

Hiring and Firing with Labor Market Frictions

by

Tim A. Maull

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Doctoral Committee:

Associate Professor Jussi S. Keppo, Chair
Professor Joseph G. Conlon
Professor Romesh Saigal
Lecturer David L. Kaufman

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List of Symbols

- $X_i^{\hat{\pi}}$ The amount of workers hired or fired at i 'th hiring/firing time in Z, L space, page 8
- δ The constant proportional quit rate for workers, page 7
- Δ Delay from firing, page 7
- η The ratio of labor to demand, page 13
- η^* The optimum ratio of labor to demand given no frictions, page 11
- η_{Δ} The value of η at time Δ at the end of the firing delay, page 14
- K_F Firm-sized firing cost, page 7
- K_H Firm-sized hiring cost, page 7
- λ Firm's market power, page 6
- \mathbb{R}^+ The set of non-negative real numbers.
- \mathbb{R}^{++} The set of positive real numbers.
- \mathbb{Z}^+ The set of non-negative integers.
- \mathcal{A} The infinitesimal generator , see equation (2.16), page 15
- μ The demand process's drift, page 6

- ω The wage paid to workers, page 7
- Π The set of admissible policies in η -space, page 13
- ρ The discount rate used under P^Z measure in η -space, page 13
- σ The demand process's volatility, page 6
- $\hat{\Pi}$ The class of admissible policies in Z, L space, page 9
- $\hat{\pi}$ A hiring and firing control policy in Z, L space, being a collection of $\{t_i^{\hat{\pi}}, s_i^{\hat{\pi}}, X_i^{\hat{\pi}}\}$, page 8
- A Productivity of its workforce, page 6
- c_F Proportional firing cost, page 7
- c_H Proportional hiring cost, page 7
- f A guess for J , the value function in η -space, page 14
- $I_{\{\}}$ The indicator function, 1 if the statement is true, zero if it is not.
- $J(\cdot)$ The value function in η -space, page 13
- L_t Number of workers at time t , page 6
- m The ratio of the change in labor to demand in η -space, page 13
- $M_F f(\eta)$ The expected discounted firing value given the continuing value is f , page 14
- $M_H f(\eta)$ The instantaneous hiring value given the continuing value is f , page 14
- P The risk-neutral probability measure.
- P^Z An equivalent probability measure used to reduce the dimensionality of V , page 12

- P_t Price goods are sold at by the firm at time t , see equation (2.1), page 6
- Q_t Quantity sold by the firm at time t , see equation (2.1), page 6
- r The continuously discounted at rate for future cash flows, page 7
- r The risk-free rate.
- $u(\eta, t)$ The discounted flow profit, page 13
- $V(Z, L)$ The value function in Z, L space.
- $V_{\hat{\pi}}(Z, L)$ The discounted expected value of the firm given policy $\hat{\pi}$, page 10
- $V_{NF}(Z_0)$ The expected firm value with no labor market frictions, page 11
- $V_{UNCON}(Z, L)$ The uncontrolled firm value, see equation (2.17), page 16
- W_t A standard Wiener process under the P measure.
- W_t The demand process's Wiener Process under P -measure, page 6
- x^+ The max of x and zero.
- x^- The min of x and zero.
- Z_t Position on the direct demand curve at time t , follows GBM, see equation (2.2), page 6
- $s_i^{\hat{\pi}}$ A sequence of the binary variable that expresses whether the decision is to hire or to fire in Z, L space, page 8
- $t_i^{\hat{\pi}}$ An increasing sequence of hiring and firing times in Z, L space, page 8

CHAPTER I

Introduction

Firms need to be able to respond to changes in consumer demand. Adjusting output by changing the number of workers in the firm is one response. Hiring and firing workers does not happen instantly and without cost. There are substantial hiring and training costs for new workers, as well as severance and layoff costs for firing workers. Regulatory constraints may create delays ranging from a few days to almost a year if the firm decides to fire workers (see Figure 1.1). This paper models the firm's optimal workforce decisions in response to changing consumer demand for the firm's product when there are costs and delays in changing its workforce.

I use stochastic control to model the firm's decisions. The objective is to maximize the firm's present value of its expected cash flows. By changing its workforce, the firm directly influences expected cash flows. The firm takes the process of consumer demand as given when calculating cash flows and follows only one variable, which is the fraction of labor to demand. In my model workers are viewed as inventory with holding costs. Costs are modelled based on the size of the firm, the number of workers changed, and the delay in firing workers. The optimal policy is to control the ratio of labor to demand, inside a pair of control barriers. Value function parts are solved analytically, while parameters and the barriers are solved numerically by using policy improvement of dynamic programming. Additionally, I show that the optimal policy

exists and is unique. Comparative statics results are then added.

To manage the risk of lost demand, the firm may keep more workers to hedge against upturns in future demand. The firm may alternately hold fewer workers to hedge against downturns in demand, especially when laying off workers is costly and time consuming. Using inventory control methods, I find the optimal strategy is a two-sided (s, S) stochastic control policy.

I first derive analytical solutions for the value of the company as a function of the fraction of labor to demand and then solve numerically for the parameters of the value functions. I also analyze how labor market frictions can impact a firm's employment decisions, profit and labor levels. The firing delay encourages firms to fire earlier. The firing delay also causes fewer workers to be fired. Raising the proportional costs of changing workers, lowers the chance that workers will be hired or fired, and also reduces the labor and profit levels. I introduce "firm-sized" hiring and firing costs that are proportional to the number of employees. This cost is different because of regulations that apply only to large companies. Firm-sized firing cost postpones a firm's firing, but raises the firing quantity. Similarly, increasing the firm-sized hiring cost reduces the chance of hiring, but raises the amount hired. The volatility of demand for the firm's good has mixed effects on the firm's labor market decisions. For low volatility, its increase reduces the chance of firing. But for large volatility values, the chance of firing increases. Over all levels of volatility, increasing it decreases the chance of hiring.

My model examines Ford Motor Company's hiring and firing policies and estimates Ford's market value. Ford has substantial firing costs and delays because of union contracts. I show the impact of these delays on Ford's market value and employment policies by comparing equal percentage changes in model parameters. My model accurately predicts the fall in Ford's employment level from 2006 to 2009. My model also accurately estimates Ford's gross value. Productivity and wages have the largest

impact on Ford's equity value, followed by the interest rate, and then the consumer demand parameters. Changing firing frictions has a negligible impact on equity value. I also give some advice about what Ford should do in union negotiations to increase its equity value.

1.1 Labor Market Discussion

Labor turnover occurs to a large segment of the U.S. economy. Davis et al. (2006) find that the job creation and destruction rates for the private sector average about 8 percent per quarter. Their analysis suggests the lumpy nature of the labor adjustments with about two-thirds of job creation and destruction occurring at places where labor changes by more than 10 percent per quarter. The lumpy nature of workforce changes is explained by additional monetary costs, above the cost proportional to the labor change. A regulatory delay involved in firing can also explain lumpy firings.

The relationship between labor policy and employment is difficult to determine. Many studies use multivariate regressions of policy parameters on employment levels over various countries (e.g., Nickell 1997). Some papers have modelled firm or industry level responses to policies and they extend those results to the macro-level economy. This method is only a partial equilibrium approach, while the data used to calibrate the models are often more applicable to general equilibrium models. Even though these papers generate interesting theoretical predictions, those predictions are difficult to validate across countries and over different time periods. Kramarz and Michaud (2010) look at French employment data and find that the firing and hiring costs each have a fixed and proportional component. The firing costs are much higher than the hiring costs. They also show that the costs depend on the size of the firm (50+ versus 49- employees). Bentolila and Saint-Paul (1994) present this type of partial equilibrium model with proportional hiring and firing costs and demand shocks. They show that the costs have a positive and negative effect on average

steady-state employment level depending on the magnitude of the costs. Bertola (1992) models a firm in a deterministic continuous time framework and shows that the effect of hiring and firing costs on employment level will depend on the size of the costs, and the interest rate and attrition rates. Shepp and Shiryaev (1996) study the employment level of a research firm given the size of the workforce and the firm capital. Bentolila and Bertola (1990) model the firm in a continuous time stochastic demand framework. Their model uses proportional hiring and firing costs. A similar model is Chen et al. (2003) which incorporates macroeconomic shocks in addition to random demand for the firm.

My work adds to the modelling of firm behavior by incorporating firing delays and firm-sized costs. Firm-sized costs extend the usual stochastic control applications that assume costs that are proportional, to the change of the state variable and/or that the costs are fixed. The firm-sized costs are proportional to the level of the state variable before it is changed (in my paper the labor quantity). Examples of these costs include: labor market regulations based on the number of employees (as in my paper), searching for a specific item in a large inventory, performing inventory counts, capital levies and wealth taxes. Firing delays are motivated and necessitated by union contracts (e.g. Ford Motor Company) and government regulation. In certain European countries, the firing delay may be almost a year (see e.g. OECD 2004). Length of the delay depends upon the country, time employed in the position, a firing categorization of individual or collective, and if the employee is classified as permanent or temporary. Figure 1.1 shows the by law minimum regulatory delay length for various OECD countries. Since this delay is substantial in some countries, it is important to analyze its effects on both the firm's hiring and firing actions and on the firm's value. Delays have been modelled in similarly in other contexts such as bank recapitalization (e.g. Peura and Keppo 2006), power generation (e.g. Blankenship and Menaldi 1984), real options (e.g. Alvarez and Keppo 2002), and in a theoretical

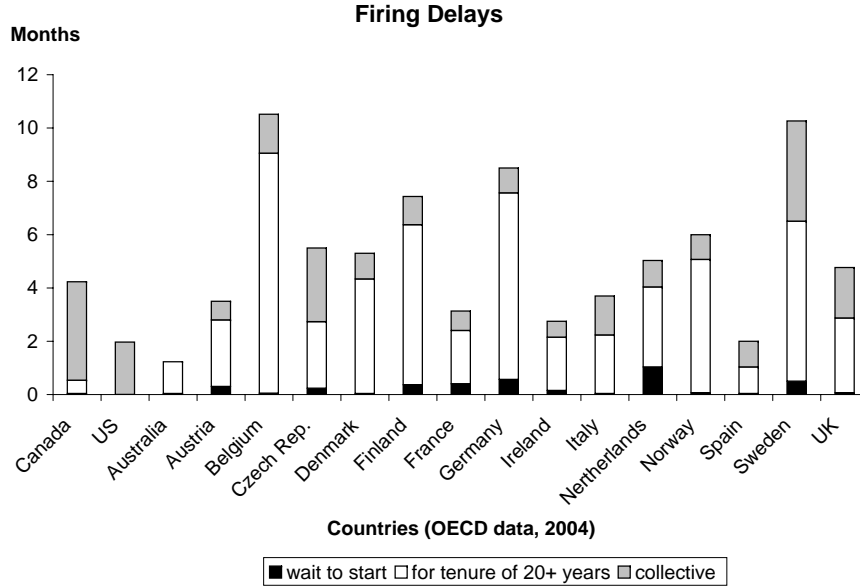


Figure 1.1: **Regulatory delay length for three types of dismissals in various OECD countries (OECD 2004).**

stochastic control framework (e.g. Robin 1977).

The rest of the paper is organized as follows. In Chapter II the firm’s problem is reduced to one control variable – the ratio of labor to demand – by using the homogeneity of the value function and the change of probability measure which is a common tool in the pricing of financial derivative instruments. Properties of the optimal policy are characterized in Chapter 2.1, and then I solve the functional form of the value function by smooth pasting and value matching in the spirit of Dixit (1991) in Chapter III. Chapter IV shows that the solution to the model exists and is unique. I also show that the numerical algorithm converges to the solution. The impacts of firm-sized costs and delay on the labor market decisions through comparative statics is explored in Chapter 5.1. My model is calibrated to Ford Motor Company in Chapter 6.1 and shows the effects of reducing the labor market frictions. Chapter VII concludes. In the Appendix I verify my main result and prove various statements made in the text.

CHAPTER II

Model

I consider a firm that faces a random demand for its product, as in Bentolila and Bertola (1990). It can only control its labor level and has a linear production technology that uses homogeneous labor as a factor of production. The firm faces a constant-elasticity demand function which is given by

$$Q_t = Z_t P_t^{\frac{1}{\lambda-1}} \quad (2.1)$$

where P_t and Q_t are the price and quantity sold by the firm at time t .

Its output is limited by the productivity of its workforce, A , and the number of workers, L_t , at a given time t : $Q_t \leq AL_t$. It has market power in the price it can charge for its product, and this market power is measured by λ which is the inverse of the markup factor. Z_t is the position on the direct demand curve, and it follows

$$dZ_t = \mu Z_t dt + \sigma Z_t dW_t \quad (2.2)$$

where the parameters μ and σ are positive, and $\{W_t : t \geq 0\}$ is a standard Wiener process¹. Demand follows a geometric Brownian motion process and therefore $Z_t =$

¹This Wiener process is on a probability space (Ω, \mathcal{F}, P) along with the standard filtration $\{\mathcal{F}_t : t \geq 0\}$. Here Ω is a set, \mathcal{F} is a σ -algebra, P is a probability measure on \mathcal{F} , and $(\mathcal{F}_t)_{0 \leq t}$ is an increasing family of σ -algebras.

$Z_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t}$. Because the marginal revenue from a constant elasticity demand function is always positive, equation (2.1) implies that the revenues are given by $Q_t P_t = Z_t^{1-\lambda} (AL_t)^\lambda$ at time t . Any future cash flows are continuously discounted at rate r .

The firm pays a wage, ω , to its workers. Workers leave the company with a constant proportional rate δ . In this case the company does not have to pay for them to leave and they leave instantly.

When the company hires new employees it pays a firm-size cost, K_H , that is proportional to the current labor size and a proportional cost c_H that is proportional to the number of employees hired. Similarly, in the case of firing it faces costs K_F and c_F . Union contracts and government regulation cause a firing delay Δ , i.e., the firm has to wait before the firing happens and would pay firm-size cost K_F at the end of the firing delay. This company controls the number of employees through hiring and firing. The costs and the firing delay are labelled as *frictions*. I assume that $K_H + K_F \geq 0$ and $c_H + c_F \geq 0$. Even though some of the costs could be negative, perhaps due to the government subsidies, there is no possibility for immediate positive cash flows by hiring and firing at the same time. The K_H and K_F costs depend on the firm size in my model. This is a linear approximation of firm size costs of hiring and/or firing workers. Many labor regulations depend on firm size (e.g. Table 2.A.9 in OECD, 1999). An example would be the definition of collective dismissal which triggers various regulations. The collective dismissal designation depends on the size of the firm and the number fired in several OECD countries.² Costs depending on firm size has also been brought up in the labor-economics literature. Lloyd (1999) cites French firing regulations that depend on the number of employees in the firm. When comparing U.S. and Portuguese labor market flows, Blanchard and Portugal (2001) make firm-size adjusted comparisons. This is justified since more volatile

² These countries are Austria, Belgium, Denmark, Finland, Germany, Greece, Korea, Poland, Portugal, Spain, and Switzerland.

employment occurs in small firms, of which the Portuguese economy has a larger share than other OECD countries

A hiring and firing control policy $\hat{\pi}$ is a collection $\{t_i^{\hat{\pi}}, s_i^{\hat{\pi}}, X_i^{\hat{\pi}}\}$ where $t_i^{\hat{\pi}}$ is an increasing sequence of hiring and firing times, $s_i^{\hat{\pi}}$ is a sequence of the binary variable that expresses whether the decision is to hire ($s_i^{\hat{\pi}} = H$) or to fire ($s_i^{\hat{\pi}} = F$), $X_i^{\hat{\pi}}$ is the amount of workers hired or fired at i 'th hiring/firing time. This set up of the control policy follows Peura and Keppo (2006). I allow for the firm to hire at the end of the firing delay, and do not count it as an additional hiring time. I assume that each $t_i^{\hat{\pi}}$ is a stopping time of the filtration \mathcal{F}_t . The binary variable $s_i^{\hat{\pi}}$ is $\mathcal{F}_{t_i^{\hat{\pi}}-}$ measurable. Further, if at time $t_i^{\hat{\pi}}$ the company decides to hire then $X_i^{\hat{\pi}}$ is also measurable with respect to $\mathcal{F}_{t_i^{\hat{\pi}}-}$ measurable, and if it decides to fire then $X_i^{\hat{\pi}}$ is measurable with respect to $\mathcal{F}_{(t_i^{\hat{\pi}}+\Delta)-}$. The firm is restricted to remaining in business; it cannot reduce its workforce to zero. It may fire an amount $X_i < L_{t_i^{\hat{\pi}}+\Delta-}$. X_i is unbounded below, since negative X represents hiring at the end of the firing delay, where the firm incurs both the firm-size costs of firing and hiring and the proportional hiring cost. For hiring process alone $X_i \geq 0$. Admissible controls must satisfy

$$t_{i+1}^{\hat{\pi}} - t_i^{\hat{\pi}} > \Delta \text{ if } s_i^{\hat{\pi}} = F \text{ for all } i \geq 1. \quad (2.3)$$

As mentioned earlier, the positive parameter Δ is the delay associated with the firing. When a firing process is started at time $t_i^{\hat{\pi}}$, the new workers leave at time $t_i^{\hat{\pi}} + \Delta$. This delay corresponds to the laws and work agreements in the region where the company operates. The measurability of X_i with respect to $\mathcal{F}_{t_i^{\hat{\pi}}+\Delta-}$ means that the company may decide on the actual amount of people to be fired at time $t_i^{\hat{\pi}} + \Delta$ based on all then available information, i.e., they do not need to precommit to any amount of people at time t_i when they start the firing process. The firm can signal that a large number of workers will be laid off when announcing the firing process, but then change its

decision to any smaller level of layoffs at the end of the firing process. Condition (2.3) states that a new hiring or firing process may not be started while a previous one is still waiting to be completed. These conditions have important technical merit and practical justification; ruling out hiring while a firing process is under way is likely to make the ongoing process successful. Thus, the constraint is defined as (2.3) a restriction set by the regulations, work agreements, and job markets. The class of admissible policies satisfying the restrictions in (2.3) along with conditions on X stated above, $s_i \in \{H, F\}$, and $t_1 \geq 0$ is denoted by $\hat{\Pi}$. The initial decision is defined as $t_0 = 0$, $s_0 = H$, $X_0 = 0$ for mathematical convenience later, but with no hiring costs. Figure 2.1 illustrates a path of (labor, demand). The firm decides to have a labor policy which tries to keep the ratio of labor to demand inside a certain range. This policy generates the two solid rays coming out of the origin where the firm hires or starts to fire. The firm's fire-down-to and hire-up-to policies have similar goals; each tries to correct a large deviation of the ratio of labor to demand by adjusting the ratio to an improved level. Large deviations occur when the labor demand hits one of the solid rays out of the origin, and the firm's labor policy corrects these by having the labor level move to a point on the dashed rays. The firm controls only labor so the path can jump vertically only in response to the firm. Hiring moves the path up vertically and firing causes the path to jump down at the end of the delay period. As indicated in Chapter III, the optimal policy is one where the firm tries to keep its labor between the two solid rays and adjusts to the dashed rays.

Given the current labor L and the current demand Z , the value of the company

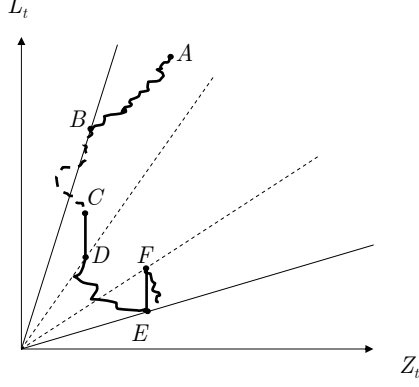


Figure 2.1: **The evolution of L, Z .** Point A is the initial point L_0, Z_0 . As labor decays and demand trends downwards, the path hits the firing trigger ray at point B and the first process is started at this time, t_1 . The firm does not select the number of workers to be fired until the end of the firing process at point C at time $(t_1 + \Delta)^-$. The firm then fires enough workers to have $L_{t_1+\Delta}$ hit the firing target ray at point D . The path continues to evolve this time with an increasing demand trend, which leads the firm to hire when the path hits the hiring trigger ray at point E . Hiring is instantaneous and the labor process increases to point F . At F the L, Z process continues to evolve.

under policy $\hat{\pi}$ is the expected net present value

$$\begin{aligned}
V_{\hat{\pi}}(Z, L) = & E \left[\int_0^{\infty} e^{-rt} [(Z_t^{1-\lambda}(AL_t)^\lambda - \omega L_t) dt] \right. \\
& - \sum_i e^{-rt_i^{\hat{\pi}}} I_{\{s_i^{\hat{\pi}}=H\}} \left(X_i^{\hat{\pi}} I_{\{X_i^{\hat{\pi}} \geq 0\}} c_H + K_H L_{t_i^{\hat{\pi}}} \right) - \sum_i e^{-r(t_i^{\hat{\pi}}+\Delta)} I_{\{s_i^{\hat{\pi}}=F\}} \\
& \left. \left([X_i^{\hat{\pi}}]^+ c_F - [X_i^{\hat{\pi}}]^- c_H + L_{t_i^{\hat{\pi}}+\Delta} (K_F + I_{\{X_i^{\hat{\pi}} < 0\}} K_H) \right) \right] \tag{2.4}
\end{aligned}$$

where $I_{\{\cdot\}}$ is the indicator function of the event in the braces, $x^+ = \max(0, x)$, $x^- = \min(0, x)$,

$$dL_t = -\delta L_t dt, L_{t_i^{\hat{\pi}}+\Delta_i} = L_{(t_i^{\hat{\pi}}+\Delta_i)^-} + X_i^{\hat{\pi}} \left(I_{\{s_i^{\hat{\pi}}=H\}} - I_{\{s_i^{\hat{\pi}}=F\}} \right), \Delta_i = \begin{cases} 0, & \text{if } s_i^{\hat{\pi}} = H \\ \Delta, & \text{if } s_i^{\hat{\pi}} = F \end{cases}$$

Equation (2.4) implies that the value of the firm is equal to the expected discounted revenues minus the expected discounted labor, hiring, and firing costs. The value

depends on both the current demand and labor. The expectation can be viewed as the risk-neutral expectation (e.g. Björk 2004) and r as the risk-free rate. In this case (2.4) is the risk-neutral pricing equation for the firm's cash flows and W_t in (2.2) is a standard Wiener process under the risk-neutral probability, i.e., P is the risk-neutral probability measure.

