Five-Dimensional Gauged Supergravity with Higher Derivatives

by

Kentaro Hanaki

A dissertation submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
(Physics)
in The University of Michigan
2012

Doctoral Committee:

Associate Professor Finn Larsen, Chair
Professor Jianming Qian
Professor Peter E. van Keken
Associate Professor James T. Liu
Associate Professor Leopoldo A. Pando-Zayas

ACKNOWLEDGEMENTS

It is my great pleasure to appreciate those who have been helping me understand physics, conduct research, and also have an enjoyable life during my graduate study. First of all, I would like to thank Finn Larsen, my thesis advisor, for his guidance, support and enlightening discussions through my graduate study.

It has been an invaluable experience to work with Sera Cremonini, Masahiro Ibe, Hai Lin, Jim Liu, Yutaka Ookouchi, Chang-soon Park, Cheng Peng and Phil Szepietowski for collaborations while I have been studying in Michigan. I have learnt a great amount of physics and also they have taught me how valuable and interesting discussing physics with other physicists is.

I would also like to thank the graduate students who have been studying together with me in the high-energy theory group: Ibou Bah, Tim Cohen, Sandeepan Gupta, Sunghoon Jung, Jing Shao, Zhichen Zhao. Not only are we good friends, but also we are good competitors, and I have been so encouraged by their efforts to trying to be good physicists. I have also enjoyed insightful discussions with excellent physicists: Ratin Akhoury, Henriette Elvang, Tom Hartman, Kenji Kadota, Ryuichiro Kitano, Jae-Hyuk Oh, Leo Pando-Zayas, and especially Keisuke Ohashi and Yuji Tachikawa, the collaborators when I was studying in Japan.

Finally, I would like to thank my family and friends, who have been always encouraging me. This was my first experience to live outside of Japan and their unconditional supports have really motivated me a lot.

TABLE OF CONTENTS

LIST OF APPENDICES

Appendix

CHAPTER I

Introduction

Quantum mechanics and general relativity are two pillars of modern physics. On one hand, quantum mechanics governs the law of physics at a short distance scale. The characteristic property of quantum mechanics is the wave-particle duality: all the matters in nature are particles whose probabilities to take some particular states are determined by their wave functions. Quantum theory is known to consistently describe three of the four known types of interactions—electromagnetic interactions, weak interactions and strong interactions—and they constitute the standard model of particle physics. On the other hand, general relativity is a classical theory of the remaining interactions—gravitational interactions. In general relativity, as opposed to physics until the nineteenth century, the spacetime is no longer a fixed object, but a dynamically changing object, and its dynamics is governed by the Einstein equation. The gravitational force is weak at currently accessible short distance scales, but gets stronger than other interactions at large distances[1]. Hence general relativity plays a key role in discussing large scale dynamics of the spacetime such as the evolution of the universe and the formation of stars.

Given these two great achievements in the twentieth century's physics, one natural

[1]Generally, the gravitational force is weaker at large distance. As opposed to other interactions, however, it cannot be shielded by anything, such as confinement in strong interactions, heavy force carriers in weak interactions or cancellation of the charges in electromagnetic interactions, so it gets the strongest at large distance.

question arises: can one combine quantum theory and general relativity to construct a quantum theory of gravity? Not only is the quantum gravity of theoretical interest, but it has several applications to the real world. One application is to discuss rigorously the beginning of the universe. It is now widely accepted that the universe started from the Big Bang, a very large density state and its subsequent explosion. In fact, one can mathematically show in the classical framework that the universe starts from a singularity at which the energy density is infinitely large [71]. However, the problem of this analysis is that one, in principle, cannot use classical theory at the initial singularity: there, everything is confined in an infinitely small volume, so the physics involved should be quantum theory. Thus, one needs a quantum theory of gravity to analyze the very beginning of the universe. Another, somewhat related application is to understand (small) black hole physics. Classically, a black hole is an object from which nothing can escape. However, one can show semi-classically that a black hole thermally emits particles and reduces its size. Again, the problem is the final stage of the reduction. If the size of the black hole gets very small, one needs to use quantum theory to describe its dynamics.

Unfortunately, naive attempts for unifying quantum theory and general relativity fail. The problem stems from the fact that quantum theory generally has a process where a particle emits a virtual particle which is re-absorbed in a short period of time by the original particle, and that process yields a divergent contribution to the probability[2]. If the virtual particles are the ones involved in electromagnetic, weak or strong interactions, we know how to regularize the divergences and how to introduce counterterms in the theory to cancel the divergences via renormalization techniques. But if the divergences are associated with graviton emission and re-absorption, one

[2]Recently, the possibility that the gravitational theory with the maximal number of supersymmetries does not have the short distance divergences has been investigated. See [48] and references therein for details.

generally needs an infinite number of counterterms, which destroys the predictability of the theory. The divergence occurs in a very short time scale, which can be translated into a short distance by Lorentz transformations, so a predictable quantum theory of gravity should require some special ingredients at short distance.

One reasonable, and the only currently known-to-work modification at short distance is to replace particles in the theory by small strings. The resulting theory is called "string theory" and has the following properties/advantages:

1. String theory includes two types of strings, open strings and closed strings, and different particles are described by different types of oscillations of the strings. Consistent string theories are known to always have a graviton in the closed string spectrum [119, 105], so a consistent string theory is naturally a quantum theory of gravity.

2. String theory, in addition to strings, has extended objects called "D-branes" in its spectrum [102]. In a weakly interacting theory, a D-brane is a static object on which open strings end. On the other hand, the mass of a D-brane gets small compared to strings in a strongly coupled theory, so it behaves as a dynamical object in the strong coupling regime.

3. All the known string theories consisting only of bosonic degrees of freedom have tachyons in their spectra, implying that the theories are expanded around unstable vacua. One can of course try to find the stable vacua for the bosonic string theories, but an easier way to eliminate tachyons is to introduce fermionic degrees of freedom. All the perturbatively consistent theories of that type have ten spacetime dimensions and supersymmetry, a symmetry exchanging bosons and fermions, and called type IIA, type IIB, type I, heterotic $SO(32)$ and heterotic

$E_8 \times E_8$, respectively. All the theories are further unified into eleven-dimensional M-theory, and all the string theories can be obtained as particular loci of M-theory [74, 117].

4. String theory is defined in ten dimensions, so one somehow needs to reduce it to four dimensions for real world applications. One can do that either by considering that six of the dimensions are very small [26] or that our universe is realized on four-dimensional D-branes [102]. A large set of theories of this type have a large gauge group, and may include standard model interactions (and many more). Hence one would expect that string theory is not just the quantum theory of gravity, but a promising candidate for the "theory of everything". Semi-realistic models have been constructed using small six extra dimensions, D-branes, seven-branes in F-theory and M-theory.

In summary, string theory is a quantum theory of gravity, and furthermore can possibly describe all the interactions in nature.

In this thesis, we focus on the large distance description of string theory, called "supergravity" [53, 47]. Supergravity is a supersymmetric field theory which describes gravitational interactions or, in other words, a supersymmetric version of Einstein gravity. The reduction from string theory to supergravity can be understood as follows. The typical length of strings in string theory is much smaller than the scale one can reach by any currently established experiment. Therefore, string theory should be effectively described as particles interacting with each other, because the string length is negligibly small. The interactions of particles are then described by using quantum field theory. One can determine the form of the theory by the invariance under general coordinate transformations and supersymmetry, and the gravitational part of the theory is uniquely determined to be that of supergravity.

In particular, we consider a special kind of corrections to supergravity from string theory, namely the higher derivative (or higher curvature, in gravitational perspective) corrections. Typically, higher derivative corrections are understood as corrections to supergravity from short distance physics. If we restrict our attention to the gravitational sector, the action should be written purely in terms of quantities associated with the spacetime which are covariant under general coordinate transformations, namely the curvature of the spacetime R. Then, a general higher derivative action is written as

$$S = \frac{1}{16\pi G_d}\int d^d x\sqrt{g}(R+\alpha' R^2+\alpha'^2 R^3+\cdots)\ , \tag{1.1}$$

where G_d is the d-dimensional Newton's constant and α' is proportional to the square of the string length. α' is an extremely small quantity, so the terms depending on the positive powers of α' do not affect physical processes unless the matters are confined in a very small region, typically at string length scale, and the extremely large density of matters curves the spacetime so that the curvature is of order $1/\alpha'$. If the curvature is not of order $1/\alpha'$, the higher derivative corrections are small, and one can treat the corrections perturbatively. In this thesis, we are considering the leading order corrections, namely four derivative or curvature squared corrections to supergravity and their supersymmetrization.

We are focusing on five-dimensional gauged supergravity in this thesis. Technically, gauged supergravity is defined as a supergravity theory in which the gravitino, the superpartner of the graviton, is charged under some internal gauge group. However, what is really important is that gauged supergravity, unlike the ungauged case, has a negative cosmological constant, so it is defined on an Anti-de Sitter (AdS) space. The gravitational theory on AdS space is believed to have a dual, non-gravitational conformal field theory[3] description living on the boundary of AdS space via the AdS/CFT correspondence [92]. Five-dimensional supergravity is of particular interest because the dual field theory is four-dimensional and realistic, and also the AdS/CFT correspondence in this case is "derivable" from string theory, as will be explained more in detail later.

As an application of five-dimensional gauged supergravity with higher derivatives, we discuss the hydrodynamic properties of gauge theory plasma, observed in heavy ion collisions, via the AdS/CFT correspondence (See [27] for a comprehensive review for this subject.). The hydrodynamics of gauge theory plasma has been studied extensively using perturbative quantum field theory, using lattice gauge theory and using the AdS/CFT correspondence. One interesting hydrodynamic quantity which characterizes the properties of gauge theory plasma is the shear viscosity to entropy density ratio. It is pretty difficult to compute this for $SU(N)$ gauge theory with $N = 3$, which describes the real quark-gluon plasma found in experiments, but it can be computed in the large N limit using the AdS/CFT duality [103, 83]. The result is

$$\frac{\eta}{s} = \frac{1}{4\pi}\ , \tag{1.2}$$

where η is the shear viscosity of a plasma and s is the entropy density. The shear viscosity to entropy density ratio was initially done in the context of the supergravity dual to $\mathcal{N} = 4$ $SU(N)$ super Yang-Mills theory, but later extended to various other theories. For example, the ratio can be calculated for

- $\mathcal{N} = 1$ SCFTs [83]
- non-conformal theories [18]

[3]Conformal field theory is a particular kind of quantum field theory which, roughly spaeking, is invariant under the change of the length scale.

- theories with fundamental matters [94]
- finite chemical potential [93, 108, 91]
- time dependent background [77]

and all of the theories yield the same result as (1.2). In this sense, the result is universal at large N. Further, based on this extremely small viscosity, it has been proposed that this ratio is the lower bound (KSS bound) of the shear viscosity to entropy density ratio.

The result (1.2) is quite interesting because it matches with the experimental measurement of the quark-gluon plasma (See, for example, [109].)

$$\frac{\eta}{s} \leq 0.2 \, . \tag{1.3}$$

In this sense, one can argue that this is the first non-trivial "experimental verification" of string theory, provided that the large N limit is a good approximation of the $N = 3$ gauge theory.

However, if one considers finite N corrections, the shear viscosity to entropy density ratio deviates from the KSS value $1/(4\pi)$. As is explained later, the finite N corrections in the CFT side correspond to higher derivative corrections in the gravity side. Inclusion of the leading order finite N corrections yields a deviation of the ratio from $1/(4\pi)$. Furthermore, one can show that the KSS bound is violated in the presence of finite N corrections [81, 17]. We discuss the effects of such finite N corrections and also the effects of introducing finite chemical potential in this thesis. In the real-world experiments, N is three and there is a finite chemical potential for the $U(1)$ baryonic charge, so our discussion is an attempt to fill the gap between the analysis in [103, 83] and the real-world experiments.

The rest of the thesis is organized as follows. In Chapter 2, we review the off-shell formulation of five-dimensional supergravity with curvature squared corrections. We first discuss the building blocks of the theory and see how one can construct the higher derivative terms in a systematic way. In Chapter 3, we gauge the supergravity theory obtained in Chapter 2 and gain a supergravity theory with a negative cosmological constant. We also discuss how to obtain the vacuum AdS_5 solution in this chapter. In Chapter 4, we apply the gauged supergravity theory to discuss the thermodynamic behavior of gauge theory plasma. The primary tool to relate the supergravity to gauge theory is the AdS/CFT correspondence, which is reviewed in this chapter. The dictionary translating the physical quantities in the AdS side into those in the CFT side and vice versa is also discussed. Finally, in Chapter 5, we compute the shear viscosity to entropy density ratio in the presence of higher derivative corrections and chemical potential and see how the KSS result is altered.

This thesis is based on the works [34, 35] in collaborations with Sera Cremonini, James T. Liu and Phillip Szepietowski. More specifically, the discussions in Chapter 3 and 4 are based on [34], and those in Chapter 5 are based on [35]. Five-dimensional gauged supergravity with higher derivatives is one of the various topics on which the author has been working during the graduate study. The other topics he has been working on are supersymmetric classical solutions of multiple M2-brane theories [67], phenomenology of gauge mediated supersymmetry breaking [69, 66] and three-dimensional supergravity with higher spin fields and its holographic dual [70].

CHAPTER II

Off-Shell Supergravity with Higher Derivatives

In this section, we review the off-shell formulation of supergravity with eight supercharges in five dimensions [121, 120, 87, 54, 12, 11][1]. The five-dimensional supergravity with eight supercharges is first formulated in an on-shell method, in which the closure of the supersymmetry algebra requires the equations of motion [63]. A shortcoming of the on-shell formulation is that one needs to consider the closure of the algebra and the invariance of the action simultaneously. The equations of motion are affected by the form of the action, so while trying to preserve the supersymmetries of the action by adding terms, one also needs to take care of the algebraic structure of the supersymmetry. On the other hand, the off-shell formulation does not require the equations of motion for the closure of the algebra. Therefore, when one adds terms to the action to preserve supersymmetry, one need not worry about the modification of the algebra. This is the reason why we employ the off-shell formulation in this thesis.

We start with the observation that general coordinate transformations and local Lorentz transformations, under which any extension of general relativity[2] is invariant, act on spacetime coordinates in the same way as local Poincaré transformations.

[1] The off-shell formulation of supergravity was first developed for theories with four supercharges in four dimensions [110, 51, 106] and extended to eight supercharges in four dimensions [16, 44, 52] and six dimensions [10] as well as five dimensions. An interested reader can refer to excellent reviews [96, 115] and references therein.

[2] We simply call this class of theory "gravitational theories" in the rest of the thesis.

This observation opens up a possibility that a gravitational theory can be regarded as a "gauge theory" of the Poincaré symmetries. In the gauge theory language, the gauge field associated with the local translations could be regarded as vielbein, and that associated with the local Lorentz transformations as spin connection in gravitational theories. Unfortunately, the situation is not so simple for the following reason: the local translations do not act on the fields in the same way as general coordinate transformations. In other words, local translational invariance in gauge theory formulation is just an internal symmetry of the theory, so one needs to realize it as a spacetime symmetry to obtain a gravitational theory. This problem is solved by imposing appropriate constraints.

In this thesis, we utilizes conformal tensor calculus [50, 80, 79], in which we impose the superconformal invariance, not just super-Poincaré invariance, on the theory. The reason why we impose this larger symmetry is quite simple: generically, the larger symmetries a theory has, the simpler it gets. By introducing this larger symmetry group, we facilitate the construction of the action. Then, we introduce expectation values for auxiliary fields to break the conformal symmetry and obtain Poincaré supergravity. At this point, the theory is off-shell, meaning that the closure of the super-Poincaré algebra requires the equations of motion. One obtains the on-shell Poincaré supergravity by eliminating the auxiliary fields by equations of motion.

So, our approach is summarized as follows:

1. Construct a gauge theory of superconformal invariance.

2. Impose constraints to obtain spacetime superconformal symmetries.

3. Introduce expectation values on the fields to break the conformal symmetries.

4. Eliminate the auxiliary fields using equations of motion.

This approach is illustrated in Section 2.1 using an example of Einstein gravity. In the context of supergravity, the first two steps are discussed in Section 2.2 and the last two are discussed in Chapter 4.

The rest of this chapter is organized as follows. For an illustration of the idea to construct the supergravity action in the superconformal setup, we start with constructing Einstein gravity in the conformal setup in Section 2.1. In Section 2.2, we go over how to construct a gauge theory of superconformal symmetries, the constraints to obtain the spacetime superconformal symmetries and the off-shell supergravity multiplet, called the Weyl multiplet. In Section 2.3, we discuss the supermultiplets needed to construct an supergravity action coupled to gauge fields. In Section 2.4, we discuss how to construct a two derivative conformal supergravity action. In Section 2.5, we discuss the construction of supersymmetric higher derivative terms, or more precisely, the supersymmetric completion of curvature squared terms. Finally, in Section 2.6, we discuss the application of the supersymmetric four derivative terms in string theory.

2.1 Einstein Gravity as a Gauge Theory

In this section, we construct Einstein gravity using the conformal gauge theory techniques discussed above. Einstein gravity is a theory which describes the dynamics of the spacetime. In other words, if the spacetime is described by a pseudo-Riemannian manifold M endowed with a metric $g_{\mu\nu}$, where the distance on the manifold is defined as $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$ for a spacetime coordinate x^μ, Einstein gravity is a theory which determines how $g_{\mu\nu}$ changes with time. A fundamental requirement on the theory is that the physical equations must be independent of the choice of coordinates, or equivalently, invariant under the general coordinate transformations (sometimes called diffeomorphism):

$$x \to f(x) \ , \tag{2.1}$$

where $f(x)$ is an arbitrary function of x^μ. The infinitesimal version for this transformation is

$$x \to x + \epsilon(x) \ , \tag{2.2}$$

where $\epsilon(x)$ is an arbitrary infinitesimal function of x^μ. Hence, the action of the theory must be written in terms of quantities consisting of the spacetime metric $g_{\mu\nu}$ and must be invariant under the general coordinate transformations (2.2). One can build such a quantity using the Riemann curvature tensor $R_{\mu\nu\rho\sigma}$. One advantage of using the Riemann curvature is that a quantity consisting of the Riemann curvatures with all the indices contracted with the inverse metric $g^{\mu\nu}$ is invariant under the general coordinate transformations. Thus, the simplest possible form of the action consists of the Ricci scalar $R = R_{\mu\nu\rho\sigma}g^{\mu\rho}g^{\nu\sigma}$ and the invariant volume form $d^5x\sqrt{\det g}$. It is given by

$$S = \frac{1}{16\pi G_5}\int d^5x\sqrt{\det g}R \ , \tag{2.3}$$

where the five-dimensional Newton's constant G_5 is introduced to make the action S dimensionless. The goal of this section is to reproduce this action in the conformal gauge theory formalism.

For this purpose, one introduces a vielbein $e^a_\mu(x)$ by

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = \eta_{ab}(e^a_\mu dx^\mu)(e^b_\nu dx^\nu) \ , \tag{2.4}$$

where η_{ab} is the metric of flat Minkowski space. In other words, the vielbein is used to define a orthonormal basis $e^a = e^a_\mu dx^\mu$ for the cotangent bundle of the spacetime

M. One can also introduce the spin connections $\omega^a{}_b(x) := \omega_\mu{}^a{}_b dx^\mu$ as a change of the orthonormal basis for two infinitesimally close points:

$$de^a = \omega^a{}_b \wedge e^b \; . \tag{2.5}$$

Then, one can write the Riemann curvature as

$$R^{ab} := R_{\mu\nu}{}^{ab} dx^\mu dx^\nu = d\omega^{ab} + \omega^{ac} \wedge \omega_c{}^b \; , \tag{2.6}$$

where the spacetime indices μ, ν and the indices for the orthonormal basis a, b can be interchanged by multiplying with the vielbeins. Since we introduced the orthonormal basis, the theory has additional local symmetries which rotate the orthonormal basis. These symmetries are called local Lorentz symmetries and, combined with the general coordinate transformations, constitute the local Poincaré invariance of the theory.

One can construct Einstein gravity as a gauge theory with the local Poincaré invariance. Here, however, we introduce a larger symmetry group, namely the conformal group, of which the Poincaré group is a subgroup. It consists of dilatations $\mathbf{D}$ and special conformal transformations $\mathbf{K}^a$ as well as translations $\mathbf{P}^a$ and Lorentz transformations $\mathbf{M}^{ab}$ in the Poincaré group. As has been discussed at the beginning of this chapter, one first constructs a gravitational theory with local conformal symmetries, and then reduce the symmetries to the Poincaré group by giving an expectation value to fields transforming under $\mathbf{D}$ and $\mathbf{K}^a$ transformations.

Now that we know the symmetry group of the theory, let us construct the action. We are considering a gauge theory of the conformal group, so the first step is to introduce the gauge fields, covariant derivatives and field strengths. For $\mathbf{P}^a$, $\mathbf{M}^{ab}$, $\mathbf{D}$ and $\mathbf{K}^a$, we introduce the gauge fields e^a_μ, $\omega_\mu{}^{ab}$, b_μ and f^a_μ, respectively. The covariant derivative is defined as $D_\mu = \partial_\mu - \sum h^A_\mu \mathbf{X}_A$, where the sum is taken for all the generators $\mathbf{X}^A$. The field strength corresponding to a generator $\mathbf{X}^A$ is denoted by $R_{\mu\nu}(\mathbf{X}^A)$.

Then, one can construct an action invariant under the local transformations, which yields a gauge theory with the local conformal symmetries being realized as internal symmetries. This completes the step one of the construction.

Once we obtain a conformal gauge theory, the next step is to identify the conformal symmetry, realized as an internal symmetry in the gauge theory, as the real spacetime symmetry. We have seen that Einstein gravity has local Poincaré symmetries, in which $\mathbf{P}^a$ generates the general coordinate transformations and $\mathbf{M}^{ab}$ generates the local Lorentz transformations. Therefore, we need to identify the local translation in our gauge theory with diffeomorphisms and Lorentz transformations in Einstein gravity. In order to illustrate this point, let us consider the local translation of the gauge field ${e_\mu}^a$:

$$\delta_P(\xi^a){e_\mu}^a = \delta_{diff}(\xi^\lambda){e_\mu}^a - \delta_A \xi^\lambda h^A_\lambda - {\hat{R}_{\mu\nu}}{}^a(P)\xi^\nu \ . \tag{2.7}$$

The first two terms are essentially the covariantized diffeomorphism, and the local translation differs from the covariantized diffeomorphism by the last term. In order to identify these two symmetries, one can impose

$${R_{\mu\nu}}^a(P) = 0 \ . \tag{2.8}$$

Here, ${R_{\mu\nu}}^a$ is, roughly speaking, in the form of $2\partial_{[\mu}{e_{\nu]}}^a - 2{\omega_{[\mu}}^{ab}e_{\nu]b}$ plus covariantization terms, so the constraint can be interpreted as the torsionless condition in general relativity. Similarly, the identification of the symmetries requires

$${R_{\mu\nu}}^{ab}(M)e^\nu_b = 0 \ . \tag{2.9}$$

This constraint allows f^a_μ to be expressed in terms of the Ricci tensor as

$${f_\mu}^a = \frac{1}{6}(R_{\mu\nu} - \frac{1}{8}g_{\mu\nu}R)e^{\nu a} \ . \tag{2.10}$$

In this way, we identified the symmetries of our gauge theory as those in Einstein gravity. We have completed the step two of the construction. Another point which is worth mentioning is that this constraint makes the covariant curvature $R_{\mu\nu}^{ab}(M)$ traceless. Thus, for a background in which the nontrivial independent field is only the vielbein, $R_{\mu\nu}^{ab}(M)$ is the Weyl tensor of the metric, i.e.

$$R_{\mu\nu}{}^{ab}(M) = R_{\mu\nu}{}^{ab} + \frac{4}{3}R_{[\mu}^{[a}e_{\nu]}^{b]} - \frac{1}{6}e_{[\mu}^{[a}e_{\nu]}^{b]}R \ , \tag{2.11}$$

where R_{abcd} is the ordinary curvature tensor constructed from the metric.

