{1}{\rho^n} = \sum_{j=0}^{\infty}\binom{j+n-1}{n-1}(1-\rho)^j. \tag{C.17}$$

Lemma C.2 implies

$$\frac{1}{|\mathcal{W}^n_m|}\Xi^n_m \le \frac{1}{|\mathcal{W}^n_{m-1}|}\Sigma^n_{m-1}. \tag{C.18}$$

From (C.16) and (C.18), we have

$$\frac{\Sigma^n_m}{|\mathcal{W}^n_m|} - \frac{\Sigma^n_{m-1}}{|\mathcal{W}^n_{m-1}|} \le \sum_{j=m+1}^{\infty} \frac{\binom{j+n-1}{n-1}}{|\mathcal{W}^n_m|}\frac{n+1-\rho}{(1-\rho)^{m+1-j}},$$

and, since this holds for all $m \in \mathbb{N}$, it follows that

$$
\begin{aligned}
\frac{\Sigma_m^n}{|\mathcal{W}_m^n|} &\le \sum_{i=1}^{m} \sum_{j=i+1}^{\infty} \frac{\binom{j+n-1}{n-1}}{|\mathcal{W}_i^n|} \frac{n+1-\rho}{(1-\rho)^{i+1-j}} + T_{1_{1:n}} \\
&= \sum_{i=0}^{m} \sum_{j=i+1}^{\infty} \frac{\binom{j+n-1}{n-1}}{\binom{i+n-1}{n-1}} \frac{n+1-\rho}{(1-\rho)^{i+1-j}} \\
&= (n+1-\rho) \sum_{i=0}^{m} \sum_{j=0}^{\infty} \frac{\binom{i+j+n}{n-1}}{\binom{i+n-1}{n-1}} (1-\rho)^j;
\end{aligned} \tag{C.19}
$$

in deriving the first equality, we used (C.12), Lemma C.1 and the fact that (see (C.9) and (C.17))

$$
T_{1_{1:n}} = \sum_{j=1}^{\infty} \binom{j+n-1}{n-1} \frac{n+1-\rho}{(1-\rho)^{1-j}}.
$$

Note that

$$
\begin{aligned}
\frac{\binom{i+j+n}{n-1}}{\binom{i+n-1}{n-1}} &= \prod_{l=0}^{n-2} \frac{i+j+n-l}{i+n-1-l} \\
&= \prod_{l=0}^{n-2} \left(1 + \frac{j+1}{i+n-1-l}\right) \le \left(1 + \frac{j+1}{i+1}\right)^{n-1}.
\end{aligned} \tag{C.20}
$$

From (C.19) and (C.20), it is straightforward that

$$
\begin{aligned}
\frac{\Sigma_m^n}{|\mathcal{W}_m^n|} &\le (n+1-\rho) \sum_{i=0}^{m} \sum_{j=0}^{\infty} \left(1 + \frac{j+1}{i+1}\right)^{n-1} (1-\rho)^j \\
&= (n+1-\rho) \sum_{i=0}^{m} \sum_{l=0}^{n-1} \binom{n-1}{l} \frac{1}{(i+1)^l} M_l(\rho),
\end{aligned} \tag{C.21}
$$

where $M_l(\rho) \equiv \sum_{j=0}^{\infty} (j+1)^l (1-\rho)^j$, which satisfies

$$
M_l(\rho) \le l! \sum_{j=0}^{\infty} \binom{j+l}{l} (1-\rho)^j = \frac{l!}{\rho^{l+1}}; \tag{C.22}
$$

the preceding inequality stems from (C.17). Combining (C.21) and (C.22) yields

$$
\frac{\Sigma_m^n}{|\mathcal{W}_m^n|} \le (n+1-\rho) \sum_{i=0}^{m} \sum_{l=0}^{n-1} \binom{n-1}{l} \frac{1}{(i+1)^l} \frac{l!}{\rho^{l+1}}, \tag{C.23}
$$

and, then, the statement of the lemma follows from (C.13) and (C.23). This completes the proof of the lemma.