The objective is to find an admissible strategy that maximizes the net present value. That is,

$$V(Z, L) = \sup_{\hat{\pi} \in \hat{\Pi}} V_{\hat{\pi}}(Z, L) \quad (2.5)$$

The function $V(Z, L)$ is called the *value function*. A sufficient condition for $V(Z, L) < \infty$ is that $r > \mu$. An upper bound of this is the value of the firm with no frictions:

Lemma II.1. (*Expected firm value with no frictions*) *Given initial demand level Z_0 and assuming $\mu < r$, the expected firm value with no labor market frictions is*

$$V_{NF}(Z_0) = \frac{(A^\lambda \eta^{*\lambda} - \omega \eta^*) Z_0}{r - \mu} \quad (2.6)$$

where $\eta^* = (A^\lambda \lambda / \omega)^{1/(1-\lambda)}$.

Value V_{NF} is independent of the initial labor level because the labor can be adjusted without frictions. Equation (2.6) is obtained by maximizing the flow profit that equals $Z_t^{1-\lambda} (A L_t)^\lambda - \omega L_t$ at time t . First order condition gives the optimal labor: $L_t^* = (A^\lambda \lambda / \omega)^{1/(1-\lambda)} Z_t$, which is equivalent to setting the firm's marginal revenue product of labor ($MRPL = \lambda A^\lambda (\frac{Z}{L})^{1-\lambda}$) equal to the wage. Constant η^* in (2.6) is the optimal ratio of labor to demand, L^*/Z , and the same ratio is used in the general case with frictions. Clearly, $V_{NF}(Z_0)$ is an upper bound for the value of the firm in the case with labor market frictions.

I use the homogeneity of value function V and a change of probability measure to reduce the dimensionality of V . An equivalent probability measure P^Z as follows $\frac{dP^Z}{dP} = M_t$ on \mathcal{F}_t , where the Radon-Nikodym derivative $M_t = e^{-\frac{1}{2}\sigma^2 t + \sigma W_t}$. This gives

$Z_t = Z_0 e^{\mu t} M_t$, i.e., the uncertainty in the demand Z_t is from M_t and therefore I call M_t as the demand's risk factor. Note that for an \mathcal{F}_t -measurable random variable Y_t I have $E^{P^Z} \left[\frac{Y_t}{M_t} \right] = E \left[M_t \frac{Y_t}{M_t} \right] = E[Y_t]$, where $t > 0$. If the dynamics of Y under P follows $dY_t = \mu_Y Y_t dt + \sigma_Y Y_t dW_t$ then the process under P^Z is $dY_t = (\mu_Y + \sigma \sigma_Y) Y_t dt + \sigma_Y Y_t d\bar{W}_t$ where \bar{W} is a Wiener process under P^Z .³ Equation (2.4) gives:

$$\begin{aligned}
V_{\hat{\pi}}(Z, L) &= E \left[\int_0^{\infty} e^{-rt} Z_t \left(\left(A \frac{L_t}{Z_t} \right)^\lambda - \omega \frac{L_t}{Z_t} \right) dt \right. \\
&\quad - \sum_i e^{-rt_i^{\hat{\pi}}} Z_{t_i^{\hat{\pi}}} I_{\{s_i^{\hat{\pi}}=H\}} \left(\frac{X_i^{\hat{\pi}} I_{\{X_i^{\hat{\pi}} \geq 0\}}}{Z_{t_i^{\hat{\pi}}}} c_H + \frac{L_{t_i^{\hat{\pi}}}}{Z_{t_i^{\hat{\pi}}}} K_H \right) \\
&\quad - \sum_i e^{-r(t_i^{\hat{\pi}} + \Delta)} Z_{t_i^{\hat{\pi}} + \Delta} I_{\{s_i^{\hat{\pi}}=F\}} \left(\frac{[X_i^{\hat{\pi}}]^+ c_F - [X_i^{\hat{\pi}}]^- c_H}{Z_{t_i^{\hat{\pi}} + \Delta}} \right. \\
&\quad \left. \left. + \frac{L_{t_i^{\hat{\pi}} + \Delta}}{Z_{t_i^{\hat{\pi}} + \Delta}} \left(K_F + I_{\{X_i^{\hat{\pi}} < 0\}} K_H \right) \right) \right] \tag{2.7}
\end{aligned}$$

where labor L_t and change in labor X_t are divided by the demand Z_t . Define ratios $\eta = L/Z$ and $m = X/Z$. Let π be a hiring and firing policy in η -space and $\hat{\pi} = \{t_i^\pi, s_i^\pi, m_i^\pi\}_{i=1}^\infty$ is defined similarly as a policy $\hat{\pi}$ in (L, Z) space, except for $m_i^\pi = X_i^{\hat{\pi}}/Z_{t_i^{\hat{\pi}} + \Delta}$. I denote the set of admissible policies in the η -space by Π and it is the same as $\hat{\Pi}$ but with m_i^π for $\hat{\pi} \in \hat{\Pi}$. The firm-sized costs become linear in η and this allows us to reduce the dimensionality.

Proposition 1. (Normalized labor problem) The value function can be written as

$$V(Z, L) = ZJ(\eta), \tag{2.8}$$

where $J(\eta) = \sup_{\pi \in \Pi} J_\pi(\eta)$,

$$\begin{aligned}
J_\pi(\eta) &= E^{P^Z} \left[\int_0^{\infty} u(\eta_t, t) dt - \sum_i e^{-\rho t_i^\pi} I_{\{s_i^\pi=H\}} \left(m_i^\pi I_{\{m_i^\pi \geq 0\}} c_H + K_H \eta_{t_i^\pi} \right) \right. \\
&\quad \left. - \sum_i e^{-\rho(t_i^\pi + \Delta)} I_{\{s_i^\pi=F\}} \left([m_i^\pi]^+ c_F - [m_i^\pi]^- c_H + \eta_{t_i^\pi + \Delta} \left(K_F + I_{\{m_i^\pi < 0\}} K_H \right) \right) \right] \tag{2.9}
\end{aligned}$$

³See e.g. Björk (2004) for more details of this technique. The use of change of measure in a labor context is rare, but Prat (2006) also uses this method in a labor search-matching model.

$u(\eta, t) = e^{-\rho t}u(\eta)$, $u(\eta) = (A\eta)^\lambda - \omega\eta$, $\rho = r - \mu$, and the P^Z -dynamics

$$d\eta_t = -(\delta + \mu)\eta_t dt - \sigma\eta_t d\bar{W}_t, \quad \eta_{t_i^\pi + \Delta_i} = \eta_{(t_i^\pi + \Delta_i)^-} + (I_{\{s_i^\pi = H\}} - I_{\{s_i^\pi = F\}})m_i^\pi. \quad (2.10)$$

Note that $MRPL = \lambda A^\lambda \frac{1}{\eta^{1-\lambda}}$, i.e., MRPL is a decreasing function of η . When η is high then labor L is high or demand Z is low. Thus, when η is high (MRPL is low) the company fires and when η is low (MRPL is high) it hires. Because $Z > 0$ and it is independent of π , the optimization of $J_\pi(\eta)$ is the same as the optimization of $V_{\bar{\pi}}(Z, L)$.

My objective is to identify $J(\cdot)$ and create a policy which achieves this value. The model has eleven parameters in total: μ and σ characterize the demand process; the market is given by the discount rate r and the inverse of the markup factor λ ; current labor is characterized by the wage ω , the labor productivity A , and the resigning rate δ ; c_H and K_H determine the frictions in the hiring process; Δ , c_F , and K_F give the firing imperfections.

2.1 Characterization of optimum

In this section I give the necessary conditions of the problem in Proposition 1. J is time-homogenous outside the firing periods, in which case the current normalized labor η_t is sufficient as a state variable for J . J is a Markovian function of η outside of firing periods. Within firing periods the value of the firm is Markovian given L , Z , and the time left to the end of the firing process. Note that J is not the value of the firm (V is) but it is the normalized value of the firm. I first make a guess for J and denote this by f . In the next section I prove $J = f$. In order to determine f , I define the following auxiliary operators.

Let \mathbb{D} be the set of real-valued functions on \mathbb{R}^+ . I define the operators $M_H : \mathbb{D} \rightarrow \mathbb{D}$

and $M_F : \mathbb{D} \rightarrow \mathbb{D}$ for all $\eta \geq 0$ and $f \in \mathbb{D}$ as follows

$$M_H f(\eta) = \sup_{m \geq 0} [f(\eta + m) - c_H m - K_H \eta] \quad (2.11a)$$

$$M_F f(\eta) = E_\eta^{P^Z} \left[\int_0^\Delta u(\eta_t, t) dt + e^{-\rho\Delta} \sup_{\eta_\Delta \geq m \geq 0} [f(\eta_\Delta - m) - c_F m] \right. \\ \left. + e^{-\rho\Delta} \sup_{m_H \geq 0} [f(\eta_\Delta + m_H) - c_H m_H - K_H \eta_\Delta] I_{\{m_H > 0\}} - e^{-\rho\Delta} K_F \eta_\Delta \right] \quad (2.11b)$$

where η_Δ is the value at time Δ of η defined by (2.10) and the expectation is conditioned on $\eta_0 = \eta$. $M_F f(\eta)$ and $M_H f(\eta)$ are the expected discounted firing value and the instantaneous hiring value when the normalized labor is η , given that the “continuing value” of the problem is f . At the end of the firing delay, the firm can fire, hire, or choose to do nothing. I have included the option to hire at the end of the delay into my firing operator, M_F .

Given a guess for the value function, f , and a policy $\pi \in \Pi$, the optimal number of normalized workers to hire at time t_i is given by

$$\check{m}_i^{f,H} = \arg \sup_{m \geq 0} \{f(\eta_{t_i} + m) - c_H m\} \quad (2.12)$$

Correspondingly, the optimal change at the end of the firing delay is

$$\check{m}_i^{f,F} = \arg \sup_{m < \eta_{t_i + \Delta_i}} \{f(\eta_{t_i + \Delta_i} - m) - c_F m^+ + c_H m^- - K_H \eta I_{\{m < 0\}}\} \quad (2.13)$$

Note that \check{m}_i^{f,s_i} depends on the conjecture for the normalized value of the firm, f , and $\check{m}_i^{f,F}$ depends also on the number of normalized workers at the end of the firing delay, $\eta_{t_i + \Delta_i}$.

By (2.10), I define the infinitesimal generator \mathcal{A} for all $\eta > 0$ and sufficiently

regular f as follows

$$\mathcal{A}f(\eta) = \frac{1}{2}\sigma^2\eta^2 f''(\eta) - (\delta + \mu)\eta f'(\eta) - \rho f(\eta) \quad (2.14)$$

The following result gives the necessary conditions for optimum by using standard arguments (see e.g. Højgaard and Taksar, 1999 or Fleming and Soner, 1993).

Proposition 2. (Optimality) Assume that the normalized-value function J satisfies Itô's formula. Then it satisfies the following set of inequalities outside of the firing process:

- (i) Hiring is an option: $J(\eta) \geq M_H J(\eta)$
- (ii) Firing is an option: $J(\eta) \geq M_F J(\eta)$
- (iii) The value is expected to fall at least by the flow profit: $\mathcal{A}J(\eta) + u(\eta) \leq 0$

where one of the inequalities is tight for all $\eta > 0$.

Proposition 2 is a system of quasi-variational inequalities, which are the first order conditions to my problem. Inequalities (i) and (ii) hold for all η because the J -value function majorizes the value of the hiring and the firing decisions. In the same way (iii) must hold since waiting is always an admissible policy. Taking no action or taking one of the admissible actions must always represent the optimal policy.

Lemma II.2. (Uncontrolled firm value) *The firm's uncontrolled firm value is*

$$V_{UNCON}(Z, L) = Z(p_1(L/Z)^\lambda + p_2(L/Z)) \quad (2.15)$$

where $p_1 = A^\lambda/c_1$, $c_1 = \rho + \lambda(\delta + \mu + \frac{1}{2}\sigma^2(1 - \lambda))$, and $p_2 = -\omega/(\delta + r)$.

Proof. The value of the uncontrolled firm is

$$V(Z, L) = ZE^{Pz} \left[\int_0^\infty e^{-\rho t} ((A\lambda\eta_t)^\lambda - \omega\eta_t) \right]$$

By interchanging the integration sign and the expectation the integral of the expectation gives 2.15. □

CHAPTER III

Value function and optimal policy

There are “optimality regions” for each of the policies. My assumption on the form of the solution outside the firing process is the following: (i) for $\eta \in (0, b_H]$, it is optimal to start a hiring process; (ii) for $\eta \in (b_H, b_F)$, it is optimal neither to hire new workers nor to fire workers; and (iii) for $\eta \in [b_F, \infty)$ it is optimal to fire workers. I expect to have $b_H < b_F$ in cases where both b_H and b_F are positive and finite. Both barriers may fail to exist when the frictions are prohibitively high. These are defined *impulse control barriers* as

$$b_H = \sup \{ \eta \geq 0 : f(\eta) = M_H f(\eta) \}$$

$$b_F = \inf \{ \eta \geq 0 : f(\eta) = M_F f(\eta) \}$$

so that f solves (i) in Proposition 2 with equality for $\eta \leq b_H$, f solves (ii) in Proposition 2 with equality for $\eta \geq b_F$, and that f solves (iii) in Proposition 2 for $\eta \in [b_H, b_F]$. The coefficients in the general solution to Proposition 2 as well as the optimal barriers b_H and b_F are found from the value matching and smooth pasting conditions that must hold at the barriers. Based on standard results (e.g. Dumas 1991 and Dixit 1991) I expect f to be continuously differentiable at the control barriers,

b_H and b_F . I define *target barriers* as

$$u_H = \sup \{ \eta \geq 0 : f'(\eta) = c_H \}$$

$$u_F = \inf \{ \eta \geq 0 : f'(\eta) = -c_F \}$$

The target barriers are the optimal η levels to which hire up or fire down. Furthermore, I expect to have $0 < b_H < u_H < u_F < b_F < \infty$.

With the four arbitrary barriers satisfying the sequence of inequalities above, I generate a policy π that follows this barrier structure. First define $t_0 = 0$, $s_0 = H$, $m_0 = 0$ without out any cost at t_0 for ease of notation. Then $t_{i+1} = \inf\{t : \eta \notin (b_H, b_F), t > t_{i+\Delta_i}\}$, $s_{i+1} = H$ if $\eta_{t_{i+1}} = b_H$, $s_{i+1} = F$ if $\eta_{t_i} = b_F$, $m_{i+1} = u_H - b_H$ if $s_{i+1} = H$, and $m_{i+1} = (\eta_{t_{i+1}+\Delta} - u_F)I_{\{\eta_{t_{i+1}+\Delta} > u_F\}} + (u_H - \eta_{t_{i+1}+\Delta})I_{\{\eta_{t_{i+1}+\Delta} \leq b_H\}}$ if $s_{i+1} = F$.

For the J -value function is

$$f(\eta) = \begin{cases} M_F f(\eta), & b_F \leq \eta \\ f_w(\eta), & b_H < \eta < b_F \\ M_H f(\eta), & 0 \leq \eta \leq b_H \end{cases} \quad (3.1)$$

where f_w is the value of waiting. The evolution of a path of η is represented in Figure 3.1 given target barriers and impulse control barriers.

I derive the J -value function in each of these three regions. Equation (3.1) generates two value matching conditions:

$$f_w(b_H) = f_w(u_H) - c_H(u_H - b_H) - K_H b_H \quad (3.2a)$$

$$M_F f(b_F) = f_w(b_F) \quad (3.2b)$$

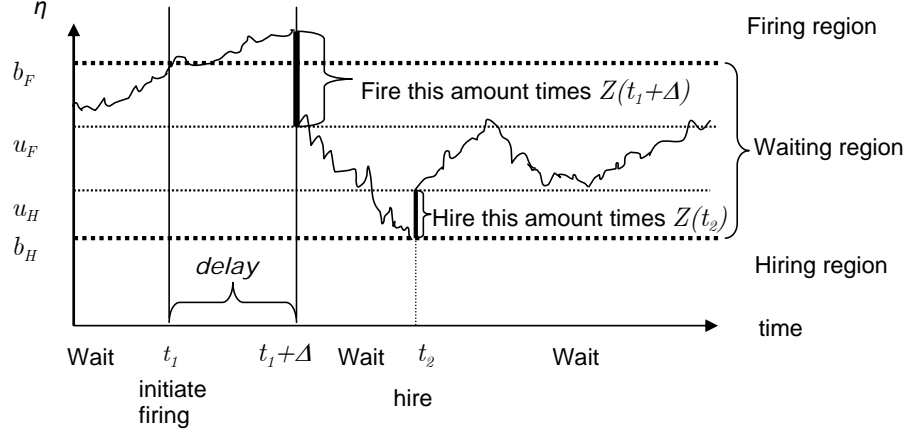


Figure 3.1: **Evolution of η .**

(3.2a) gives me a pair of smooth pasting conditions:

$$f'_w(b_H) = c_H - K_H \quad (3.3a)$$

$$f'_w(u_H) = c_H \quad (3.3b)$$

The other value matching condition, (3.2b), gives another smooth pasting condition:

$$M_F f'(\eta)|_{\eta=b_F} = f'_w(b_F) \quad (3.4)$$

This gives three smooth pasting conditions. Using the assumption that the second-order condition holds when the firm fires at the end of the delay, i.e. it is optimal to fire at the end of the firing period, the first-order condition in $M_F f$ gives the remaining barrier condition of

$$f'_w(u_F) = -c_F \quad (3.5)$$

Now I have six equations and four unknown barriers. There are two unknown coef-

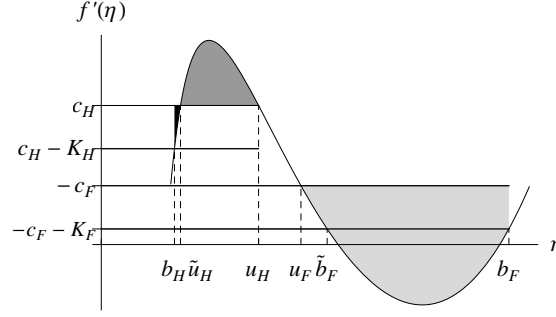


Figure 3.2: **Graphical Representation of Traditional Proportional and Fixed Costs.**

ficients on the terms of f_w , which I show below. I then solve the unknown barriers and coefficients.

I compare my model of firm-sized and proportional costs to the more well known model of fixed and proportional costs (Dixit 1991). In Dixit's model, the state variable X follows a two-sided (s, S) inventory policy. Stating if the state variable X falls below the lower barrier s then the policy raises the state variable to a refill level $S > s$, and if $X \geq r$ then the policy decreases the state variable to a drawdown level $R \in (S, r)$. By using the value-matching condition ((26) in Dixit 1991) for increasing X I get $F(S) - F(s) = a_s + b_s(S - s)$, where by Dixit's notation $F(X)$ is the value function, the fixed cost of increasing is a_s and marginal cost is b_s . This is written as $\int_s^S (F'(x) - b_s) dx = a_s$, which means that the area between $F'(x)$ and b_s from s to S equals a_s . By taking the derivatives with respect to S and s gives $F'(S) = b_s$ and $F'(s) = b_s$. Similarly the value-matching condition ((28) in Dixit 1991) for decreasing X is $F(R) - F(r) = a_r + b_r(r - R)$, where the fixed cost of decreasing is a_r and marginal cost is $-b_r$. This gives $\int_R^r (-b_r - F'(x)) dx = a_r$. Thus, the area between $-b_r$ and $F'(x)$ from R to r is a_r . Taking the derivatives of R and r gives $F'(R) = -b_r$ and $F'(r) = -b_r$. Figure 3.2 shows this graphically. The upper shaded area equals the fixed cost of increasing, a_s , and the lower shaded area is the fixed cost of decreasing, a_r . The smooth-pasting conditions of $F'(S) = F'(s) = b_s$ and $F'(R) = F'(r) = -b_r$.