So far, we have not specified the form of the action. In order to obtain the Einstein-Hilbert action, we take our action to be

$$S = \int d^5xe\phi D^\mu D_\mu\phi \ , \tag{2.12}$$

where e is an abbreviation of $\det e_\mu^a$ and may be written as $\sqrt{\det g_{\mu\nu}}$ in terms of the metric, and ϕ is a scalar field with conformal weight $3/2$. If one writes the covariant derivatives explicitly, one obtains

$$S = \int d^5xe\phi e_a^\mu[(\partial_\mu - 3b_\mu)D^a\phi - \omega_\mu{}^{ab}D_b\phi + f_\mu{}^a\phi] \ , \tag{2.13}$$

where $D_\mu\phi = \partial_\mu\phi - \frac{3}{2}b_\mu\phi$. To obtain Einstein gravity, one first must reduce the conformal symmetries to Poincaré symmetries by giving expectation values to the fields. To break the special conformal symmetry, we note that b_μ is not invariant for any value of b_μ. Therefore, we take $b_\mu = 0$ to break the special conformal symmetry. The dilatation can be fixed by taking $\phi = M$, where M is some dimensionful constant. Then, all the terms but the last term vanishes in (2.13), so we obtain

$$S = \frac{M^3}{16}\int d^5x\sqrt{g}R \ . \tag{2.14}$$

By setting $M^3 = (\pi G_5)^{-1}$, one obtains Einstein gravity. This procedure completes step three of the construction. Since there is no auxiliary field to be eliminated in this theory, we have completed step four as well.

2.2 Conformal Supergravity as a Gauge Theory

As in the bosonic example, the first step to construct a supergravity action is to define a gauge theory of the superconformal group[3]. The superconformal algebra consists of translation $\mathbf{P}^a$, Lorentz transformation $\mathbf{M}^{ab}$, dilatation $\mathbf{D}$, special conformal transformation $\mathbf{K}^a$, $SU(2)_R$ transformation $\mathbf{U}^{ij}$, supercharge $\mathbf{Q}^i$ and superconformal charge $\mathbf{S}^i$, where $i, j, \cdots$ are indices for $SU(2)_R$ symmetry. In gauge theory, one defines the gauge fields corresponding to these generators, which are denoted as

$$e_\mu{}^a, \quad \omega_\mu{}^{ab}, \quad b_\mu, \quad f_\mu{}^a, \quad V_\mu^{ij}, \quad \psi_\mu^i, \quad \phi_\mu^i \ . \tag{2.15}$$

We use two types of covariant derivatives in this thesis, and they are defined as

$$\begin{aligned} \mathcal{D}_\mu &= \partial_\mu - \sum_{\mathbf{X}^A = \mathbf{M}^{ab}, \mathbf{D}, \mathbf{U}^{ij}} h_\mu^A \mathbf{X}_A, \\ \hat{\mathcal{D}}_\mu &= \mathcal{D}_\mu - \sum_{\mathbf{X}^A = \mathbf{Q}^i, \mathbf{S}^i, \mathbf{K}^a} h_\mu^A \mathbf{X}_A, \end{aligned} \tag{2.16}$$

where $\mathbf{X}_A$ denote the generators and h_μ^A the corresponding gauge fields. As is obvious, $\mathcal{D}_\mu$ is covariantized with respect to $\mathbf{M}^{ab}$, $\mathbf{D}$ and $\mathbf{U}^{ij}$, and $\hat{\mathcal{D}}_\mu$ is fully covariantized under the superconformal symmetry. The fully covariantized field strengths are defined as

$$[\hat{\mathcal{D}}_\mu, \hat{\mathcal{D}}_\nu] = - \sum_{\mathbf{X}^A = \mathbf{Q}^i, \mathbf{M}^{ab}, \mathbf{D}, \mathbf{U}^{ij}, \mathbf{S}^i, \mathbf{K}^a} \hat{R}_{\mu\nu}^A \mathbf{X}_A \ . \tag{2.17}$$

The explicit form of the field strengths and the covariant derivatives acting on the transformation parameters are given in Appendix B.

[3] The detailed algebraic structure is not important in this thesis. An interested reader can refer to Section 2.5 of [115].

The superconformal symmetries of this gauge theory can be identified, as in the bosonic case, as the spacetime superconformal symmetries by the following constraints:

$$\hat{R}_{\mu\nu}{}^{a}(P) = 0, \quad \gamma^{\mu}\hat{R}^{i}_{\mu\nu}(Q) = 0, \quad \hat{R}^{ab}_{\mu\nu}(M)e^{\nu}{}_{b} = 0\ . \tag{2.18}$$

These are called "conventional constraints", and we use this set of constraints throughout this thesis.

The conventional constraints are not invariant under the original supersymmetry transformations. One way to see that is to notice the fact that the conventional constraints, as can be seen from (B.6), let us solve the gauge fields $\omega_{\mu}{}^{ab}$, ϕ^{i}_{μ} and $f_{\mu}{}^{a}$ in terms of other gauge fields[4]. This reduces the number of degrees of freedom and yields a mismatch between the bosonic and fermionic degrees of freedom. Hence, one needs to introduce auxiliary fields to match the number of bosonic and fermionic degrees of freedom, and also modify the supersymmetry transformations so that the constraints are invariant. We do not know of any systematic way to carry out this procedure, but the resulting multiplet is known as the Weyl multiplet[87, 12], which consists of the independent gauge fields and auxiliary fields

$$v^{ab}, \quad \xi^{i}, \quad D, \tag{2.19}$$

where v^{ab} is an antisymmetric tensor, ξ^{i} is an $SU(2)_{R}$-Majorana spinor and D is a scalar field. The commutation relation between two **Q**s and that for **S** and **Q** are

[4]The explicit expressions for dependent fields in terms of independent fields are given in (B.1)

given by

$$(2.20)\quad [\delta_Q(\varepsilon_1),\ \delta_Q(\varepsilon_2)] = \delta_P(2i\bar{\varepsilon}_1\gamma_a\varepsilon_2) + \delta_M(2i\bar{\varepsilon}_1\gamma^{abcd}\varepsilon_2 v_{ab}) + \delta_U(-4i\bar{\varepsilon}_1^i\gamma\cdot v\varepsilon_2^j) + \delta_S(\cdots) + \delta_K(\cdots)\,,$$

$$(2.21)\quad [\delta_S(\eta), \delta_Q(\varepsilon)] = \delta_D(-2i\bar{\varepsilon}\eta) + \delta_M(2i\bar{\varepsilon}\gamma^{ab}\eta) + \delta_U(-6i\bar{\varepsilon}^{(i}\eta^{j)}) + \delta_K(\cdots)\,.$$

For other commutators, it would be more useful to write them in terms of the gauge transformations, rather than commutation relations. The transformation properties under Q^i, S^i and K^a are given in the form of $\delta = \bar{\epsilon}^i Q_i + \bar{\eta}^i S_i + \xi_K^a K_a$ by

(2.22)

$$\begin{aligned}
\delta e_\mu{}^a &= -2i\bar{\varepsilon}\gamma^a\psi_\mu, \\
\delta\psi_\mu^i &= \mathcal{D}_\mu\varepsilon^i + \frac{1}{2}v^{ab}\gamma_{\mu ab}\varepsilon^i - \gamma_\mu\eta^i, \\
\delta b_\mu &= -2i\bar{\varepsilon}\phi_\mu - 2i\bar{\eta}\psi_\mu - 2\xi_{K\mu}, \\
\delta V_\mu^{ij} &= -6i\bar{\varepsilon}^{(i}\phi_\mu^{j)} + 4i\bar{\varepsilon}^{(i}\gamma\cdot v\psi_\mu^{j)} - \frac{i}{4}\bar{\varepsilon}^{(i}\gamma_\mu\chi^{j)} + 6i\bar{\eta}^{(i}\psi_\mu^{j)}, \\
\delta v_{ab} &= -\frac{i}{8}\bar{\varepsilon}\gamma_{ab}\chi - \frac{32}{i}\bar{\varepsilon}\hat{R}_{ab}(Q), \\
\delta\chi^i &= D\varepsilon^i - 2\gamma^c\gamma^{ab}\varepsilon^i\hat{\mathcal{D}}_a v_{bc} + \gamma\cdot\hat{R}(U)^i{}_j\varepsilon^j - 2\gamma^a\varepsilon^i\epsilon_{abcde}v^{bc}v^{de} + 4\gamma\cdot v\eta^i, \\
\delta D &= -i\bar{\varepsilon}\hat{\not{\mathcal{D}}}\chi - 8i\bar{\varepsilon}\hat{R}_{ab}(Q)v^{ab} + i\bar{\eta}\chi.
\end{aligned}$$

This Weyl multiplet includes a graviton $e_\mu{}^a$ and gravitinos ψ^i, and constitute a part of the on-shell supergravity multiplet. Note that $\hat{R}(M)$ should satisfy a similar relation as (2.11):

$$(2.23)\qquad R_{\mu\nu}{}^{ab}(M) = R_{\mu\nu}{}^{ab} + \frac{4}{3}R^{[a}_{[\mu}e^{b]}_{\nu]} - \frac{1}{6}e^{[a}_{[\mu}e^{b]}_{\nu]}R\,.$$

2.3 Supermultiplets

In this section, we introduce three particular types of supermultiplets, namely, vector multiplets, hypermultiplets and linear multiplets. Vector and hypermultiplets are not optional, but are required to construct an off-shell Poincaré supergravity multiplet[5], as is discussed later on. The linear multiplet is used to construct an invariant action.

2.3.1 Vector Multiplet

A vector multiplet is a multiplet which has gauge fields W_μ^I, $SU(2)$-Majorana gauginos Ω_i^I, scalar fields M^I and auxiliary fields Y_{ij}^I, whose i and j indices are symmetric, as its components. I is the index for a gauge group **G** and the fields are conveniently denoted as, for instance, $W_\mu = W_\mu^I T^I$ in terms of the generators of the gauge group T^I. The **Q**- and **S**-transformations of the components are given by

(2.24)

$$\begin{aligned}
\delta W_\mu &= -2i\bar{\varepsilon}\gamma_\mu\Omega + 2i\bar{\varepsilon}\psi_\mu M, \\
\delta M &= 2i\bar{\varepsilon}\Omega, \\
\delta\Omega^i &= -\frac{1}{4}\gamma\cdot\hat{F}(W)\varepsilon^i - \frac{1}{2}\hat{\not{\mathcal{D}}}M\varepsilon^i + Y^i{}_j\varepsilon^j - M\eta^i, \\
\delta Y^{ij} &= 2i\bar{\varepsilon}^{(i}\hat{\not{\mathcal{D}}}\Omega^{j)} - i\bar{\varepsilon}^{(i}\gamma\cdot v\Omega^{j)} - \frac{i}{4}\bar{\varepsilon}^{(i}\chi^{j)}M - 2ig\bar{\varepsilon}^{(i}[M,\Omega^{j)}] - 2i\bar{\eta}^{(i}\Omega^{j)}.
\end{aligned}$$

The transformation laws above shows that the supersymmetry transformations are not separable from the gauge transformations under the gauge group **G**. It is thus required to modify the supersymmetry transformations as

$$[\delta_Q(\varepsilon_1), \delta_Q(\varepsilon_2)] = (\text{R.H.S. of } (2.20)) + \delta_G(-2i\bar{\varepsilon}_1\epsilon_2 M). \tag{2.25}$$

[5]One can use, instead of a hypermultiplet, another multiplet to form an on-shell supergravity multiplet, but we use a hypermultiplet in this thesis.

Therefore, the supercovariant curvature is given by

$$\hat{F}_{\mu\nu}(W) = 2\partial_{[\mu}W_{\nu]} - g[W_\mu, W_\nu] + 4i\bar{\psi}_{[\mu}\gamma_{\nu]}\Omega - 2i\bar{\psi}_\mu\psi_\nu M. \tag{2.26}$$

Not only does one need the vector multiplets to couple supergravity to gauge fields, but also it is required as a part of the on-shell gravity multiplet. The on-shell supergravity multiplet should contain a vector field, graviphoton, as well as a graviton and gravitinos[63]. Since there is no vector field in the Weyl multiplet, one needs to combine the Weyl multiplet with a supermultiplet including a vector field to construct a supergravity multiplet. One can also see that the gauge transformations showing up in the right hand side of the supersymmetry algebra (2.25) is, from the definition, the gauge transformation associated with the graviphoton on-shell.

2.3.2 Hypermultiplet

A hypermultiplet consists of scalars $\mathcal{A}_\alpha^i$, spinors ζ_α and auxiliary fields $\mathcal{F}_\alpha^i$. They carry the index α $(= 1, 2, \dots, 2r)$ of $USp(2r)$. The scalars satisfy the reality condition $\mathcal{A}_\alpha^i = -(\mathcal{A}_i^\alpha)^*$, and ζ_α are $USp(2r)$-Majorana spinors. A subgroup G' of the gauge group G can act on the index α as a subgroup of $USp(2r)$. The Q and S transformations of $\mathcal{A}_\alpha^i$ and ζ_α are given by

$$\begin{aligned} \delta\mathcal{A}_\alpha^i &= 2i\bar{\varepsilon}^i\zeta_\alpha, \\ \delta\zeta^\alpha &= \hat{\not{\mathcal{D}}}\mathcal{A}_j^\alpha\varepsilon^j - \gamma\cdot v\varepsilon^j\mathcal{A}_j^\alpha - gM_*\mathcal{A}_j^\alpha\varepsilon^j + 3\mathcal{A}_j^\alpha\eta^j, \end{aligned} \tag{2.27}$$

where $\hat{\not{\mathcal{D}}}$ and M_* include the 'central charge' gauge transformation Z. The quantity g is the coupling constant, and the notation X_*Y represents the action of generators of the gauge transformation,

$$(X_*Y)^\alpha = X^I t_I{}^\alpha{}_\beta Y^\beta + X^0 Z Y^\alpha, \tag{2.28}$$

where X takes values in a Lie algebra, Y takes values in its representation, and ${t_I}^\alpha{}_\beta$ is the representation matrix. The closure of the algebra thus determines the 'central charge' gauge transformation of $\mathcal{A}^i_\alpha$ via $\mathcal{F}^i_\alpha$, though we set $Z = 0, (\mathcal{F}^i_\alpha = 0)$ in this thesis[6].

2.3.3 Linear Multiplet

A linear multiplet consists of a real scalar L^{ij}, whose i and j indices are symmetric, a $SU(2)$-Majorana spinor φ^i, a vector E_a, and a scalar N. The **Q** and **S** transformations on the components yield

$$
\begin{aligned}
(2.29)\qquad \delta L^{ij} &= 2i\bar\varepsilon^{(i}\varphi^{j)}, \\
\delta\varphi^i &= -\hat{\not{\mathcal{D}}} L^{ij}\varepsilon_j + \frac{1}{2}\gamma^a\varepsilon^i E_a + \frac{1}{2}\varepsilon^i N \\
&\quad +2\gamma\cdot v\varepsilon_j L^{ij} + gM_* L^{ij}\varepsilon_j - 6L^{ij}\eta_j, \\
\delta E^a &= 2i\bar\varepsilon\gamma^{ab}\hat{\mathcal{D}}_b\varphi - 2i\bar\varepsilon\gamma^{abc}\varphi v_{bc} + 6i\bar\varepsilon\gamma_b\varphi v^{ab} \\
&\quad +2ig\bar\varepsilon\gamma^a M_*\varphi - 4ig\bar\varepsilon^i\gamma^a\Omega^j_* L_{ij} - 8i\bar\eta\gamma^a\varphi, \\
\delta N &= -2i\bar\varepsilon\hat{\not{\mathcal{D}}}\varphi - 3i\bar\varepsilon\gamma\cdot v\varphi + \frac{1}{2}i\bar\varepsilon^i\chi^j L_{ij} + 4ig\bar\varepsilon^{(i}\Omega^{j)}_* L_{ij} - 6i\bar\eta\varphi.
\end{aligned}
$$

The algebra closes if E^a satisfies the following Q- and S-invariant constraint:

$$
(2.30)\qquad \hat{\mathcal{D}}_a E^a + gM_* N + 4ig\bar\Omega_*\varphi + 2gY^{ij}_* L_{ij} = 0.
$$

One important property concerning the linear multiplet is that any symmetric, real composite bosonic field L^{ij}, which is invariant under S transformations, automatically leads to the above transformation law with suitable choices of φ^i, E^a and N. Thus, the construction of a linear multiplet can be carried out by repeated supersymmetric transformations starting from the lowest component, L^{ij}.

[6]Interested readers can refer to [87] and [54] for details.

2.4 Invariant Action at Two Derivatives

In this section, we construct the invariant action for supergravity coupled to n_V+1 $U(1)$ vector multiplets. The starting point of the construction of an invariant action is to realize that the following Lagrangian is invariant under the superconformal transformations:

$$
\begin{aligned}
e^{-1}\mathcal{L}(\mathbf{V}\cdot\mathbf{L}) \equiv\; & Y^{ij}\cdot L_{ij} + 2i\bar{\Omega}\cdot\varphi + 2i\bar{\psi}^{a}_{i}\gamma_a\Omega_j\cdot L^{ij} \\
& -\frac{1}{2}W_a\cdot\left(E^a - 2i\bar{\psi}_b\gamma^{ba}\varphi + 2i\bar{\psi}^{(i}_{b}\gamma^{abc}\psi^{j)}_{c}L_{ij}\right) \\
& +\frac{1}{2}M\cdot\left(N - 2i\bar{\psi}_b\gamma^{b}\varphi - 2i\bar{\psi}^{(i}_{a}\gamma^{ab}\psi^{j)}_{b}L_{ij}\right).
\end{aligned} \tag{2.31}
$$

Here, we restrict our consideration to neutral L_{ij} for simplicity. Then, one can embed two sets of vector multiplets, V^I and V^J, into the linear multiplet to obtain the action quadratic in the gauge field strengths. More concretely, we identify the lowest component of the linear multiplet L_{ij} as

$$
L_{ij}(\mathbf{V}\cdot\mathbf{V}) = Y^I_{ij}f_I - i\bar{\Omega}^I_i\Omega^J_j f_{IJ}, \tag{2.32}
$$

where $f_{IJ} = \partial^2 f(M)/\partial M^I\partial M^J$ and $f(M)$ is an arbitrary quadratic homogeneous function of M^I. Then, one can obtain the higher components by carrying out the supersymmetry transformations and comparing both sides. The result is

$$
\begin{aligned}
(2.33)\quad \varphi_i(\mathbf{V}\cdot\mathbf{V}) &= -\frac{1}{4}\chi_i f \\
&\quad + \left(\hat{\not{\mathcal{D}}}\Omega_i^I - \frac{1}{2}\gamma\cdot v\Omega_i^I - g[M,\Omega]^I\right) f_I \\
&\quad + \left(-\frac{1}{4}\gamma\cdot\hat{F}^I(W)\Omega^J + \frac{1}{2}\hat{\not{\mathcal{D}}}M^I\Omega^J - Y^I\Omega^J\right) f_{IJ}, \\
E_a(\mathbf{V}\cdot\mathbf{V}) &= \hat{\mathcal{D}}^b\left(4v_{ab}f + \hat{F}^I_{ab}(W)f_I + i\bar{\Omega}^I\gamma_{ab}\Omega^J f_{IJ}\right) \\
&\quad + \left(-2ig[\bar{\Omega},\gamma_a\Omega]^I + g[M,\hat{\mathcal{D}}_a M]^I\right) f_I \\
&\quad + \left(-2ig\bar{\Omega}^I\gamma_a[M,\Omega]^J + \frac{1}{8}\epsilon_{abcde}\hat{F}^{bcI}(W)\hat{F}^{deJ}(W)\right) f_{IJ}, \\
N(\mathbf{V}\cdot\mathbf{V}) &= -\hat{\mathcal{D}}^a\hat{\mathcal{D}}_a f + \left(-\frac{1}{2}D - 3v^2\right) f \\
&\quad + \left(-2\hat{F}_{ab}(W)v^{ab} + i\bar{\chi}\Omega^I + 2ig[\bar{\Omega},\Omega]^I\right) f_I \\
&\quad + \left(\begin{array}{c} -\frac{1}{4}\hat{F}^I_{ab}(W)\hat{F}^{abJ}(W) + \frac{1}{2}\hat{\mathcal{D}}^a M^I\hat{\mathcal{D}}_a M^J \\ +2i\bar{\Omega}^I\hat{\not{\mathcal{D}}}\Omega^J - i\bar{\Omega}^I\gamma\cdot v\Omega^J + Y^I_{ij}Y^{Jij}\end{array}\right) f_{IJ}.
\end{aligned}
$$

By plugging this into (2.31), one can obtain the invariant two derivative action. The resulting action is completely symmetric in I, J and K, so we obtain an invariant action $\mathcal{L}_V$ given a gauge-invariant cubic polynomial $\mathcal{N} = c_{IJK}M^I M^J M^K$. The fermionic part does not matter in this thesis, so we just present the bosonic part of the action:

(2.34)

$$
\begin{aligned}
e^{-1}\mathcal{L}_V\Big|_{\text{bosonic}} &= \mathcal{N}\left(-\frac{1}{2}D + \frac{1}{4}R(M) - 3v^2\right) + \mathcal{N}_I\left(-2v^{ab}F^I_{ab}(W)\right) \\
&\quad + \mathcal{N}_{IJ}\left(-\frac{1}{4}F^I_{ab}(W)F^{abJ}(W) + \frac{1}{2}\mathcal{D}^a M^I\mathcal{D}_a M^J + Y^I_{ij}Y^{Jij}\right) \\
&\quad - e^{-1}\frac{1}{24}\epsilon^{\lambda\mu\nu\rho\sigma}\mathcal{N}_{IJK}W^I_\lambda F^J_{\mu\nu}(W)F^K_{\rho\sigma}(W).
\end{aligned}
$$

Note that there is a Chern-Simons interaction, $W\wedge F\wedge F$, which stems from the $W_a\cdot E^a$ term in the invariant action formula (2.31). The strength of the interaction is $\mathcal{N}$, implying that it governs the entire vector multiplet Lagrangian.

One thing which we need to mention for the vector multiplet Lagrangian (2.34) is that the equation of motion of the auxiliary field D imposes $\mathcal{N}=0$. This is obviously problematic, because it forces the coefficient of the Einstein-Hilbert term to vanish. A way out is to introduce a compensator hypermultiplet and add the compensator action to the vector multiplet action. The compensator action is obtained by embedding the square of hypermultiplets into the linear multiplet and using the invariant action (2.31) as

$$e^{-1}\mathcal{L}_H\big|_{\text{bosonic}} = \mathcal{D}^a\mathcal{A}_i^{\bar{\alpha}}\mathcal{D}_a\mathcal{A}_\alpha^i + \mathcal{A}_i^{\bar{\alpha}}(gM)^2\mathcal{A}_\alpha^i + \mathcal{A}^2\left(\frac{1}{8}D + \frac{3}{16}R(M) - \frac{1}{4}v^2\right) + 2gY^{ij}_{\alpha\beta}\mathcal{A}_i^{\bar{\alpha}}\mathcal{A}_j^\beta, \tag{2.35}$$

where $\mathcal{A}^2 \equiv \mathcal{A}_i^{\bar{\alpha}}\mathcal{A}_\alpha^i = \mathcal{A}_i^\beta d_\beta{}^\alpha\mathcal{A}_\alpha^i$ with the metric $d_\alpha{}^\beta$ arranged to be $\delta_\beta{}^\alpha$ for a compensator. Here, we have already eliminated the auxiliary fields $\mathcal{F}_{i\alpha}$ using their equations of motion. Let us now consider a system coupled to n_V+1 conformal vector multiplets, $I=0,\ldots,n_V$, and one conformal hypermultiplet, $\mathcal{A}_\alpha^i$ $(i,\alpha=1,2)$, as a compensator. We let its action be $\mathcal{L}_0=\mathcal{L}_H-\frac{1}{2}\mathcal{L}_V$. The equation of motion for D gives $\mathcal{A}^2+2\mathcal{N}=0$, while the scalar curvature appears in the Lagrangian in the form

$$(\frac{3}{16}\mathcal{A}^2-\frac{1}{8}\mathcal{N})R(M). \tag{2.36}$$

Thus, we can make the Einstein-Hilbert term canonical by fixing the dilatational gauge transformation $\mathbf{D}$ via the condition $\mathcal{A}^2=-2$. It also fixes $\mathcal{N}=c_{IJK}M^IM^JM^K=1$ via the equation of motion for D. The target space of the scalar fields with this constraint is called a very special manifold and has been studied in the literature [46].