C.2 Proof of Lemma 4.2

It is straightforward from (A1) and (A2) that

$$\mathbb{E}\tau_{0_{1:n}} = \sum_{m=1}^{\infty} \frac{1}{|\mathcal{V}_m^n|} \sum_{x_{1:n}\in\mathcal{V}_m^n} T_{x_{1:n}} \mathbb{P}[X_n(0)=m];$$

recall that $T_{x_{1:n}}$ is defined in (C.10). Combining (4.5), the preceding equality and Lemma 4.1 yields

$$\mathbb{E}\xi(\tau_{0_{1:n}}) \le \sum_{m=1}^{\infty}\sum_{i=1}^{m}\sum_{j=1}^{n} \binom{n-1}{j-1} \frac{(j-1)!}{i^{j-1}} \frac{1}{\rho^j} \mathbb{P}[X_n(0)=m]. \tag{C.24}$$

Note that

$$\sum_{i=1}^{m} \frac{1}{i} \le 1 + \int_1^m \frac{1}{x} dx = 1 + \log m. \tag{C.25}$$

Similarly, for $j \ge 3$, we have

$$\sum_{i=1}^{m} \frac{1}{i^{j-1}} \le 1 + \int_1^m \frac{1}{x^{j-1}} dx \le 1 + \int_1^{\infty} \frac{1}{x^{j-1}} dx = 1 + \frac{1}{j-2}. \tag{C.26}$$

From (C.24)–(C.26), it follows that, for $n \ge 2$,

$$\mathbb{E}\xi(\tau_{0_{1:n}}) \le \frac{1}{\rho}\mathbb{E}X_n(0) + \frac{n-1}{\rho^2}(1+\mathbb{E}\log X_n(0)) + \sum_{j=3}^{n} \binom{n-1}{j-1} (j-1)! \frac{j-1}{j-2}\frac{1}{\rho^j}.$$

For $n=2$, one obtains

$$\frac{\rho^2}{\log\rho^{-1}}\mathbb{E}\xi(\tau_{0_{1:2}}) \le \frac{\rho}{\log\rho^{-1}}\mathbb{E}X_2(0) + \frac{1}{\log\rho^{-1}}(1+\log\mathbb{E}X_2(0)); \tag{C.27}$$

note that we used the fact that $\mathbb{E}\log X_n(0) \le \log\mathbb{E}X_n(0)$, which follows from Jensen's inequality. Similarly, for $n \ge 3$, it can be shown that

$$\begin{aligned} \rho^n\mathbb{E}\xi(\tau_{0_{1:n}}) \le \rho^{n-1}\mathbb{E}X_n(0) &+ (n-1)\rho^{n-2}(1+\log\mathbb{E}X_n(0)) \\ &+ \sum_{j=3}^{n} \binom{n-1}{j-1} (j-1)! \frac{j-1}{j-2}\rho^{n-j}. \end{aligned} \tag{C.28}$$

Under the assumption of the lemma (see (A3)), we have

$$\lim_{\rho\downarrow 0} \rho\mathbb{E}X_n(0) = c \quad \text{and} \quad \lim_{\rho\downarrow 0} \frac{\log \mathbb{E}X_n(0)}{\log \rho^{-1}} = 1.$$

The statement of the lemma follows from (C.27), (C.28) and the preceding two limits, and this completes the proof of the lemma.

C.3 Proof of Lemma 4.3

Let Q_i, $-m+1 \le i \le 0$, be the number of packets at node i when packet 0 departs from node 0. Assume that the system starts in stationarity. By Burke's theorem and the PASTA property, the system is in stationarity when packet 0 departs from node 0 (e.g., see [9, Section 3.7]); note that the departure process from node 0 is Poisson. Thus, it follows that (e.g., see [11, p. 146])

$$\begin{aligned} \mathbb{P}[Q_{-m+1:0} = q_{-m+1:0}] &= \prod_{i=-m+1}^{0} (1-\lambda_k)\lambda_k^{q_i} \\ &= \lambda_k^{\sum_{i=-m+1}^{0} q_i} (1-\lambda_k)^m, \end{aligned} \tag{C.29}$$

for $q_i \ge 0$, $-m+1 \le i \le 0$.

Now consider some $x_{1:b+1} \in \mathcal{W}^{b+1}$. If $x_{b+1} \le m-1$, then

$$\mathbb{P}[L^m_{1:b+1} = -x_{1:b+1}] = \mathbb{P}[Q_{-x_{b+1}} \ge r_{-x_{b+1}}, \, Q_{-x_{b+1}+1:0} = r_{-x_{b+1}+1:0}],$$

where r_i, $-m+1 \le i \le 0$, are given by

$$r_i = \sum_{j=1}^{b+1} 1\{-x_j = i\}; \tag{C.30}$$

using (C.29) further results in

$$\begin{aligned} \mathbb{P}[L^m_{1:b+1} = -x_{1:b+1}] &= \lambda_k^{r_{-x_{b+1}}} \prod_{i=-x_{b+1}+1}^{0} (1-\lambda_k)\lambda_k^{r_i} \\ &= \lambda_k^{\sum_{i=-x_{b+1}}^{0} r_i} (1-\lambda_k)^{x_{b+1}}. \end{aligned} \tag{C.31}$$