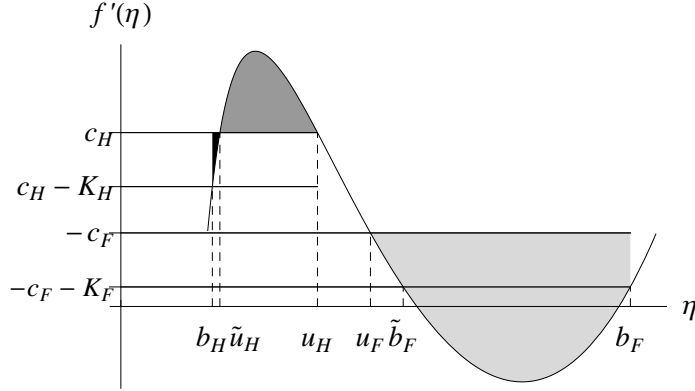


Figure 3.3: **Graphical Representation of Proportional and Firm-Sized Costs.** Parameters: $\mu=5\%$, $\sigma=25\%$, $\delta=5\%$, $r=10\%$, $\lambda=0.7$, $c_H=0.10518$, $K_H=0.1$, $c_F=0.08049$, $K_F=0.1$, $A=1$, and $\omega=1$. Barriers: $b_H=0.14128$, $u_H=0.30366$, $u_F=0.38545$, and $b_F=0.78686$. Coefficients: $h_1=0.445146$ and $h_2=0.662328$.

are obvious.

Figure 3.3 shows the effects of proportional and firm-sized costs in my model. The value-matching equation for hiring (3.2a) gives $f(b_H) = f(u_H) - c_H(u_H - b_H) - K_H b_H$ which can be written as

$$K_H b_H = \int_{b_H}^{u_H} (f'(\eta) - c_H) d\eta \quad (3.6)$$

The firm-sized cost, $K_H b_H$, then equals the area between $f'(\eta)$ and c_H . The left-hand side of equation (3.6) is the rectangle from 0 to b_H and from c_H to $c_H - K_H$.

By the first-order conditions at b_H and u_H I get $f'(b_H) = c_H - K_H$ and $f'(u_H) = c_H$ shown in Figure 3.3. The first one implies $f'(b_H) - c_H < 0$. Therefore, the integral in (3.6) starts negative at b_H . Each Δ value that results in a negative contribution to (3.6) is represented by the dark sliver from b_H to the point where $f'(\eta)$ intersects c_H . When $\eta > \tilde{u}_H = \inf\{\eta : f'(\eta) = c_H\}$ the difference of $f'(\eta)$ and c_H is positive, and the integral becomes positive as η increases. This is the upper medium gray area. By the definition of $M_H f(\eta)$ in (2.11a), there may be η values that give $f'(\eta) = c_H$ (i.e. satisfies the first order condition), but does not maximize $M_H f(\eta)$. \tilde{u}_H minimizes

$M_H f(\eta)$, since if I integrate (2.11a) only up to \tilde{u}_H then the integral is negative. Since \tilde{u}_H is the first η where the first order condition holds, I have defined u_H to be the supremum. This is shown in Figure 3.3 where $f'(\eta)$ is downward sloping, which gives the concavity of $f(\eta)$. The rectangle from 0 to b_H and from c_H to $c_H - K_H$ is the graphical representation of the firm-sized cost, $K_H b_H$. These observations are shown in Figure 3.3. The negative area is shown as the sliver of black, while the upper medium gray area is the positive area.

For simplicity the graphical representation of firing in Figure 3.3 is with zero delay. The value-matching equation for firing (3.2b with zero delay) gives $f(b_F) = f(u_F) - c_F(b_F - u_F) - K_F b_F$, which can be written as $K_F b_F = \int_{u_F}^{b_F} (-c_F - f'(\eta)) d\eta$. The left-hand side of this equation can be shown graphically by the rectangle from 0 to b_F and between the $-c_F$ and $-c_F - K_F$ lines. The right-hand side is the area between $-c_F$ and $f'(\eta)$ from u_F to b_F . This is the light gray shaded area on the bottom right of Figure 3.3. By first-order conditions at b_F and u_F I get $f'(b_F) = -c_F - K_F$ and $f'(u_F) = -c_F$. The first order conditions imply that $-c_F > f'(\eta)$ over this range, so the integral is positive. There are two values of η where $f'(\eta) = -c_F - K_F$, $\tilde{b}_F = \inf\{\eta : f'(\eta) = -c_F - K_F\}$ and b_F . Verification of the optimal policy requires that f is convex at b_F , so I choose the second η . This is where $f'(\eta)$ is upward sloping insuring convexity.

Given the η process and the barriers, the lemma below gives the probability that the firm's first action is firing. Define the hitting time of an interval $T_{a,b} = \inf\{t \geq 0 | \eta(t) = a \text{ or } \eta(t) = b\}$. Now Karlin and Taylor (1975, see Theorem 5.2) give

Proposition 3. (Probability of firing before hiring) The probability that the firm's first action is firing is given by

$$Pr\{\eta(T_{b_H, b_F}) = b_F | \eta(0) = \eta_0\} = \frac{\exp(2(\delta + \mu) \log(\eta_0/\sigma^2)) - \exp(2(\delta + \mu) \log(b_H/\sigma^2))}{\exp(2(\delta + \mu) \log(b_F/\sigma^2)) - \exp(2(\delta + \mu) \log(b_H/\sigma^2))}$$

3.0.1 Value function parts

The value of waiting is the solution of the PDE (iii) in Proposition 2: for $\eta \in (b_H, b_F)$ the flow profit is equal to the change in the value function. Thus, f_w satisfies

$$\frac{1}{2}\sigma^2\eta^2 f_w''(\eta) - (\delta + \mu)\eta f_w'(\eta) - \rho f_w(\eta) + v(\eta) = 0 \quad (3.7)$$

Note the following: $v(\eta) = p_1 \eta^\lambda + p_2 \eta$, $d_\pm = \frac{1}{2} \pm (\delta + \mu) / \sigma^2 \pm \sqrt{((\delta + \mu) / \sigma^2 + \frac{1}{2})^2 + 2\rho / \sigma^2}$. The following lemma, which is proven in Subsection III.1, gives the value of waiting and hiring.

Lemma III.1. (*Value of waiting and hiring*) For a policy characterized by barriers $0 < b_H < u_H < u_F < b_F$ the following holds

$$(i) \text{ Value of waiting: } f_w(\eta) = h_1 \eta^{d_+} + h_2 \eta^{d_-} + v(\eta)$$

$$(ii) \text{ Value of hiring: } M_H(\eta) = (f_w(u_H) - c_H(u_H - \eta)) I_{\{\eta < u_H\}} + f_w(\eta) I_{\{\eta \geq u_H\}} - K_H \eta$$

Condition (ii) of Lemma III.1 gives the value of hiring. Once the firm pays the firm-size hiring cost, it is optimal for them to hire a positive amount of workers that would change η to u_H . However, if $\eta > u_H$ the firm should not hire any workers, because it is already overstaffed.

I use notation $M_F f(\eta) = M_F f(\eta; u_F, u_H, b_H)$ since the value of firing depends on the current level of η , hiring trigger b_H and the target levels u_H and u_F . From (2.11b) and my conjecture of the optimality regions gets

$$\begin{aligned} M_F f(\eta; u_F, u_H, b_H) &= E_\eta^{PZ} \left[\int_0^\Delta u(\eta_t, t) dt + e^{-\rho\Delta} [-K_F \eta_\Delta + f(\eta_\Delta) I_{\{b_H \leq \eta_\Delta < u_F\}} \right. \\ &+ \left. ((f(u_F) + c_F u_F) - c_F \eta_\Delta) I_{\{\eta_\Delta \geq u_F\}} + ((f(u_H) - c_H u_H) + (c_H - K_H) \eta_\Delta) I_{\{\eta_\Delta < b_H\}} \right] \end{aligned} \quad (3.8)$$

Finding the analytic form of M_F requires computing the above expectation. As an

intermediate step I write

$$\begin{aligned}
M_F f(\eta; u_F, u_H, b_H) &= E_\eta^{P^Z} \left[\int_0^\Delta u(\eta_t, t) dt \right. \\
&+ e^{-\rho\Delta} \left[h_1 u_H^{d+} + h_2 u_H^{d-} + p_1 u_H^\lambda + p_2 u_H - c_H(u_H - \eta_\Delta) - K_H \eta_\Delta \right] I_{\{\eta_\Delta \leq b_H\}} \\
&+ e^{-\rho\Delta} \left[h_1 u_F^{d+} + h_2 u_F^{d-} + p_1 u_F^\lambda + p_2 u_F - c_F(\eta_\Delta - u_F) \right] I_{\{\eta_\Delta \geq u_F\}} \\
&+ e^{-\rho\Delta} \left[h_1 \eta_\Delta^{d+} + h_2 \eta_\Delta^{d-} + p_1 \eta_\Delta^\lambda + p_2 \eta_\Delta \right] I_{\{b_H < \eta_\Delta < u_F\}} - e^{-\rho\Delta} K_F \eta_\Delta \left. \right], \tag{3.9}
\end{aligned}$$

which uses the function $f_w(\eta)$. Section A.3 shows that

Lemma III.2. (Value of firing) *The value of the firing is given by*

$$\begin{aligned}
M_F f(\eta) &= p_1 \eta^\lambda (1 - e^{-c_1 \Delta}) + p_2 \eta (1 - e^{-(\delta+r)\Delta}) - c_F \eta e^{-(\delta+r)\Delta} \Phi(-z_F + \sigma\sqrt{\Delta}) \\
&+ e^{-\rho\Delta} \left[\left(h_1 u_H^{d+} + h_2 u_H^{d-} + p_1 u_H^\lambda + u_H(p_2 - c_H) \right) \Phi(z_H) \right. \\
&+ (c_H - K_H) \eta e^{-(\delta+r)\Delta} \Phi(-z_H + \sigma\sqrt{\Delta}) \\
&+ \left. \left(h_1 u_F^{d+} + h_2 u_F^{d-} + p_1 u_F^\lambda + u_F(p_2 + c_F) \right) \Phi(-z_F) \right] \\
&+ g(\eta; h_1(u_H), d_+) + g(\eta; h_2(u_H), d_-) + g(\eta; p_1, \lambda) + g(\eta; p_2, 1) \\
&- K_F \eta e^{-(\delta+r)\Delta}
\end{aligned}$$

where $z(u) = \frac{\log(u/\eta) + (\delta + \mu + \frac{1}{2}\sigma^2)\Delta}{\sigma\sqrt{\Delta}}$, $z_F = z(u_F)$, $z_H = z(b_H)$,

$g(\eta; k, n) = k \eta^n e^{-(\rho + (\delta + \mu + \frac{1}{2}\sigma^2(1-n))n)\Delta} \left[\Phi(z_F - \sigma n \sqrt{\Delta}) - \Phi(z_H - \sigma n \sqrt{\Delta}) \right]$, and

$\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^z e^{-\frac{1}{2}y^2} dy$ is the cumulative normal distribution.

The main result of the paper is proved in Appendix B.

Theorem III.3. (Optimal policy) *Assume that $c_H, K_H, c_F, K_F, \Delta > 0$, and that there exists barriers $0 < b_H < u_H < u_F < b_F < \infty$ and coefficients h_1, h_2 which give f_w and policy π . Let J_π be a solution to (3.1), such that (3.2) - (3.5) hold. Let*

$$\lim_{\eta^+ \rightarrow b_H} \frac{\partial^2 J_\pi(\eta)}{\partial \eta^2} > 0, \quad \lim_{\eta^- \rightarrow b_F} \frac{\partial^2 J_\pi(\eta)}{\partial \eta^2} \Big|_{\Delta=0} > 0. \tag{3.10}$$

and

$$J'_\pi(\eta) > c_H - K_H \text{ for } \eta \in (b_H, u_H). \tag{3.11}$$

Then J_π is a solution to Proposition 2 and the optimal hiring and firing policy is to hire $\check{m}_i^\pi Z_{t_i^\pi}$ when $\eta_{t_i^\pi} = b_H$, start firing at t_i^π when $\eta_{t_i^\pi} = b_F$, and fire $\check{m}_i^\pi Z_{t_i^\pi + \Delta}$ at $t_i^\pi + \Delta$. Moreover, the value function outside firing times is $V(Z, L) = ZJ_\pi(\eta)$.

Theorem III.3 says that if for a given set of frictions I find barriers that give a f_w that satisfies (3.1) and (3.2) - (3.5), then those barriers satisfy first-order conditions that give an optimal hiring and firing policy. However, I need to check if the first-order conditions maximize the firm value. Conditions (3.10) - (3.11) ensure that this is true. I then verify if the converse is also true: if I have an optimal policy in the form of four barriers, then the conditions mentioned in Theorem III.3 for the J function is true.

In Subsection B.2 I prove the following proposition.

Proposition 4. (Hiring and firing barriers) The hiring and firing trigger levels satisfy $b_H < \eta^*$ and $b_F|_{\Delta=0} > \eta^*$, where $\eta^* = \left(\frac{\lambda A^\lambda}{\omega}\right)^{\frac{1}{1-\lambda}}$, which is the optimal η -level in the no-frictions case of Lemma II.1.

This proposition gives bounds on the target barriers, which are then used in the numerical algorithm for solving this problem.

CHAPTER IV

Existence and numerical algorithm

This chapter shows my model's solution exists and is unique. The continuous time model is written as a discrete time dynamic program. In discrete time the Bellman equation is shown to be a contraction and thus has a unique solution. The solution to the discrete time dynamic program converges to the solution to the continuous time problem. I then explain the numerical algorithm used to find the solution.

4.1 Dynamic Programming

The discretization is done as follows. τ is the time interval. Let $\pi(\eta) = ((A\eta)^\lambda - \omega\eta)\tau$ is the flow profits over this interval, τ . The discount factor over the time interval is defined as $\beta = e^{-\rho\tau} < 1$. For a given η , the uncontrolled normalized labor process evolves according to Cox et al. (1979), $u = e^{\sigma\sqrt{\tau}}$, $d = e^{-\sigma\sqrt{\tau}}$, $p = \frac{1}{2} - \frac{\delta+\mu}{2\sigma}\sqrt{\tau}$. The state variable η can go to ηu with probability p and ηd with probability $q = 1 - p$. This describes how the normalized labor process evolves over time τ .

4.1.1 Operators for the firm's labor adjustments

The state space for the firm's problem is the initial normalized labor, η . In the continuous time set up in Section II, hiring was handled by the M_H operator, which is defined for discrete time. First, I set up the convention that the decision to hire,

fire or do nothing is done at the beginning of the epoch. The cash flows are incurred at the beginning of the epoch. The discrete time hiring operator is defined as

$$M_H^D f(\eta) = -K_H \eta + \max_{m_d^H \in \mathbb{Z}^+} \{-c_H \eta (u^{m_d^H} - 1) + \beta(p f(\eta u^{m_d^H} u) + q f(\eta u^{m_d^H} d))\} \quad (4.1)$$

A version of the continuous time firing operator (2.11b) is used to create the discrete time one. Using Lemma III.2, I solve for the integral and the firm sized firing cost. They are $p_1 \eta^\lambda (1 - e^{-c_1 \Delta}) + p_2 \eta (1 - e^{-(\delta+r)\Delta})$ and $-K_F \eta e^{-(\delta+r)\Delta}$. The discrete time firing operator at η is

$$\begin{aligned} M_F^D f(\eta) = & p_1 \eta^\lambda (1 - e^{-c_1 \Delta}) + p_2 \eta (1 - e^{-(\delta+r)\Delta}) \\ & + e^{-\rho \Delta} E_\eta^{Pz} \left[\max_{m_d \in \mathbb{Z}^+} f(\eta_\Delta d^{m_d}) - c_F \eta_\Delta (1 - d^{m_d}) \right] \\ & + e^{-\rho \Delta} E_{\eta_i}^{Pz} \left[\left(\max_{m_H^d \in \mathbb{Z}^+} f(\eta_\Delta u^{m_H^d}) - c_H \eta (u^{m_H^d} - 1) \right. \right. \\ & \left. \left. - K_H \eta_\Delta \right) I\{m_H^d > 0\} \right] - K_F \eta e^{-(\delta+r)\Delta} \end{aligned} \quad (4.2)$$

In subsection 4.1.3 these operators converge to their continuous time counterparts. I use the operators to show that the dynamic program is a contraction mapping.

4.1.2 Contraction Mapping

In this subsection the dynamic program is a contraction mapping, and thus has a unique solution. To show this, the operator needs to be operating on bounded functions. The optimal function is bounded above by the value of the firm without any frictions (see Lemma II.1) and below by the uncontrolled value of the firm (see Lemma II.2). Let $\mathcal{B}(X)$ be the space of functions bounded above by (2.6) and below by (2.15), where $X = \mathbb{R}^{++}$.

I define the operator $T : \mathcal{B}(X) \rightarrow \mathcal{B}(X)$ as

$$(Tf)(\eta) = \pi(\eta) + \max\{\beta(pf(\eta u) + qf(\eta d)), M_H^D f(\eta), M_F^D f(\eta)\} \quad (4.3)$$

Over the time interval the firm gets $\pi(\eta)$ regardless of its actions. It will maximize over the choices of doing nothing, hiring, or firing. This operator gives the Bellman equation.

Blackwell's sufficient conditions for a contraction (see e.g. Stokey et al. (1989, Theorem 3.3)), states that if an operator satisfies both the monotonicity and discounting conditions, then it is a contraction.

The monotonicity condition for T is defined as for any $f, g \in \mathcal{B}(X)$ and $f(x) \leq g(x)$, for all $x \in X$ implies that $(Tf)(x) \leq (Tg)(x)$, for all $x \in X$. The discounting condition means that there exists some $\alpha \in (0, 1)$ such that

$$[T(f + a)](x) \leq (Tf)(x) + \alpha a, \text{ all } f \in \mathcal{B}(X), a \geq 0, x \in X$$

where $(f + a)(x)$ is defined by $(f + a)(x) = f(x) + a$.

Lemma IV.1. (*Monotonicity and discounting*) *The operator T satisfies the monotonicity and discounting conditions.*

Proof. Consider equation (4.3). For a given η , one of the three actions maximizes the right-hand side. Each of these cases is considered below and show that monotonicity holds for each of them.

Suppose $f, g \in \mathcal{B}(X)$ and $f(\eta) \leq g(\eta)$, for all $\eta \in X$. Suppose that for a specific $\eta \in X$, $(Tf)(\eta)$ is maximized by $\beta(pf(\eta u) + qf(\eta d))$. Then

$$\begin{aligned} (Tf)(\eta) &= \pi(\eta) + \beta(pf(\eta u) + qf(\eta d)) \\ &\leq \pi(\eta) + \beta(pg(\eta u) + qg(\eta d)) \\ &\leq (Tg)(\eta) \end{aligned} \quad (4.4)$$

Suppose that for a specific $\eta \in X$, $(Tf)(\eta)$ is maximized by $M_H^D f(\eta)$, and that m^* maximizes m in $M_H^D f(\eta)$. Then

$$\begin{aligned}
(Tf)(\eta) &= \pi(\eta) - K_H \eta - c_H \eta (u^{m^*} - 1) + \beta(pf(\eta u^{m^*} u) + qf(\eta u^{m^*} d)) \\
&\leq \pi(\eta) - K_H \eta - c_H \eta (u^{m^*} - 1) + \beta(pg(\eta u^{m^*} u) + qg(\eta u^{m^*} d)) \\
&\leq \pi(\eta) + M_H^D g(\eta) \\
&\leq (Tg)(\eta)
\end{aligned} \tag{4.5}$$

The second to last inequality holds since for firing given g its best action leads to results weakly better than using m^* . The last inequality is since firing might not be the best action at η . Suppose that for a specific $\eta \in X$, $(Tf)(\eta)$ is maximized by $M_F^D f(\eta)$ with $m(\eta(\Delta))$ being the optimal choice at $\eta(\Delta)$. By using the definition of M_F^D , that $f \leq g$, and that $m(\eta(\Delta))$ is not necessarily the best choice for $M_F^D g(\eta)$, I get:

$$\begin{aligned}
(Tf)(\eta) &= \pi(\eta) + M_F^D f(\eta) \\
&= \pi(\eta) + p_1 \eta^\lambda (1 - e^{-c_1 \Delta}) + p_2 \eta (1 - e^{-(\delta+r)\Delta}) \\
&\quad + e^{-\rho \Delta} E_\eta^{Pz} [f(\eta_\Delta d^{m(\eta(\Delta))}) - c_F \eta_\Delta (1 - d^{m(\eta(\Delta))})] \\
&\quad + e^{-\rho \Delta} E_{\eta_i}^{Pz} [(f(\eta_\Delta u^{m(\eta(\Delta))}) - c_H \eta (u^{m(\eta(\Delta))} - 1) \\
&\quad - K_H \eta_\Delta) I\{m(\eta(\Delta)) > 0\}] - K_F \eta e^{-(\delta+r)\Delta} \\
&\leq \pi(\eta) + p_1 \eta^\lambda (1 - e^{-c_1 \Delta}) + p_2 \eta (1 - e^{-(\delta+r)\Delta}) \\
&\quad + e^{-\rho \Delta} E_\eta^{Pz} [g(\eta_\Delta d^{m(\eta(\Delta))}) - c_F \eta_\Delta (1 - d^{m(\eta(\Delta))})] \\
&\quad + e^{-\rho \Delta} E_{\eta_i}^{Pz} [(g(\eta_\Delta u^{m(\eta(\Delta))}) - c_H \eta (u^{m(\eta(\Delta))} - 1) \\
&\quad - K_H \eta_\Delta) I\{m(\eta(\Delta)) > 0\}] - K_F \eta e^{-(\delta+r)\Delta} \\
&\leq \pi(\eta) + M_F^D g(\eta) \\
&\leq (Tg)(\eta)
\end{aligned} \tag{4.6}$$

Thus, T satisfies monotonicity. By (4.3), and $(f + a)(x) = f(x) + a$, the operator T

also satisfies discounting:

$$\begin{aligned}
(T(f + a))(\eta) &= \pi(\eta) + \max\{\beta(p(f(\eta u) + a) + q(f(\eta d) + a)), (M_H^D(f + a))(\eta), \\
&\quad (M_F^D(f + a))(\eta)\} \\
&= \pi(\eta) + \max\{\beta(pf(\eta u) + qf(\eta d)) + \beta a, M_H^D f(\eta) + \beta a, \\
&\quad M_F^D f(\eta) + e^{-\rho\Delta} a\} \\
&\leq (Tf)(x) + \max(\beta, e^{-\rho\Delta})a
\end{aligned} \tag{4.7}$$

where $a \geq 0$, $\max(\beta, e^{-\rho\Delta}) \in (0, 1)$. \square

Define the metric $E : X \times X \rightarrow \mathbb{R}^+$, $E(g, h) = | \|g\| - \|h\| |$, where $g, h \in \mathcal{B}(X)$, where $\| \cdot \|$ is the sup norm, $\|h\| = \sup_{x \in X} |h(x)|$.