2.5 Supersymmetric Completion of R^2 Terms

2.5.1 Strategy

Before moving on, we need to make a few comments on the physical interpretation of the higher derivative terms, in particular in the off-shell formalism. Firstly, if we naively apply the variational method to obtain the equation of motion from a higher derivative theory, it results in a differential equation which is higher than second order. This means that giving the value and the first derivative of a field does not suffice as initial values. In other words, there are 'extra modes' in addition to the modes of the two-derivative Lagrangian. This is inevitable if we take the Lagrangian as giving an ultra-violet definition.

However, we regard our Lagrangian as the effective low-energy description in a derivative expansion with a small expansion parameter α'. Thus, the solution to the equation of motion should take the form of a perturbative expansion in α', and, in particular, its $\alpha' \to 0$ limit should exist. Such solutions are known to be determined by the value and the first derivative of a field at $t = 0$, just as in the case with the two-derivative Lagrangian, making the 'extra modes' mentioned above unphysical[7].

Secondly, it is readily checked that the auxiliary fields would appear with physical kinetic terms and begin to propagate when one constructs higher derivative terms in the off-shell formalism. It is known, however, that the auxiliary fields can be eliminated perturbatively in α' (see e.g. the introduction of Ref.[4]) to produce many higher derivative terms in the physical fields. The resulting Lagrangian is to be understood as explained in the previous paragraph. Thus, the would-be propagating auxiliary fields are just the 'extra modes' associated with the higher derivative terms, and they are not to be regarded as physical fields.

[7] The details can be found, for example, in [49] and [32].

The third comment is of a slightly different nature. In the higher derivative theory of gravity, one can redefine the metric as

$$g_{\mu\nu} \to g_{\mu\nu} + aRg_{\mu\nu} + bR_{\mu\nu} + \cdots . \tag{2.37}$$

with a and b small parameters. This leaves the leading-order Einstein-Hilbert term intact, while changing the form of the higher-order derivative terms. For example, it can be used to arbitrarily shift the coefficients of $R^{\mu\nu}R_{\mu\nu}$ and R^2, while that of $R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}$ cannot be shifted. It should also change the supersymmetry transformation law. The physics described by the Lagrangian, of course, remains the same under the redefinition. We need to use a redefinition to compare our results to those in the literature.

Below, we construct a very specific higher derivative term, whose form is not preserved by (2.37). This is because we use a very specific form of the supersymmetry transformation dictated by the superconformal formalism. Change in the conventional constraints (2.18) also induces a field redefinition among the fields in the Weyl multiplets without altering the physical contents of the theory. Our choice of the constraint $\hat{R}_\mu^a = 0$ is a convenient one because it greatly reduces the number of higher derivative terms to consider by forbidding the appearance of terms like $\hat{R}_{ab}\hat{R}^{ab}$ or $\hat{R}^2$.

With these preliminary remarks, we set out to construct a supersymmetric curvature-squared term in 5d $\mathcal{N} = 2$ supergravity [68]. More precisely, we obtain the supersymmetric completion of the mixed gauge-gravitational Chern-Simons term,

$$\epsilon^{abcde} W_a^I R_{bcfg} R_{de}{}^{fg}. \tag{2.38}$$

We recall that the gauge Chern-Simons term in (2.34) arises from the $W_a \cdot E^a$ term in the $\mathbf{V} \cdot \mathbf{L}$ invariant action formula. Judging from the similarity of the roles played by the gauge curvature F^I_{ab} and the metric curvature $R_{ab}{}^{cd}$, a natural guess would be to first embed the Weyl multiplet into a vector multiplet $V^{cd}[\mathbf{W}]$ with extra antisymmetric Lorentz indices c and d, and then to construct a linear multiplet from the $\mathbf{L}(V^I, V^J)$ embedding formula. However, we have found that this method is not significantly better than the direct construction of the linear multiplet. Therefore, our strategy is as follows:

1. Embed the Weyl multiplet to the linear multiplet.

2. Use the $\mathcal{L}(\mathbf{V}\cdot\mathbf{L})$ invariant action formula.

3. Gauge-fix down to the Poincaré supergravity.

The strong restriction in five dimensions, of course, already appears in the two-derivative Lagrangian. Indeed, the structure of the four-dimensional $\mathcal{N}=2$ vector multiplet is determined by a holomorphic function $F(X^I)$, but in five dimensions, the corresponding object $\mathcal{N}$ must be a purely cubic function. This restriction comes from the gauge invariance of the gauge Chern-Simons terms, just as in the case considered above.

2.5.2 Embedding and an Invariant Action

The linear multiplet should have $E_a \ni \epsilon_{abcde} R^{bc}{}_{fg}(M) R^{defg}(M)$ to be used in the invariant action formula in order to obtain the gravitational Chern-Simons term (2.38). The supertransformation law (B.9) for $\hat{R}(Q)$ reveals that we need the following structure:

$$E_a \ni R(M)^2 \quad \leftarrow \quad \varphi \ni R(M)R(Q) \quad \leftarrow L_{ij} \ni R(Q)^2. \tag{2.39}$$

Thus, L^{ij} is of Weyl-weight 3 and an $SU(2)_R$ triplet, constructed solely from the Weyl multiplet. Hence L^{ij} should be given by

$$L^{ij}[\mathbf{W}^2] = i\bar{\hat{R}}_{ab}{}^{(i}(Q)\hat{R}^{abj)}(Q) + A_1 i\bar{\chi}^{(i}\chi^{j)} + A_2 v^{ab}\hat{R}_{ab}{}^{ij}(U) \tag{2.40}$$

for suitable coefficients $A_{1,2}$. This quantity must be invariant under S transformations to be the lowest component of a linear multiplet. The transformation

$$\begin{aligned} \delta_S(\eta)L^{ij}[\mathbf{W}^2] &= 8i\bar{\eta}^{(i}\hat{R}^{j)}_{ab}(Q)v^{ab} - 8i\bar{\eta}^{(i}\gamma_{ab}v^{ab}\chi^{j)}A_1 \\ &\quad + \left(6i\bar{\eta}^{(i}\hat{R}^{j)}_{ab}(Q)v^{ab} - \frac{i}{2}\bar{\eta}^{(i}\gamma_{ab}\chi^{j)}\right)A_2 \end{aligned} \tag{2.41}$$

fixes $A_2 = -4/3$ and $A_1 = 1/12$. Then, the embedding formula is determined by a straightforward but tedious and lengthy repeated application of the supersymmetry transformation:

(2.42)

$$\begin{aligned} L^{ij}[\mathbf{W}^2] &= i\bar{\hat{R}}_{ab}{}^{(i}(Q)\hat{R}^{abj)}(Q) + \frac{1}{12}i\bar{\chi}^{(i}\chi^{j)} - \frac{4}{3}v^{ab}\hat{R}_{ab}{}^{ij}(U), \\ \varphi^i[\mathbf{W}^2] &= \frac{1}{12}\chi^i D + \frac{1}{4}\gamma_{ab}\hat{R}_{cd}{}^i(Q)\hat{R}^{abcd}(M) - \hat{R}^{abi}{}_j(U)\left(\hat{R}^j_{ab}(Q) + \frac{1}{12}\gamma_{ab}\chi^j\right) \\ &\quad + 8\gamma_{[c}\hat{\mathcal{D}}^c\hat{R}_{a]b}{}^i(Q)v^{ab} - 2\gamma_c\hat{R}_{ab}{}^i(Q)\hat{\mathcal{D}}^a v^{bc} \\ &\quad - \frac{1}{3}\gamma_{[a}\hat{\mathcal{D}}_{b]}\chi^i v^{ab} + \frac{1}{6}\gamma^{ab}\gamma^c\chi^i\hat{\mathcal{D}}_a v_{bc} - \frac{2}{3}\gamma_{ab}\hat{R}_{cd}{}^i(Q)v^{ac}v^{bd}, \\ E_a[\mathbf{W}^2] &= -\frac{1}{8}\epsilon_{abcde}\hat{R}^{bcfg}(M)\hat{R}^{de}{}_{fg}(M) + \frac{1}{6}\epsilon_{abcde}\hat{R}^{bcij}(U)\hat{R}^{de}{}_{ij}(U) \\ &\quad + \hat{\mathcal{D}}^b\left(-\frac{2}{3}v_{ab}D + 2\hat{R}_{abcd}(M)v^{cd} - \frac{8}{3}\epsilon_{abcde}v^{cf}\hat{\mathcal{D}}_f v^{de}\right. \\ &\quad \left. - 4\epsilon_{abcde}v^c{}_f\hat{\mathcal{D}}^d v^{ef} + \frac{16}{3}v_{ac}v^{cd}v_{db} + \frac{4}{3}v_{ab}v^2\right), \\ N[\mathbf{W}^2] &= \frac{1}{6}D^2 + \frac{1}{4}\hat{R}^{abcd}(M)\hat{R}_{abcd}(M) - \frac{2}{3}\hat{R}_{abij}(U)\hat{R}^{abij}(U) \\ &\quad - \frac{2}{3}\hat{R}_{abcd}(M)v^{ab}v^{cd} + \frac{16}{3}v_{ab}\hat{\mathcal{D}}^b\hat{\mathcal{D}}_c v^{ac} + \frac{8}{3}\hat{\mathcal{D}}^a v^{bc}\hat{\mathcal{D}}_a v_{bc} + \frac{8}{3}\hat{\mathcal{D}}^a v^{bc}\hat{\mathcal{D}}_b v_{ca} \\ &\quad - \frac{4}{3}\epsilon_{abcde}v^{ab}v^{cd}\hat{\mathcal{D}}_f v^{ef} + 8v_{ab}v^{bc}v_{cd}v^{da} - 2(v_{ab}v^{ab})^2. \end{aligned}$$

Here, we have omitted the terms trilinear in fermions in the expression of φ^i and the terms including fermions in the expressions of E_a and N. The first non-trivial check comes from the constraint (2.30), indicating that the divergence of E_a vanishes. This holds because the divergence of the first line in E_a vanishes, by the Bianchi identity, while the second and third lines vanish if we use the identity $\hat{\mathcal{D}}^a\hat{\mathcal{D}}^b A_{ab} = 0$ for a **K**-invariant, $SU(2)$-singlet, antisymmetric tensor A_{ab}. Another non-trivial check is the **K**-invariance of E_a and N, and we can see that E_a and N are invariant under **K** transformations.

We form an invariant action for off-shell conformal supergravity from the linear multiplet constructed above, using the $V \cdot L$ formula. The bosonic term is

(2.43)

$$
\begin{aligned}
\mathcal{L}(\mathbf{V}\cdot\mathbf{L}[\mathbf{W}^2])\big|_{\text{bosonic}} &= c_I Y^I_{ij} L^{ij}[\mathbf{W}^2] - \frac{1}{2} c_I W^I_a E^a[\mathbf{W}^2] + \frac{1}{2} c_I M^I N[\mathbf{W}^2] \\
&= -\frac{4}{3} c_I Y^I_{ij} v^{ab} \hat{R}^{ij}_{ab}(U) \\
&\quad + \frac{1}{16}\epsilon_{abcde} c_I W^{aI} \hat{R}^{bcfg}(M) \hat{R}^{de}{}_{fg}(M) \\
&\quad - \frac{1}{12}\epsilon_{abcde} c_I W^{aI} \hat{R}^{bc}{}_{jk}(U) \hat{R}^{dejk}(U) \\
&\quad + \frac{1}{8} c_I M^I \hat{R}^{abcd}(M) \hat{R}_{abcd}(M) - \frac{1}{3} c_I M^I \hat{R}_{abjk}(U) \hat{R}^{abjk}(U) \\
&\quad + \frac{1}{12} c_I M^I D^2 + \frac{1}{6} c_I \hat{F}^{Iab} v_{ab} D - \frac{1}{3} c_I M^I \hat{R}_{abcd}(M) v^{ab} v^{cd} \\
&\quad - \frac{1}{2} c_I \hat{F}^{Iab} \hat{R}_{abcd}(M) v^{cd} + \frac{8}{3} c_I M^I v_{ab} \hat{\mathcal{D}}^b \hat{\mathcal{D}}_c v^{ac} \\
&\quad + \frac{4}{3} c_I M^I \hat{\mathcal{D}}^a v^{bc} \hat{\mathcal{D}}_a v_{bc} + \frac{4}{3} c_I M^I \hat{\mathcal{D}}^a v^{bc} \hat{\mathcal{D}}_b v_{ca} \\
&\quad - \frac{2}{3} c_I M^I \epsilon_{abcde} v^{ab} v^{cd} \hat{\mathcal{D}}_f v^{ef} + \frac{2}{3} c_I \hat{F}^{Iab} \epsilon_{abcde} v^{cf} \hat{\mathcal{D}}_f v^{de} \\
&\quad + c_I \hat{F}^{Iab} \epsilon_{abcde} v^c{}_f \hat{\mathcal{D}}^d v^{ef} - \frac{4}{3} c_I \hat{F}^{Iab} v_{ac} v^{cd} v_{db} \\
&\quad - \frac{1}{3} c_I \hat{F}^{Iab} v_{ab} v_{cd} v^{cd} + 4 c_I M^I v_{ab} v^{bc} v_{cd} v^{da} - c_I M^I (v_{ab} v^{ab})^2
\end{aligned}
$$

for constants c_I [68]. Note that the term containing the second-order supercovariant derivative of v depends on the Ricci tensor through the **K**-gauge field given in (B.1), because $\hat{\mathcal{D}}_a v_{bc}$ includes the terms $\sim b_a v_{bc}$ and $\sim \omega_{a[b}{}^d v_{c]d}$, and the supercovariant derivative of b_a and $\omega_a{}^{bc}$ yields f_{ab} [see (B.2), (B.4) and (B.6)]. The result is

$$v_{ab}\hat{\mathcal{D}}^b\hat{\mathcal{D}}_c v^{ac} = v_{ab}\mathcal{D}^b\mathcal{D}_c v^{ac} - \frac{2}{3}v^{ac}v_{cb}R_a{}^b - \frac{1}{12}v_{ab}v^{ab}R \tag{2.44}$$

modulo terms including fermions.

To conclude this section, we should mention that another set of supersymmetric curvature squared terms has recently been constructed [13]. The newly constructed terms are apparently different from (2.43) and it would be interesting to see if these terms play a physically important role when embedded in string/M-theory as (2.43).

2.6 Applications to String Theory

We summarize the applications of the higher derivative action (2.43) to string theory in this section. Although the primary goal of this thesis is to discuss the gauged supergravity with higher derivative terms and its applications, we believe this subject is of interest for most of the readers of this thesis.

But before discussing the applications, we relate the five-dimensional supergravity with (more than) eight supercharges and a vanishing cosmological constant to string/M-theory. This type of supergravity theories can be obtained as a low-energy effective theory of type II or heterotic string theory compactified on $K3 \times S^1$ or T^5, or that of M-theory compactified on a Calabi-Yau three-fold. It can be understood by simply counting the number of supersymmetries: type IIB string and M-theory have 32 supercharges, and T^5, $K3 \times S^1$ and a Calabi-Yau three-fold preserve all, half and one-quarter of the supersymmetries of the theory, respectively. As was discussed in the Introduction, the low-energy effective theory of string/M-theory is supergravity, so one, after the compactifications, ends up with having five-dimensional supergravity with at least eight supercharges. This relation is why the five-dimensional supergravity has numerous applications to string/M-theory.

One direction of applications is to use the higher derivative action to understand black hole physics. A supersymmetric black hole, or BPS black hole in a more common terminology, in string/M-theory can be realized by wrapping D-branes on cycles on the internal compact manifold. Since the D-branes have masses, one can increase the Schwarzschild radius of the D-branes by increasing the number of D-branes wrapping the internal manifold. Then, a black hole is formed when the Schwarzschild radius surpass the compton wave length. This type of black hole has been extensively studied for over 15 years, and several properties of black holes in Einstein gravity are now understood microscopically, including the thermodynamic entropy [111] and Hawking radiation [25, 41].

Most of the analyses explained in the previous paragraph, however, have been validated using the two-derivative approximation of the supergravity action. The two-derivative action is applicable for any infinitely large black holes, but one needs to consider the higher derivative corrections to compute the finite size effects on the physical quantities. The higher derivative action (2.43) opens up a possibility to discuss the finite size effects to the five-dimensional black holes, and they have actually been investigated by several authors [28, 24, 29, 3, 30, 38, 45, 88, 7]. It has also been argued that the thermodynamic entropy computed using the higher derivative terms (2.43) is exact at the classical level, and cannot be modified by higher order corrections [84, 85, 42].

Another direction, closely related to the previous example, is to figure out the worldsheet theory of multiple heterotic strings [40, 78, 89, 86]. The worldsheet theory of N coincident fundamental strings has not yet been known except that the theory should be described by 2d CFT with central charges $(c_L, c_R) = (24N, 12N)$. One may try to understand it from the gravity dual description, but the gravitational solution of N coincident strings is known to be singular [39], and so unreliable at this level. Here is where the higher derivative terms play a role. The higher derivative corrections in (2.43) resolve this singularity [28], and the analyses of the gravity dual suggests that the multiple heterotic strings should be described by nonlinear $\mathcal{N} = 8$ superconformal field theory in two dimensions.

CHAPTER III

Gauged Supergravity with Higher Derivatives

This chapter in devoted to obtaining the gauged supergravity action with higher derivatives. Gauged supergravity is a particular kind of supergravity theory in which the gravitino is charged under the internal gauge group G. It was proposed as an attempt to construct a maximally supersymmetric $\mathcal{N} = 8$ supergravity in four dimensions [43], and extended to five dimensional supergravity soon later [65, 64, 62]. A qualitative difference from the ungauged supergravity is that gauged supergravity has a negative cosmological constant and the vacuum solution is a maximally supersymmetric AdS space. Later in this thesis, gauged supergravity is used to analyze the AdS/CFT correspondence.

We start this chapter with explaining how one obtains gauged supergravity from ungauged supergravity in Section 3.1. Then, we proceed to constructing minimal gauged supergravity action at two-derivative order in Section 3.2 and at four-derivative order in Section 3.4. For the construction of minimal gauged supergravity with higher derivatives, one needs to know the maximally supersymmetric vacuum solution of the theory. Hence the solution is discussed in Section 3.3. The discussions on this chapter are based on the paper [34].

3.1 Gauged Supergravity

In this section, we briefly summarize how to obtain the gauged supergravity action. The detailed calculation is given in the next section. As has been mentioned earlier, gauged supergravity is defined as a supergravity theory in which the gravitino is charged under the gauge group G. Here in this thesis, we only consider $G = U(1)^{n_V+1}$. Since supergravity is a gauge theory of super-Poincaré symmetries and in particular the gravitino is the gauge field associated with the supercharge, one can express the condition that the gravitino is charged under G in terms of the commutation relation:

$$[G_I, \mathbf{Q}_\alpha] = P_I \mathbf{Q}_\alpha, \tag{3.1}$$

where G_I is the generator of G and P_I is the charge vector of the gravitino under G.

To obtain the commutation relation (3.1), we utilize the fact that the gravitino in ungauged supergravity is an $SU(2)_R$ doublet. Then, if one gives an expectation value to a field which is charged under both $SU(2)_R$ and G, the symmetry group is broken as

$$SU(2)_R \times G \quad \rightarrow \quad G_{diag.}, \tag{3.2}$$

and the gravitino acquires charges under the $G_{diag.}$ symmetry. More concretely, we introduce the charged compensators, whose transformation properties are defined by

$$G_I \mathcal{A}_i^\alpha = P_I (i\sigma^3)^\alpha{}_\beta \mathcal{A}_i^\beta. \tag{3.3}$$

Then, only the following linear combination of $\mathbf{U}$ and G remains to be the symmetry of the theory:

$$\delta'_G(\Lambda^I) = \delta_G(\Lambda^I) + \delta_U(\Lambda^I P_I i\sigma^3). \tag{3.4}$$

The equation of motion for V_μ^{ij} requires

$$V_\mu^{ij} = P_I (i\sigma^3)^{ij} W_\mu^I. \tag{3.5}$$

In this way, all the $SU(2)_R$ non-singlet fields, including the gravitino, are now coupled to the gauge group G' with charge vector given by P_I.

3.2 Two-Derivative Lagrangian

We construct the on-shell two-derivative Lagrangian in this section. We start with the off-shell Lagrangian $\mathcal{L}_H - \mathcal{L}_V/2$, where $\mathcal{L}_V$ and $\mathcal{L}_H$ are given in (2.34) and (2.35), respectively, break the superconformal symmetries to super-Poincaré symmetries, and eliminate the auxiliary fields.

In addition to the gauge symmetries of Poincaré supergravity, we have $SU(2)_R$ symmetries U, dilatational invariance D, special conformal transformations K and S-invariance. We focus on the bosonic part of the action in this thesis, and break the bosonic symmetries by giving appropriate expectation values to the fields. As has been discussed in the previous section, we break D and $SU(2)_R$ symmetries by

$$\mathcal{A}_i^\alpha = \delta_i^\alpha. \tag{3.6}$$

This expectation value mixes the $U(1)_R \in SU(2)_R$ and $U(1)$ internal gauge symmetries and consequently assigns $U(1)$ charges to the gravitino as discussed previously. To break the K-invariance, note that the dilatational gauge field b_μ is not invariant under K-transformations for any value of b_μ. Thus, one can set

$$b_\mu = 0. \tag{3.7}$$

which is basically irrelevant to the form of the action. These two conditions are what we impose to obtain the (bosonic part of) off-shell Poincaré supergravity action. The explicit form of the action is given by

(3.8)

$$
\begin{aligned}
e^{-1}\mathcal{L}_0 &= \tfrac{1}{2}D(\mathcal{N}-1) - R\left(\tfrac{3}{4}+\tfrac{1}{4}\mathcal{N}\right) + v^2(3\mathcal{N}+1) + \tfrac{1}{24}c_{IJK}\epsilon^{\mu\nu\rho\lambda\sigma}A^I_\mu F^J_{\nu\rho}F^K_{\lambda\sigma} \\
&\quad +2\mathcal{N}_I v^{\mu\nu}F^I_{\mu\nu} + \mathcal{N}_{IJ}(\tfrac{1}{4}F^I_{\mu\nu}F^{J\,\mu\nu} - \tfrac{1}{2}\mathcal{D}^\mu M^I\mathcal{D}_\mu M^J) - \mathcal{N}_{IJ}Y^I_{ij}Y^{J\,ij} \\
&\quad +2\left[|(V^{ij}_\mu - gW^I_\mu P_I(i\sigma^3)^{ij}|^2 + 4g^2(P_I M^I)^2 + 2gY^{Iij}P_I(i\sigma^3)_{ij}\right],
\end{aligned}
$$

where we used $\mathcal{D}_\mu\mathcal{A}^\alpha_i = (-gW^I_\mu P_I(i\sigma^3)^\alpha_\beta + V_{\mu\beta})\delta^\beta_i$.