If $x_{b+1} = m$, then

$$\begin{aligned} \mathbb{P}[L^m_{1:b+1} = -x_{1:b+1}] &= \mathbb{P}[Q_{-m+1:0} = r_{-m+1:0}] \\ &= \lambda_k^{\sum_{i=-m+1}^{0} r_i} (1-\lambda_k)^m. \end{aligned} \tag{C.32}$$

From (C.30), we have

$$\sum_{i=-x_{b+1}}^{0} r_i = \sum_{i=-m+1}^{0} r_i = \sum_{i=1}^{b+1} 1_{\{x_i \le m-1\}}.$$

Thus, the statement of the lemma follows from (C.31) and (C.32) and the preceding equality.

C.4 Proof of Lemma 4.4

Consider two identical k-node linear networks, labeled by 1 and 2, respectively. Let $\mathscr{A}_i \equiv \{A_{i,j}\}_{j\in\mathbb{N}}$, $i = 1, 2$, be the increasing sequence of arrival times of packets at node 1 in network i. Suppose that $\mathscr{A}_2$ is Poisson with rate λ and $\mathscr{A}_1$ is generated by thinning $\mathscr{A}_2$ such that each arrival is selected randomly with probability λ'/λ, independently of the others. Then the thinned process $\mathscr{A}_2$ is also Poisson with rate λ' (e.g., see [25, Section 2.3]). Under this setup, it is straightforward that

$$\mathscr{A}_1 \subseteq \mathscr{A}_2. \tag{C.33}$$

Suppose that the system starts with empty nodes. Using [29, Lemma 1], it can be obtained from (C.33) recursively for all $1 \le l \le k$ that

$$\{D^l_{1,j}\}_{j\in\mathbb{N}} \subseteq \{D^l_{2,j}\}_{j\in\mathbb{N}}, \tag{C.34}$$

where $\{D^l_{i,j}\}_{j\in\mathbb{N}}$, $i = 1, 2$, is the increasing sequence of departure times of packets at node l in network i. The the statement of the lemma simply follows from (C.34).

C.5 Proof of Lemma 4.5

Under Approximation 4.1, it is straightforward that the process $\{T_0(n),\, n \geq 0\}$ is Poisson with rate $1-\rho$ (see (4.9)). By definition, the processes $\{T_i(n),\, n \geq 0\}$, $1 \leq i \leq b+1$, are independent and Poisson with unit rate (see (4.11)); they are also independent of the process $\{T_0(n),\, n \geq 0\}$ since the sequences $\{W_0(i)\}_{i \in \mathbb{N}}$ and $\{S_i(j)\}_{1 \leq i \leq b+1,\, j \in \mathbb{N}}$ are independent; this follows from Burke's theorem. Note that the process $\{T(n),\, n \geq 0\}$ is the superposition of these independent processes. From (4.8)–(4.12) and by the properties of superposed Poisson processes (e.g., see [15, p. 327-329]), the discrete-time process $\{Y^m_{1:b+1}(i),\, i \geq 0\}$ can be shown to behave as the random walk introduced in Section 4.3.1, given that $L^m_{b+1} \geq -m+1$. Thus, from (4.13) and (4.18), we have

$$\tilde{r}_k = 1 - \lim_{m \to \infty} \mathbb{P}[\mathcal{E}^m_k \,|\, X_{1:b+1}(0) = Y^m_{1:b+1}(0)], \tag{C.35}$$

where

$$\mathcal{E}^m_k \equiv \{\xi(\tau_{0_{1:n}}) \leq k-1,\, X_{b+1}(0) \leq m\};$$

note that $Y^m_{b+1}(0) = 1_{1:b+1} - L^m_{1:b+1}$ (see (4.12)). Then, it follows from Corollary 4.1 that

$$\mathbb{P}[\mathcal{E}^m_k \,|\, X_{1:b+1}(0) = Y^m_{1:b+1}(0)] = \mathbb{P}[\mathcal{E}^m_k \,|\, X_{1:b+1}(0) = 1_{1:b+1} - L^\infty_{1:b+1}]. \tag{C.36}$$

It is not difficult to show that

$$\lim_{m \to \infty} \mathbb{P}[\mathcal{E}^m_k \,|\, X_{b+1}(0) = 1_{1:b+1} - L^\infty_{1:b+1}] = \mathbb{P}[\xi(\tau_{0_{1:n}}) \leq k-1 \,|\, X_{b+1}(0) = 1_{1:b+1} - L^\infty_{1:b+1}]. \tag{C.37}$$

Combining (C.35)–(C.37) and using Markov's inequality render the statement of the lemma.

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