Lemma IV.2. (*Complete metric space*) *The metric space (X, E) is a complete metric space.*

Proof. The following fact is stated on page 47 of Stokey et al. (1989), “The set of real numbers \mathbb{R} with the metric $\rho(x, y) = |x - y|$ is a complete metric space.” Then by using the above fact, (X, E) is a complete metric space. \square

Theorem IV.3. (*Existence and uniqueness*) *The firm’s problem has a solution and it is unique.*

Proof. Stokey et al. (1989, Theorem 3.2) shows that if an operator on a complete metric space is a contraction mapping then it has exactly one fixed point. The space $\mathcal{B}(X)$ is a closed bounded space and is a complete metric space under the sup norm. Since T is a contraction and $\mathcal{B}(X)$ is a complete metric space, T has a unique fixed point. \square

4.1.3 Optimal dynamic programming policy and continuous time optimal policy

The optimal policy for the dynamic program is the same policy as the continuous time problem in Chapter III. This approach is based on Dixit (1991). He starts out with arithmetic Brownian motion, which he approximates as a random walk. He gets a differential equation, with some change to his notation, by doing the following.

His state variable is X , and moves up or down an amount ϵ with probabilities p and q . Using the known mean (μ) and variance (σ^2) of Brownian motion, he selects the time step and then the probabilities to match the mean and variance. The mean of X over τ is $\mu\tau = p\epsilon + q(-\epsilon)$ and its variance over τ is $\sigma^2\tau = p(\epsilon - \mu\tau)^2 + q(\epsilon + \mu\tau)^2$. By using $p + q = 1$ and keeping only those terms that are not $o(\tau)$ he gets a relationship between the time step, state-space step and variance, $\sigma^2\tau = \epsilon^2$.

The random walk has the same mean and variance of the continuous time Brownian motion. He then considers an arbitrary four-barrier control policy, $s < S < R < r$ and finds the points in discrete state space that corresponds to those barriers, X_s, X_S, X_R , and X_r . Here each of the two outer barriers, (s and r) take the state variable to one of the two inner barriers, (S and R) at a known cost. Given this control, he modifies the random walk one step away from each of the two outer barriers. Instead of continuing to move in the direction of the outer barrier, the state variable would jump to the inner barrier with the given probability. For example, at X_{r-1} the state will jump to X_R with probability p and to X_{r-2} with probability q . I call this an “edge case” state, as opposed to an inner state. For an inner state, X_i , where $i \in \{s + 2, \dots, r - 2\}$, X_i moves to X_{i+1} with probability p and to X_{i-1} with probability q . With this change Dixit knows how each state transitions and their probabilities. He then incorporates these transition probabilities into a transition matrix α .

Next he solves for the value function for a given control. This can be written in

vector notation, F , where the vector value of states today is the infinite sum of the discounted expected profits over the time intervals, or $F = \sum_{k=0}^{\infty} e^{-k\rho\tau} \alpha^k f$, where k is the k th time step, and ρ is the discount rate over τ . Here the vector f is the flow profit over time step τ . Letting $B = e^{-\rho\tau} \alpha$ the Bellman equation is $F = f + BF$. In the Bellman equation form, the value of the firm, F is the profit over the time step, f , plus the discounted expected continuation values at the new states, BF . At a given state, the firm would get the flow profit times the time step as the profit now. Also at that given state, the state variable would evolve to two possible states. The discounted expected future values for an inner state is the sum of the two discounted continuation values of those two possible states weighted by the probabilities. If the state is an edge case, one movement will hit an outer barrier and cause the state to move to an inner barrier. For an edge case the discounted expected future values have two terms. The first is the discounted continuation value of staying in an inner state weighted by its probability. There is a probability that the state will evolve to the outer barrier triggering the control. Then the discounted continuation value weighted by its probability of moving to an inner barrier minus the cost of moving is the second term. For given continuation values, the Bellman equation is known for all states within the outer barriers.

For a given inner state X_i , its Bellman equation, $F(X_i) = f(X_i)\tau + e^{-\rho\tau}qF(X_{i-1}) + e^{-\rho\tau}qF(X_{i+1})$ can be transformed into a differential equation. Dixit first multiplies by $e^{\rho\tau}$ and rearranges by adding the terms $-F(X_i) + qF(X_i) + pF(X_i)$ (these terms are zero since $p + q = 1$) to get

$$[e^{\rho\tau} - 1]F(X_i) - q[F(X_{i-1}) - F(X_i)] - p[F(X_{i+1}) - F(X_i)] = e^{\rho\tau} f(X_i)\tau$$

By doing a Taylor expansion, the right-hand side is just $\tau f(X_i) + o(\tau)$. By plugging in the values of the probabilities and the relationship between the time step and the

state step, Dixit gets the following for the left-hand side

$$\rho\tau F(X_i) - \mu\tau F'(X_i) - \frac{1}{2}\sigma^2\tau F''(X_i) + o(\tau)$$

The expansion gives first and second derivatives of the continuation value and terms that go to zero faster than the time step, $o(\tau)$. By dividing by the time step, and taking the limit as the time step goes to zero, Dixit gets a differential equation, for any $X \in [s, r]$:

$$\frac{1}{2}\sigma^2 F''(X) + \mu F'(X) - \rho F(X) + f(X) = 0$$

The solution to this differential equation is the general solution to the homogeneous equation and the particular solution depending on the flow profit function. The general solution is $C_1 e^{\alpha_1 X} + C_2 e^{\alpha_2 X}$ with unknown coefficients C_i and constants, α_i . The unknown coefficients are similar to my h_1 and h_2 . The constants are the roots of the characteristic equation, similar to my model's d_+ and d_- .

By setting $x = \ln(\eta)$ the solution is transformed from two exponentials to two power functions with the powers being the solution to the characteristic equation. By choosing the drift of the arithmetic Brownian motion correctly and the other parameters, the constants become d_+ and d_- of my model. The particular solution of the differential equation depends on the flow profit. By choosing the flow profit function correctly, I get the last two terms of the solution to my differential equation (see Lemma III.1).

I now show that as τ gets small the discrete time hiring and firing operators converge to the continuous time ones. For a hiring decision m_d , the proportional hiring cost is $c_H \eta (u^{m_d} - 1)$. Setting $u^{m_d} = m_H / \eta + 1$, the proportional hiring cost

becomes $c_H m_H$. The hiring operator is written as

$$\begin{aligned}
M_H^D f(\eta) &= -K_H \eta + \max_{m_d \in \mathbb{Z}^+} \{-c_H \eta (u^{m_d} - 1) + \beta(p f(\eta u^{m_d} u) + q f(\eta u^{m_d} d))\} \\
&= -K_H \eta + \max_{m_H \in \mathbb{R}^+} \{-c_H m_H + \beta(p f((\eta + m_H)u) + q f((\eta + m_H)d))\}
\end{aligned} \tag{4.8}$$

As τ goes to zero, $p \rightarrow q \rightarrow 0.5$, $u \rightarrow d \rightarrow 1$ and $\beta \rightarrow 1$. As u gets arbitrarily close to 1, for any choice of m_H the value of m_d that gives m_H can be found. When τ is zero, the sum of the continuation values collapse to $f(\eta + m_H)$, and $M_H^D f(\eta)$ becomes $M_H f(\eta)$.

The firing operator similarly converges to the continuous time one. Only the two expectation terms need to be shown to be the same as the continuous time operator's terms. Using the argument above for M_H^D the first expecting term is the same as the continuous time operator's term, where η is now η_Δ . For the expectation with the firing at the end of the delay, let $d^{m_d} = 1 - m/\eta_\Delta$, but confine $d^{m_d} \in [0, \eta_\Delta)$. Thus, the term in the expectation becomes $\sup_{m \in [0, \eta_\Delta)} f(\eta_\Delta - m) - c_F m$. And the discrete time operator converges to the continuous time one.

4.1.4 Numerical Approach

For each implementation of a policy, the value of that policy is updated with the value of the previous policy. When the improvement in the policies is small enough, the implementation stops. I denote the hiring and firing operators used for coding with the ‘‘C’’ superscript.

In order to keep the state space bounded and computable, η , is divided into $2M$ values, with $\eta^* = \eta_M$ and $\eta_i = \eta^* u^{(i-M)^+} d^{(M-i)^+}$. At η_{2M} the state variable stays at η_{2M} with probability p , and goes to η_{2M-1} with probability q . Similarly, at η_1 the state variable stays at η_1 with probability q , and goes to η_2 with probability p .

In discrete time there isn't a value function, f , but a vector of points that correspond to a value function, f_i , for $i = 1, \dots, 2M$. For the do-noting option, if $i = 1$

the down node's continuation value with probability q is f_1 . If $i = 2M$ the up node's continuation value with probability p is f_{2M} .

The discrete time hiring operator for coding at η_i is

$$M_H^C f_i = -K_H \eta_i + \max_{m_H \in \{1, \dots, 2M-i\}} \{-c_H \eta_i (u^{m_H} - 1) + \beta(p f_{i+m_H+1} + q f_{i+m_H-1})\} \quad (4.9)$$

At the end of the delay the following happens: Given that firing started at η_i , the state variable at the end of the delay, $\eta_i(\Delta)$, is a random variable that can take on any value on \mathbb{R}^+ . In discrete time, when firing is started at η_i the η_j that the η process gets closest to at the end of the delay is a random variable, $\eta_{i,j}$. It has the following probabilities: $P(i, j) = \Pr(\eta_i(\Delta) \in ((\eta_j + \eta_{j-1})/2, (\eta_j + \eta_{j+1})/2)) = \Phi(z((\eta_j + \eta_{j+1})/2)) - \Phi(z((\eta_j + \eta_{j-1})/2))$, for $j \in [2, 2M - 1]$, $P(i, 1) = \Pr(\eta_i(\Delta) < (\eta_1 + \eta_2)/2) = \Phi(z((\eta_1 + \eta_2)/2))$, and $P(i, 2M) = \Pr(\eta_i(\Delta) > (\eta_{2M-1} + \eta_{2M})/2) = 1 - \Phi(z((\eta_{2M-1} + \eta_{2M})/2))$, where z is defined in Lemma III.2.

The discrete time firing operator for coding at η_i with continuation value vector f is

$$\begin{aligned} M_F^C f_i = & p_1 \eta^\lambda (1 - e^{-c_1 \Delta}) + p_2 \eta (1 - e^{-(\delta+r)\Delta}) \\ & - e^{-\rho \Delta} E_{\eta_i}^{P_Z} \left[\max_{m^j \in \{0, \dots, (j-1)\}} f_{j-m^j} - c_F \eta^* u^{j-m^j} (u^{m^j} - 1) \right] \\ & - e^{-\rho \Delta} E_{\eta_i}^{P_Z} \left[\left(\max_{m_H^j \in \{0, \dots, (2M-j)\}} f_{j+m_H^j} - c_H \eta^* d^{M-j-m_H^j} (1 - d^{m_H^j}) \right) \right. \\ & \left. - K_H \eta_j \right] I\{m_H^j > 0\} - K_F \eta e^{-(\delta+r)\Delta} \end{aligned} \quad (4.10)$$

If the firm fires and changes η from η_j to η_{j-m} and gets continuation value f_{i-m^j} , it pays $c_F(\eta_j - \eta_{j-m^j}) = c_F(\eta^* u^{j-M} - \eta^* u^{j-m^j-M}) = c_F \eta^* u^{j-m^j-M} (u^{m^j} - 1)$. This is the second term in the first expectation. Similarly, hiring from η_j to $\eta_{j+m_H^j}$ means the firm pays $c_H \eta^* d^{M-j-m_H^j} (1 - d^{m_H^j})$ for the proportional cost. Since an expectation is

the summed possible outcomes weighted by their probabilities I get:

$$\begin{aligned}
M_F^C f_i &= p_1 \eta^\lambda (1 - e^{-c_1 \Delta}) + p_2 \eta (1 - e^{-(\delta+r)\Delta}) \\
&- e^{-\rho \Delta} \sum_{j=M}^{2M} \left[\max_{m^j \in \{0, \dots, (j-1)\}} f_{j-m^j} - c_F \eta^* u^{j-m^j} (u^{m^j} - 1) \right] P(i, j) \\
&- e^{-\rho \Delta} \sum_{j=1}^{M-1} \left[\left(\max_{m_H^j \in \{0, \dots, (2M-j)\}} f_{j+m_H^j} - c_H \eta^* d^{M-j-m_H^j} (1 - d^{m_H^j}) \right) \right. \\
&\quad \left. - K_H \eta_j I\{m_H^j > 0\} \right] P(i, j) - K_F \eta e^{-(\delta+r)\Delta}
\end{aligned} \tag{4.11}$$

Thus, for a given f vector, $M_F^C f_i$ can be found for all $i \in \{1, \dots, 2M\}$.

4.1.4.1 No Delay

I briefly consider the no delay case (denoted with the subscript 0). For no delay, the new operator for policy implementation is

$$(T_0 f)(\eta_i) = \pi(\eta_i) + \max\{\beta(pf(\eta_i u) + qf(\eta_i d)), M_H^D f(\eta_i), M_{F,0}^D f(\eta_i)\} \tag{4.12}$$

where, the no firing delay firing operator is

$$\begin{aligned}
M_{F,0}^C f_i &= K_F \eta_i + \max_{m \in \{0, \dots, i-2\}} \{-c_F \eta_i (1 - d^m) \\
&\quad + \beta(pf_{\min(2M, i+1-m)} + qf_{\min(2M, i-1-m)})\}
\end{aligned} \tag{4.13}$$

The operator T_0 is also related to the optimization at the end of the delay without the firm-sized firing cost term.

4.1.4.2 Coding Description

Ideally, the code would return for each η_i , the best decision: hire, do nothing, start firing, and the best state to move to, if any. From this I get the optimal barriers. Given these best decisions, the largest η_i whose best decision is to hire is b_H . The smallest η_i whose best decision is to start firing is b_F . The best state to goto at b_H

would then be defined as u_H . The η that is the best state to goto when firing at the end of the delay is defined as u_F . My program updates the value function and solves for these barriers.

I initialize variables, including the starting value function vector, v , the starting metric, iteration number, the metric tolerance that will stop the implementation, and $P(i, j)$. The metric measures the differences in the value functions between this policy and the previous policy. Then I start the while loop that continues as long as the metric is above its tolerance, and the max iterations have not been met.

In the while loop, I initialize the temporary value function vector, tv , based on the action of doing nothing. For the middle nodes, $1 < i < 2M$, $tv(i) = \pi(\eta_i) + \beta(p \cdot v(i+1) + q \cdot v(i-1))$. The edge nodes, $i = 1, i = 2M$, the equation is adjusted so that v 's vector indices are $1, 2$ or $2M - 1, 2M$. For each i , I have a “best state” variable that tells the index of best state to go to at η_i . The best state for η_i is set at i , meaning doing nothing is best. I also have a “best decision” variable that states what the best decision is for all states i . It is initialized in the while loop as “do nothing”.

Then I consider the value of hiring. For each η_i I consider hiring to any level between η_{i+1} and η_{M-1} to find the best amount to hire at η_i . If the value of hiring, $M_H^C v(i)$ is more than the $tv(i)$ then I update $tv(i) = \pi(\eta_i) + M_H^C v(i)$, the best state as the index of the state best to hire to, and the best decision as “hiring” at i .

Next I consider the optimization at the end of the delay. At the end of the delay, there is a positive probability that η takes on any i value. The optimal decision and action for values of i were considered in the value of hiring. For each i , I solve $M_{F,0}^C v(i)$, as well as the best state. I also store the “best value” of $v(i-m^*) - c_F \eta^* u^{i-m^*} (u^{m^*} - 1)$, where m^* solves the maximization in the first expectation in (4.11). I solve $M_{F,0}^C v(i)$ to help compute M_F^C .

With the best states and values solved for at the end of the delay, I now solve for $M_F^C v(i)$. Only two terms have yet to be computed for M_F^C : the two sums. These

two sums considers each possible $j \in \{1, \dots, 2M\}$ that η can be near at the end of the delay. I assume that doing “nothing” is optimal at $j = M$. For the first sum, where firing at the end of the delay is considered, the optimal firing amount and the best value has already been calculated. By summing these best values by their probabilities, $P(i, j)$, plus $v(M) \cdot P(i, M)$, the first sum is found. Each term in the second sum related to where hiring has been calculated by comparing $v(j)$ to $M_H^C v(j)$ and getting $tv(j)$, as described above. This sum term is just summed $tv(j)$ weighted by $P(i, j)$. Therefore, for each $i > M$, $M_F^C v(i)$ has been found. Now by comparing $M_F^C v(i)$ versus the value of doing nothing, $tv(i)$, I update the best decision as “firing,” if $M_F^C v(i) > tv(i)$.

Now at the end of the loop, I update the metric as the maximum absolute difference between v and tv at any point, set $v = tv$, and increment the iteration count.

When the while loop exits due to the metric being below the tolerance, or max iterations reached, the barriers are returned.

The results the numerical algorithm are shown in Chapter 5.1.

4.2 Convergence of Value Function and Optimal Policy

Here I show that the discrete time dynamic program converges to the continuous-time solution to the firm’s problem. First I show results of the value-matching and smooth-pasting conditions as the time step, τ , gets smaller. Then based on Kushner (1977) I show that for this problem, that both the discrete time value function and optimal policy converge to the continuous time value function and optimal policy.

Figure 4.1 shows the value function, barriers and value-matching and smooth-pasting conditions as a function of τ . The value function converges as the time step gets smaller. The barriers do not change substantially. From the construction of b_F in the algorithm, the value-matching and smooth-pasting equations hold at b_F . For the other three barriers (b_H, u_H, u_F) the differences between the left and right-hand

sides of the equations are presented in the bottom four panels. The metrics used to evaluate how close the barriers are to optimal are the difference between left and right-hand sides of equations (3.2a), (3.3a), (3.3b), and (3.5). The graphs show that the metrics gets closer to zero as τ falls.

(Kushner 1977, Theorems 8.5.2 and 8.5.3) shows that for an impulse control problem with one under Lipschitz continuous condition for the drift and volatility coefficient, the discrete time value function and optimal policy converge to the continuous time value function and optimal policy. (Kushner 1990, Assumption (A2.2)) also suggests that the cost functions needs to be continuous and bounded for convergence.

Theorem IV.4. *(Convergence of the value function and optimal policy) The discrete time value function and optimal policy converge to the continuous time value function and optimal policy.*

Proof. For $x, y > 0$, $|-(\delta + \mu)x - (-\delta + \mu)y| \leq K_d|x - y|$, where $K_d = |(\delta + \mu)| + 1$, and $|-\sigma x - (-\sigma y)| \leq K_v|x - y|$, where $K_v = \sigma + 1$. Thus, the drift and the volatility coefficients on the diffusion are Lipschitz continuous, then the theorems (Kushner 1977, Theorems 8.5.2 and 8.5.3) hold. The cost functions including $M_F f(\eta)$ are continuous and bounded. $M_F f(\eta)$ is bounded because it is a weighted sum of f functions, which are bounded by Theorem IV.3. Thus, the discrete time value function and optimal policy converge to the continuous time value function and optimal policy.