Integrating Out the Auxiliary Fields

We can move onto integrating out the auxiliary fields. In the action (3.8), V^{ij}_μ shows up in the second term in the third line in the form of a magnitude square, so one can readily integrate it out and obtain

$$V^{ij}_\mu = gP_I(i\sigma^3)^{ij}W^I_\mu. \tag{3.9}$$

As discussed in the previous section, the $SU(2)_R$ gauge field is identified as a linear combination of $U(1)$ gauge fields, and now all the $SU(2)_R$ non-singlet fields, including the gravitino, are coupled to G.

We then integrate out Y^I_{ij}. The equation of motion is given by

$$-2\mathcal{N}_{IJ}Y^J_{ij} + 4gP_J(i\sigma^3)^{ij} = 0. \tag{3.10}$$

Assuming that $\mathcal{N}_{IJ}$ can be inverted, one obtains

$$Y^I_{ij} = 2(\mathcal{N}^{-1})^{IJ}P_J(i\sigma^3)_{ij}. \tag{3.11}$$

Before we move on, let us discuss the form of the potential at this point. After eliminating Y^I_{ij}, one obtains the scalar potential of the form

$$V = -4g^2[2(\mathcal{N}^{-1})^{IJ}P_IP_J + (P_IM^I)^2], \tag{3.12}$$

where we used the fact that the Abelian gauge group G acts trivially on the vector multiplet scalars M^I and so $\mathcal{D}_\mu M^I = \partial_\mu M^I$. Note that $\mathcal{N}_{IJ}$ turns out to be the coefficient of the gauge kinetic term, so the first term must be negative definite. The second term is the negative of the real value squared, so it also must be negative. This implies that the theory must have negative cosmological constant at the vacuum.

The auxiliary fields $v_{\mu\nu}$ can be eliminated using the equation of motion for $v_{\mu\nu}$. To make the form of the solution simpler, we first solve the equation of motion for D:

$$\tfrac{1}{2}(\mathcal{N} - 1) = 0. \tag{3.13}$$

The equation of motion for D imposes the constraint $\mathcal{N} = 1$ on the scalar manifold. This constrained scalar manifold is called "very special geometry" and is extensively studied in the literature. The equations of motion for $v_{\mu\nu}$ yields

$$2(3\mathcal{N} + 1)v_{\mu\nu} + 2\mathcal{N}_I F^I_{\mu\nu} = 0, \tag{3.14}$$

which, in turn, implies

$$v_{\mu\nu} = -\frac{1}{4}\mathcal{N}_I F^I_{\mu\nu}. \tag{3.15}$$

Note that this $v_{\mu\nu}$ is originally from the Weyl multiplet and related to the graviton and gravitino by supersymmetry. Therefore, by definition, this particular linear combination of the gauge fields should be identified as the graviphoton in the on-shell gravity multiplet.

Finally, we solve for D. It can be primarily done by using the equation of motion for M^I:

$$\begin{aligned} &\tfrac{1}{2}\mathcal{N}_I(D - \tfrac{1}{2}R + 6v_{\mu\nu}v^{\mu\nu}) + 2\,\mathcal{N}_{IJ}F^J_{\mu\nu}v^{\mu\nu} + \tfrac{1}{4}\,c_{IJK}\,F^J_{\mu\nu}F^{K\,\mu\nu} + \mathcal{N}_{IJ}\Box M^J \\ &+\tfrac{1}{2}\,c_{IJK}\,\partial_\mu M^J\partial^\mu M^K - g^2\frac{\delta V}{\delta M^I} = 0. \end{aligned} \tag{3.16}$$

Multiplying M^I to the equation of motion for M^I yields

(3.17)

$$\begin{aligned} D - \tfrac{1}{2}R + 6v_{\mu\nu}v^{\mu\nu} &= -\tfrac{8}{3}\mathcal{N}_I F^I_{\mu\nu}v^{\mu\nu} - \tfrac{1}{6}\mathcal{N}_{IJ}F^I_{\mu\nu}F^{J\,\mu\nu} - \tfrac{1}{3}\mathcal{N}_{IJ}\partial_\mu M^I\partial^\mu M^J \\ &\quad -\tfrac{4}{3}\mathcal{N}_I\Box M^I + \tfrac{2}{3}M^I\frac{\delta V}{\delta M^I}. \end{aligned}$$

One can readily solve for D using this equation. However, the result includes R, which changes the correctly normalized coefficient of the Einstein-Hilbert term. To avoid this problem we add the trace of the Einstein equation

(3.18)

$$\begin{aligned} R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R &= -\tfrac{1}{2}\mathcal{N}_{IJ}(\partial_\mu M^I\partial_\nu M^J - \tfrac{1}{2}g_{\mu\nu}\partial_\lambda M^I\partial^\lambda M^J) \\ &\quad -\tfrac{1}{2}(\mathcal{N}_I\mathcal{N}_J - \mathcal{N}_{IJ})(F^I_{\mu\lambda}F^{J\lambda}_{\nu} - \tfrac{1}{4}g_{\mu\nu}F^I_{\lambda\sigma}F^{J\,\lambda\sigma}) + \tfrac{1}{2}g_{\mu\nu}V\,. \end{aligned}$$

One can eliminate the explicit R dependence using the trace of this equation. Plugging in the on-shell value of $v_{\mu\nu}$ and the trace of the Einstein equation, one obtains the on-shell form of D

(3.19)

$$\begin{aligned} D &= -\tfrac{7}{12}\mathcal{N}_{IJ}\partial_\mu M^I\partial^\mu M^J - \tfrac{4}{3}\mathcal{N}_I\Box M^I + \tfrac{1}{4}(\mathcal{N}_I\mathcal{N}_J - \tfrac{1}{2}\mathcal{N}_{IJ})F^I_{\mu\nu}F^{J\,\mu\nu} \\ &\quad -\tfrac{5}{6}V + \tfrac{2}{3}M^I\frac{\delta V}{\delta M^I} \\ &= -\tfrac{7}{12}\mathcal{N}_{IJ}\partial_\mu M^I\partial^\mu M^J - \tfrac{4}{3}\mathcal{N}_I\Box M^I + \tfrac{1}{4}(\mathcal{N}_I\mathcal{N}_J - \tfrac{1}{2}\mathcal{N}_{IJ})F^I_{\mu\nu}F^{J\,\mu\nu} \\ &\quad +2g^2[6P_IP_J(\mathcal{N}^{-1})^{IJ} - P_IP_JM^IM^J]\,. \end{aligned}$$

Now that we have solved for all the auxiliary fields, the on-shell action can be obtained by plugging the solutions of the auxiliary fields into the off-shell action (3.8). It yields

$$\begin{aligned} (3.20)\quad e^{-1}\mathcal{L} &= -R - \tfrac{1}{2}\mathcal{N}_{IJ}\partial_\mu M^I\partial^\mu M^J - \tfrac{1}{4}(\mathcal{N}_I\mathcal{N}_J - \mathcal{N}_{IJ})F^I_{\mu\nu}F^{J\,\mu\nu} \\ &\quad +\tfrac{1}{24}\,c_{IJK}\,\epsilon^{\mu\nu\rho\lambda\sigma}A^I_\mu F^J_{\nu\rho}F^K_{\lambda\sigma} + 4g^2[2(\mathcal{N}^{-1})^{IJ}P_IP_J + (P_IM^I)^2]\,, \end{aligned}$$

where now the M^I are a set of constrained scalars satisfying the very special geometry condition $\mathcal{N} = 1$. The Lagrangian perfectly matches the bosonic sector of the standard two-derivative $\mathcal{N} = 2$ supergravity action coupled to n_V vector multiplets.

3.3 Supersymmetric AdS_5 Vacuum Solution

In this section, we consider the maximally supersymmetric vacuum solution of our supergravity theory and see how **Q** and **S** transformations in the conformal supergravity reduces to the physical supersymmetry **Q′** in Poincaré supergravity. We will be finding the vacuum solution using the BPS equations, which does not depend on the action in the off-shell formulation, so most of the analysis in this section is applicable even in the presence of higher derivative corrections.

We make an ansatz for the AdS space as

$$L^2 \left(u^2 \eta_{\alpha\beta} dx^\alpha dx^\beta - \frac{du^2}{u^2} \right) , \tag{3.21}$$

where $\alpha, \beta = 0, 1, 2, 3$, $\eta = \text{diag}(+, -, -, -)$, $u = x^4$ and L is the curvature radius. We further suppose that any field with non-zero spin is zero. We start with the fact that in such a background, the equation

$$D_\mu \varepsilon - \frac{i}{2L} \gamma_\mu \varepsilon = 0 \tag{3.22}$$

has eight linearly-independent solutions, corresponding to the physical supersymmetries of the theory. Here, D_μ denotes the derivative covariant with respect to local Lorentz transformations, and ε is a spinor without the $SU(2)$-Majorana condition. If the $i = 1$ component of an $SU(2)$-Majorana spinor ε^i satisfies (3.22), then the $i = 2$ component instead satisfies

$$D_\mu \varepsilon^{i=2} + \frac{i}{2L} \gamma_\mu \varepsilon^{i=2} = 0. \tag{3.23}$$

Thus, to express it covariantly under $SU(2)_R$, one needs to introduce a unit three-vector $\vec{q}$ so that

$$D_\mu \varepsilon^i - \frac{1}{2L}\gamma_\mu i(\vec{q}\cdot\vec{\sigma})^i{}_j \varepsilon^j = 0. \tag{3.24}$$

The supersymmetry transformation of the gravitino (2.22) can then be made zero by choosing

$$\eta^i = \frac{1}{2L}(i\vec{q}\cdot\vec{\sigma})^i{}_j \varepsilon^j. \tag{3.25}$$

The supersymmetric transformation which remains after the gauge fixing is

$$\delta'_Q(\varepsilon) = \delta_Q(\varepsilon) + \delta_S\left(\frac{1}{2L}(i\vec{q}\cdot\vec{\sigma})\varepsilon\right). \tag{3.26}$$

The vanishing of $\delta_Q \chi^i$ implies that $D = 0$.

Next, the vanishing of the gaugino transformation (2.24) requires

$$Y^{Ii}{}_j \varepsilon^j - \frac{1}{2L}M^I(i\vec{q}\cdot\vec{\sigma})^i{}_j \varepsilon^j = 0 \tag{3.27}$$

for all I. This relation is satisfied for the maximal number of ε^i if and only if

$$Y^I_{ij} = \frac{1}{2L}(i\vec{q}\cdot\vec{\sigma})_{ij} M^I. \tag{3.28}$$

We can set $i\vec{q}\cdot\vec{\sigma} = i\sigma^3$ without loss of generality. The vanishing of the transformation of the hyperino $\delta\zeta^\alpha = 0$ under the gauge fixing $\mathcal{A}_i^\alpha \propto \delta_i^\alpha$ determines the curvature radius as

$$L = \frac{3}{2}(P_I M^I)^{-1}. \tag{3.29}$$

Another interesting condition comes from the $[\delta'_Q, \delta'_Q]$ commutator. From (2.20), (2.22) and (2.25), it is

$$\begin{aligned}[\delta'_Q(\varepsilon), \delta'_Q(\varepsilon')] &= \delta_U\left(-\frac{6}{L}\bar{\varepsilon}^{(i}(i\sigma^3)^{j)}{}_k \varepsilon'^k\right) + \delta_G(-2iM^I\bar{\varepsilon}\varepsilon') \\ &= \delta_U\left(2P_I M^I(i\sigma^3)^{ij}\bar{\varepsilon}\varepsilon'\right) + \delta_G(-2iM^I\bar{\varepsilon}\varepsilon') \\ &= \delta'_G(-2iM^I\bar{\varepsilon}\varepsilon'),\end{aligned} \tag{3.30}$$

where δ'_G is the surviving gauge transformation under the condition $\mathcal{A}^{\alpha}{}_i \propto \delta^{\alpha}_i$ defined in (3.4). This implies that $\delta'_G(M^I)$ should leave the scalar VEVs invariant if we consider additional charged matter fields.

The reader can check that the analysis up to this point does not use any specific property of the action. Thus it is applicable to any $d = 5$ $\mathcal{N} = 2$ supergravity Lagrangian with arbitrarily complicated higher derivative terms. Now, let us write down the condition (3.28) for our Lagrangian. Since our higher derivative Lagrangian is independent of Y^I_{ij}, the solution of Y^I_{ij} does not change in the presence of four derivative terms and is

$$Y^I_{ij} = 2(\mathcal{N}^{-1})^{IJ} P_J (i\sigma^3)_{ij}. \tag{3.31}$$

Substituting this into (3.28), we obtain

$$P_I = \frac{1}{4}\mathcal{N}_{IJ} M^J / L = \frac{3}{2} c_{IJK} M^J M^K / L. \tag{3.32}$$

This is the attractor equation in 5d gauged supergravity first found in [33]. By multiplying this equation by M^I we find the condition $\mathcal{N} = c_{IJK} M^I M^J M^K = 1$ again. One can check that it solves the modified equations of motion which follows from $\mathcal{L}_0 + \mathcal{L}_1$. The correction to the potential $(\mathcal{N} - 1)^2$ does not shift the VEVs of the scalars, since the solution before considering higher derivative corrections satisfies $\mathcal{N} = 1$, minimizing the added potential. Note that higher terms with respect to the hatted curvature $\hat{R}_{abcd}(M)$ do not change the AdS solution, since the AdS background gives $\hat{R}_{abcd}(M) = 0$.

In this thesis, we are mainly interested in the R-charged black branes, so let us truncate the two-derivative supergravity action coupled to vector multiplets into the minimal supergravity. Note that the linear combination of $U(1)$ charges in the right hand side of (3.30) should be identified as the $U(1)_R$ charge in the superalgebra by definition. The truncation, therefore, can be done by taking [1]

$$M^I = \bar{M}^I, \qquad W^I_\mu = \bar{M}^I W_\mu. \tag{3.33}$$

As we have seen, $\bar{M}^I$ is fixed in gauged supergravity by the attractor equation (3.32), which yields

$$P_I \bar{M}^I = \frac{3}{2}\,, \qquad (\bar{\mathcal{N}}^{-1})^{IJ} P_I P_J = \frac{3}{8}\,, \tag{3.34}$$

in which case the potential becomes $\bar{V} = -12g^2$. The resulting Lagrangian for the bosonic fields of the supergravity multiplet $(g_{\mu\nu}, W_\mu)$ then reads

$$e^{-1}\mathcal{L} = -R - \tfrac{3}{4}F^2_{\mu\nu} + \tfrac{1}{4}\epsilon^{\mu\nu\rho\lambda\sigma} W_\mu F_{\nu\rho} F_{\lambda\sigma} + 12g^2\,, \tag{3.35}$$

which reproduces the conventional on-shell supergravity Lagrangian [65] once the graviphoton is rescaled according to $W_\mu \to W_\mu/\sqrt{3}$.

While this completes the analysis relevant to the leading, two-derivative action, we note that the expression for D simplifies further in the case of constant scalars. Substituting (3.33) and (3.34) into the expression (3.19) for D yields the simple result

$$D = \tfrac{1}{4}(\bar{\mathcal{N}}_I \bar{\mathcal{N}}_J - \tfrac{1}{2}\bar{\mathcal{N}}_{IJ}) F^I_{\mu\nu} F^{J\,\mu\nu} = \tfrac{3}{2}F^2_{\mu\nu}\,. \tag{3.36}$$

By taking $\mathcal{N} = 1$, we see that this explicit form of D does not play a role in the leading expression for the two-derivative Lagrangian. However, it will become relevant in the discussion of higher derivative corrections, which we turn to next.

3.4 Higher Derivative Lagrangian

Generally, obtaining the on-shell supergravity action with higher derivative terms is a difficult task. To obtain the on-shell action from the off-shell action, as we saw

[1] Note that our definition differs by a factor of 1/3 from the conventional one where $W_\mu = W^I_\mu \mathcal{N}_I$.

in the previous section, one needs to integrate out the auxiliary fields. This is easy at two-derivative order because the equations of motion do not have any derivatives for the auxiliary fields and hence are algebraically solved. If one includes the higher derivative terms, however, the terms including the derivatives shows up in the action and the equations of motion can no longer be solved algebraically.

Nevertheless, one can still obtain the leading order higher derivative corrections to the two-derivative action [4]. To see that, let us expand the off-shell action and the solutions of the auxiliary fields in terms of the number of derivatives as

$$S_{off-shell} = S^{(2)} + S^{(4)} + \cdots \,, \quad \phi_{sol} = \phi^{(2)} + \phi^{(4)} + \cdots \,, \tag{3.37}$$

where the numbers in the superscripts in the expansion of the action represent the number of derivatives and those in the expansion of the solution indicate that $\phi^{(n)}$ is the terms in the solution which show up when one solves the equation of motion up to n derivatives. The difficulty we described in the previous paragraph was that it is hard to obtain $\phi^{(4)}$, which apparently matters when one tries to derive the four-derivative on-shell action. However, plugging the solution of the auxiliary fields into the off-shell action yields

$$\begin{aligned} S_{on-shell} &= S^{(2)}(\phi^{(2)}) + S^{(4)}(\phi^{(2)}) + \phi^{(4)}\frac{\delta S^{(2)}}{\delta\phi}(\phi^{(2)}) + \cdots \,, \\ &= S^{(2)}(\phi^{(2)}) + S^{(4)}(\phi^{(2)}) + \cdots \,, \end{aligned} \tag{3.38}$$

where from the first line to the second line, we used the two-derivative equation of motion. Therefore, one does not really need to find $\phi^{(4)}$ and one can plug the solution of the two-derivative equations of motion for auxiliary fields into the four-derivative off-shell action to obtain the four-derivative on-shell action. This is the strategy taken in this section.

We now turn to the four-derivative corrections to the action, which we parameterize by $\mathcal{L}_1$. For convenience, we separate the contributions to $\mathcal{L}_1$ present in the ungauged theory from those coming strictly from the gauging, $\mathcal{L}_1 = \mathcal{L}_1^{\text{ungauged}} + \mathcal{L}_1^{\text{gauged}}$. The two are given by:

(3.39)

$$\begin{aligned} e^{-1}\mathcal{L}_1^{\text{ungauged}} &= \tfrac{1}{24}c_{2I}\Big[\tfrac{1}{16}\epsilon_{\mu\nu\rho\lambda\sigma}A^{I\,\mu}R^{\nu\rho\alpha\beta}R^{\lambda\sigma}{}_{\alpha\beta} + \tfrac{1}{8}M^I C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} + \tfrac{1}{12}M^I D^2 \\ &+\tfrac{1}{6}F^I_{\mu\nu}v^{\mu\nu} - \tfrac{1}{3}M^I C_{\mu\nu\rho\sigma}v^{\mu\nu}v^{\rho\sigma} - \tfrac{1}{2}F^{I\,\mu\nu}C_{\mu\nu\rho\sigma}v^{\rho\sigma} \\ &+\tfrac{8}{3}M^I v_{\mu\nu}\nabla^\nu\nabla_\rho v^{\mu\rho} - \tfrac{16}{9}M^I v^{\mu\rho}v_{\rho\nu}R^\nu_\mu - \tfrac{2}{9}M^I v^2 R \\ &+\tfrac{4}{3}M^I\nabla^\mu v^{\nu\rho}\nabla_\mu v_{\nu\rho} + \tfrac{4}{3}M^I\nabla^\mu v^{\nu\rho}\nabla_\nu v_{\rho\mu} \\ &-\tfrac{2}{3}M^I\epsilon_{\mu\nu\rho\lambda\sigma}v^{\mu\nu}v^{\rho\lambda}\nabla_\delta v^{\sigma\delta} + \tfrac{2}{3}F^{I\,\mu\nu}\epsilon_{\mu\nu\rho\lambda\sigma}v^{\rho\delta}\nabla_\delta v^{\lambda\sigma} \\ &+F^{I\,\mu\nu}\epsilon_{\mu\nu\rho\lambda\sigma}v^\rho{}_\delta\nabla^\lambda v^{\sigma\delta} - \tfrac{4}{3}F^{I\,\mu\nu}v_{\mu\rho}v^{\rho\lambda}v_{\lambda\nu} \\ &-\tfrac{1}{3}F^{I\,\mu\nu}v_{\mu\nu}v^2 + 4M^I v_{\mu\nu}v^{\nu\rho}v_{\rho\lambda}v^{\lambda\mu} - M^I(v^2)^2\Big]\,, \end{aligned}$$

(3.40)

$$\begin{aligned} e^{-1}\mathcal{L}_1^{\text{gauged}} &= \tfrac{1}{24}c_{2I}\Big[-\tfrac{1}{12}\epsilon_{\mu\nu\rho\lambda\sigma}\,A^{I\,\mu}R^{\nu\rho\,ij}(U)R^{\lambda\sigma}_{ij}(U) \\ &-\tfrac{1}{3}\,M^I R^{\mu\nu\,ij}(U)R_{\mu\nu\,ij}(U) - \tfrac{4}{3}\,Y^I_{ij}v_{\mu\nu}R^{\mu\nu\,ij}(U)\Big]\,, \end{aligned}$$

where

$$R_{\mu\nu}{}^{ij}(U) = \partial_\mu V_\nu^{ij} - V^i_{\mu k}V^{kj}_\nu - (\mu\leftrightarrow\nu)\,. \quad (3.41)$$

As we can see, the constants c_{2I} parameterize the magnitude of these contributions. Notice that the scalar D no longer acts as a Lagrange multiplier, since it now appears quadratically in $\mathcal{L}_1$. In fact, by varying the full action $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1$ with respect to D, we obtain the modified very special geometry constraint

$$\mathcal{N} = 1 - \frac{c_{2I}}{72}(DM^I + F^{I\,\mu\nu}v_{\mu\nu})\,, \quad (3.42)$$

which encodes information about how the scalars M^I are affected by higher-derivative corrections.

Integrating out the auxiliary fields

As in the two-derivative case, in order to obtain a Lagrangian written solely in terms of the physical fields of the theory we need to eliminate the auxiliary fields D, $v_{\mu\nu}$, $V^i_{\mu\nu}$ and Y^I_{ij} from $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1$. In Section 3.2, we solved for the auxiliary fields by neglecting higher order corrections, and then integrated them out of the two-derivative action. It turns out that the lowest order expressions for the auxiliary fields are sufficient when working to linear order in the c_{2I} [4]. This allows us to reuse the results of the previous section for the auxiliary fields, which we summarize here:

$$V^{ij}_\mu = gP_I(i\sigma^3)^{ij}A^I_\mu\,, \tag{3.43}$$

$$Y^I_{ij} = 2(\mathcal{N}^{-1})^{IJ}P_J(i\sigma^3)_{ij}\,, \tag{3.44}$$

$$v_{\mu\nu} = -\tfrac{1}{4}\mathcal{N}_I F^I_{\mu\nu}\,, \tag{3.45}$$

$$D = \tfrac{1}{4}(\mathcal{N}_I\mathcal{N}_J - \tfrac{1}{2}\mathcal{N}_{IJ})F^I_{\mu\nu}F^{J\,\mu\nu}\,. \tag{3.46}$$

While it is valid to use these lowest order expressions, it is important to realize that the scalar fields are modified because of (3.42). This modification leads to additional contributions to the two-derivative on-shell action (3.21), which combines with $\mathcal{L}_1$ to yield the complete action at linear order in c_{2I}.