□

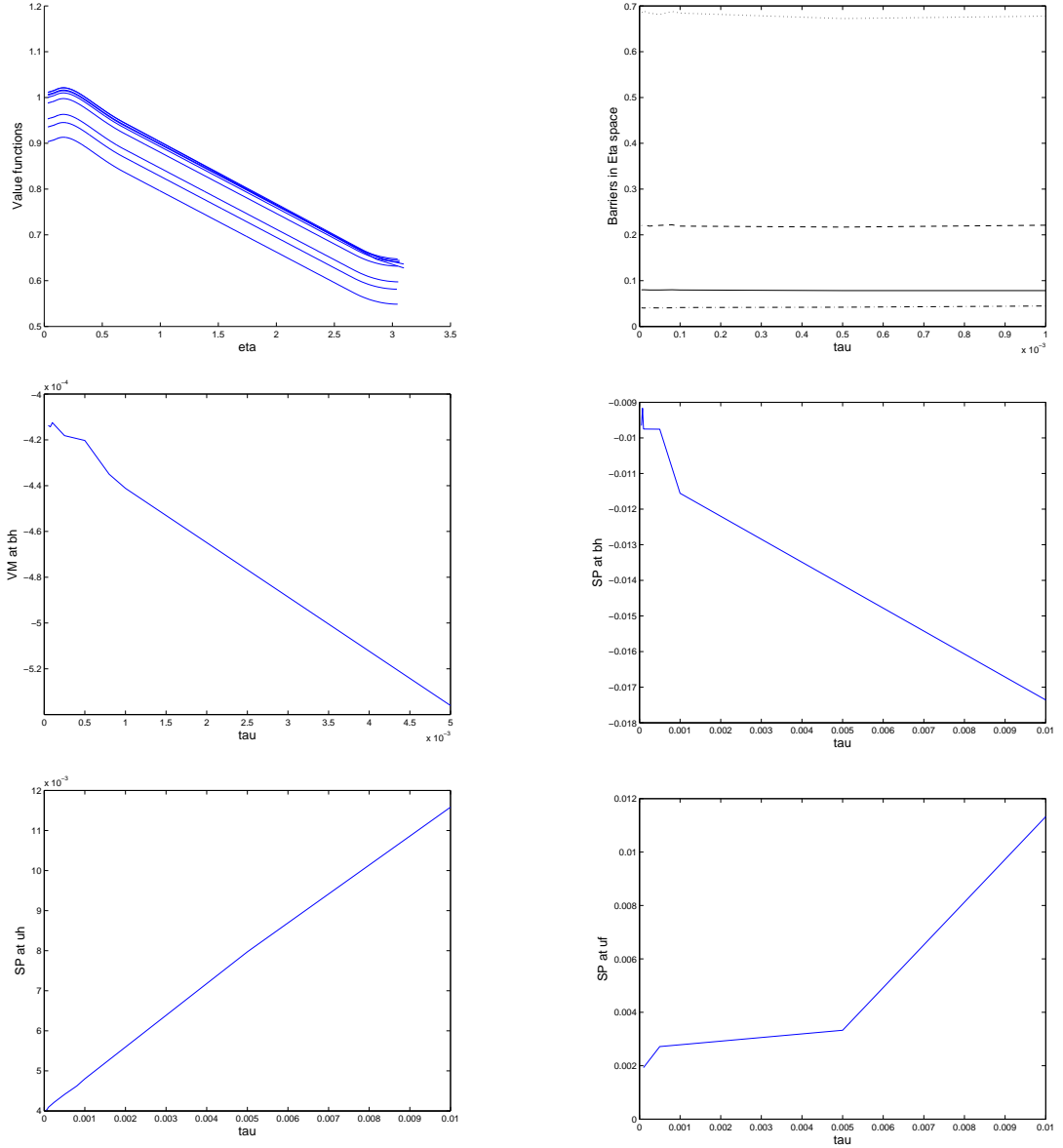


Figure 4.1: **Effect from time step, τ .** Parameters: $\sigma = 0.8$, $c_F = 1/12$, $K_F = 0.01$, $c_H = 1/12$, $K_H = 0.05$, $\mu = 0\%$, $\delta = 10\%$, $r = 10\%$, $\lambda = 0.7$, $A = 1$, and $\omega = 1$. Units are years and annual wage. The metrics used to evaluate how close the barriers are to optimal are the difference between left and right-hand sides of equations (3.2a), (3.3a), (3.3b), and (3.5). From the construction of b_F in the algorithm, the value-matching and smooth-pasting equations hold at b_F , and their metrics are identical to zero, and thus are not included in this figure.

CHAPTER V

Comparative statics

5.1 Comparative statics

This chapter analyzes the impact from the firing delay, the firing and hiring costs, and the demand volatility on the optimal barrier policies and simulated profits and labor level. I find the optimal barriers by the algorithm in Chapter IV. Parameters used in this chapter were inspired by OECD (2004) for market frictions, by Bentolila and Bertola (1990) for productivity, wage, and market power, and by market data for r and demand parameters. Kramarz and Michaud (2010) find that proportional firing costs can be up to ten months of wages. They also state that proportional hiring cost can be as much as 2.84 annual wages. Robertson (1979) examines Ontario industries and finds that all Ontario industries have a turnover rate of 15.1%, and can be as high as 39% for mining. Since “firm-sized” costs in this paper is new, I know of no estimations of these parameters.

Given the firm’s optimal policy, I use Monte Carlo simulation to calculate the firm’s expected profit and labor levels and their standard deviations. For each Monte Carlo path the firm starts at the labor level of η^* , with initial demand of $Z_0 = 1$ and zero profits. I then simulate monthly demand, η , as a discretized geometric Brownian motion for twenty years, and allow for the firm to follow its optimal labor policy. Each month the previous month’s accumulated profits, plus interest, are added to

the current month's profit. At the end of the twenty years the simulation reports the labor level and accumulated profit with interest. Simulating over a large number of paths generates the expected and standard deviation of the profit and labor levels.

For each set of model parameters, I double the simulations using antithetic variates (see e.g. Brandimarte, 2006). This reduces the standard deviations of the estimates. I present figures showing how the estimated labor, profit and their standard deviations change as the parameters and optimal barriers change. In the simulation for each change in parameters and barriers, I use the same random draws.

The impact of the parameter changes on profit and labor also depends on the sign of the drift of η . Recall from (2.10) that the drift of η is $-(\delta + \mu)$. If $-(\delta + \mu) > 0$, the firm expects to fire more often than hire, and vice-versa if the drift is negative. I have chosen different values of δ and μ to show how the different drifts of η impact the results.

5.1.1 Firing delay

Figure 5.1 shows the effect of increasing the firing delay on the barriers and the firm's profit and labor. Here, the drift of η is -0.1 . Therefore, the firm will be expecting to hire often. Since the delay restricts the firm's actions, requiring it to wait longer, the firm chooses to get the firing process started early and the firing trigger barrier, b_F , falls. Lowering the trigger level, while the other barriers remain roughly flat, lowers the average labor. Firing more often, before η can get too large, means that there is less variation in labor and profit. Since the delay does not explicitly increase the cost, it does not directly impact the profit level. However, firing more frequently, and incurring the firing cost means that average profit decreases, especially as the delay gets substantially larger, while smaller delay levels have a small effect on profits. The slight change in the hiring barrier for small delay may explain why the standard deviations fall dramatically over the first few values of delay. Additionally,

the changes in the standard deviations amount to only one-percent changes.

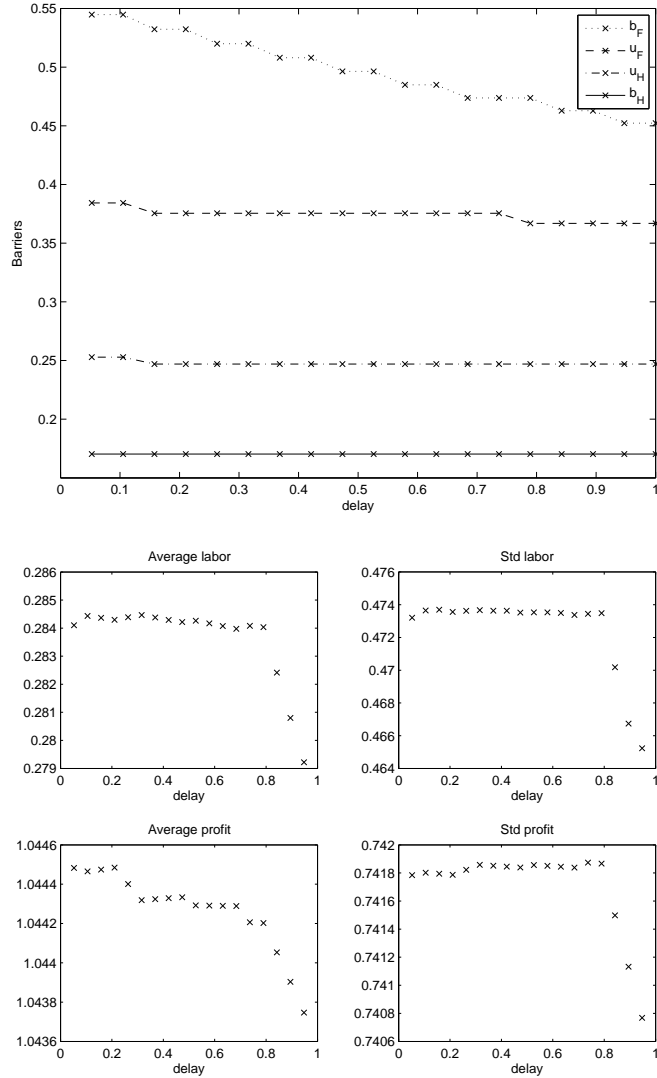


Figure 5.1: **Effect from delay Δ .** Parameters: $\sigma = 0.3$, $c_F = 1/12$, $K_F = 0.01$, $c_H = 1/12$, $K_H = 0.01$, $\mu = 0\%$, $\delta = 10\%$, $r = 10\%$, $\lambda = 0.7$, $A = 1$, and $\omega = 1$. Units are years and annual wage. The panels were generated by Monte Carlo simulation with 100,000 runs.

5.1.2 Proportional firing cost

The effects from proportional firing cost c_F are illustrated in Figure 5.2. The firm avoids the firing costs by raising the firing barriers. By raising b_F , there is a smaller probability of firing being started. Since each additional worker fired becomes more

expensive, the firm is less willing to fire more workers, so u_F also shifts up. Firing cost raises the cost of firing new workers, therefore the value of hiring falls and decreases the hiring barriers. Since $-(\delta + \mu) = -10\%$, the drift of η is negative. This draft means that hiring is more likely, and more time is spent between the hiring barriers, u_H and b_H . Since hiring is likely, the increase in hiring cost lowers profit. More time is spent in a lower profit area away from the optimal level without frictions ($\eta^* \sim 0.3$, see Lemma II.1), which also lowers profit. The shrinking of the difference between the hiring barriers, u_H and b_H , means that more time is spent in a narrower η interval, which lowers the standard deviation of labor. Since η is expected to spend more time in a smaller interval, the profit the firm gets from being in this interval is less varied.

5.1.3 Proportional hiring cost

The impact of increasing the proportional hiring cost on the barriers, profit and labor are shown in Figure 5.3. All four barriers widen when the cost rises. Hiring is more costly, so the firm waits until it has fewer workers to trigger hiring. When it does hire, it hires fewer workers to lower the hiring cost. The increasing cost of hiring an additional worker means that the cost of replacing a fired worker increases. To avoid this cost, the firm increases the firing barriers.

For the impact of the increasing cost on the labor and profit statistics, notice that $\delta + \mu > 0$, which means that the drift of η is negative. The same reason that explains the lowering of the labor and profit levels and their standard deviations in Subsection 5.1.2 holds here.

5.1.4 Firm-sized firing cost

The effects of increasing the firm-sized firing cost, K_F , are illustrated in Figure 5.4. To avoid paying this increasing cost, the firm raises the firing trigger barrier b_F .

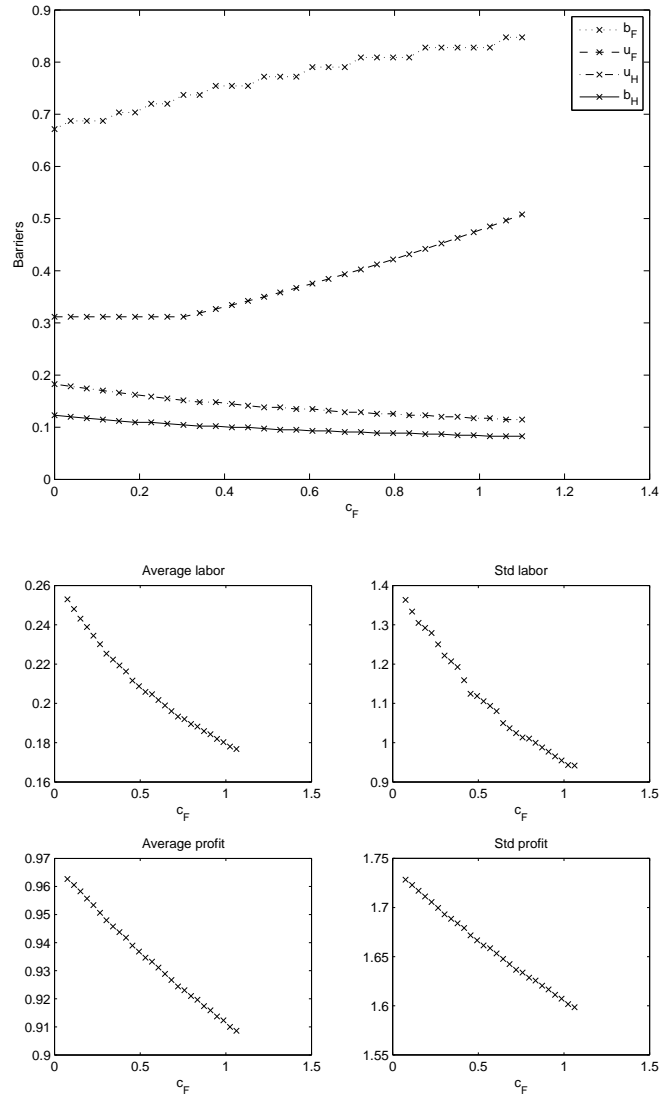


Figure 5.2: **Effect from proportional firing costs c_F .** Parameters: $\sigma = 0.5$, $K_F = 0.01$, $c_H = 1/12$, $K_H = 0.01$, $\mu = 0$, $\delta = 10\%$, $r = 10\%$, $\lambda = 0.7$, $\Delta = 0.5$, $A = 1$, and $\omega = 1$. Units are years and annual wage. The panels were generated by Monte Carlo simulation with 100,000 runs.

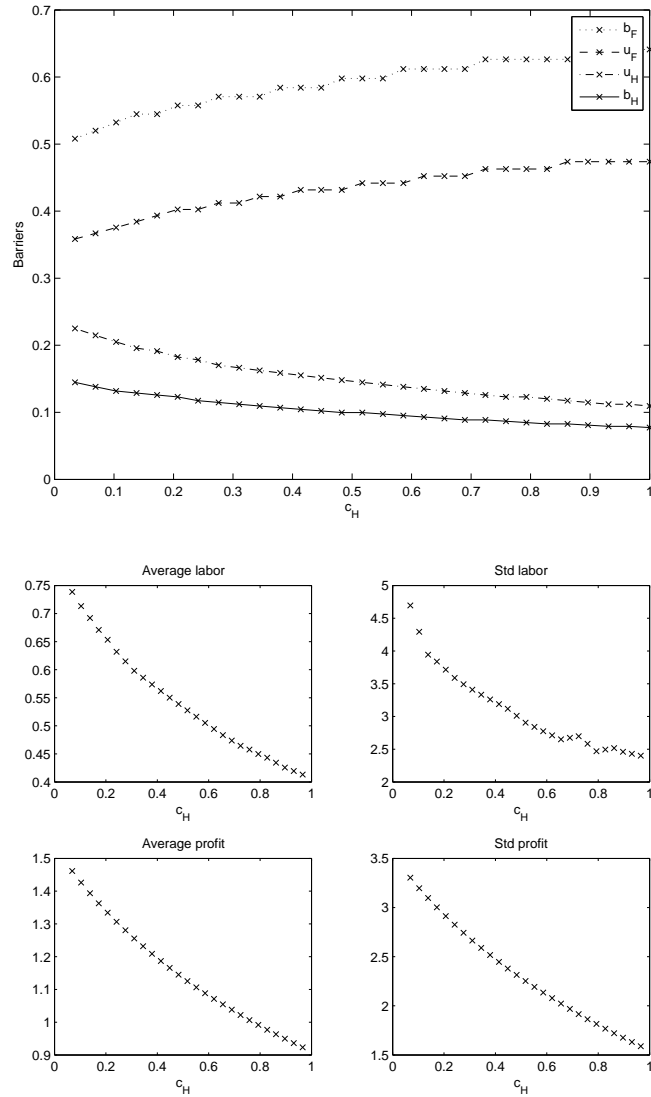


Figure 5.3: **Effect from proportional hiring costs c_H .** Parameters: $\sigma = 0.5$, $c_F = 1/12$, $K_F = 0.01$, $K_H = 0.01$, $\mu = 5\%$, $\delta = 10\%$, $r = 10\%$, $\lambda = 0.7$, $\Delta = 0.25$, $A = 1$, and $\omega = 1$. Units are years and annual wage. The panels were generated by Monte Carlo simulation with 100,000 runs.

This lowers the chance of b_F being hit. Since K_F is a sunk cost once a firing process has started, it does not have an impact on u_F . Thus, the u_F barrier is flat. Increasing the firing cost lowers the value of hiring, and lowers its barriers. These lower barriers with the negative drift of η lead to the lowering of the labor as well as profit levels and their standard deviations, similar to Subsection 5.1.2.

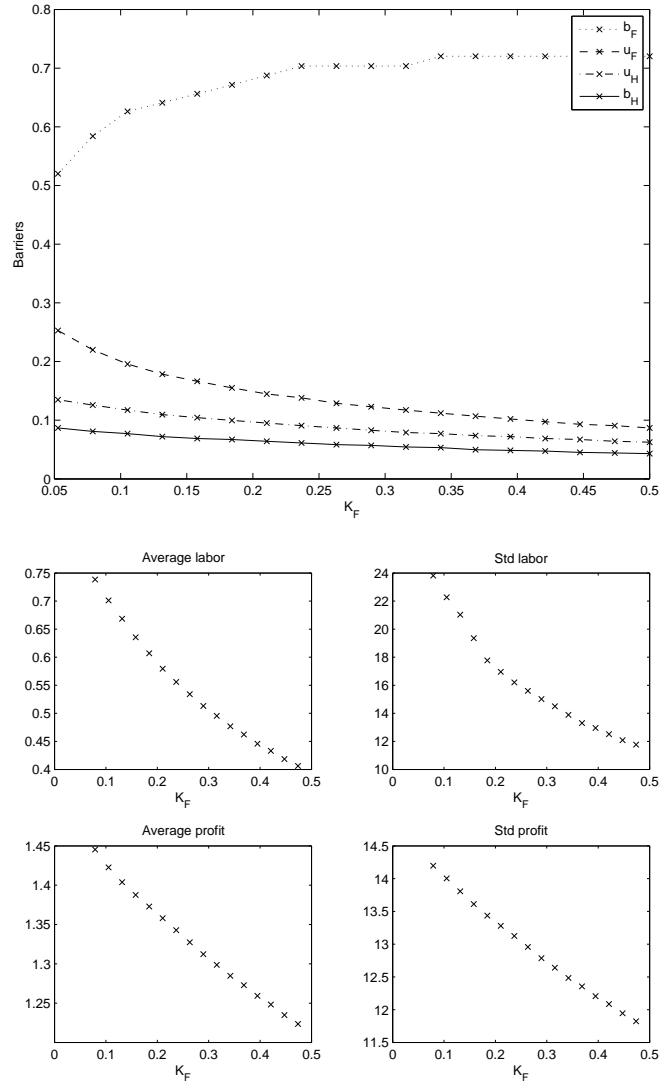


Figure 5.4: **Effect from firm-sized firing costs K_F .** Parameters: $\sigma = 0.80$, $c_F = 1/12$, $c_H = 1/12$, $K_H = 0.01$, $\mu = 7\%$, $\delta = 5\%$, $r = 10\%$, $\lambda = 0.7$, $\Delta = 0.25$, $A = 1$, and $\omega = 1$. Units are years and annual wage. The panels were generated by Monte Carlo simulation with 100,000 runs.

5.1.5 Firm-sized hiring cost

Figure 5.5 shows the effects from increasing the firm-sized hiring costs, K_H . Increasing the hiring cost lowers the net value of starting to hire, thus the hiring trigger barrier b_H falls. This way, the firm reduces the probability of hiring. Since the firm-sized hiring cost is linear in b_H , this lowers the cost paid. The firm-sized hiring cost is a sunk cost as soon as hiring is started. Therefore, the firm should not reduce the amount of workers it hires. In fact, to avoid incurring this increasing cost in the future, the firm hires more workers. Thus, the amount hired, $u_H - b_H$, increases.

Increasing the cost of hiring lowers the value of firing workers, since replacing the fired worker is more expensive. Therefore, the firing barriers increase, since firing a worker is less valuable to the firm.

Since the drift of η is negative, the firm is expecting to hire. This means more hiring cost is incurred and this cost lowers the mean profit. This lowers the average profit. Profit stays closer to the average profit as the cost increases. This is because the mean profit is decreasing, the labor level is farther from optimal before triggering hiring, and the proportional hiring is more expensive causing the standard deviation of profit to fall.

5.1.6 Demand volatility

The effects of demand volatility is shown in Figure 6.2. As the volatility increases, the chance of hitting the trigger barriers b_H and b_F and incurring those hiring and firing costs increase. To reduce this increased expected cost, the firm expands its trigger barriers. This is why b_H falls and b_F rises in the volatility. As the volatility increases, the expected firm-sized cost of firing increases, since this cost term is linear in b_F . To avoid this cost of firing, the firm curves back the firing barrier. This means the firm fires more often, but with a smaller firm-sized firing cost. These two competing effects, reducing the probability of firing and reducing the cost of firing,

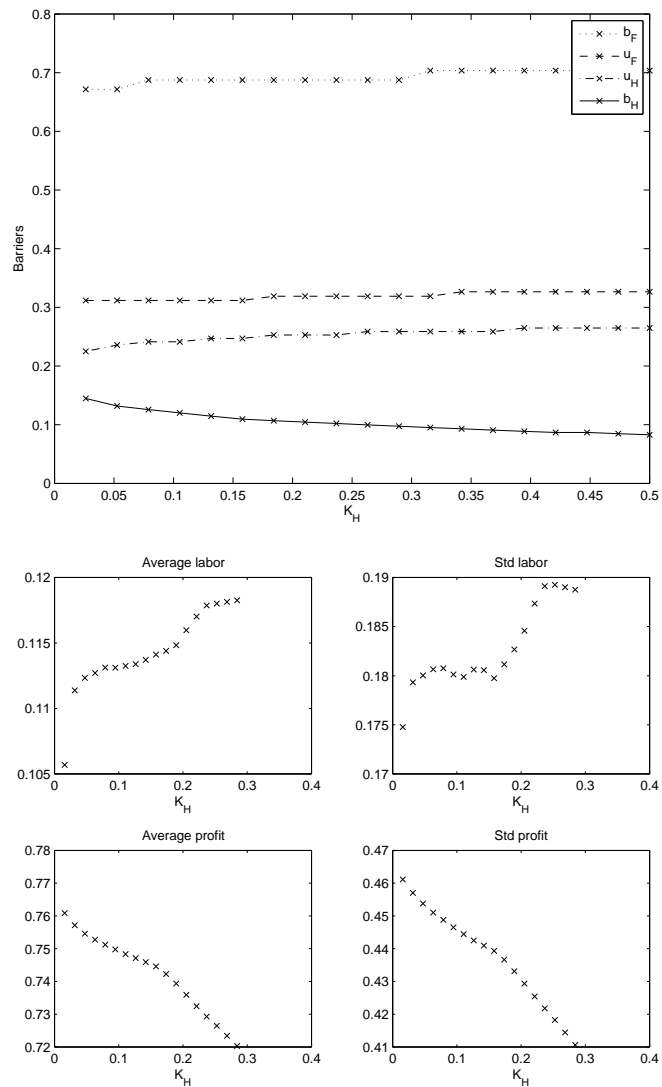


Figure 5.5: **Effect from firm-sized hiring costs K_H .** Parameters: $\sigma = 0.3$, $c_F = 1/12$, $K_F = 0.01$, $c_H = 1/12$, $\mu = -5\%$, $\delta = 10\%$, $r = 10\%$, $\lambda = 0.7$, $\Delta = 0.5$, $A = 1$, and $\omega = 1$. Units are years and annual wage. The panels were generated by Monte Carlo simulation with 100,000 runs.

act in opposite ways. For smaller volatility reducing the probability of firing by rising b_F dominates. For larger volatilities, there is a greater chance of hitting b_F . When the firm reduces the expected firing cost by lowering b_F , which dominates increasing b_F for decreasing the probability that firing is triggered.

As the volatility increases, each additional worker is less valuable, since it is more likely that the worker may have to be fired. This lowers the hiring trigger barrier b_H and the refill level u_H .

Average labor increases, decreases, and then increases again as the volatility increases. As the barriers widen, especially as the firing barrier increases, the labor increases. As the firing barrier levels off, the decrease in the hiring barrier starts to dominate the average labor causing it to fall. As the firing barrier starts to fall again and the decrease of b_H slows down, the average labor turns up.

The average profit falls as volatility increases. The normalized labor level spends less time near its optimal level due to the volatility. Standard deviations of both labor and profit increase as the volatility of demand increases. This occurs because the η variable is more volatile and drives both labor and profits.

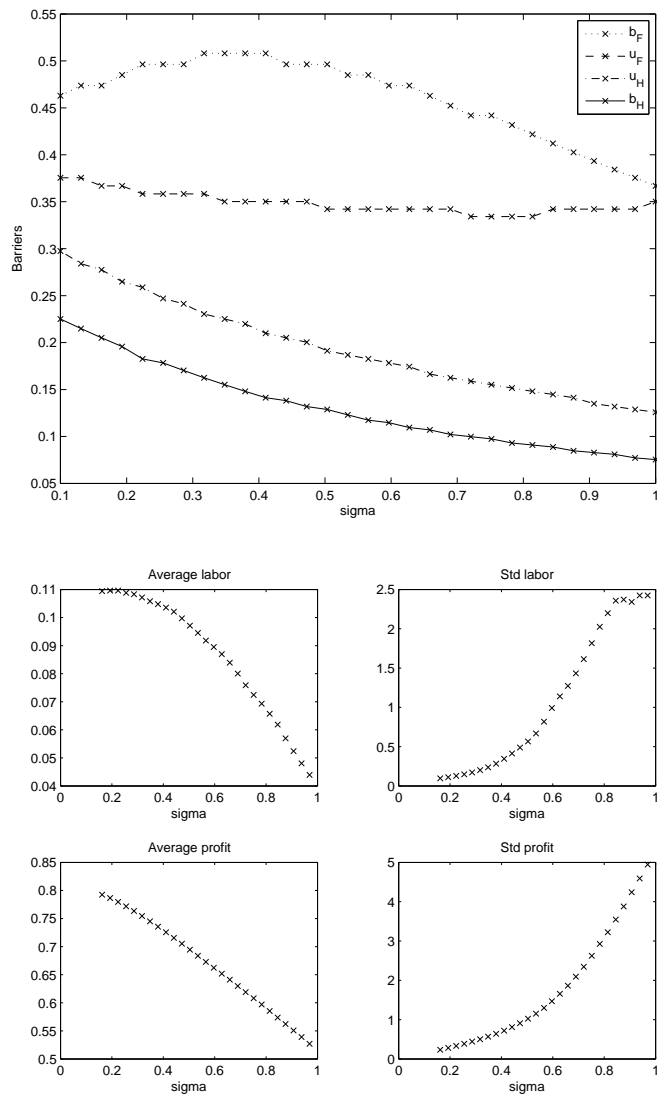


Figure 5.6: **Effect from demand volatility σ .** Parameters: $c_F = 1/12$, $K_F = 0.01$, $\Delta = 0.25$, $c_H = 1/12$, $K_H = 0.01$, $\mu = -5\%$, $\delta = 10\%$, $r = 10\%$, $\lambda = 0.7$, $A = 1$, and $\omega = 1$. Units are years and annual wage. The panels were generated by Monte Carlo simulation with 100,000 runs.

CHAPTER VI

Ford Motor Company

6.1 Ford Motors Discussion

I apply the model to Ford Motor Company. In this chapter I explain parameter estimation for Ford and give comparative statics. Then I discuss my model's results for Ford, including predicting the change in its workforce and the value of Ford.

The model is highly stylized. It does not directly model Ford's less successful management decisions over multiple decades (relative to e.g. Toyota). Ford has the characteristics of my model: random demand, market power, wage costs, and most importantly costly and delayed firings. Model parameters for Ford can be estimated from publicly available data. I use two sets of parameters: A "stable" regime from 2003-2005 and a "recession" regime from 2008 to the end of 2010. I chose these regimes, because they represent a clear change in Ford's business strategy, including a draw down of \$10.1 billion from its credit lines starting in January 2009, but obviously considered in 2008 (Reed 2009). The demand parameters changed by a significant amount across the regimes. This change allows me to test predictions of my model.

To calibrate my model for Ford, some adjustments must be made. Ford has significant non-labor costs. The total liabilities, including pensions and health care costs, for Ford in September 2008 amounts to \$244 billion (Google 2009). Since this was 86.9% of the total assets in 2008, I have to consider this. At the time Ford

had a significant bankruptcy risk. This results in yields for its bonds as high as 45% (InvestingInBonds 2009), and I use this as the recession regime's interest rate. During the stable 2003-2005 time period I use 11% as the discount rate (Pittman 2006). The corporate bond yield is a lower bound for the weighted average cost of capital (WACC), since the return on equity is even higher. However, Ford's debt to equity is 6.63 (Forbes.com 2012) and, therefore, one could expect that the WACC is not much higher.

In the recession, Ford was offered government stimulus money. The expected value of this money, or the government taking over Ford's pension and health care obligations might have increased Ford's market value, but I do not model this effect. During the stable 2003-2005 time period I use \$282 billion as Ford's total liabilities (Ford 2005). There are other non-labor cash flows, such as Ford Credit income, capital leases, and construction costs. The discount rates discussed above give the present value of the non-labor cash flows of -\$10 and -\$2 billion in the stable and recession regimes, respectively. Assuming that Ford's current labor decisions do not have a large influence on the liabilities, I write Ford's market capitalization as the sum of the model value minus total liabilities: $V(L, Z; r = \text{bond yield}) - \text{total liabilities} + \text{present value of non-labor cash flows}$. I also call $V(L, Z)$ Ford's "gross value". As mentioned before, the expected impact of government policy is much harder to quantify, the potential bailout of Ford during the recession that started in December 2007 is not addressed.

I focus on finding model estimates of the equity value and Ford's employment changes. Then I compare the estimates to the actual values, and I also compare the two regimes. Finally I analyze the sensitivity of the gross value with respect to the model parameters. To estimate Ford's gross market value I use Monte Carlo simulation as in Chapter 5.1.¹

¹ In the Monte Carlo simulations 100,000 independent paths with antithetic variates giving a total of 200,000 paths were used. The time step was one month, and the time length of each path

6.2 Ford Parameter Estimation

This section discusses in detail how I estimate Ford’s parameters. Table 6.1 gives the exact estimates.

I estimated the annual drift and volatility of the demand process, assuming it follows a geometric Brownian motion process. Monthly sales data between 2003 and 2005 was obtained from WardsAuto (2009) and personal communication with Ford. Sales data from 2007 to 2010 was gathered from Ford press releases (Ford Media 2008, 2009, 2010). I estimate stable regime demand parameters using monthly data from January 2003 to December 2005 and recession parameters from January 2008 to December 2010.²

Since Ford has been offering early retirement packages and buy out offers to workers, I assume that workers would rather take these than quit. Therefore, I set the quit rate, δ , to zero.

Ford maintained “job banks” where retired or laid-off workers are paid to do nothing, while they can get rehired instantly (see e.g. McCracken, 2006). I set the costs of hiring, c_H and K_H , at zero, since there are no search or training costs.

For the market power, λ , I use 0.88. This estimate is from the flexible nested logit specification of demand elasticity for automobiles (Brenkers and Verboven 2006). More specifically, Brenkers and Verboven (2006) has an estimate of 7.734 for the logit specification and, by the relationship of λ to elasticity, ε of $\lambda = \frac{\varepsilon}{1+\varepsilon}$ we got 0.88 from the logit estimate.

For the delay in firing I use the maximum length of time a plant would be open after it was scheduled to close under the “Way Forward” plan announced by Ford in January 2006 (Ford, 2006). This gives a delay, Δ , of 18 months.

was 20 years.

² The hypothesis that the μ estimates are the same fails to be rejected at the 10% level. The t-statistic for 54 degrees of freedom is $t_{10\%/2,54} = 1.6736$, while the test statistic value is 1.398. The hypothesis that the σ estimates are the same is rejected the 1% level. The F-statistic for 34 and 34 degrees of freedom is $F_{1\%,34,34} = 2.2583$, while the test statistic value is 3.0272.

I estimate the two firing cost parameters (c_F and K_F) from (2.4). I assume that these costs are constant over time and across divisions time and divisions in Ford. The annual reports give the expected costs of restructuring the Premier Automotive Group (England) in 2004 and 2005 to be \$175 million laying off 1,500 workers (Ford, 2005). The costs of implementing the Ford Europe Improvement Plan in 2003 and 2004 was \$605 million resulting in 7,000 workers laid off (Ford, 2005). Using these two data points I found that the proportional cost of firing each worker is nine months of the the worker's wage, and the firing firm-size cost is 1.3% of the total annual wage. This proportional cost is in line with the amount Ford is offering to employees to leave the company (Reuters 2012).

From Ford's annual reports (2005 and 2006) and a DaimlerChrysler (2007) document, I obtained estimates on wages and productivity. Using the number of vehicles produced in North America and Ford's number of hourly North American employees, I estimated the annual productivity of 32 vehicles per worker. I do not consider managers or other employees in this calculation, since they do not physically produce cars. A document from the 2007 Daimler Chrysler - UAW labor talks states that the hourly rate of a Ford assembler is \$26.10 (DaimlerChrysler 2007). Assuming an assembler gets paid for forty-hour weeks, 52 weeks a year, and after adjusting for the increases due to the UAW Ford negotiations UAW (2007), I get an annual wage of around \$60,000.

To estimate Ford's value with my model, I also need the initial point on the demand curve, Z_0 , since $V(L, Z_0) = Z_0 J(L/Z_0)$. To estimate Z_0 , I assume that Toyota is the ideal automobile company, and operates without any labor market frictions. For instance, National Labor Committee (2008) reports that one-third of its assembly line workers are temporary and subcontract workers with contracts as short as under half a year. This makes it easy to layoff workers at little firing cost or delay. Since Toyota and Ford are both global automobile companies, I assume that their products are

interchangeable. They face the same demand level, Z_0 , for their products. I estimate Toyota's $V(L, Z_0)$ from Toyota's gross value (Google 2009). I then estimate the no-friction value of Toyota given the parameters for Ford, $J_{NF,Toyota}$. Thus, the initial point on the demand curve can be found by $Z_0 = \text{Toyota's gross value}/J_{NF,Toyota}$. I do this for both the stable and the recession regimes. ³

6.3 Ford Comparative Statics

Ford's parameters are different than the values used in Chapter 5.1. Therefore, I run comparative statics also for Ford with respect to parameters μ and σ . These parameters are analyzed here because they changed due to the recession. For the comparative statics, I run the algorithm for a range of values for μ and σ . The range for μ was selected since it covered the μ values in the two regimes. The range for σ was determined since it was the σ value $\pm 25\%$. The estimates of μ and σ for each regime fall in the range of the comparative statics.

Figure 6.2 shows the effect of μ on the barriers, equity, profits, and labor. As the demand drift parameter increases, the hiring barriers increase. Increasing demand means that a worker is more valuable, since there is a higher demand for the cars he produces. The higher demand also explains the increasing average labor and average profit. Looking at the effect of changing μ on equity, a -20% drift would lead to -\$96.5 billion equity value, and a drift of -1% would lead to an equity value of \$105.2 billion. Ford's estimated equity becomes zero near a drift of -7.5%. A negative equity value would mean that Ford would be bankrupt. This suggests that Ford should do everything it can to make sure that the demand drift does not fall below -7.5%. However, the model does not take into account the possibility of bankruptcy.

The effects of demand volatility can be seen in Figure 6.2. Increasing the demand volatility lowers the value of an additional worker, since the value of his output is

³ $Z_{0,stable} = 4.88 \times 10^{33}$ and $Z_{0,recession} = 2.7 \times 10^{34}$.

more uncertain. This lowers the hiring barriers, and thus the average labor. Since the η process spends more time away from the optimal level (η^*), the average profit goes down. The equity value ranges from \$50.7 billion to \$42.0 billion. Hence, in this sense, the equity value is impacted more by μ than by σ . This will also be seen in the following section. The increasing volatility also increases the standard deviations of profit and labor.

Figures 6.1 and 6.2 suggest that changing μ and σ affects the profit, labor and hiring barriers. However, the firing barriers b_F and u_F are less sensitive against the changes in the parameters.

6.4 Ford Results

In this section, I explain the changes in Ford's workforce during the recession, market value under the two regimes, as well how changes in the parameters effect Ford's value by using the model.

Ford was under a stable business regime during the mid-2000's. During the recession starting in December 2007 through June 2009 (National Bureau of Economic Research 2010) Ford's business changed drastically. Ford may have continued its employment policies for the stable regime, but had to adapt and reevaluate its employment levels during the recession. From the end of the stable regime, the end of 2006, to the end of the recession, the end of 2009, Ford reduced its workforce in North America from 128,000 to 74,000, or by 42.2% (Ford 2008, 2010).

Here is my stylized story explaining Ford's actions. When Ford saw a decline in vehicle demand in 2007, it triggered its stable regime firing process, $b_F(stable)$. As the process started Ford realized that the decline in demand was throughout the economy, and adapted to the new recession regime. It thus updated its beliefs on the parameter values, and found new optimal barriers, including $u_F(recession)$. At the end of the firing process that started in the stable regime, it fired down

to the u_F barrier under the recession regime. The estimated percentage change is $\frac{b_F(stable) - u_F(recession)}{b_F(stable)} = 38.5\%^4$. My model predicts quite well Ford's employment policy from 2006 to 2009 of a 42.2% employment reduction.

I now explain how each parameter effects Ford's gross and net values. Table 6.1 shows my results comparing the two regimes and how a small change in a parameter effects the net value. The actual net value is Ford's market capitalization. The actual gross value is Ford's market capitalization plus Ford's debt and other obligations. The estimated gross value is my model's $V(L, Z_0) = Z_0 J(\eta^*)$, while the estimated equity value is $V(L, Z_0)$ minus the debt and other obligations.

To measure the impact of the various parameters on the model's net value, I use elasticity. Elasticity is frequently used in economics to compare the impacts of changing parameters. Elasticity explains how a one percent change in a parameter causes a resulting percent change in the net value. Since the parameters have many different units, it is hard to make direct comparisons. However, I use elasticity, because the units do not affect the elasticity. I use the net value in the elasticity, since this is an estimate of the firm's market capitalization, which is what the shareholders care about the most.

In the stable regime Ford's estimated gross value is \$282 billion, which is close to its actual gross value of \$257.7 billion.⁵ Its estimated equity value is \$47.5 billion, which is roughly double the 2006 market capitalization of \$22.7 billion. Looking at the recession regime the estimated gross value is \$263 billion, compared to the actual firm value of \$250.5 billion. The net value of Ford in the recession regime is \$17.4 billion,

⁴ The four barriers in the stable regime are $b_H = 8.7571 \times 10^{-028}$, $u_H = 8.9632 \times 10^{-028}$, $u_F = 1.1576 \times 10^{-027}$, $b_F = 1.1849 \times 10^{-027}$, and the barriers in the recession regime are $b_H = 5.3885 \times 10^{-028}$, $u_H = 5.5153 \times 10^{-028}$, $u_F = 7.2909 \times 10^{-028}$, $b_F = 7.4625 \times 10^{-028}$. Note that this is in η space, which due to the parameters are of this magnitude.

⁵ The hypothesis that the estimated gross value and the actual gross value are the same fails to be rejected at the 1% confidence level. The $J_{estimated} = 5.7834 \times 10^{-23}$, and its standard deviation is 3.0434×10^{-23} . This is from the Monte Carlo simulation. For the values, I took the simulated $J_{estimated}$ and its standard deviation, and compared it to the actual J_{actual} , where actual $J_{actual} = V_{actual}/Z_0$, since $V = Z_0 \times J$. Here $J_{actual} = 5.2747 \times 10^{-23}$.

Table 6.1: **Ford Results.** Value in billions of USD. Elasticities are in absolute value and are calculated relative to the net value, i.e. net of debt and other obligations. For instance, the elasticity of 40.96 for A in the stable regime means that a one-percent change in productivity results in a 40% change in the net value of the firm. Parameter values shown are the estimated values. Parameter value were estimated from the many sources (WardsAuto 2009, Ford Media 2008, 2009, 2010, InvestingInBonds 2009, Forbes.com 2012, Brenkers and Verboven 2006, McCracken 2006, Ford 2005, 2007).

Stable Regime, 2006, Firm Value 257.7, Equity Value is 22.7								
	None	μ	σ	r	c_F	Δ	A	ω
parameter value	-	-0.040	0.27	0.11	0.75	1.5	32	56500
gross value	282	282	282	281	282	282	302	264
net value	47.5	46.8	47.4	45.7	47.5	47.5	66.9	29.3
elasticity	-	1.38	0.27	3.72	0.00	0.00	40.96	38.32
Recession Regime, 2009, Firm Value 250.5, Equity Value is 4.5								
parameter value	-	-0.17	0.46	0.45	0.75	1.5	32	60000
gross value	263	263	263	261	263	263	282	246
net value	17.4	16.7	17.2	15.3	17.4	17.4	35.5	0.4
elasticity	-	4.05	0.90	12.18	0.00	0.00	104.41	97.68

while Ford’s January 2009 market capitalization is \$4.5 billion.⁶ This difference in the recession’s net value can be explained by the market incorporating the probability of Ford’s bankruptcy. One can see this in Ford’s corporate bond yields of 45%. Stock investors may have discounted Ford’s cash flows by an even larger discount rate due to the bankruptcy risk and, thus, driving the equity value below the model’s net value. Hence, in the model I use the bond yields as the estimate for the WACC and this is just a lower bound for the discount rate. Since most likely the actual WACC is higher, the model overstates Ford’s value, especially the net value. This is because equity should be discounted with the highest rate. Ford’s bond yields give a lower bound on its weighted average cost of capital (WACC).