In principle, we may work with the full system of supergravity coupled to n_V vector multiplets. However, here we focus on the truncation to pure supergravity, where the scalars M^I are taken to be non-dynamical. Even so, they are not entirely trivial. While at the two-derivative level, we may simply set them to constants according to (3.33), here we must allow for the modification (3.42) by defining

(3.47) $$M^I = \bar{M}^I + c_2\hat{M}^I, \qquad A^I_\mu = \bar{M}^I A_\mu, \qquad c_2 \equiv c_{2I}\bar{M}^I\,,$$

where $\hat{M}^I$ are possible scalar fluctuations that enter at $\mathcal{O}(c_2)$. Substituting this into the expressions (3.45) and (3.46) for the auxiliary fields then yields

(3.48) $$v_{\mu\nu} = -\tfrac{3}{4}F_{\mu\nu} + \mathcal{O}(c_2), \qquad D = \tfrac{3}{2}F^2 + \mathcal{O}(c_2)\,,$$

which match the lowest order expressions for constant scalars. The modified very special geometry constraint (3.42) can now be simplified further, and becomes

(3.49) $$\mathcal{N} = 1 - \frac{c_2}{96}F^2 + \mathcal{O}(c_2^2).$$

In general, a solution to the fluctuating scalars $\hat{M}^I$ ought to come from the equations of motion. However, as a shortcut, we make the ansatz that $\hat{M}^I$ is proportional to $\bar{M}^I$. The modified constraint (3.49) is then enough to fix the correction to the scalars to be

(3.50) $$M^I = \bar{M}^I\left[1 - \frac{c_2}{288}F^2 + \mathcal{O}(c_2^2)\right]\,.$$

Consistency with the equations of motion will presumably demand an appropriate relation between the various c_{2I} coefficients. However, since the vectors will be truncated out, we only care about the combination c_2 given in (3.47), and will not work out this relation explicitly.

We are now ready to integrate out both the scalars M^I and the auxiliary fields from the two-derivative action $\mathcal{L}_0$. By making use of the corrections[2] to the leading order scalar expressions (3.34)

(3.51) $$P_I M^I = \frac{3}{2}\left[1 - \frac{c_2}{288}F^2\right]\,, \qquad (\mathcal{N}^{-1})^{IJ}P_I P_J = \frac{3}{8}\left[1 + \frac{c_2}{288}F^2\right]\,,$$

[2]These can be easily verified using $P_I = \frac{1}{4}\bar{\mathcal{N}}_{IJ}\bar{M}^J$.

we find that the contribution coming from $\mathcal{L}_0$ yields the following terms:

(3.52)
$$e^{-1}\mathcal{L}_0 = -R - \frac{3}{4}F^2 + \frac{1}{4}\epsilon^{\mu\nu\rho\lambda\sigma}A_\mu F_{\nu\rho}F_{\lambda\sigma} + 12g^2 + \frac{c_2}{24}\left[\frac{1}{16}RF^2 + \frac{1}{64}(F^2)^2 - \frac{5}{4}g^2F^2\right].$$

Turning next to the four-derivative contributions, we note that, since such terms are already linear in c_2, we may simply use the leading order solution for the scalars. The gauging contribution (3.40) is then particularly simple

$$e^{-1}\mathcal{L}_1^{\text{gauged}} = -\frac{c_2}{64}g^2\,\epsilon_{\mu\nu\rho\lambda\sigma}A^\mu F^{\nu\rho}F^{\lambda\sigma}. \tag{3.53}$$

On the other hand, the contribution to $\mathcal{L}_1^{\text{ungauged}}$ is given by:

(3.54)
$$\begin{aligned} e^{-1}\mathcal{L}_1^{\text{ungauged}} \;=\; & \frac{c_2}{24}\Big[\frac{1}{16}\,\epsilon_{\mu\nu\rho\lambda\sigma}A^\mu R^{\nu\rho\delta\gamma}R^{\lambda\sigma}{}_{\delta\gamma} + \frac{1}{8}\,C^2_{\mu\nu\rho\sigma} \\ & + \frac{3}{16}C_{\mu\nu\rho\lambda}F^{\mu\nu}F^{\rho\lambda} - F^{\mu\rho}F_{\rho\nu}R^\nu_\mu - \frac{1}{8}RF^2 \\ & + \frac{3}{2}\,F_{\mu\nu}\nabla^\nu\nabla_\rho F^{\mu\rho} + \frac{3}{4}\,\nabla^\mu F^{\nu\rho}\nabla_\mu F_{\nu\rho} + \frac{3}{4}\,\nabla^\mu F^{\nu\rho}\nabla_\nu F_{\rho\mu} \\ & + \frac{1}{8}\,\epsilon_{\mu\nu\rho\lambda\sigma}F^{\mu\nu}(3F^{\rho\lambda}\nabla_\delta F^{\sigma\delta} + 4F^{\rho\delta}\nabla_\delta F^{\lambda\sigma} + 6F^\rho{}_\delta\nabla^\lambda F^{\sigma\delta}) \\ & + \frac{45}{64}F_{\mu\nu}F^{\nu\rho}F_{\rho\lambda}F^{\lambda\mu} - \frac{45}{256}(F^2)^2\Big]. \end{aligned}$$

The full on-shell Lagrangian is thus given by

(3.55)
$$\begin{aligned} e^{-1}\mathcal{L} \;=\; & -R - \frac{3}{4}F^2\Big(1 + \frac{5}{72}c_2g^2\Big) + \frac{1}{4}\Big(1 - \frac{1}{16}c_2g^2\Big)\epsilon^{\mu\nu\rho\lambda\sigma}A_\mu F_{\nu\rho}F_{\lambda\sigma} + 12g^2 \\ & + \frac{c_2}{24}\Big[\frac{1}{16}RF^2 + \frac{1}{64}(F^2)^2\Big] + \mathcal{L}_1^{\text{ungauged}}. \end{aligned}$$

Finally, we may redefine A_μ to write the kinetic term in canonical form:

$$A_\mu^{\text{final}} = \sqrt{3}\Big(1 + \frac{5}{144}c_2g^2\Big)\,A_\mu^{\text{old}}. \tag{3.56}$$

The Lagrangian then becomes:

$$
\begin{aligned}
(3.57) \qquad \mathcal{L} &= -R - \frac{1}{4}F^2 + \frac{1}{12\sqrt{3}}\Big(1 - \frac{1}{6}c_2 g^2\Big)\epsilon^{\mu\nu\rho\lambda\sigma}A_\mu F_{\nu\rho}F_{\lambda\sigma} + 12g^2 \\
&\quad + \frac{c_2}{24}\Big[\frac{1}{48}RF^2 + \frac{1}{576}(F^2)^2\Big] + \mathcal{L}_1^{\text{ungauged}}\,,
\end{aligned}
$$

with

(3.58)

$$
\begin{aligned}
e^{-1}\mathcal{L}_1^{\text{ungauged}} &= \frac{c_2}{24}\Big[\frac{1}{16\sqrt{3}}\epsilon_{\mu\nu\rho\lambda\sigma}A^\mu R^{\nu\rho\delta\gamma}R^{\lambda\sigma}{}_{\delta\gamma} + \frac{1}{8}C^2_{\mu\nu\rho\sigma} \\
&\quad + \frac{1}{16}C_{\mu\nu\rho\lambda}F^{\mu\nu}F^{\rho\lambda} - \frac{1}{3}F^{\mu\rho}F_{\rho\nu}R^\nu_\mu - \frac{1}{24}RF^2 \\
&\quad + \frac{1}{2}F_{\mu\nu}\nabla^\nu\nabla_\rho F^{\mu\rho} + \frac{1}{4}\nabla^\mu F^{\nu\rho}\nabla_\mu F_{\nu\rho} + \frac{1}{4}\nabla^\mu F^{\nu\rho}\nabla_\nu F_{\rho\mu} \\
&\quad + \frac{1}{32\sqrt{3}}\epsilon_{\mu\nu\rho\lambda\sigma}F^{\mu\nu}(3F^{\rho\lambda}\nabla_\delta F^{\sigma\delta} + 4F^{\rho\delta}\nabla_\delta F^{\lambda\sigma} + 6F^\rho{}_\delta\nabla^\lambda F^{\sigma\delta}) \\
&\quad + \frac{5}{64}F_{\mu\nu}F^{\nu\rho}F_{\rho\lambda}F^{\lambda\mu} - \frac{5}{256}(F^2)^2\Big]\,.
\end{aligned}
$$

This completes the construction of the curvature squared action in five-dimensional gauged supergravity.

CHAPTER IV

AdS/CFT Correspondence and Holographic Thermodynamics

In this chapter, we discuss the AdS/CFT correspondence and holographic thermodynamics as its application. The AdS/CFT correspondence[1] [92] is a strong-weak type duality between $(D+1)$-dimensional gravitational theory in AdS_{D+1} and conformal field theory in D dimensions. The gravitational theory at strong coupling corresponds to CFT at weak coupling and vice versa, so it has been used to find the strong coupling dynamics on the various CFTs. This chapter is devoted to introducing the AdS/CFT correspondence with particular emphasis on the subjects which are used in the thesis and figuring out some thermodynamic properties of CFTs using the duality with higher derivative terms.

The rest of this chapter is organized as follows. In Section 4.1, we briefly go over the AdS/CFT correspondence, with a particular emphasis on the subjects which we need to use in this thesis. In Section 4.3, we discuss the thermodynamic properties of black branes, which corresponds to introducing finite temperature in the CFT side. The discussions on this chapter are based on our work on [34].

[1]For a comprehensive review of AdS/CFT correspondence, see [1]. AdS/CFT correspondence is a concrete realization of the holographic principle [113, 112]. For a review of holographic principle, see [15].

4.1 AdS/CFT Correspondence

4.1.1 AdS/CFT Correspondence with Eight Supercharges

The AdS/CFT correspondence [92] was first proposed as an equivalence between type IIB supergravity on $AdS_5 \times S^5$ and $\mathcal{N} = 4$ superconformal field theory. This initial proposal, however, is not useful when we discuss the application to "real world" physics, since the theory has too many supersymmetries. We therefore consider a more realistic extension of the AdS/CFT correspondence, namely the correspondence between theories with eight supercharges[2]. In order to see how we obtain the correspondence. We briefly review the original "derivation" of the correspondence [92] with 32 supercharges.

Initially, the AdS/CFT correspondence has been derived as two equivalent descriptions of theories near D3-branes. D3-branes, $(3+1)$-dimensional supersymmetric massive extended objects in string theory, have two types of descriptions. If the coupling constant is small, one can describe a D-brane as an object on which open strings can end [102]. Also in this coupling regime, the interactions between the closed sting degrees of freedom in the bulk and open string degrees of freedom are negligibly small compared to the interactions of the open strings. Hence the D-branes can be described purely by open strings, and the resulting theory at low-energy is $\mathcal{N} = 4$ $SU(N)$ super-Yang-Mills theory in four dimensions.

On the other hand, D-branes can also be seen as the source of gravitational forces due to their masses. Therefore, if the coupling constant is large, the interactions between D-branes and gravitons get large and they form a black hole at some point. In this description, the open string degrees of freedom are hidden behind the black hole horizon, and the D-branes can be regarded just as a background geometry. More

[2]The number of fermionic charges is doubled in superconformal field theories. This theory has four supercharges and thus possesses $\mathcal{N} = 1$ supersymmetry.

concretely, the geometry is given [73] by

$$ds^2 = \left(1 + \frac{L^4}{r^4}\right)^{-1/2} ds^2_{M_4} - \left(1 + \frac{L^4}{r^4}\right)^{1/2} ds_{\mathbf{R}^6} \ , \tag{4.1}$$

where r parametrizes the distance from the branes, L is a parameter with dimension of length, M_4 spans in the directions parallel to the branes and $\mathbf{R}^6$ is the space transverse to the branes. Note that the degrees of freedom near the horizon are redshifted, and so have smaller energy compared to the degrees of freedom in the bulk. We therefore expect that only the near-horizon degrees of freedom survive at low-energy. This proposal can be made more concrete by taking the near-horizon limit of the geometry $r \to 0$. Then the constant term in the warp factor can be neglected and the resulting geometry is

$$\begin{aligned} ds^2 &\sim \frac{r^2}{L^2} ds^2_{M_4} - \frac{L^2}{r^2}(dr^2 + r^2 d\Omega_5^2) \\ &= L^2 \left(\frac{ds^2_{M_4} - dz^2}{z^2}\right) - L^2 d\Omega_5^2 \ . \end{aligned} \tag{4.2}$$

The metric in the parentheses in the second line is that of AdS_5, and $d\Omega_5$ defines the line element on S^5. Therefore, this is $AdS_5 \times S^5$ geometry and the theory reduces to type IIB supergravity on $AdS_5 \times S^5$. In this way, we have obtained the equivalence between $\mathcal{N} = 4$ super-YM theory in four dimensions and type IIB supergravity on $AdS_5 \times S^5$.

The number of supercharges in both sides of the theory can be seen as follows. Initially, type IIB string theory 32 supercharges. We defined the type IIB theory on flat spacetime, so the background does not break any supersymmetries. Then, the N parallel D-branes are known to break half of the supersymmetries, so we have 16 supercharges at this point. Finally, the number of supersymmetries is enhanced to twice near the horizon, so we end up with having 32 supercharges.

In order to just keep eight supercharges, we have to break three quarters of supersymmetries in such a way that the near-horizon geometry of the D-branes still have AdS_5. To keep the AdS_5 factor, note from the derivation of $AdS_5 \times S^5$ as a near-horizon geometry in (4.2) that the radial direction r is combined with the directions parallel to the D-branes to form the AdS_5 factor. Therefore, the transverse directions to the D-branes should have a cone-type geometry

$$ds^2 = dr^2 + r^2 ds^2_{X_5} \tag{4.3}$$

and the D-branes should be placed at the tip of the cone. One also needs to break three quarters of supersymmetries, which requires that the cone geometry should be a six-dimensional Calabi-Yau manifold. The cone is Calabi-Yau if and only if the five-dimensional base space X_5 is a Sasaki-Einstein manifold [82, 61]. In conclusion, the AdS/CFT correspondence with eight supercharges can be obtained by placing N parallel D-branes on the tip of a Calabi-Yau cone and the resulting theory is type IIB supergravity on $AdS_5 \times X_5$, where X_5 is a five-dimensional Sasaki-Einstein manifold, on the gravity side and $\mathcal{N} = 1$ SCFT in the CFT side. Then, one can reduce the supergravity on X_5, and one obtains five-dimensional gauged supergravity with eight supercharges.

4.1.2 GKP-Witten Relation and Two-Point Functions

Now that we know how to obtain theories on both sides of correspondence and compared the symmetries, we would like to know the concrete correspondence between the physical quantities. One can derive them using the defining equation of the AdS/CFT correspondence, called the GKP-Witten relation [60, 118]:

$$e^{-I_{sugra}[\phi(x,y)|_{\partial AdS_5}=\phi_0(x)]} = \langle \exp(-\int d^4\phi_0(x)\mathcal{O}(x))\rangle, \tag{4.4}$$

where y is the radial coordinate of the AdS space, ϕ denotes a field in general, and $\mathcal{O}$ is an operator corresponding to ϕ. Intuitively, this relation suggests that the partition function of supergravity with a fixed boundary condition is equal to the partition function of CFT with an operator sourced by the boundary value of the field. This relation is used several times in the following discussion.

For an application of this GKP-Witten relation, let us consider how to compute the two-point function in CFT from the corresponding gravitational description. What the GKP-Witten relation implies is that the generating function of the N-point correlators in CFT can be derived using the corresponding gravitational description. Especially, the Green's function can be computed by taking the second order derivative of ϕ_0 for the gravitational partition function. In this subsection, we focus on the scalar fields living on the AdS space and compute the Euclidean Green's function for the corresponding operator. This result can be applied to computing the shear viscosity of gauge theory plasma in the next chapter.

We consider the following coordinates of an Euclidean AdS_5 in five dimensions:

$$ds^2 = \frac{(\pi T L)^2}{u}(f(u)dt^2 + ds^2_{\mathbf{R}^3}) + \frac{L^2}{4u^2 f(u)}du^2, \tag{4.5}$$

where T is the Hawking temperature of the black brane and $f(u) = 1 - u^2$. The horizon is located at $u = 1$ and the boundary of the AdS_5 at $u = 0$. Since we are interested in Green's function, only the quadratic terms matter. A general invariant action at two-derivative order is given by

$$S = \int d^5x \sqrt{g} \frac{1}{2}(\partial_\mu \phi(x) \partial^\mu \phi(x) - m^2 \phi(x)^2). \tag{4.6}$$

On (4.5), the action can be rewritten as

$$S = L^3 \int dz d^x z^{-3} \left[(\partial_z \phi)^2 + (\partial_i \phi)^2 + \frac{m^2 L^2}{z^2} \phi^2 \right]. \tag{4.7}$$

One can Fourier transform ϕ as

$$\phi(z,x) = \int \frac{d^4k}{(2\pi)^4} e^{ik\cdot x} f_k(z)\phi_0(k). \tag{4.8}$$

Then, the action is expressed as

(4.9)

$$S = L^3 \int_\epsilon^\infty dz \int \frac{d^4k d^4k'}{(2\pi)^4} \frac{\delta(k+k')}{z^3} \left[\partial_z f_k \partial_z f_{k'} - \left(kk' - \frac{m^2L^2}{z^2}\right) f_k f_{k'}\right] \phi_0(k)\phi_0(k').$$

At quadratic order, one can always rewrite the action as a sum of boundary term and terms proportional to the equation of motion. The equation of motion for $f_{k'}$ is

$$f_k'' - \frac{3}{z} f_k' - \left(k^2 + \frac{m^2L^2}{z^2}\right) f_k = 0. \tag{4.10}$$

Using this, one obtains the on-shell boundary action

$$S = L^3 \int \frac{d^4k d^4k'}{(2\pi)^8} \delta^4(k+k')\phi_0(k)\phi_0(k') \frac{f_{k'}(z)\partial_z f_k(z)}{z^3}. \tag{4.11}$$

Then, the two-point function is expressed as

$$\langle \mathcal{O}(k)\mathcal{O}(k')\rangle = -2L^3(2\pi)^4\delta(k+k') \frac{f_{k'}(z)\partial_z f_k(z)}{z^3}. \tag{4.12}$$

To obtain the explicit form of the two-point function, one needs to solve the equation of motion to find $f_k(z)$. A regular solution is given by

$$f_k(z) = \frac{z^2 K_\nu(kz)}{\epsilon^2 K_\nu(k\epsilon)}. \tag{4.13}$$

Plugging this into (4.12) yields the two-point function in the gravity perspective.

4.1.3 Relation Between the Parameters

To compute physical quantities in the CFT using the gravity description, one first needs to express the parameters in supergravity using those defined in the corresponding CFT. Especially, in our supergravity with four derivative terms, there are two parameters

$$\kappa^2 \quad \text{and} \quad c_I M^I. \tag{4.14}$$

On the other hand, the CFT has two central charges

$$a \quad \text{and} \quad c. \tag{4.15}$$

To see the relation between these parameter, one can utilize the correspondence in symmetries [2]. Let us carry out $U(1)$ transformations on both side of (4.4). In the AdS side, the action transforms as

$$I_{bulk} = \cdots + \int \left[\frac{\mathrm{tr}(G_I G_J G_K)}{24\,\pi^2} A^I \wedge F^J \wedge F^K + \frac{\mathrm{tr}\, G_I}{192\,\pi^2} A^I \wedge R_{ab} \wedge R^{ab} \right] . \tag{4.16}$$

Therefore, we have non-invariance at the boundary of AdS space. This corresponds to the anomalies on the CFT side. The $U(1)$ transformations in the CFT side yields

$$\delta_I(\Lambda) Z_{CFT} = \int \Lambda^I \left[\frac{\mathrm{tr}(G_I G_J G_K)}{24\,\pi^2} F^J \wedge F^K + \frac{\mathrm{tr}\, G_I}{192\,\pi^2} R_{ab} \wedge R^{ab} \right], \tag{4.17}$$

where G_I is a global $U(1)_I$ generator, and the trace is taken to be a sum over all the fermion loops. By comparison, we obtain the relation

$$\mathrm{tr}\, G_I = -\frac{\pi c_{2I}}{8 G_5}. \tag{4.18}$$

To relate $c_2 \equiv c_{2I} \bar{M}^I$ to the central charges, we can use the relation

$$a = \frac{3}{32}(3\mathrm{tr}R^3 - \mathrm{tr}R), \quad c = \frac{1}{32}(9\mathrm{tr}R^3 - 5\mathrm{tr}R)\,, \tag{4.19}$$

provided we can relate G_I appropriately to the U(1) charges R. A few comments are needed to explain how to identify the R-charge correctly. First of all, the R-charge is a particular linear combination of the G_I, proportional to $\bar{M}^I G_I$. Also, the supercharge Q_α should have R-charge one. The U(1) charges of Q_α can be read off from the coupling between the gauge fields and the graviphoton in the gravity side, and the algebra is given by $[G_I, Q_\alpha] = P_I Q_\alpha$. This uniquely determines the R-charge as

$$R = \frac{\bar{M}^I G_I L}{P_I \bar{M}^I} \quad \rightarrow \quad \mathrm{tr}R = -\frac{1}{P_I \bar{M}^I} \frac{\pi c_2 L}{8 G_5}\,. \tag{4.20}$$

Recall that the combination $P_I \bar{M}^I = 3/2$ can be determined from the vacuum solution, (3.34). By plugging this equation into (4.19), we obtain

$$\frac{c_2}{24} = \frac{8G_5}{\pi L}(c-a)\,. \tag{4.21}$$

In addition, the gravitational constant also can be determined from the U(1) anomaly. Eq. (4.16) implies

$$\mathrm{tr}(G_I G_J G_K) = \frac{\pi}{8G_5}\left(12c_{IJK} - \frac{g^2}{3}c_{(I}P_J P_{K)}\right)\,. \tag{4.22}$$

By multiplying $\bar{M}^I \bar{M}^J \bar{M}^K$ on both sides, we obtain

$$\frac{27}{8L^3}\mathrm{tr}R^3 = \frac{\pi}{8G_5}\left(12 - \frac{3c_2}{4L^2}\right)\,. \tag{4.23}$$

The formula for the central charges (4.19) and (4.21) then gives

$$\frac{1}{16\pi G_5} = \frac{a}{2\pi^2 L^3}\,. \tag{4.24}$$

Using this relation, (4.21) can be rewritten as

$$\frac{c_2}{24L^2} = \frac{c-a}{a}\,. \tag{4.25}$$

These relations can be computed using the holographic Weyl anomaly [72, 14, 100, 55] and the results agree with each other.

4.2 R-Charged Black Branes

The embedding of the leading order five-dimensional $\mathcal{N} = 2$ gauged $U(1)^3$ supergravity into IIB supergravity was done in [36]. If the three $U(1)$ charges are taken to be equal, we end up with the minimal supergravity system that we have considered above. The static stationary non-extremal solutions are well known, and were found in [9]. For the truncation to minimal supergravity, they take the form

$$\begin{aligned} ds^2 &= H^{-2} f dt^2 - H\Big(f^{-1}dr^2 + r^2 d\Omega_{3,k}^2\Big), \\ A &= \sqrt{\frac{3(kQ+\mu)}{Q}}\Big(1 - \frac{1}{H}\Big)dt, \end{aligned} \tag{4.26}$$

where the metric functions H and f are:

$$
\begin{aligned}
H(r) &= 1+\frac{Q}{r^2}, \\
f(r) &= k-\frac{\mu}{r^2}+g^2r^2H^3\,.
\end{aligned}
\tag{4.27}
$$

Here μ is a non-extremality parameter and $d\Omega^2_{3,k}$ for $k=1$, 0, or -1 corresponds to the unit metric of a spherical, flat, or hyperbolic 3-dimensional geometry, respectively.