Next I compare the effects of changing the parameters across the two regimes. Productivity, A , and wages, ω , have the biggest impact. A one-percent change in one of these parameters would increase the net value by 38% and 100%, respectively. This

⁶ The hypothesis that the estimated gross value and the actual gross value are the same fails to be rejected at the 1% confidence level. The hypothesis that the estimated gross value is zero, is rejected at the 1% level. The $J_{estimated} = 9.7683 \times 10^{-24}$, and its standard deviation is 2.9083×10^{-24} . Here $J_{actual} = 9.2778 \times 10^{-24}$.

is because, since most of the Ford's value is from the flow profits, where productivity and wages are the main drivers. The interest rate plays the next biggest role, it is used to discount all the cash flows. The Table 6.1 shows that a one-percent reduction in the interest rate during the recession regime would increase Ford's net value by 12%. When the interest rate quadrupled from stable regime to recession regime, it also increased the value of the demand parameters. The demand parameters play a minor role, since their impact on demand occurs slowly over time compared to changes in productivity or wages, which effect cash flows instantly. The firing frictions have a negligible impact, since firing costs happen infrequently and in the future.

Improving Ford's market value and financial viability would be good for its employees, Ford shareholders and bondholders, and Ford automobile owners. As we have seen, a one percent increase in the productivity or decrease in wage costs would have the biggest impact for Ford. In union negotiations this could be emphasized. Table 6.2 shows the results for simulating Ford's optimal employment policies for changes in the firing frictions, wages, and productivity. I choose these parameters, because Ford has the ability to change these parameters when negotiating with the union. Small reductions in the delay in shutting down factories or reducing firing costs would not have much of an impact, so Ford could offer raising the delay and severance packages to the union in negotiations, in exchange for wage reductions or productivity increases. These productivity increases could be from increasing capital or changing management and union policies. Table 6.2 shows that increasing severance pay from 0.75 years to 2.25 years would not have an impact on Ford's equity value. However, increasing firing delay to 3 or 4.5 years would harm Ford's equity substantially. Being able to reduce wages by one-percent would double Ford's equity, and even larger wage reductions would have extremely large impacts, assuming the same productivity. Increasing productivity by one-percent would also double Ford's equity, and even larger changes would have substantial effects. These results are con-

sistent with Table 6.1 for the firing frictions, and one-percent changes in wages and productivity. Ford has been following the strategy discussed above by having offered individual workers early retirement packages that are equivalent to up to twice the annual wage in severance packages for firing, without any delay (Reuters 2012).

Table 6.2: **Ford Negotiation Options.** Value in billions of USD.

parameter	base	Δ	Δ	c_F	c_F	ω	ω	ω	A	A	A
change (%)	-	100	200	100	200	-1	-5	-10	1	5	10
Equity value	17.4	7.2	-1.4	17.4	17.4	35.7	125.2	287.1	35.5	119.1	252.4

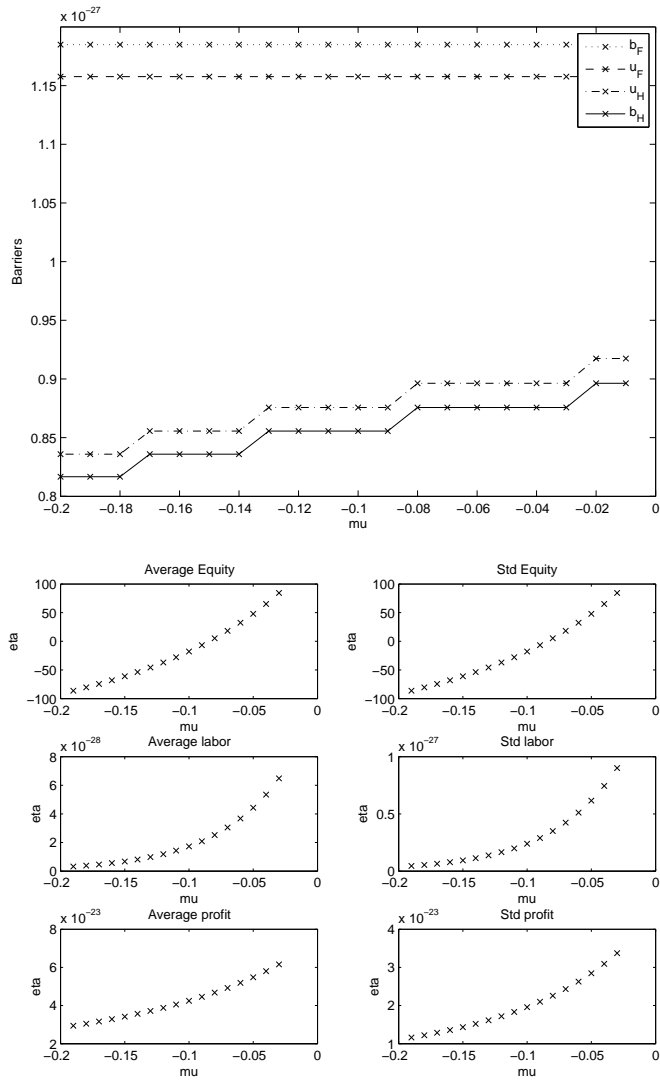


Figure 6.1: **Effect from demand drift μ on Ford.** Parameters are the same as in the stable regime (Table 6.1). The panels were generated by Monte Carlo simulation with 100,000 runs.

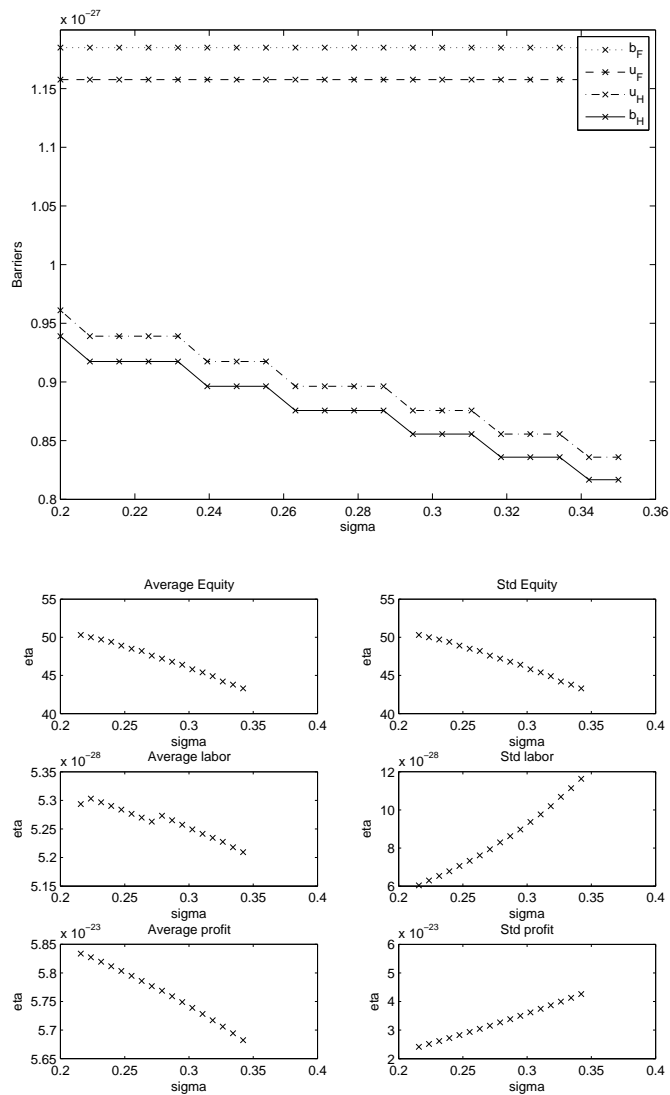


Figure 6.2: **Effect from demand volatility σ on Ford.** Parameters are the same as in the stable regime (Table 6.1). The panels were generated by Monte Carlo simulation with 100,000 runs.

CHAPTER VII

Conclusion

In this thesis I model a firm's optimal employment policy under demand uncertainty and labor market frictions. I model the cost of hiring and firing workers when there is a delay in firing. The optimal conditions are found for the firm's employment policy and I show through contraction mapping, that an optimal policy exists and is unique. Using dynamic programming, I created an algorithm that solves the optimal employment policy. With the optimal labor policies and common parameter values, I simulated the company's profits and labor characteristics. I then perform comparative statics showing the impact of various parameters on the optimal policies, profit and labor levels. Finally, I apply my model to Ford Motors.

I model the firm's cash flows from its production as well as the costly labor adjustments. Additionally, I solve for the value of firing when there is a delay between when the firing is started, and when it is completed. I also introduce the concept of "firm-sized" costs when workers are hired or fired. The other adjustment cost is a proportional cost based on the number of workers changed. I find the necessary conditions on the labor policy from the smooth pasting and value matching conditions that give the optimal policy. I also compare these optimal conditions with "firm-sized" costs with what is used in the literature of proportional and fixed costs. I also verify that the necessary conditions for optimality are sufficient conditions.

Using the model set up I prove that the optimal labor policy exists and is unique. I use contraction mapping of a discrete time model to show existence and uniqueness. I also show that the discrete time model converges to the continuous time model that I derive. Finally, I use the discrete time model to derive an algorithm that uses policy iteration to solve for the optimal policy.

I use comparative statics to examine the impacts of the parameters on the optimal policy. The firing delay encourages firms to fire earlier. It also causes fewer workers to be fired. Raising the proportional costs of changing workers, lowers the chance that workers will be hired or fired, and also reduces the labor and profit levels. The demand volatility has a non-monotonic impact on the firing decisions.

The parameter estimates for Ford Motor Company with my model explains changes in Ford's actual employment level and equity price. I find that the firing costs and firing delay have a negligible impact on Ford's value. Ford's value is dominated by labor's productivity and wages. These insights can help Ford when negotiating with its unions.

APPENDICES

APPENDIX A

Omitted Proofs

A.1 Proof of Proposition 1

Proof. Proof of Proposition 1. The dynamics of η under P is given by

$$d\eta_t = -(\delta + \mu - \sigma^2)\eta_t dt - \sigma\eta_t dW_t, \quad \eta_{t_i^\pi + \Delta_i} = \eta_{(t_i^\pi + \Delta_i)^-} + (I_{\{s_i^\pi = H\}} - I_{\{s_i^\pi = F\}})m_i^\pi$$

which with the change of measure in Section II gives (2.10). I transform the policy $\hat{\pi}$ corresponding to $V(Z, L)$ to a policy, π , in η -space: $t_i^\pi = t_i^{\hat{\pi}}$, $s_i^\pi = s_i^{\hat{\pi}}$ and the normalized amount hired or fired, $m_i^\pi = X_i^{\hat{\pi}}/Z_{t_i^{\hat{\pi}} + \Delta_i}$. This gives a policy $\pi \in \Pi$ for the control of the J -function, where $\pi = \{t_i^\pi, s_i^\pi, m_i^\pi\}$. By the change of probability measure, $V(Z, L) = ZJ(\eta)$ where $J(\eta) = \sup_{\pi \in \Pi} J_\pi(\eta)$. \square

A.2 Proof of Lemma III.1

Proof. Proof of Lemma III.1. The solution of the homogenous part of (3.7) is the power function

$$f_h(\eta) = h_1\eta^{d^+} + h_2\eta^{d^-} \tag{A.1}$$

where $d_{\pm} = \frac{1}{2} + (\delta + \mu)/\sigma^2 \pm \sqrt{((\delta + \mu)/\sigma^2 + \frac{1}{2})^2 + 2\rho/\sigma^2}$. I add any particular solution of (3.7) and I select

$$f_p(\eta) = p_1\eta^\lambda + p_2\eta \quad (\text{A.2})$$

where $p_1 = A^\lambda/c_1$, $p_2 = -\omega/(\delta + r)$, and $c_1 = \rho + (\delta + \mu + \frac{1}{2}\sigma^2(1 - \lambda))\lambda$. Let $c_1 \neq 0$, $\delta \neq -r$, and $\mu \neq r$. \square

A.3 Proof of Lemma III.2

Proof. Proof of Lemma III.2. In equation (3.9) the integral term inside the expectation is a follows

$$E_\eta^{PZ} \left[\int_0^\Delta u(\eta_t, t) dt \right] = p_1\eta^\lambda (1 - e^{-c_1\Delta}) + p_2\eta (1 - e^{-(\delta+r)\Delta}) \quad (\text{A.3})$$

By (2.10), the second term in (3.9) can be written as

$$\begin{aligned} & E_\eta^{PZ} \left[e^{-\rho\Delta} \left[(c_H - K_H) \eta_\Delta + h_1 u_H^{d+} + h_2 u_H^{d-} + p_1 u_H^\lambda + p_2 u_H - c_H u_H \right] \mathbf{I}_{\{\eta_\Delta \leq b_H\}} \right] \\ &= (c_H - K_H) \eta e^{(-\delta - \mu - \frac{1}{2}\sigma^2 - \rho)\Delta} \int_{-\infty}^{z_H} e^{\sigma\sqrt{\Delta}z} \varphi(z) dz \\ &+ \left(h_1 u_H^{d+} + h_2 u_H^{d-} + p_1 u_H^\lambda + p_2 u_H - c_H u_H \right) e^{-\rho\Delta} \int_{-\infty}^{z_H} \varphi(z) dz, \end{aligned} \quad (\text{A.4})$$

where $\varphi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$ is the density of a standard normal distribution,

$b_H = \eta e^{(-\delta - \mu - \frac{1}{2}\sigma^2)\Delta + \sigma\sqrt{\Delta}z_H}$, and this gives $z_H = \frac{\log\left(\frac{b_H}{\eta}\right) + (\delta + \mu + \frac{1}{2}\sigma^2)\Delta}{\sigma\sqrt{\Delta}}$. The first term

on the right-hand-side of (A.4) equals

$$\begin{aligned}
& (c_H - K_H) \eta e^{(-\delta - \mu - \frac{1}{2}\sigma^2 - \rho)\Delta} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z_H} e^{\sigma\sqrt{\Delta}z - \frac{1}{2}z^2} dz \\
&= (c_H - K_H) \eta e^{(-\delta - \mu - \frac{1}{2}\sigma^2 - \rho)\Delta} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z_H} e^{-\frac{1}{2}(z - \sigma\sqrt{\Delta})^2 + \frac{1}{2}\sigma^2\Delta} dz \\
&= (c_H - K_H) \eta e^{(-\delta - \mu - \rho)\Delta} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z_H} e^{-\frac{1}{2}(z - \sigma\sqrt{\Delta})^2} dz = (c_H - K_H) \eta e^{(-\delta - \mu - \rho)\Delta} \\
&\quad \cdot \Phi\left(z_H - \sigma\sqrt{\Delta}\right),
\end{aligned} \tag{A.5}$$

where $\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^z e^{-\frac{1}{2}y^2} dy$ is the cumulative normal distribution and in the last equality I used $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z_H} e^{-\frac{1}{2}(z - \sigma)^2} dz = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z_H - \sigma} e^{-\frac{1}{2}z^2} dz$. From (A.4) and (A.5) I get

$$\begin{aligned}
& E_\eta^{PZ} \left[e^{-\rho\Delta} \left[(c_H - K_H) \eta_\Delta + h_1 u_H^{d+} + h_2 u_H^{d-} + p_1 u_H^\lambda + p_2 u_H - c_H u_H \right] \mathbf{I}_{\{\eta_\Delta \leq b_H\}} \right] = \\
& (c_H - K_H) \eta e^{(-\delta - \mu - \rho)\Delta} \Phi\left(z_H - \sigma\sqrt{\Delta}\right) + \left(h_1 u_H^{d+} + h_2 u_H^{d-} + p_1 u_H^\lambda + p_2 u_H - c_H u_H \right) \\
& \cdot e^{-\rho\Delta} \Phi(z_H).
\end{aligned} \tag{A.6}$$

Similarly, the third term in (3.9) equals

$$\begin{aligned}
& E_\eta^{PZ} \left[e^{-\rho\Delta} \left[-c_F \eta_\Delta + h_1 u_F^{d+} + h_2 u_F^{d-} + c_F u_F \right] \mathbf{I}_{\{\eta_\Delta \geq u_F\}} \right] = \\
& -c_F \eta e^{(-\delta - \mu - \rho)\Delta} \Phi\left(-z_F + \sigma\sqrt{\Delta}\right) + \left(h_1 u_F^{d+} + h_2 u_F^{d-} + p_1 u_F^\lambda + p_2 u_F + c_F u_F \right) \\
& \cdot e^{-\rho\Delta} \Phi(-z_F),
\end{aligned} \tag{A.7}$$

where $z_F = \frac{\log(\frac{u_F}{\eta}) + (\delta + \mu + \frac{1}{2}\sigma^2)\Delta}{\sigma\sqrt{\Delta}}$. The fourth term in (3.9):

$$\begin{aligned}
E_\eta^{PZ} & \left[e^{-\rho\Delta} \left(h_1\eta_\Delta^{d_+} + h_2\eta_\Delta^{d_-} + p_1\eta_\Delta^\lambda + p_2\eta_\Delta \right) \mathbf{I}_{\{b_H < \eta_\Delta < u_F\}} \right] = \\
& h_1\eta^{d_+} e^{(-\delta - \mu - \frac{1}{2}\sigma^2)d_+ \Delta - \rho\Delta} \frac{1}{\sqrt{2\pi}} \int_{z_H}^{z_F} e^{\sigma d_+ \sqrt{\Delta} z - \frac{1}{2}z^2} dz \\
& + h_2\eta^{d_-} e^{(-\delta - \mu - \frac{1}{2}\sigma^2)d_- \Delta - \rho\Delta} \frac{1}{\sqrt{2\pi}} \int_{z_H}^{z_F} e^{\sigma d_- \sqrt{\Delta} z - \frac{1}{2}z^2} dz \\
& + p_1\eta^\lambda e^{(-\delta - \mu - \frac{1}{2}\sigma^2)\lambda \Delta - \rho\Delta} \frac{1}{\sqrt{2\pi}} \int_{z_H}^{z_F} e^{\sigma \lambda \sqrt{\Delta} z - \frac{1}{2}z^2} dz \\
& + p_2\eta e^{(-\delta - \mu - \frac{1}{2}\sigma^2 - \rho)\Delta} \frac{1}{\sqrt{2\pi}} \int_{z_H}^{z_F} e^{\sigma \sqrt{\Delta} z - \frac{1}{2}z^2} dz \\
& = h_1\eta^{d_+} e^{(-\delta - \mu - \frac{1}{2}\sigma^2)d_+ \Delta + (\frac{1}{2}\sigma^2 d_+^2 - \rho)\Delta} \left[\Phi(z_F - \sigma d_+ \sqrt{\Delta}) - \Phi(z_H - \sigma d_+ \sqrt{\Delta}) \right] \\
& + h_2\eta^{d_-} e^{(-\delta - \mu - \frac{1}{2}\sigma^2)d_- \Delta + (\frac{1}{2}\sigma^2 d_-^2 - \rho)\Delta} \left[\Phi(z_F - \sigma d_- \sqrt{\Delta}) - \Phi(z_H - \sigma d_- \sqrt{\Delta}) \right] \\
& + p_1\eta^\lambda e^{(-\delta - \mu - \frac{1}{2}\sigma^2)\lambda \Delta + (\frac{1}{2}\sigma^2 \lambda^2 - \rho)\Delta} \left[\Phi(z_F - \sigma \lambda \sqrt{\Delta}) - \Phi(z_H - \sigma \lambda \sqrt{\Delta}) \right] \\
& + p_2\eta e^{-(\delta+r)\Delta} \left[\Phi(z_F - \sigma \sqrt{\Delta}) - \Phi(z_H - \sigma \sqrt{\Delta}) \right].
\end{aligned} \tag{A.8}$$

The last term in (3.9) can be written as

$$E_\eta^{PZ} [e^{-\rho\Delta} K_F \eta_\Delta] = K_F \eta e^{-(\delta+r)\Delta}. \tag{A.9}$$

Collecting (A.3), (A.6), (A.7), (A.8), and (A.9) I get Lemma III.2. \square

APPENDIX B

Verification Theorem

In this chapter I prove Theorem III.3.

B.1 Preparatory Material

Definition B.1. A hiring or firing policy $\pi = \{(\tau_i, s_i, m_i)\}_{i=0}^\infty \in \Pi$ is a sequence of hiring/firing times, τ_i , decisions, s_i , to either hire (H) or fire (F), and amounts, m_i , such that for all $j \in \mathbb{N}$,

1. $0 \leq \tau_i + \Delta_i < \tau_{i+1}$ a.s.
2. τ_i is a stopping time, s_i is \mathcal{F}_{τ_i} measurable, and m_i is $\mathcal{F}_{\tau_i + \Delta_i^-}$ measurable,

where \mathbb{N} denotes the set of natural numbers.

Definition B.2. A hiring or firing policy is called admissible if $\tau_{i+1} > \tau_i + \Delta_i$, $s_i \in \{F, H\}$, if $s_i = F$ then $m_i < \eta_{(\tau_i + \Delta_i)^-}$, and if $s_i = H$, then $m_i \geq 0$. The set of all admissible policies is denoted by Π . Note that m_i can be negative for firing, i.e. the firm hires at the end of the firing delay and does not count as an additional decision.

The verification of my optimal strategy is based on Liu (2004). The steps of the proof are as follows:

STEP 1: Lemma B.3(a) shows that if there exists a function J that follows Proposition 2 with some technical conditions then it is superior value function of any admissible policy.

STEP 2: In Lemma B.3(b) I show that my policy is optimal.

STEP 3: I show in Lemma B.3(a) that my policy satisfies technical conditions 4-6.

STEP 4: Finally I show in Lemma B.4 that this policy is admissible.

Augment the value function from taking only one variable η to also include time t , i.e., $J(\eta) = J(\eta, t)$. Define

$$\Gamma^{J,\pi}(\eta_{t_i-}, t_{i-}, s_i, \bar{W}) = \begin{cases} J(\eta_{t_i} + m_i^\pi, t_i) - J(\eta_{t_i-}, t_{i-}), & \text{if } s_i = H \\ e^{-\rho\Delta} J(\eta_{t_i+\Delta} + m_i^\pi, t_i + \Delta) - J(\eta_{t_i-}, t_{i-}), & \text{if } s_i = F \end{cases}$$

which is the realized change in the value function under policy π from the i^{th} hiring or firing decision. Since m for hiring does not depend on the path of η I rewrite $\Gamma^{J,\pi}(\eta_{t_i-}, t_{i-}, H, \bar{W})$ as $\Gamma^{J,\pi}(\eta_{t_i-}, t_{i-}, H)$. For firing the decision is made Δ time units later. This value needs the path of \bar{W} during the firing process to find the realized flow profit, since η is not sufficient within the delay. I also need this because arbitrary policies for m may depend on the path of η during the firing delay.

Let $\tilde{T} \in [0, \infty)$ be fixed. Since I do not know if \tilde{T} is in a firing delay, I define

$$T(\tilde{T}) = \begin{cases} \tilde{T}, & \text{if } t_i + \Delta_i \leq \tilde{T} < t_{i+1} \\ t_{i-}, & \text{if } t_i \leq \tilde{T} < t_i + \Delta_i \end{cases} \quad (\text{B.1})$$

which is obviously outside of the firing delay.

Lemma B.3. (Necessary and sufficient conditions)

(a) Suppose there exists a C^1 function $J : \mathbb{R}^+ \rightarrow \mathbb{R}$ which is C^2 except over a Lebesgue measure zero subset of \mathbb{R}^+ such that

1. $\mathcal{A}J(\eta) + u(\eta) \leq 0$

2. $J(\eta) \geq M_H J(\eta)$
3. $J(\eta) \geq M_F J(\eta)$
4. $E^{P^Z} \left[\sum_{i=0}^{\infty} \int_{t_i+\Delta_i}^{t_{i+1}-\wedge T} | -e^{-\rho s} \sigma \eta \frac{\partial J}{\partial \eta} |^2 ds \right] < \infty, \forall T \in [0, \infty)$
5. $\lim_{\tilde{T} \rightarrow \infty} E^{P^Z} [e^{-\rho T(\tilde{T})} J(\eta_{T(\tilde{T})})] = 0$ for any $\eta_{T(\tilde{T})}$ following an admissible policy
6. $\{e^{-\rho T} J(\eta_T)\}$ is uniformly integrable

where one of the conditions 1, 2, or 3 must hold with equality for any η outside firing processes. T is defined in (B.1) as outside the firing process for conditions 4-6. Then

$$J(\eta_t) \geq J_\pi(\eta_t), \forall \pi \in \Pi, \eta_t > 0, t \text{ is outside firing delays} \quad (\text{B.2})$$

where J_π is the value function from following policy π .

(b) Define the no-action set as

$$\text{NA} = \{\eta_t : J(\eta_t) > M_F J(\eta_t) \text{ or } J(\eta_t) > M_H J(\eta_t); t \text{ is outside firing delays}\}$$

If $J(\eta_t) = M_F J(\eta_t)$ then $\check{s}_t = F$, and if $J(\eta_t) = M_H J(\eta_t)$ then $\check{s}_t = H$. Also define $b_H = \sup \{\eta \geq 0 : J(\eta) = M_H J(\eta)\}$ and $b_F = \inf \{\eta \geq 0 : J(\eta) = M_F J(\eta)\}$.

Define J as in (3.1) with the analytical function replacing my guess f :

$$J(\eta) = \begin{cases} M_F J(\eta), & b_F \leq \eta \\ J_w(\eta), & b_H < \eta < b_F \\ M_H J(\eta), & 0 \leq \eta \leq b_H \end{cases} \quad (\text{B.3})$$

Define the hiring and firing policy

$$\tilde{\pi} \equiv (\tilde{\tau}_1, \tilde{\tau}_2, \dots; \check{s}_1, \check{s}_2, \dots; \check{m}_1, \check{m}_2, \dots)$$

inductively as follows: $\check{\tau}_0 = 0, s_0 = H, m_0 = 0$ and for all $k \in \{0, 1, 2, \dots\}$ I have

$$\check{\tau}_{k+1} = \inf\{t > \check{\tau}_k + \Delta_k : \eta_t^{(k)} \notin \text{NA}\}, \quad \check{m}^k = \check{m}^{J, \check{s}_k}$$

where $\eta_t^{(k)}$ results from policy $\check{\pi}_k \equiv (\check{\tau}_1, \check{\tau}_2, \dots, \check{\tau}_k; \check{s}_1, \check{s}_2, \dots, \check{s}_k; \check{m}_1, \check{m}_2, \dots, \check{m}_k)$ and \check{m}^k is defined above for J . If $\check{\pi}$ is admissible, then $J(\eta) = J_{\check{\pi}}(\eta)$ and the hiring and firing policy $\check{\pi}$ is optimal, where $J_{\check{\pi}}(\eta)$ is the value function defined in (2.9).

Proof. (a) Assuming that J satisfies the conditions in part (a), I let $\pi \in \Pi$ be any admissible Markovian hiring and firing policy, where $\pi \equiv (\tau_1, \tau_2, \dots; s_1, s_2, \dots; m_1, m_2, \dots)$. By Øksendal (1998, Theorem 11.2.3) Markov policies can maximize firm value, and thus I only consider them in this paper. For all $k \geq 0$, define $\theta_k = (\tau_k + \Delta_k) \wedge T$ with $\tau_0 = 0, s_0 = H$, and $m_0 = 0$ with zero hiring cost. For notational convenience let $\eta_t = \eta_t^\pi, \tau_i = \tau_i^\pi$, and $s_i = s_i^\pi$. Then I can write for every $n \in \mathbb{N}$:

$$\begin{aligned} e^{-\rho\theta_n} J(\eta_{\theta_n}) - J(\eta) &= \\ \sum_{i=1}^n [e^{-\rho(T \wedge \tau_i^-)} J(\eta_{T \wedge \tau_i^-}) - e^{-\rho\theta_{i-1}} J(\eta_{\theta_{i-1}})] &+ \sum_{i=1}^n I_{\{\tau_{i-} < T\}} e^{-\rho\tau_{i-}} \Gamma^{J, \pi}(\eta_{\tau_{i-}}, \tau_{i-}, s_i, \bar{W}) \end{aligned} \quad (\text{B.4})$$

I know that η_t is continuous semi-martingale in the stochastic interval $[\theta_k, \tau_{k+1})$ and J is C^2 except over Lebesgue measure zero subset of \mathbb{R}^+ . Thus, Lemma (45.9) of Rogers and Williams (2000) holds, and for all $i \in \mathbb{N}$ I have

$$e^{-\rho(T \wedge \tau_i^-)} J(\eta_{T \wedge \tau_i^-}) - e^{-\rho\theta_{i-1}} J(\eta_{\theta_{i-1}}) = \int_{\theta_{i-1}}^{\tau_i \wedge T} e^{-\rho s} \mathcal{A}J(\eta_s) ds - \int_{\theta_{i-1}}^{\tau_i \wedge T} e^{-\rho s} \sigma \eta \frac{\partial J}{\partial \eta} d\bar{W}_s. \quad (\text{B.5})$$

By condition 1 of the Lemma B.3 (a), I have

$$e^{-\rho(T \wedge \tau_i^-)} J(\eta_{T \wedge \tau_i^-}) - e^{-\rho\theta_{i-1}} J(\eta_{\theta_{i-1}}) \leq - \int_{\theta_{i-1}}^{\tau_i \wedge T} u(\eta_s, s) ds - \int_{\theta_{i-1}}^{\tau_i \wedge T} e^{-\rho s} \sigma \eta \frac{\partial J}{\partial \eta} d\bar{W}_s. \quad (\text{B.6})$$

Combining (B.4) - (B.6) and taking expectations, I have

$$J(\eta) \geq E^{PZ} \left[e^{-\rho\theta_n} J(\eta_{\theta_n}) + \sum_{i=1}^n \left(\int_{\theta_{i-1}}^{\tau_i - \wedge T} u(\eta_s, s) ds + \int_{\theta_{i-1}}^{\tau_i - \wedge T} e^{-\rho s} \sigma \eta \frac{\partial J}{\partial \eta} d\bar{W}_s - I_{\{\tau_{j_-} < T\}} e^{-\rho\tau_i} \Gamma^{J, \pi}(\eta_{\tau_i}, \tau_{i-}, s_i, \bar{W}) \right) \right] \quad (\text{B.7})$$

By condition 4 of Lemma B.3 (a), I get $E^{PZ} \left[\sum_{i=1}^n \int_{\theta_{i-1}}^{\tau_i - \wedge T} \sigma \eta \frac{\partial J}{\partial \eta} d\bar{W}_s \right] = 0$ for any fixed n . From condition 6 in Lemma B.3 (a), I get

$$\lim_{n \rightarrow \infty} E^{PZ} [e^{-\rho\theta_n} J(\eta_{\theta_n})] = E^{PZ} [e^{-\rho T} J(\eta_T)].$$

By limit $n \rightarrow \infty$ in (B.7) and the monotone convergence theorem, I have

$$J(\eta) \geq E^{PZ} \left[e^{-\rho T} J(\eta_T) + \sum_{i=1}^{\infty} \left(\int_{\theta_{i-1}}^{\tau_i - \wedge T} u(\eta_s, s) ds - I_{\{\tau_{j_-} < T\}} e^{-\rho\tau_i} \Gamma^{J, \pi}(\eta_{\tau_i}, \tau_{i-}, s_i, \bar{W}) \right) \right]$$

Taking limit $T \rightarrow \infty$ and using condition 5 in Lemma B.3 and the monotone convergence theorem, I obtain

$$J(\eta) \geq E^{PZ} \left[\sum_{i=1}^{\infty} \left(\int_{\theta_{i-1}}^{\tau_i -} u(\eta_s, s) ds - e^{-\rho\tau_i} \Gamma^{J, \pi}(\eta_{\tau_i}, \tau_{i-}, s_i, \bar{W}) \right) \right] \quad (\text{B.8})$$

If policy π has a finite number of decisions, k , then let $\tau_{k+i} = \infty$, $i > 0$. Consider the second sum in (B.4). Condition 2 and the definition of M_H give

$$J(\eta, \tau_{i-}) \geq M_H J(\eta, \tau_{i-}) \geq J(\eta + m_i, \tau_i) - m_i c_H - K_H \eta$$

where the second inequality is from the of the supremum in the definition M_H . I can rewrite this as follows

$$m_i c_H + K_H \eta \geq J(\eta + m_i, \tau_i) - J(\eta, \tau_{i-}) = \Gamma^{J, \pi}(\eta_{\tau_i}, \tau_{i-}, H) \quad (\text{B.9})$$

Similarly with condition 3 and by taking conditional expectation over the path of $\bar{W}(t)$ during the delay I get $E^{PZ} [J(\eta, \tau_{i-})|\bar{W}(t)] \geq E^{PZ} [M_F J(\eta, \tau_{i-})|\bar{W}(t)]$, where the right hand side is greater than $E^{PZ} \left[\int_{\tau_i}^{\tau_i+\Delta} u(\eta_s, s) ds + e^{-\rho\Delta} \{J(\eta_{\tau_i+\Delta-} - m_i) - (K_F + K_H I_{\{m_i < 0\}}) \eta_{\tau_i+\Delta-} - [m_i]^+ c_F + [m_i]^- c_H \} |\bar{W}(t) \right]$. Rearranging I get

$$\begin{aligned} & E^{PZ} \left[- \int_{\tau_i}^{\tau_i+\Delta} u(\eta_s, s) ds + e^{-\rho\Delta} \{ (K_F + K_H I_{\{m_i < 0\}}) \eta_{\tau_i+\Delta-} + [m_i]^+ c_F - [m_i]^- c_H \} \right] \\ & \geq E^{PZ} \left[e^{-\rho\Delta} J(\eta_{\tau_i+\Delta} + m_i, \tau_i + \Delta) - J(\eta, \tau_{i-}) \right] = E^{PZ} \left[\Gamma^{J, \pi}(\eta_{\tau_{i-}}, \tau_{i-}, F, \bar{W}) \right] \end{aligned} \quad (\text{B.10})$$

Using (B.8)-(B.10) I get

$$\begin{aligned} J(\eta) \geq & E^{PZ} \left[\int_0^\infty u(\eta_s, s) ds - \sum_{i=1}^\infty e^{-\rho\tau_{i-}} I_{\{s_i=H\}} \{ [m_i]^+ c_H + K_H \eta_{t_i} \} \right. \\ & \left. - \sum_{i=1}^\infty e^{-\rho(\tau_i+\Delta-)} I_{\{s_i=F\}} \{ [m_i]^+ c_F - [m_i]^- c_H + (K_F + K_H I_{\{m_i < 0\}}) \eta_{\tau_i+\Delta-} \} \right] \end{aligned} \quad (\text{B.11})$$

Thus, $J(\eta) \geq J_\pi(\eta)$ since the right-hand-side of (B.11) equals (2.9). In other words the value function J as defined in part (a) majorizes any hiring/firing policy outside the firing process, and m is optimally selected by the realized η at the end of the delay.

(b) By the definition of NA, (B.6) must hold with equality. By (2.12) and (2.13) and the definitions of M_H and M_F , (B.9) and (B.10) hold with equality.

Combining with (B.2), I get

$$J(\eta) \geq \sup_{\pi \in \Pi} J_\pi(\eta) \geq J_{\tilde{\pi}}(\eta) = J(\eta).$$

Hence, $J(\eta) = J_{\tilde{\pi}}(\eta)$ and $\tilde{\pi}$ is optimal. It remains to be shown for this lemma that the policy constructed in (b) satisfies the conditions in (a).

CONDITIONS 1-3 OF LEMMA B.3: Define $J_w \equiv J_{\tilde{\pi}}(\eta)$ for $\eta \in [b_H, b_F]$. By construction of J , condition 1 holds with equality for $\eta \in [b_H, b_F]$, condition 2 holds with equality for $\eta \in (0, b_H)$, and condition 3 holds with equality for $\eta \geq b_F$. I now

need to show that conditions 2 and 3 hold with inequality in this no-action region. Consider the hiring function. For $\eta \geq u_H$ it is clearly not optimal to hire, since the firm would not hire anyone, and still pay the firm-size cost $K_H\eta$. Thus, $M_HJ(\eta) < J_w(\eta)$ for $\eta \geq u_H$. For $\eta \in [b_H, u_H]$ I get $M_HJ(\eta)|_{\eta < u_H} = J(u_H) - c_Hu_H + (c_H - K_H)\eta < J(u_H)$. This is a linear function of η . By (3.2a) and (3.3a), M_HJ passes through J_w at b_H with the same slope. Given (3.10), J_w is convex at b_H . Thus, its tangent line, M_HJ , starts below J_w at b_H and remains below by concavity of J_w up to u_H and from $M_HJ(\eta) < J(u_H)$. Therefore, $J > M_HJ$ for $\eta \in \text{NA}$.

I show that $M_FJ(\eta) < J(\eta)$ holds for $\eta \in \text{NA}$ by contradiction. Assume $M_FJ(\eta) > J(\eta) \Leftrightarrow H(\eta) = J_w(\eta) - M_FJ(\eta) > 0$, for some $\eta \in \text{NA}$. $H \in C^1$ since $J_w, M_FJ \in C^1$. The assumption can hold for either all $\eta \in \text{NA}$ or some $\eta \in \text{NA}$. If the assumption holds for only some $\eta \in \text{NA}$ by continuity there exists $\eta \in \text{NA}$ such that $H(\eta) = 0$. But by the definition of b_F being the infimum where this happens, I have a new b_F , which is a contradiction. It is also possible that the assumption holds for all $\eta \in \text{NA}$, or $H(\eta) < 0$ over $[b_H, b_F)$. Consider the endpoint b_H . If $H(b_H) < 0$, then since both M_FJ and M_HJ are continuous, $H(b_H) = J_w(b_H) - M_FJ(b_H) = M_HJ(b_H) - M_FJ(b_H) < 0 \Rightarrow M_FJ(b_H) > M_HJ(b_H)$. This means that in the hiring region, it is optimal to fire which is a contradiction of the definition of the hiring region. Otherwise, $H(b_H) = 0$ and then $b_F = b_H$, which is another contradiction. Thus, $M_FJ < J$ for $\eta \in (b_H, b_F)$.

CONDITION 4 OF LEMMA B.3: I wish to show that the expected discounted sum of $\eta \frac{\partial J}{\partial \eta}$ squared is finite. J is defined outside of firing processes and thus $b_H \leq \eta \leq b_F < \infty$. Now I show that $\frac{\partial J}{\partial \eta}$ is finite. The partial derivative of this is bounded since J_w is the sum of constants times η raised to non-zero powers (see Lemma III.1).

CONDITIONS 5 AND 6 OF LEMMA B.3: By the definition of T , optimal policy $\eta_T \in [b_H, b_F]$ and J is equal to the value of waiting. This is the sum of constants times η raised to non-zero powers (see Lemma III.1). Thus, its maximum value is

bounded. Discounting this finite value as \tilde{T} goes to infinity means that the value goes to zero and condition 5 holds.

For uniform integrability it is sufficient to show that there exists $p > 1$ and constant B such that $E[|e^{-\rho T} J_{\tilde{\pi}}(\eta_T)|^p] \leq B$ (see e.g. Steele 2001, p. 49). Let $B = \max_{\eta \in [b_H, b_F]} J_w(\eta)^p$ which is bounded by the argument above. Thus, $E[|e^{-\rho T} J_{\tilde{\pi}}(\eta_T)|^p] \leq B$. Thus, condition 6 of Lemma B.3 holds. \square

Lemma B.4. (Admissibility) *Let $\tilde{\pi}$ be the hiring and firing policy of Theorem III.3. Then $\tilde{\pi}$ is an admissible hiring and firing policy.*

Proof. Proof of Lemma B.4. Let $\{\tau_j\}$ be the times the firm hires or fires according to policy $\tilde{\pi}$. These times are when η exits NA outside the firing process. $\{\tau_j\}$ are clearly stopping times with $0 \leq \tau_j + \Delta_j < \tau_{j+1}$, a.s., $\forall j \in \mathbb{N}$. Further, $\check{s}_i \in \mathcal{F}_{\tau_i}$ and $\check{m}_i^{J, s_i} \in \mathcal{F}_{(\tau_i + \Delta_i)^-}$. Thus $\tilde{\pi}$ is admissible. \square

B.2 Proof of Proposition 4

Proof. Proof of Proposition 4. Consider b_H first. I know that $J'(b_H) = c_H - K_H$ and $J'(u_H) = c_H$, and that J is convex to the right of b_H by (3.10). Since u_H maximizes the value of hiring, the hiring decision's second order condition means that J is concave at u_H . This implies that there exists $\hat{\eta} \in (b_H, u_H)$ such that $J''(\hat{\eta}) = 0$ and $J'''(\hat{\eta}) < 0$. Note that $J'''(\hat{\eta}) > 0$ would contradict (3.11). Taking the derivative of PDE (3.7) with $J = f_w$ I get

$$\frac{1}{2}\sigma^2\eta^2J'''(\eta) + (\sigma^2 - \delta - \mu)\eta J''(\eta) - (\delta + r)J'(\eta) + v'(\eta) = 0$$

At $\hat{\eta}$ I get $-(\delta + r)J'(\hat{\eta}) + v'(\hat{\eta}) > 0$ or $v'(\hat{\eta}) > (\delta + r)J'(\hat{\eta}) > 0$. Since $v'(\hat{\eta}) > 0$, $\hat{\eta} < \eta^*$, since $v(\cdot)$ is concave and finds its maximum at η^* . Therefore $b_H < \eta^*$. Similarly for b_F with no delay. \square

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