4.2.1 Higher order corrected R-charged Solutions

We find corrections to the R-charged solutions (4.26) given the higher derivative Lagrangian (3.57). The discussions on this section is primarily based on [34]. To this end, as in [90] we treat c_2 as a small parameter and expand the metric and gauge field as follows:

$$
\begin{aligned}
H(r) &= 1+\frac{Q}{r^2}+c_2h_1(r)\,, \\
f(r) &= k-\frac{\mu}{r^2}+g^2r^2H^3+c_2f_1(r)\,, \\
A &= \sqrt{\frac{3(kQ+\mu)}{Q}}\Big(1-\frac{1+c_2a_1(r)}{H}\Big)dt\,,
\end{aligned}
\tag{4.28}
$$

where h_1, f_1, and a_1 parameterize the corrections to the background geometry. Solving the equations of motion for the theory, we arrive at:

$$
\begin{aligned}
h_1 &= -\frac{Q(kQ+\mu)}{72r^6H_0^2}\,, \\
f_1 &= \frac{-5g^2Q(kQ+\mu)}{72r^4}+\frac{\mu^2}{96r^6H_0}\,, \\
a_1 &= \frac{Q}{144r^6H_0^3}\big[4(kQ+\mu)-3\mu-\frac{3Q\mu}{r^2}\big]\,.
\end{aligned}
\tag{4.29}
$$

The new corrected geometry is therefore given by

$$
\begin{aligned}
H(r) &= H_0(r) + \frac{c_2}{24}\left[\frac{-Q(kQ+\mu)}{3r^6 H_0^2}\right], \\
f(r) &= f_0(r) + \frac{c_2}{24}\left[-\frac{8g^2 Q(kQ+\mu)}{3r^4} + \frac{\mu^2}{4r^6 H_0}\right], \\
A_t(r) &= A_{t\,0}(r) - \frac{c_2}{24}\frac{\sqrt{3Q(kQ+\mu)}}{2r^8 H_0^4}\left[2(kQ+\mu)r^2 - \mu r^2 H_0\right],
\end{aligned} \tag{4.30}
$$

where H_0, f_0, and A_0 refer to the background solutions (4.26) and (4.27). Finally, we should note that in the literature Q and μ are sometimes written in terms of a parameter β, defined by $\sinh^2\beta = kQ/\mu^2$.

We will state the $k = 0$ and $k = 1$ solutions explicitly, since they have several interesting applications: the former to studies of the hydrodynamic regime of the theory, and the latter to the issue of horizon formation for small black holes. For $k = 0$, the solution is given by

$$
\begin{aligned}
H(r) &= H_0(r) + \frac{c_2}{24}\left[\frac{-Q\mu}{3\,r^6 H_0^2}\right], \\
f(r) &= f_0(r) + \frac{c_2}{24}\left[-\frac{8g^2\mu Q}{3r^4} + \frac{\mu^2}{4\,r^6 H_0}\right], \\
A_t(r) &= A_{t\,0}(r) - \frac{c_2}{24}\left[\frac{\sqrt{3Q\mu}}{2r^8 H_0^4}(\mu r^2 - Q\mu)\right].
\end{aligned} \tag{4.31}
$$

while for $k = 1$ it is given by

$$
\begin{aligned}
H(r) &= H_0(r) - \frac{c_2}{24}\left[\frac{Q(Q+\mu)}{3r^2(r^2+Q)^2}\right], \\
f(r) &= f_0(r) + \frac{c_2}{24}\left[-\frac{8g^2 Q(Q+\mu)}{3r^4} + \frac{\mu^2}{4r^6 H_0}\right], \\
A_t(r) &= A_{t\,0}(r) - \frac{c_2}{24}\left[\frac{\sqrt{3Q(Q+\mu)}}{2r^8 H_0^4}\Big((2Q+\mu)r^2 - Q\mu\Big)\right].
\end{aligned} \tag{4.32}
$$

4.2.2 Conditions for Horizon Formation

We would like to conclude this section with some comments on the structure of the horizon for the solutions that we have found. In particular, we are interested in whether higher derivative corrections will facilitate or hinder the formation of a horizon. In the standard two-derivative theory, the BPS-saturated limit ($\mu = 0$) of the $k = 1$ solution (4.26)-(4.27) describes a geometry with a naked singularity, the so-called superstar [99]. Furthermore, even if the non-extremality parameter is turned on, one finds that a horizon develops only given a certain critical amount, $\mu \geq \mu_c$ [9]. It is therefore natural to ask what happens to such geometries once we start incorporating curvature corrections. For the superstar, we would like to see hints of horizon formation. In the non-extremal case, on the other hand, it would be nice to determine whether the inclusion of higher-derivative corrections leads to a smaller (larger) critical value μ_c, increasing (decreasing) the parameter space for the appearance of a horizon. However, one should keep in mind that our arguments are only suggestive, since our analysis is perturbative, while the formation of a horizon is a non-perturbative process. Moreover, given that even in the non-extremal case turning on μ does not guarantee the presence of a horizon, it is not clear at all whether higher derivative corrections can be enough to push the superstar to develop a horizon. A more proper analysis would involve looking directly at the SUSY conditions, and asking whether they are compatible with having a superstar solution with a finite horizon. In fact, there are already studies which seem to indicate [97] that this may not be possible.

The spherically symmetric solutions presented in (4.32) are of the form:

$$ds^2 = F_1(r)\, dt^2 - F_2(r)\, dr^2 - F_3(r)\, d\Omega_3^2\,. \tag{4.33}$$

Horizons appear at zeroes of the function $F_1(r)$. One can make arguments about their existence without having to solve explicitly for their exact location. Notice that $F_1(r)$ is a positive function for large r. Thus, a sufficient condition for having at least one horizon is

$$F_1(r_{min}) \leq 0\,, \tag{4.34}$$

where r_{min} is a (positive) minimum of $F_1(r)$. This was the reasoning used in [9] to study the properties of the horizon of the non-extremal solution.

For the corrected superstar solution we have, expanding in c_2:

$$F_1 \equiv \frac{f}{H^2} = \frac{f_0 + c_2(f_1 - 2f_0 h_1 H_0^{-1})}{H_0^2} + \mathcal{O}(c_2^2)\,. \tag{4.35}$$

It is easy to see that, to leading order, the numerator does not vanish. With the inclusion of higher-derivative terms, however, it picks up a negative contribution, hinting at the possibility of a horizon. Furthermore, the minimum of the function $F \equiv f_0 + c_2(f_1 - 2f_0h_1H_0^{-1})$ will shift. Let's see precisely how that happens. To lowest order, its minimum is given by $x^{(0)}_{min} = 2Q$, which in turn gives us $F(x^{(0)}_{min}) = 1 + 27g^2Q/4$. Including higher order corrections, we find

$$x_{min} = x^{(0)}_{min} + c_2 x^{(1)}_{min} = 2Q - c_2\frac{81g^2Q - 4}{4374Qg^2}\,. \tag{4.36}$$

Now we have

$$F(x_{min}) = 1 + 27g^2Q/4 + c_2\left(\frac{1}{972Q} - \frac{g^2}{48}\right), \tag{4.37}$$

which tells us that the minimum of the function will be slightly closer to zero as long as $g^2Q > 4/81$.

The analysis of the conditions for the existence of a horizon in the non-extremal case ($\mu \neq 0$) is significantly more involved. The expression for the corrected horizon radius in terms of the original, two-derivative horizon radius r_0 is:

(4.38)

$$r_H = r_0\left(1 + \frac{c_2}{24}\left\{\frac{g^4H_0^4(3Q^2 - 26Qr_0^2 + 3r_0^4) - 2g^2H_0^2(13Q - 3r_0^2) + 3}{24H_0r_0[g^2H_0^2(Q - 2r_0^2) - 1]}\right\}\right).$$

Notice that we traded μ in favor or r_0 in the expression above by making use of $f_0(r_0) = 0$, *i.e.* the relation $\mu/r_0^2 = 1 + g^2 r_0^2 H_0^3$. As we mentioned above, in the two-derivative case one finds a critical value μ_{crit} above which a horizon will form. It would certainly be interesting to explore for which parameter values r_H decreases or increases, and more importantly, how the (corrected) critical value of μ is affected by the curvature corrections. We leave this to future studies.

4.3 Holographic Thermodynamics with Higher Derivatives

We may now study some of the basic thermodynamic properties of the non-extremal solutions constructed above. With an eye towards AdS/CFT in the Poincaré patch, we will focus on the $k = 0$ solution (4.31), although the analysis may easily be carried out for the other cases as well. We begin with the entropy, which for Einstein gravity is characterized by the area of the event horizon. In the presence of higher derivative terms, however, this relation is modified, and the entropy is no longer given by the area law. Instead, we may turn to the Noether charge method developed in [116] (see also [76, 75]).

The original Noether charge method is only applicable to a theory with general covariance, but has been extended to a theory with gravitational Chern-Simons terms in [114]. Our action includes a mixed Chern-Simons term of the form $A \wedge R \wedge R$. But as long as we keep this term as it is, with a bare gauge potential, the general covariance is unbroken and we can still use the original formulation. In the absence of covariant derivatives of the Riemann tensor, the entropy formula is given by [116]

$$S = -2\pi \int_\Sigma d^{D-2}x \sqrt{-h} \frac{\delta \mathcal{L}}{\delta R_{\mu\nu\rho\sigma}} \epsilon_{\mu\nu} \epsilon_{\rho\sigma} \, , \tag{4.39}$$

where Σ denotes the horizon cross section, h is the induced metric on the it and $\epsilon_{\mu\nu}$ is the binormal to the horizon cross section.

For the metric ansatz (4.26) the only non-vanishing component of the binormal $\epsilon_{\mu\nu}$ is

(4.40) $$\epsilon_{tr} = -\epsilon_{rt} = H^{-1/2}\,.$$

Applying the prescription (4.39) to the action (3.57), we obtain, to linear order in c_2,

(4.41)
$$\begin{aligned} S &= \frac{A}{8G_5}\Big[-g^{\mu\rho}g^{\nu\sigma} \\ &\quad + \frac{c_2}{24}\big(-\tfrac{1}{4}C^{\mu\nu\rho\sigma} - \tfrac{1}{32}g^{\mu\rho}g^{\nu\sigma}F^2 + \tfrac{5}{12}g^{\nu\sigma}F^{\mu\lambda}F^{\rho}{}_{\lambda} - \tfrac{1}{16}F^{\mu\nu}F^{\rho\sigma}\big)\Big]\epsilon_{\mu\nu}\epsilon_{\rho\sigma}\Big|_{r=r_+} \\ &= \frac{A}{4G_5}\left[1 + c_2\,\frac{\mu(Q+3r_0^2)}{48(r_0^2+Q)^3}\right], \end{aligned}$$

where $A = \int\sqrt{-h}\ d\Omega_{3,0}$ is the area of the horizon for the solution to the higher derivative theory. Also, r_+ denotes the radius of the event horizon for the corrected black brane solution, while r_0 is the horizon location for the original, two-derivative solution (4.27). The former can be found by requiring that the $g_{tt} = f(r)/H(r)^2$ component of the corrected metric vanishes[3]. Similarly, r_0 satisfies $f_0(r_0) = 0$. Notice that the non-extremality parameter μ can be expressed entirely in terms of r_0 and Q:

(4.42) $$f_0(r_0) = 0 \;\;\Rightarrow\;\; \mu = \frac{g^2(r_0^2+Q)^3}{r_0^2}\,.$$

We can therefore eliminate μ from (4.41), and write the entropy in the following form:

(4.43) $$S = \frac{A}{4G_5}\left[1 + c_2 g^2\frac{Q+3r_0^2}{48\,r_0^2}\right].$$

[3]To linear order in the expansion parameter c_2, this coincides with demanding that $f(r)$ vanishes.

The first term above is simply the contribution coming from the area, while the remaining $\mathcal{O}(c_2)$ term is the expected deviation from the area law.

In order to arrive at the entropy density, we need one more ingredient, which is the relation between the corrected and uncorrected horizon radii r_+ and r_0:

$$r_+ = r_0 \left(1 + \frac{c_2}{24} \frac{g^2(r_0^2+Q)(3Q^2 - 26Qr_0^2 + 3r_0^4)}{24r_0^4(Q-2r_0^2)} \right). \tag{4.44}$$

This is because the area A appearing in (4.43) is computed using r_+. This expression allows us to write the entropy per unit three-brane spatial volume entirely in terms of r_0 as well as the physical parameters of the theory

$$\begin{aligned} s &= \frac{(r_0^2+Q)^{3/2}}{4G_5L^3} \left(1 + \frac{c_2}{24} \frac{g^2(3Q^2 - 14Qr_0^2 - 21r_0^4)}{8r_0^2(Q-2r_0^2)} \right) \\ &= \frac{2(r_0^2+Q)^{3/2}}{\pi L^6} \left(a + (c-a) \frac{3Q^2 - 14Qr_0^2 - 21r_0^4}{8r_0^2(Q-2r_0^2)} \right). \end{aligned} \tag{4.45}$$

In the second line we have used the relations (4.24) and (4.25) to replace the gravitational quantities G_5 and c_2 by the central charges of the dual CFT. Notice that the lowest order term above matches the two-derivative entropy computation of [108].

While r_0 is the coordinate location of the horizon in the lowest order computation, it is not in itself a physically relevant parameter. Instead, it may be viewed as a proxy for the Hawking temperature associated with the non-extremal solution. A simple way of computing this temperature is to identify it with the inverse of the periodicity of Euclidean time τ. The relevant components of the metric are given by

$$ds^2 = H^{-2} f d\tau^2 + H f^{-1} dr^2 + \cdots, \tag{4.46}$$

and the horizon is located at $f(r_+) = 0$. Expanding near the horizon and identifying the proper period of τ to remove the conical singularity yields the temperature

$$T_H = \frac{(r_0^2+Q)^{1/2}}{2\pi L^2} \left[\frac{(2r_0^2-Q)}{r_0^2} + \frac{c_2}{24L^2} \frac{(3Q^3 + 4Q^2r_0^2 + 59Qr_0^4 - 10r_0^6)}{8r_0^4(2r_0^2-Q)} \right]. \tag{4.47}$$

In principle, we may invert this expression to obtain r_0 as a function of temperature T_H and charge Q. This then allows us to rewrite the entropy density as a function of charge and temperature, $s = s(T_H, Q)$. In practice, however, non-trivial R-charge introduces a new scale, so that the entropy density/temperature relation no longer takes the simple form $s \sim T^3$ resulting from simple dimensional analysis.

CHAPTER V

Holographic Hydrodynamics

In this chapter, we apply the AdS/CFT correspondence to analyzing the hydrodynamic properties of gauge theory plasma[1]. The primary goal of this chapter is to compute the shear viscosity, denoted by η in this thesis, over the entropy density ratio of gauge theory plasma observed in heavy ion collisions. Experimentally, this ratio is given, for example in [109], by

$$\frac{\eta}{s} \leq 0.2, \tag{5.1}$$

Theoretically, however, it is pretty hard to compute this quantity reliably. The main problem is that at the energy scale at which gauge theory plasma is formed, the theory is somewhat strongly coupled. Hence, the standard perturbative calculations are unreliable. Actually, perturbative calculations [5] yield

$$\frac{\eta}{s} \sim \frac{1}{\alpha_s^2} \ln \alpha_s^{-1} \sim 1, \tag{5.2}$$

at the relevant temperatures, which is quite off from the experimental value. Another problem is that the shear viscosity is a real-time quantity, meaning that the viscosity is associated with the time evolution of the fluid. So, the Euclidean computations such as lattice gauge theory calculations also do not work very well[2].

[1] For a comprehensive review of this chapter, see [27] and references therein.
[2] Nevertheless, there are attempts to obtain the η/s from lattice computations. See [95] for details.

The AdS/CFT correspondence overcomes these two difficulties. First of all, the strong coupling in the CFT side corresponds to a gravitational theory at weak coupling, so one can reliably compute the strong coupling quantities in the CFT using the gravitational description. Also, Lorentzian AdS/CFT techniques have been extensively studied, so one can compute real-time quantities in contrast to the lattice gauge theory. The shear viscosity computation was first done by Policastro, Son and Starinets in [103], and the result is

$$\eta = \frac{\pi}{8} N^2 T^3. \tag{5.3}$$

When combined with the entropy density computed also in using AdS/CFT, one obtains the shear viscosity over entropy ratio [83]

$$\frac{\eta}{s} = \frac{1}{4\pi}. \tag{5.4}$$

This result is universal for any large N gauge theory and within the interval allowed by experiments [18, 94, 93, 108, 91, 77].

Based on this extremely small ratio, Kovtun, Son and Starinets conjectured the following KSS bound:

$$\frac{\eta}{s} \leq \frac{1}{4\pi} \tag{5.5}$$

In this chapter, however, we show that this bound is violated by finite N corrections or, in gravity language, in the presence of higher derivative corrections. We introduce the finite chemical potential for R-charge, but it worsens the violation of the bound [35, 98].

Before our work [35] has been published, there have been numerous works published on the computations of the shear viscosity over entropy density ratio with higher derivative terms. The first example is the inclusion of finite 't Hooft coupling corrections, which corresponds to higher derivative terms in ten-dimensional supergravity [20]. Then, the four derivative terms in five-dimensional Einstein gravity, which, as discussed in Section 4.1.3, correspond to finite N corrections, have been taken into account [81, 17]. It yields a lower ratio than the KSS result for theories with $c - a > 0$, showing that the KSS bound can be violated in the presence of finite N corrections. Later, it has also been shown that the CFTs with gravity duals generally satisfy $c-a > 0$ [21], although there are some counter-examples in theories with no Lagrangian descriptions [56, 57]. The inclusions of both finite N corrections and chemical potential have also been considered in [58, 23], but the include only a particular subset of four-derivative terms. Our work is the first result which considers all the four derivative corrections in a way consistent with supersymmetry.

The rest of this chapter is organized as follows. In Section 5.1, we briefly review how to compute shear viscosity using Lorentzian AdS/CFT correspondence and reproduce the KSS result for η/s. In Section 5.2, we extend the analysis of Section 5.1 with higher derivative corrections in the gravity side, and see the KSS bound is violated. As discussed in the previous chapter, these higher derivative effects are interpreted as finite N corrections in the CFT side.

5.1 Holographic Shear Viscosity at Two-Derivatives

In this section, we compute the shear viscosity over entropy density ratio using Lorentzian AdS/CFT techniques. First of all, shear viscosity is formally defined as a parameter in the stress energy tensor of a fluid:

$$T_{ij} = \delta_{ij}\rho - \eta\left(\partial_i u_j + \partial_j u_i - \frac{2}{3}\delta_{ij}\partial_k u_k\right) - \zeta\delta_{ij}\partial_k u_k, \tag{5.6}$$

where η is shear viscosity and ζ is bulk viscosity. In order to clarify the meaning of the shear viscosity, let us decompose a fluid into layers. Then, the shear viscosity is the coefficient of the terms which represent the change in fluid velocity in two adjacent layers. As the shear viscosity increases, the amount of energy associated with the change in fluid velocities in adjacent layers increases. Therefore, the shear viscosity can be thought of as the coefficient of "friction" between two layers, or equivalently, as the resistance of a fluid under shear stress.

Shear viscosity of a fluid can be generally computed using the Kubo formula:

$$\eta = \lim_{\omega \to 0} \frac{1}{\omega} \mathrm{Im} \left(-i \int dt d\vec{x} e^{i\omega t} \theta(t) \langle [T_{xy}(t, \vec{x}), T_{xy}(0,0)] \rangle \right), \tag{5.7}$$

where the quantity in the parentheses is the retarded Green's function for the shear modes of stress tensors. The goal of this section is to evaluate this formula using AdS/CFT techniques. As has been discussed in Section 4.1.2, the Euclidean two point function is calculated as the coefficient of the terms quadratic in the boundary fields ϕ_0. We extend this technique to the Lorentzian case.

The detailed procedure to obtain the retarded Green's function has been discussed in [107, 104]. At quadratic order, the Lorentzian on-shell action, as in the Euclidean case, can be written as

$$S = \int \frac{d^4k}{(2\pi)^4} \phi_0(-k) \mathcal{F}(k,z) \phi_0(k), \tag{5.8}$$

where a field ϕ is expanded as

$$\phi(x,z) = \int \frac{d^4k}{(2\pi)^4} f_k(z) \phi_0(k). \tag{5.9}$$

There are two differences from the Euclidean case. In the Euclidean case, the classical action for ϕ is uniquely determined by its value at the boundary ϕ_0 and imposing regularity at the horizon. However, regularity at the horizon is not sufficient to determine the solution in the Lorentzian case, so one needs more refined boundary condition at the horizon [6]. The authors of [107, 104] imposed the incoming wave boundary condition, in which nothing can escape from the black hole horizon. Another difference is that if one naively applies the GKP-Witten procedure, one obtains

$$G(k) = -\mathcal{F}(k,z)|_{z_B}^{z_H} - \mathcal{F}(-k,z)|_{z_B}^{z_H}, \tag{5.10}$$

which is a real quantity, and cannot be a candidate for the retarded Green's function. A way out given in [107, 104] is to take the retarded Green's function as

$$G_R(k) = -2\mathcal{F}(k,z)|_{z_B} \ . \tag{5.11}$$

Although there is no rigorous justification of this procedure from first principles, it passes several consistency checks. So, we apply this procedure to compute the shear viscosity of gauge theory plasma in this section.

So, let us compute the shear viscosity at two-derivative order without any non-trivial gauge field. Since we are interested in the retarded Green's function for T_{xy}, we consider the metric perturbation

$$g_{xy} = g_{xy}^{b.g.} + h_{xy} \ , \tag{5.12}$$

where $b.g.$ in the superscript implies the background geometry, which is given by

$$ds^2 \ = \ \frac{g^2 r_0^2}{u}\left[\frac{f(u)}{H(u)}dt^2 - H(u)ds^2_{\mathbf{R}^3}\right] - \frac{H(u)}{4g^2u^2f(u)}du^2 \ , \tag{5.13}$$

where $f(u) = 1 - u^2$ and $H(u) = 1$. We introduce $H(u)$ although it is trivial in this case for later use. Since we are just interested in the gravitational part of the action, one can safely set the gauge field to vanish. Then, the action is just the Einstein action

$$S = -\frac{1}{16\pi G_5}\left[\int du \int d^x\sqrt{g}\left(R - 12g^2\right) + 2\int d^4x\sqrt{-h}K\right] \ , \tag{5.14}$$

where K is the extrinsic curvature of the boundary and the last term is the Gibbons-Hawking term, which is added to cancel the boundary terms in the variation of the action. One also needs to add counterterms to keep the action finite, but we omitted them because it does not matter in the following computation. If one considers the metric perturbation (5.12) and defines $\phi = h^x{}_y$, then the equation of motion is given by

$$\phi_k'' - \frac{1+u^2}{uf}\phi_k' + \frac{\omega^2 - q^2 f}{uf^2}\phi_k = 0. \tag{5.15}$$

The solution representing the incoming wave boundary condition at the horizon is

$$\phi_k(u) = (1-u)^{-i\omega/2} F_k(u), \tag{5.16}$$

where $F(u)$ is regular at $u = 1$. One can express $F_k(u)$ as

$$F_k(u) = 1 - \frac{i\omega}{2} \ln \frac{1+u}{2} + \cdots , \tag{5.17}$$

where the terms with higher order in ω do not contribute to the calculation of the shear viscosity and thus are omitted. Then, $\mathcal{F}(k, z)$ is given by

$$\mathcal{F}(k, z) = \frac{N^2 T^2}{32}(2\pi i T\omega + k^2) + \cdots , \tag{5.18}$$

Plugging this into the Kubo formula (5.7) yields the shear viscosity [103]

$$\eta = \frac{\pi}{8} N^2 T^3. \tag{5.19}$$

On the other hand, the entropy density of the black brane is given by $s = \pi^2 N^2 T^3/2$. Hence the shear viscosity over entropy ratio is given by

$$\frac{\eta}{s} = \frac{1}{4\pi}. \tag{5.20}$$

This is how KSS derived the formula [83].

5.2 Holographic Shear Viscosity with Higher Derivatives

We turn to the calculation of shear viscosity with four derivative terms in this section. One difficulty in the calculation with higher derivative terms is that the equation of motion includes higher order in derivatives. If one expand the Lagrangian to second order in the perturbation $\phi = h^x{}_y$, one finds

(5.21)
$$S = \frac{1}{16\pi G_5}\int \frac{d^4k}{(2\pi)^4}\int_0^1 du\Big[A\phi''_k\phi_{-k} + B\phi'_k\phi'_{-k} + C\phi'_k\phi_{-k} + D\phi_k\phi_{-k} + E\phi''_k\phi''_{-k} + F\phi''_k\phi'_{-k}\Big].$$

This action involves fourth order derivatives of ϕ; thus the equation of motion also includes terms proportional to ϕ''''. To solve the equation, one needs to impose four independent boundary conditions, but we do not know what those conditions should be.

However, if one assumes that c_2, the coefficient of the four derivative terms, is a small parameter, which is the case for sufficiently large N, one can reduce the order of derivatives using the leading order equation of motion [20]. The tree level equation of motion is given by

$$\phi'' + \left(\frac{f_0'}{f_0} - \frac{1}{u}\right)\phi' + \frac{\bar{\omega}^2 H_0^3}{u f_0^2}\phi = 0\,, \tag{5.22}$$

where we have defined the dimensionless frequency

$$\bar{\omega}^2 = \frac{\omega^2}{4g^4 r_0^2}\,. \tag{5.23}$$

The lowest order metric functions are

$$f_0 = (1+qu)^3 - (1+q)^3u^2, \qquad H_0 = 1+qu. \tag{5.24}$$

Taking additional derivatives of (5.22) allows us to eliminate ϕ''' and ϕ'''' terms in the full equation of motion. The result is rather simple:

$$\phi'' + \left(\frac{f'}{f} - \frac{1}{u} - c_2\frac{(1+q)^3 u}{(1+qu)^3}\right)\phi' + \frac{\bar{\omega}^2 H^3}{uf^2}\,\phi = 0. \tag{5.25}$$

Notice that the form of this equation is almost identical to that of (5.22), the lowest order equation of motion, modified only by the presence of the corrected metric functions f and H as well as one new term, which is explicitly $\mathcal{O}(c_2)$.

Since the function $f(u)$ vanishes linearly at the horizon u_+, the point $u = u_+$ is a regular singular point of the equation of motion (5.25). This suggests that we write

$$\phi(u) = f(u)^\nu F(u), \tag{5.26}$$

where $F(u)$ is assumed to be regular at the horizon. The exponent ν is then obtained by solving the indicial equation. In the hydrodynamic limit, the lowest order solution is known [93, 108] and is given by:

(5.27)

$$\phi_0 = f_0(u)^{\nu_0}\left\{1 - \frac{\nu_0}{2}\left[\Delta \ln \frac{(\Xi - \alpha_1 - 1 + 2\alpha_3 u)(\Xi + \alpha_1 + 1)}{(\Xi + \alpha_1 + 1 - 2\alpha_3 u)(\Xi - \alpha_1 - 1)} + 3\ln\left(1 + (\alpha_1 + 1)u - \alpha_3 u^2\right)\right]\right\},$$

where

(5.28)

$$\alpha_1 \equiv 3q, \qquad \alpha_2 \equiv 3q^2, \qquad \alpha_3 \equiv q^3, \qquad \Xi \equiv (1+q)(1+4q)^{1/2}, \qquad \Delta \equiv -3\frac{q+1}{\Xi}.$$

The exponent ν_0 is given by

$$\nu_0 = -\frac{i\bar{\omega}}{(2-q)(1+q)^{1/2}}, \tag{5.29}$$

and may be re-expressed as $\nu_0 = -i\omega/4\pi T_0$, where T_0 is the lowest order temperature. Note that we have chosen incoming wave boundary conditions at the horizon as appropriate to the shear viscosity calculation.

Adding higher derivative terms will have two effects on this solution, one being a correction to the function $F(u)$ and the other a modification of the exponent ν defined above. For the exponent, solving the indicial equation gives

$$\nu = -\frac{i\bar{\omega}}{(2-q)(1+q)^{1/2}}\Big(1+\frac{c_2}{8}\frac{10-59q-4q^2-3q^3}{(q-2)^2}\Big) = -\frac{i\omega}{4\pi T}, \tag{5.30}$$

where the relation to the temperature is valid to linear order in c_2. We may now substitute $\phi(u) = f(u)^\nu F(u)$ into the equation of motion (5.25) and linearize in c_2 to obtain an equation for $F(u)$. While this is difficult to solve exactly, since we only need a solution in the hydrodynamic regime, it is sufficient to work to first order in ω (or equivalently ν). The solution for $F(u)$ is quite complicated and can be found in Appendix C.

Given this solution, it remains to evaluate the on-shell value of the action. As explained in [20], the bulk action (5.21) must be paired with an appropriate generalization of the Gibbons-Hawking term. In general, the fourth order equation of motion yields a boundary value problem for the two-point function where additional data must be specified (*e.g.* fields and their first derivatives at the endpoints). However, when working perturbatively in $\bar{c}_2$, the equation of motion reduces to a second order one, given by (5.25). This allows us to use a generalized Gibbons-Hawking term of the form

$$\mathcal{K} = -A\phi_k\phi'_{-k} - \frac{F}{2}\phi'_k\phi'_{-k} + E(p_1\phi'_k + 2p_0\phi_k)\phi'_{-k}, \tag{5.31}$$

where

$$p_1 = \frac{f'_0}{f_0} - \frac{1}{u}, \qquad p_2 = \frac{\bar{\omega}^2 H_0^3}{u f_0^2} \tag{5.32}$$

are the coefficients in the lowest order equation of motion (5.22).

Evaluating the on-shell action then amounts to evaluating a boundary term

$$S = \int \frac{d^4k}{(2\pi)^4} \mathcal{F}_k \Big|_0^1, \tag{5.33}$$

where

$$\begin{aligned} \mathcal{F}_k &= \frac{1}{16\pi G_5}\Bigg[\Big(B - A - \frac{F'}{2}\Big)\phi'_k\phi_{-k} + \frac{1}{2}(C - A')\phi_k\phi_{-k} - E'\phi''_k\phi_{-k} \\ &\quad + E\phi''_k\phi'_{-k} - E\phi'''_k\phi_{-k} - E\Big(\frac{f'_0}{f_0} - \frac{1}{u}\Big)\phi'_k\phi'_{-k} + 2E\frac{\bar{\omega}^2 H_0^3}{uf_0^2}\phi'_k\phi_{-k}\Bigg]. \end{aligned} \tag{5.34}$$

In order to compute the shear viscosity we need only the limit of the above action as u approaches the AdS boundary (*i.e.* $u \to 0$). It turns out that only the first and third terms contribute. This yields a value for the shear viscosity via the Kubo relation

$$\eta = \lim_{\omega\to 0} \frac{1}{\omega} \lim_{u\to 0}(2\,\mathrm{Im}\,\mathcal{F}_k) = \frac{(gr_0)^3}{16\pi G_5}(q+1)^{3/2}\Big(1 + \frac{c_2}{8}\frac{5 + 6q + 5q^2}{2 - q}\Big). \tag{5.35}$$

Finally, dividing this by the entropy density (4.45) gives a value for the shear viscosity to entropy density ratio of

$$\frac{\eta}{s} = \frac{1}{4\pi}\Big[1 - c_2(1+q)\Big] = \frac{1}{4\pi}\Big[1 - \frac{c-a}{a}(1+q)\Big], \tag{5.36}$$

where we have rewritten c_2 in terms of the anomaly coefficients c and a using (4.25).

The shear viscosity over entropy density ratio (5.36) can be expressed purely in terms of the CFT quantities. To do that, one uses the fact that the boundary value (the difference from the value at the horizon, more precisely) of the time-component of a gauge field should be identified as the chemical potential for the charge associated to the gauge field [31, 37]. More concretely, the R-charge in the bulk q is, at the leading order, related to the R-charge chemical potential Φ in the CFT as

$$\Phi = gr_0\sqrt{3q(1+q)} \tag{5.37}$$

This relation, together with the temperature at the leading order $T_0 = \frac{g^2 r_0}{2\pi}(2-q)(1+q)^{1/2}$, yields

$$q = \frac{3}{2\bar{\Phi}}\left(1 + \frac{4}{3}\bar{\Phi}^2 - \sqrt{1 + \frac{8}{3}\bar{\Phi}^2}\right), \tag{5.38}$$

where $\bar{\Phi} = g\Phi/2\pi T$ is the dimensionless chemical potential. Note that q is an increasing function with respect to $\bar{\Phi}$, which ranges from 0 to 2. Substituting this into (5.38) into (5.36) yields

$$\frac{\eta}{s} = \frac{1}{4\pi}\left[1 - \frac{c-a}{c}\left(1 + \frac{3}{2\bar{\Phi}^2}\left(1 + \frac{4}{3}\bar{\Phi}^2 - \sqrt{1 + \frac{8}{3}\bar{\Phi}^2}\right)\right)\right] \tag{5.39}$$

As is obvious from the expression for η/s in (5.36) and the fact that q is defined to be positive, the introduction to the chemical potential worsens the violation of the KSS bound, provided $c - a > 0$. Taking the range $0 \leq q \leq 2$ into account, one sees that the ratio takes the following range:

$$\frac{1}{4\pi}\left(1 - 3\frac{c-a}{a}\right) \leq \frac{\eta}{s} \leq \frac{1}{4\pi}\left(1 - \frac{c-a}{a}\right) \tag{5.40}$$

for $c - a > 0$.

After the work [35], there have been numerous works computing the shear viscosity to entropy density ratio in various types of theories. A partial list includes a dilaton gravity with four derivative terms [22], Einstein gravity with four and six derivative terms [8], that with up to ten derivatives [101], Lovelock gravity [59], and Gauss-Bonnet gravity with dimensions greater than five [19].

APPENDICES

APPENDIX A

Notations

A.1 Notations

We summarize our notational conventions in this appendix. Firstly, the components of various multiplets and their basic properties are summarized in Table. The gamma matrices γ^a satisfy $\{\gamma^a, \gamma^b\} = 2\eta^{ab}$ and $(\gamma^a)^\dagger = \eta_{ab}\gamma^b$, where $\eta^{ab} = \mathrm{d}iag(+,-,-,-,-)$. $\gamma_{a\ldots b}$ represents an antisymmetrized product of gamma matrices:

(A.1) $$\gamma_{a\ldots b} = \gamma_{[a} \cdots \gamma_{b]},$$

where the square brackets denote complete antisymmetrization with weight 1. Similarly $(\ldots)$ denote complete symmetrization with weight 1. We choose the Dirac matrices to satisfy

(A.2) $$\gamma^{a_1 \ldots a_5} = \epsilon^{a_1 \ldots a_5},$$

where $\epsilon^{a_1 \ldots a_5}$ is a totally antisymmetric tensor with $\epsilon^{01234} = 1$.

The $SU(2)$ index i (i=1,2) is raised and lowered with ϵ_{ij}, where $\epsilon_{12} = \epsilon^{12} = 1$, in the northwest-southeast (NW-SE) convention:

(A.3) $$A^i = \epsilon^{ij} A_j, \qquad A_i = A^j \epsilon_{ji}.$$

The charge conjugation matrix C in 5D has the properties

$$C^T = -C, \qquad C^\dagger C = 1, \qquad C\gamma_a C^{-1} = \gamma_a^T. \tag{A.4}$$

Our five-dimensional spinors satisfy the $SU(2)$-Majorana condition

$$\bar{\psi}^i \equiv \psi_i^\dagger \gamma^0 = \psi^{iT} C, \tag{A.5}$$

where the spinor indices are omitted. When the $SU(2)$ indices are suppressed in the bilinear terms of spinors, the NW-SE contraction is understood, e.g. $\bar{\psi}\gamma^{a_1\ldots a_n}\lambda = \bar{\psi}^i\gamma^{a_1\ldots a_n}\lambda_i$. Changing the order of the spinors in a bilinear leads to the following signs:

$$\bar{\psi}\gamma^{a_1\ldots a_n}\lambda = (-1)^{(n+1)(n+2)/2}\bar{\lambda}\gamma^{a_1\ldots a_n}\psi. \tag{A.6}$$

If the $SU(2)$ indices are not contracted, the sign switches. We often use the Fierz identity, which in 5D reads

$$\psi^i\bar{\lambda}^j = -\frac{1}{4}(\bar{\lambda}^j\psi^i) - \frac{1}{4}(\bar{\lambda}^j\gamma^a\psi^i)\gamma_a + \frac{1}{8}(\bar{\lambda}^j\gamma^{ab}\psi^i)\gamma_{ab}. \tag{A.7}$$

Table A.1: Multiplets in 5D superconformal gravity.

field	type	remarks	$SU(2)$	Weyl-weight
$e_\mu{}^a$	boson	fünfbein	$\mathbf{1}$	-1
ψ^i_μ	fermion	$SU(2)$-Majorana	$\mathbf{2}$	$-\frac{1}{2}$
b_μ	boson	real	$\mathbf{1}$	0
V^{ij}_μ	boson	$V^{ij}_\mu = V^{ji}_\mu = (V_{\mu ij})^*$	$\mathbf{3}$	0
v_{ab}	boson	real, antisymmetric	$\mathbf{1}$	1
χ^i	fermion	$SU(2)$-Majorana	$\mathbf{2}$	$\frac{3}{2}$
D	boson	real	$\mathbf{1}$	2
		dependent gauge fields		
$\omega_\mu{}^{ab}$	boson	spin connection	$\mathbf{1}$	0
ϕ^i_μ	fermion	$SU(2)$-Majorana	$\mathbf{2}$	$\frac{1}{2}$
$f_\mu{}^a$	boson	real	$\mathbf{1}$	1
		Vector multiplet		
W_μ	boson	real gauge field	$\mathbf{1}$	0
M	boson	real	$\mathbf{1}$	1
Ω^i	fermion	$SU(2)$-Majorana	$\mathbf{2}$	$\frac{3}{2}$
Y_{ij}	boson	$Y^{ij} = Y^{ji} = (Y_{ij})^*$	$\mathbf{3}$	2
		Hypermultiplet		
$\mathcal{A}^\alpha_i$	boson	$\mathcal{A}^i_\alpha = \epsilon^{ij}\mathcal{A}^\beta_j \rho_{\beta\alpha} = -(\mathcal{A}^\alpha_i)^*$	$\mathbf{2}$	$\frac{3}{2}$
ζ^α	fermion	$\bar{\zeta}^\alpha \equiv (\zeta_\alpha)^\dagger \gamma_0 = \zeta^{\alpha\mathrm{T}} C$	$\mathbf{1}$	2
$\mathcal{F}^\alpha_i$	boson	$\mathcal{F}^i_\alpha = -(\mathcal{F}^\alpha_i)^*$	$\mathbf{2}$	$\frac{5}{2}$
		Linear multiplet		
L^{ij}	boson	$L^{ij} = L^{ji} = (L_{ij})^*$	$\mathbf{3}$	3
φ^i	fermion	$SU(2)$-Majorana	$\mathbf{2}$	$\frac{7}{2}$
E_a	boson	real, constrained by (2.30)	$\mathbf{1}$	4
N	boson	real	$\mathbf{1}$	4

APPENDIX B

Definitions and Useful Formulae for the Weyl Multiplet

In this appendix, we summarize useful formulae for the Weyl multiplet. Firstly, the solution to the constraints (2.18) is given by the following:

$$
\begin{aligned}
&\omega_\mu{}^{ab} = \omega_\mu^{0\,ab} + i(2\bar{\psi}_\mu\gamma^{[a}\psi^{b]} + \bar{\psi}^a\gamma_\mu\psi^b) - 2e_\mu{}^{[a}b^{b]}, \\
&\quad\text{with}\quad \omega_\mu^{0\,ab} \equiv -2e^{\nu[a}\partial_{[\mu}e_{\nu]}{}^{b]} + e^{\rho[a}e^{b]\sigma}e_\mu{}^c\partial_\rho e_{\sigma c}, \\
&\phi_\mu^i = \left(-\frac{1}{3}e_\mu^a\gamma^b + \frac{1}{24}\gamma_\mu\gamma^{ab}\right)\hat{R}'^{\,i}_{ab}(Q), \\
&f_\mu{}^a = \left(\frac{1}{6}\delta_\mu^\nu\delta_b^a - \frac{1}{48}e_\mu^a e_b^\nu\right)\hat{R}'_\nu{}^b(M).
\end{aligned} \tag{B.1}
$$

Here, $\hat{R}_\mu{}^a(M) \equiv \hat{R}_{\mu\nu}{}^{ba}(M)e_b^\nu$, and the primes on the curvatures indicate that $\hat{R}'^{\,i}_{ab}(Q) = \hat{R}^i_{ab}(Q)|_{\phi_\mu=0}$ and $\hat{R}'_\mu{}^a(M) = \hat{R}_\mu{}^a(M)|_{f_\nu{}^b=0}$. The transformation laws of their dependent gauge fields can be obtained by using those of the independent fields of the Weyl multiplet, in principle. The explicit **K**-transformation laws of the gauge field $\omega_\mu{}^{ab}$,

$$
\delta_K(\xi_K^a)\omega_\mu{}^{ab} = -4\xi_K^{[a}e_\mu{}^{b]} \tag{B.2}
$$

are needed to check the **K**-invariance of the embedding formulae in (2.42).

We used two types of covariant derivatives in the main text. The first one is the 'unhatted' derivative $\mathcal{D}_\mu$, which is covariant only with respect to the homogeneous transformations $\mathbf{M}_{ab}, \mathbf{D}$ and $\mathbf{U}^{ij}$ and the $\mathbf{G}$ transformation for non-singlet fields under the Yang-Mills group G. The other is the 'hatted' derivative $\hat{\mathcal{D}}_\mu$, which denotes the fully superconformal covariant derivative. With h_μ^A denoting the gauge fields of the transformation $\mathbf{X}_A$, they are defined as

(B.3) $$\mathcal{D}_\mu \equiv \partial_\mu - \sum_{\mathbf{X}_A=\mathbf{M}_{ab},\mathbf{D},\,\mathbf{U}^{ij}(,\mathbf{G})} h_\mu^A \mathbf{X}_A, \qquad \hat{\mathcal{D}}_\mu = \mathcal{D}_\mu - \sum_{\mathbf{X}_A=\mathbf{Q}^i,\mathbf{S}^i,\mathbf{K}_a} h_\mu^A \mathbf{X}_A.$$

Below we give the explicit forms of the covariant derivatives appearing in Eq.(2.22) for convenience:

(B.4) $$\begin{aligned}
\mathcal{D}_\mu \varepsilon^i &= \left(\partial_\mu - \frac{1}{4}\omega_\mu{}^{ab}\gamma_{ab} + \tfrac{1}{2}b_\mu\right)\varepsilon^i - V_\mu{}^i{}_j\varepsilon^j, \\
\mathcal{D}_\mu \eta^i &= \left(\partial_\mu - \frac{1}{4}\omega_\mu{}^{ab}\gamma_{ab} - \frac{1}{2}b_\mu\right)\eta^i - V_\mu{}^i{}_j\eta^j, \\
\mathcal{D}_\mu \xi_K^a &= \left(\partial_\mu - b_\mu\right)\xi_K^a - \omega_\mu{}^{ab}\xi_{Kb}, \\
\hat{\mathcal{D}}_\mu v_{ab} &= \partial_\mu v_{ab} + 2\omega_{\mu[a}{}^c v_{b]c} - b_\mu v_{ab} + \frac{i}{8}\bar{\psi}_\mu\gamma_{ab}\chi + \frac{3}{2}i\bar{\psi}_\mu\hat{R}_{ab}(Q), \\
\hat{\mathcal{D}}_\mu \chi^i &= \mathcal{D}_\mu\chi^i - D\psi_\mu^i + 2\gamma^c\gamma^{ab}\psi_\mu^i\hat{\mathcal{D}}_a v_{bc} - \gamma\cdot\hat{R}(U)^i{}_j\psi_\mu^j \\
&\quad +2\gamma^a\psi_\mu^i\epsilon_{abcde}v^{bc}v^{de} - 4\gamma\cdot v\phi_\mu^i, \\
\mathcal{D}_\mu \chi^i &= \left(\partial_\mu - \frac{1}{4}\omega_\mu{}^{ab}\gamma_{ab} - \frac{3}{2}b_\mu\right)\chi^i - V_\mu{}^i{}_j\chi^j\,.
\end{aligned}$$

The superconformally covariant curvatures $\hat{R}_{\mu\nu}{}^A$ are defined as the commutator of the covariant derivatives:

(B.5) $$[\hat{\mathcal{D}}_a, \hat{\mathcal{D}}_b] = -\sum_{A=\mathbf{Q}^i,\mathbf{M}_{ab},\mathbf{D},\mathbf{U}_{ij},\mathbf{S}^i,\mathbf{K}_a} \hat{R}_{ab}{}^A\mathbf{X}_A\,.$$

They are given explicitly by the following expressions:

(B.6)

$$\begin{aligned}
\hat{R}_{\mu\nu}{}^{a}(P) &= 2\partial_{[\mu}e_{\nu]}{}^{a} - 2\omega_{[\mu}{}^{ab}e_{\nu]b} + 2b_{[\mu}e_{\nu]}{}^{a} + 2i\bar{\psi}_{\mu}\gamma^{a}\psi_{\nu},\\
\hat{R}_{\mu\nu}{}^{i}(Q) &= 2\partial_{[\mu}\psi^{i}_{\nu]} - \frac{1}{2}\omega_{[\mu}{}^{ab}\gamma_{ab}\psi^{i}_{\nu]} + b_{[\mu}\psi^{i}_{\nu]} - 2V^{i}_{[\mu j}\psi^{j}_{\nu]} + \gamma_{ab[\mu}\psi_{\nu]}v^{ab} - 2\gamma_{[\mu}\phi^{i}_{\nu]},\\
\hat{R}_{\mu\nu}{}^{ab}(M) &= 2\partial_{[\mu}\omega_{\nu]} - 2\omega_{[\mu}{}^{a}{}_{c}\omega_{\nu]}{}^{cb} - 4i\bar{\psi}_{[\mu}\gamma^{ab}\phi_{\nu]} + 2i\bar{\psi}_{[\mu}\gamma^{abcd}\psi_{\nu]}v_{cd}\\
&\quad +4i\bar{\psi}_{[\mu}\gamma^{[a}\hat{R}_{\nu]}{}^{b]}(Q) + 2i\bar{\psi}_{[\mu}\gamma_{\nu]}\hat{R}^{ab}(Q) + 8f_{[\mu}{}^{[a}e_{\nu]}{}^{b]},\\
\hat{R}_{\mu\nu}(D) &= 2\partial_{[\mu}b_{\nu]} + 4i\bar{\psi}_{[\mu}\phi_{\nu]} + 4f_{[\mu\nu]},\\
\hat{R}_{\mu\nu}{}^{ij}(U) &= 2\partial_{[\mu}V^{ij}_{\nu]} - 2V^{i}_{[\mu k}V^{kj}_{\nu]} + 12i\bar{\psi}^{(i}_{[\mu}\phi^{j)}_{\nu]} - 4i\bar{\psi}^{i}_{[\mu}\gamma\cdot v\psi^{j}_{\nu]} + \frac{i}{2}\bar{\psi}^{(i}_{[\mu}\gamma_{\nu]}\chi^{j)}\,,\\
\hat{R}_{\mu\nu}{}^{i}(S) &= 2\partial_{[\mu}\phi^{i}_{\nu]} - \frac{1}{2}\omega_{[\mu}{}^{ab}\gamma_{ab}\phi^{i}_{\nu]} - b_{[\mu}\phi^{i}_{\nu]} - 2V^{i}_{[\mu j}\phi^{j}_{\nu]} + 2f_{[\mu}{}^{a}\gamma_{a}\psi^{i}_{\nu]} + \cdots\,,\\
\hat{R}_{\mu\nu}{}^{a}(K) &= 2\partial_{[\mu}f_{\nu]}{}^{a} - 2\omega_{[\mu}{}^{ab}f_{\nu]b} - 2b_{[\mu}f_{\nu]}{}^{a} + 2i\bar{\phi}_{\mu}\gamma^{a}\phi_{\nu} + \cdots\,,
\end{aligned}$$

where the dots in the $\mathbf{S}^i$ and $\mathbf{K}^a$ curvature expressions denote terms containing other curvatures.

To compute the **Q**-variation of the covariant derivatives of some fields, the following formula is useful:

$$[\delta_Q, \hat{\mathcal{D}}_a] = -\delta_Q([\delta_Q\psi^i_a]_{\rm cov})) - \delta_S([\delta_Q\phi^i_a]_{\rm cov})) + \cdots\,. \tag{B.7}$$

Here, the fermionic terms are omitted and $[\cdots]_{cov}$ denotes the covariant part of the variations, namely,

$$\begin{aligned}
[\delta_Q\psi^i_a]_{\rm cov} &= \frac{1}{2}\gamma_{abc}v^{bc}\varepsilon^i,\\
[\delta_Q\phi^i_a]_{\rm cov} &= \frac{1}{3}\Big(\hat{R}_{ab}{}^{i}{}_{j}(U)\gamma^b - \frac{1}{8}\gamma_a\gamma\cdot\hat{R}^{i}{}_{j}(U)\Big)\varepsilon^j\\
&\quad - \frac{1}{12}\Big(3\hat{\mathcal{D}}_\mu\gamma\cdot v\varepsilon^i + \gamma_{abcd}\hat{\mathcal{D}}^b v^{cd}\varepsilon^i + \gamma_{ab}\hat{\mathcal{D}}_c v^{cb}\varepsilon^i - 2\gamma^{bc}\varepsilon^i\hat{\mathcal{D}}_b v_{ca} - 3\varepsilon^i\hat{\mathcal{D}}^b v_{ba}\\
&\quad - \gamma_{abcde}\varepsilon^i v^{bc}v^{de} + 4v_{ab}v_{cd}\gamma^{bcd}\varepsilon^i + 16v_{ab}v^{bc}\gamma_c\varepsilon^i + 5v_{bc}v^{bc}\gamma_a\varepsilon^i\Big).
\end{aligned} \tag{B.8}$$

Using this, we can verify that the variations of the supercovariant curvatures not contain any term non-covariant with respect to the superconformal transformations.

Finally, we present the explicit forms of the variations of the supercovariant curvatures $\hat{R}^i(Q)$ and $\hat{R}^{ij}(U)$:

(B.9)

$$\begin{aligned}
\delta\hat{R}^i_{ab}(Q) \;=\;& -\frac{1}{4}\hat{R}_{ab}{}^{cd}(M)\gamma_{cd}\varepsilon^i - \frac{1}{3}\hat{R}_{ab}{}^i{}_j(U)\varepsilon^j + \frac{1}{12}\gamma\cdot\hat{R}^i{}_j(U)\gamma_{ab}\varepsilon^j \\
&+ \frac{1}{6}\gamma_{abcde}\hat{\mathcal{D}}^c v^{de}\varepsilon^i + \frac{1}{3}\hat{\mathcal{D}}_{[a}v^{cd}\gamma_{b]cd}\varepsilon^i + \frac{1}{6}\gamma_{abc}\hat{\mathcal{D}}_d v^{dc}\varepsilon^i \\
&- \frac{2}{3}\hat{\mathcal{D}}_{[a}v_{b]c}\gamma^c\varepsilon^i + \frac{1}{3}\hat{\mathcal{D}}^c v_{c[a}\gamma_{b]}\varepsilon^i + \frac{1}{3}\hat{\not{\mathcal{D}}} v_{ab}\varepsilon^i \\
&- \frac{2}{3}v_{ab}\gamma\cdot v\varepsilon^i - \frac{2}{3}\gamma_{cd}v_{[a}{}^c v_{b]}{}^d\varepsilon^i + \frac{4}{3}\gamma_{[a}{}^c v_{b]}{}^d v_{cd}\varepsilon^i - \frac{1}{6}\gamma_{ab}v^2\varepsilon^i + \cdots \\
&- \frac{1}{3}\gamma_{abcd}v^{cd}\eta^i + \frac{4}{3}\gamma_{[a}{}^c v_{b]c}\eta^i + 2v_{ab}\eta^i,
\end{aligned}$$

(B.10)

$$\begin{aligned}
\delta\hat{R}_{ab}{}^{ij}(U) \;=\;& -6i\bar{\varepsilon}^{(i}\hat{R}_{ab}{}^{j)}(S) + 4i\bar{\varepsilon}^{(i}\gamma\cdot v\hat{R}_{ab}{}^{j)}(Q) + \frac{i}{2}\bar{\varepsilon}^{(i}\gamma_{[a}\hat{\mathcal{D}}_{b]}\chi^{j)}, \\
&- \frac{i}{4}\bar{\varepsilon}^{(i}\gamma_{abcd}\chi^{j)}v^{cd} - \frac{i}{2}\bar{\varepsilon}^{(i}\gamma_{c[a}\chi^{j)}v_{b]}{}^c \\
&+ 6i\bar{\eta}^{(i}\hat{R}^{j)}_{ab}(Q) - \frac{i}{2}\bar{\eta}^{(i}\gamma_{ab}\chi^{j)}.
\end{aligned}$$

The ellipsis in (B.9) represents terms trilinear in fermions in $\delta_Q\hat{R}(Q)$. No term of $\delta_S\hat{R}(Q)$ is omitted.

APPENDIX C

Some Detailed Computations Related to η/s

The quadratic action for the scalar channel perturbation ϕ is given in (5.21) in terms of six coefficients $A, \ldots, F$. Here we present their explicit forms:

(C.1)

$$
\begin{aligned}
A(u) &= \frac{4}{u}f_0 \\
&+c_2\Big[\frac{2uf_0(1+q)^3(5qu-1)}{H_0^3} - \frac{32g^2qu^2(1+q)^3}{3} + \frac{g^2u^3(1+q)^6}{H_0} - \frac{\omega^2}{g^2}\frac{H_0^2}{3}\Big], \\
B(u) &= \frac{3f_0}{u} \\
&+c_2\Big[-\frac{\omega^2}{g^2}H_0^2 + \frac{g^2(4qu+1)^2H_0^3}{3u} - \frac{g^2u(1+q)^3(56q^2u^2+7qu+11)}{6} \\
&\quad + \frac{g^2u^3(1+q)^6(26q^2u^2-17qu+17)}{6H_0^3} - 8g^2(1+q)^3qu^2 + \frac{3g^2u^3(1+q)^6}{4H_0}\Big], \\
C(u) &= \frac{2g^2(4qu-3)H_0^2}{u^2} - \frac{2g^2(1+q)^3(2qu+1)}{H_0} \\
&+c_2\Big[-\frac{\omega^2}{6uf_0}\Big((4qu+1)H_0^4 - (1+q)^3(-11qu^3+13u^2)H_0\Big) \\
&\quad - \frac{g^2(1+q)^3(4q^2u^2+45qu+3)}{3H_0} \\
&\quad + \frac{g^2u^2(1+q)^6(4q^3u^3-7q^2u^2-32qu+15)}{2H_0^4}\Big], \\
D(u) &= \frac{2g^2H_0^3 - g^2qu^3(1+q)^3}{u^3H_0^2} + \omega^2\frac{H_0^3}{4u^2f_0} + c_2\Big[\frac{\omega^4}{g^2}\frac{H_0^5}{12uf_0^2} \\
&+\frac{\omega^2g^2(1+q)^3}{48f_0^2}\Big(2(31qu-9)H_0^3 - 3u^2(1+q)^3(5q^2u^2-4qu+11)\Big) \\
&\quad - \frac{19g^2q(1+q)^3}{3H_0^2} - \frac{3g^2u(1+q)^6(6q^2u^2-17qu+1)}{2H_0^5}\Big], \\
E(u) &= c_2\frac{4uf_0^2}{3g^2H_0}, \\
F(u) &= c_2\, f_0\, \frac{2(2(4qu+1)H_0^3 - u^2(1+q)^3(7qu+4))}{3H_0^2}.
\end{aligned}
$$

Here we also present the $\mathcal{O}(c_2)$ solution for ϕ. Writing $\phi(u) = f(u)^\nu F(u)$, we may expand $F(u)$ to first order in both c_2 and ω

(C.2) $$F(u) = F_0(u,\omega) + c_2(F_{10}(u) + \omega F_{11}(u)).$$

Since $F(u)$ satisfies a second order equation (after linearizing in c_2 and using the lowest order equation of motion), it is consistent to choose the boundary conditions such that $F(u)$ is normalized at the boundary ($F(0) = 1$) and is regular at the horizon.

The function $F_0(u,\omega)$ is given by the expression in the curly brackets in (5.27), while the remaining functions are

(C.3)

$$
\begin{aligned}
F_{10}(u) &= 0, \\
F_{11}(u) &= \frac{(1+q)^{3/2}(11q^5+4q^4+179q^3-10q^2-8q-16)}{32q^2(1+q)^2(q-2)^3} \\
&\qquad \times \left[i\ln(q^3u^2-3qu-u-1)+\pi \right] \\
&\quad + \frac{i(q+1)^{3/2}(60q^6+99q^5+648q^4-69q^3-154q^2-104q-16)}{16(4q+1)^{3/2}(q+1)^2(q-2)^3} \\
&\qquad \times \left[\tanh^{-1}\frac{-(1+3q)}{(4q+1)^{1/2}(q+1)} - \tanh^{-1}\frac{2q^3u-(1+3q)}{(4q+1)^{1/2}(q+1)} \right] \\
&\quad - \frac{i\ln(1+qu)(1+q)^{3/2}}{8q^2} \\
&\quad - \frac{i(q+1)^{3/2}(-4q^5+21q^4+143q^3-21q^2-39q-6)}{8q^4(4q+1)(q-2)^2} \\
&\quad - \frac{i(q+1)^{3/2}(4q^7-27q^6+64q^5+511q^4+137q^3-128q^2-57q-6)qu^2}{8(1+qu)q^4(q^3u^2-3qu-u-1)(4q+1)(q-2)^2} \\
&\quad + \frac{i(-12q^6+102q^5+605q^4+63q^3-177q^2-63q-6)u}{8(1+qu)q^4(q^3u^2-3qu-u-1)(4q+1)(q-2)^2} \\
&\quad - \frac{i(4q^5+21q^4+143q^3-21q^2-39q-6)}{8(1+qu)q^4(q^3u^2-3qu-u-1)(4q+1)(q-2)^2}.
\end{aligned}
$$

BIBLIOGRAPHY

BIBLIOGRAPHY

[1] Ofer Aharony, Steven S. Gubser, Juan Martin Maldacena, Hirosi Ooguri, and Yaron Oz. Large N field theories, string theory and gravity. *Phys. Rept.*, 323:183–386, 2000.

[2] Ofer Aharony, Jacek Pawelczyk, Stefan Theisen, and Shimon Yankielowicz. A note on anomalies in the AdS/CFT correspondence. *Phys. Rev.*, D60:066001, 1999.

[3] Mohsen Alishahiha. On R**2 corrections for 5D black holes. *JHEP*, 0708:094, 2007.

[4] Philip C. Argyres, Adel M. Awad, Gregory A. Braun, and F. Paul Esposito. Higher derivative terms in N=2 supersymmetric effective actions. *JHEP*, 0307:060, 2003. 30 pages. Added references and simplified final form of WZ term Report-no: UCTP-106-03.

[5] Peter Brockway Arnold, Guy D Moore, and Laurence G. Yaffe. Transport coefficients in high temperature gauge theories. 2. Beyond leading log. *JHEP*, 0305:051, 2003.

[6] Vijay Balasubramanian, Per Kraus, Albion E. Lawrence, and Sandip P. Trivedi. Holographic probes of anti-de Sitter space-times. *Phys. Rev.*, D59:104021, 1999.

[7] Nabamita Banerjee, Bernard de Wit, and Stefanos Katmadas. The Off-Shell 4D/5D Connection. 2011. 35 pages.

[8] Nabamita Banerjee and Suvankar Dutta. Shear Viscosity to Entropy Density Ratio in Six Derivative Gravity. *JHEP*, 0907:024, 2009.

[9] K. Behrndt, Mirjam Cvetic, and W.A. Sabra. Nonextreme black holes of five-dimensional N=2 AdS supergravity. *Nucl.Phys.*, B553:317–332, 1999.

[10] E. Bergshoeff, E. Sezgin, and Antoine Van Proeyen. SUPERCONFORMAL TENSOR CALCULUS AND MATTER COUPLINGS IN SIX-DIMENSIONS. *Nucl.Phys.*, B264:653, 1986.

[11] Eric Bergshoeff, Sorin Cucu, Tim de Wit, Jos Gheerardyn, Stefan Vandoren, et al. N = 2 supergravity in five-dimensions revisited. *Class.Quant.Grav.*, 21:3015–3042, 2004.

[12] Eric Bergshoeff, Sorin Cucu, Martijn Derix, Tim de Wit, Rein Halbersma, et al. Weyl multiplets of N=2 conformal supergravity in five-dimensions. *JHEP*, 0106:051, 2001.

[13] Eric A. Bergshoeff, Jan Rosseel, and Ergin Sezgin. Off-shell D=5, N=2 Riemann Squared Supergravity. *Class.Quant.Grav.*, 28:225016, 2011.

[14] Matthias Blau, K.S. Narain, and Edi Gava. On subleading contributions to the AdS / CFT trace anomaly. *JHEP*, 9909:018, 1999.

[15] Raphael Bousso. The Holographic principle. *Rev.Mod.Phys.*, 74:825–874, 2002.

[16] Peter Breitenlohner and Martin F. Sohnius. SUPERFIELDS, AUXILIARY FIELDS, AND TENSOR CALCULUS FOR N=2 EXTENDED SUPERGRAVITY. *Nucl.Phys.*, B165:483, 1980.

[17] Mauro Brigante, Hong Liu, Robert C. Myers, Stephen Shenker, and Sho Yaida. Viscosity Bound Violation in Higher Derivative Gravity. *Phys.Rev.*, D77:126006, 2008.

[18] Alex Buchel, Stan Deakin, Patrick Kerner, and James T. Liu. Thermodynamics of the N=2* strongly coupled plasma. *Nucl.Phys.*, B784:72–102, 2007.

[19] Alex Buchel, Jorge Escobedo, Robert C. Myers, Miguel F. Paulos, Aninda Sinha, et al. Holographic GB gravity in arbitrary dimensions. *JHEP*, 1003:111, 2010.

[20] Alex Buchel, James T. Liu, and Andrei O. Starinets. Coupling constant dependence of the shear viscosity in N=4 supersymmetric Yang-Mills theory. *Nucl.Phys.*, B707:56–68, 2005.

[21] Alex Buchel, Robert C. Myers, and Aninda Sinha. Beyond eta/s = 1/4 pi. *JHEP*, 0903:084, 2009.

[22] Rong-Gen Cai, Zhang-Yu Nie, Nobuyoshi Ohta, and Ya-Wen Sun. Shear Viscosity from Gauss-Bonnet Gravity with a Dilaton Coupling. *Phys.Rev.*, D79:066004, 2009.

[23] Rong-Gen Cai, Zhang-Yu Nie, and Ya-Wen Sun. Shear Viscosity from Effective Couplings of Gravitons. *Phys.Rev.*, D78:126007, 2008.

[24] Rong-Gen Cai and Da-Wei Pang. On Entropy Function for Supersymmetric Black Rings. *JHEP*, 0704:027, 2007.

[25] Curtis G. Callan and Juan Martin Maldacena. D-brane approach to black hole quantum mechanics. *Nucl.Phys.*, B472:591–610, 1996. 18 pages, uses harvmac and psfig Report-no: PUPT-1591.

[26] P. Candelas, Gary T. Horowitz, Andrew Strominger, and Edward Witten. Vacuum Configurations for Superstrings. *Nucl.Phys.*, B258:46–74, 1985.

[27] Jorge Casalderrey-Solana, Hong Liu, David Mateos, Krishna Rajagopal, and Urs Achim Wiedemann. Gauge/String Duality, Hot QCD and Heavy Ion Collisions. 2011.

[28] Alejandra Castro, Joshua L. Davis, Per Kraus, and Finn Larsen. 5D attractors with higher derivatives. *JHEP*, 0704:091, 2007.

[29] Alejandra Castro, Joshua L. Davis, Per Kraus, and Finn Larsen. 5D Black Holes and Strings with Higher Derivatives. *JHEP*, 0706:007, 2007.

[30] Alejandra Castro, Joshua L. Davis, Per Kraus, and Finn Larsen. Precision Entropy of Spinning Black Holes. *JHEP*, 0709:003, 2007.

[31] Andrew Chamblin, Roberto Emparan, Clifford V. Johnson, and Robert C. Myers. Charged AdS black holes and catastrophic holography. *Phys.Rev.*, D60:064018, 1999.

[32] Tai-Chung Cheng, Pei-Ming Ho, and Mao-Chuang Yeh. Perturbative approach to higher derivative and nonlocal theories. *Nucl.Phys.*, B625:151–165, 2002.

[33] Aaron S. Chou, R. Kallosh, J. Rahmfeld, Soo-Jong Rey, M. Shmakova, et al. Critical points and phase transitions in 5-D compactifications of M theory. *Nucl.Phys.*, B508:147–180, 1997. 38 pages, LaTex, epsf, 8 figures (using 18.eps files), a version with figures of higher quality is available at http://www-leland.stanford.edu/ rahmfeld/5d.uu, typographical errors and one figure are corrected.

[34] Sera Cremonini, Kentaro Hanaki, James T. Liu, and Phillip Szepietowski. Black holes in five-dimensional gauged supergravity with higher derivatives. *JHEP*, 0912:045, 2009.

[35] Sera Cremonini, Kentaro Hanaki, James T. Liu, and Phillip Szepietowski. Higher derivative effects on eta/s at finite chemical potential. *Phys.Rev.*, D80:025002, 2009.

[36] Mirjam Cvetic, M.J. Duff, P. Hoxha, James T. Liu, Hong Lu, et al. Embedding AdS black holes in ten-dimensions and eleven-dimensions. *Nucl.Phys.*, B558:96–126, 1999.

[37] Mirjam Cvetic and Steven S. Gubser. Phases of R charged black holes, spinning branes and strongly coupled gauge theories. *JHEP*, 9904:024, 1999.

[38] Maro Cvitan, Predrag Dominis Prester, Silvio Pallua, and Ivica Smolic. Extremal black holes in D=5: SUSY vs. Gauss-Bonnet corrections. *JHEP*, 0711:043, 2007.

[39] Atish Dabholkar, Gary W. Gibbons, Jeffrey A. Harvey, and Fernando Ruiz Ruiz. SUPERSTRINGS AND SOLITONS. *Nucl.Phys.*, B340:33–55, 1990.

[40] Atish Dabholkar and Sameer Murthy. Fundamental Superstrings as Holograms. *JHEP*, 0802:034, 2008.

[41] Sumit R. Das and Samir D. Mathur. Comparing decay rates for black holes and D-branes. *Nucl.Phys.*, B478:561–576, 1996.

[42] Justin R. David, Bindusar Sahoo, and Ashoke Sen. AdS(3), black holes and higher derivative corrections. *JHEP*, 0707:058, 2007.

[43] B. de Wit and H. Nicolai. N=8 Supergravity. *Nucl.Phys.*, B208:323, 1982.

[44] B. de Wit, J.W. van Holten, and Antoine Van Proeyen. Transformation Rules of N=2 Supergravity Multiplets. *Nucl.Phys.*, B167:186, 1980.

[45] Bernard de Wit and Stefanos Katmadas. Near-Horizon Analysis of D=5 BPS Black Holes and Rings. *JHEP*, 1002:056, 2010.

[46] Bernard de Wit and Antoine Van Proeyen. Broken sigma model isometries in very special geometry. *Phys.Lett.*, B293:94–99, 1992.

[47] Stanley Deser and B. Zumino. Consistent Supergravity. *Phys.Lett.*, B62:335, 1976.

[48] Lance J. Dixon. Ultraviolet Behavior of N = 8 Supergravity. pages 1–40, 2010. 30 pages, 12 figures/ lectures presented at International School of Subnuclear Physics, 47th Course, Erice Sicily, August 29-September 7, 2009.

[49] D.A. Eliezer and R.P. Woodard. The Problem of Nonlocality in String Theory. *Nucl.Phys.*, B325:389, 1989.

[50] S. Ferrara, M. Kaku, P.K. Townsend, and P. van Nieuwenhuizen. Gauging the Graded Conformal Group with Unitary Internal Symmetries. *Nucl.Phys.*, B129:125, 1977.

[51] S. Ferrara and P. van Nieuwenhuizen. The Auxiliary Fields of Supergravity. *Phys.Lett.*, B74:333, 1978.

[52] E.S. Fradkin and Mikhail A. Vasiliev. MINIMAL SET OF AUXILIARY FIELDS IN SO(2) EXTENDED SUPERGRAVITY. *Phys.Lett.*, B85:47–51, 1979.

[53] Daniel Z. Freedman, P. van Nieuwenhuizen, and S. Ferrara. Progress Toward a Theory of Supergravity. *Phys.Rev.*, D13:3214–3218, 1976.

[54] Tomoyuki Fujita and Keisuke Ohashi. Superconformal tensor calculus in five-dimensions. *Prog.Theor.Phys.*, 106:221–247, 2001.

[55] Masafumi Fukuma, So Matsuura, and Tadakatsu Sakai. Higher derivative gravity and the AdS / CFT correspondence. *Prog.Theor.Phys.*, 105:1017–1044, 2001.

[56] Davide Gaiotto. N=2 dualities. 2009.

[57] Davide Gaiotto and Juan Maldacena. The Gravity duals of N=2 superconformal field theories. 2009.

[58] Xian-Hui Ge, Yoshinori Matsuo, Fu-Wen Shu, Sang-Jin Sin, and Takuya Tsukioka. Viscosity Bound, Causality Violation and Instability with Stringy Correction and Charge. *JHEP*, 0810:009, 2008.

[59] Xian-Hui Ge, Sang-Jin Sin, Shao-Feng Wu, and Guo-Hong Yang. Shear viscosity and instability from third order Lovelock gravity. *Phys.Rev.*, D80:104019, 2009. 15 pages, 1 figure+3 tables.

[60] S.S. Gubser, Igor R. Klebanov, and Alexander M. Polyakov. Gauge theory correlators from noncritical string theory. *Phys.Lett.*, B428:105–114, 1998.

[61] Steven S. Gubser. Einstein manifolds and conformal field theories. *Phys. Rev.*, D59:025006, 1999.

[62] M. Gunaydin, L.J. Romans, and N.P. Warner. Compact and Noncompact Gauged Supergravity Theories in Five-Dimensions. *Nucl.Phys.*, B272:598, 1986.

[63] M. Gunaydin, G. Sierra, and P.K. Townsend. The Geometry of N=2 Maxwell-Einstein Supergravity and Jordan Algebras. *Nucl.Phys.*, B242:244, 1984.

[64] M. Gunaydin, G. Sierra, and P.K. Townsend. VANISHING POTENTIALS IN GAUGED N=2 SUPERGRAVITY: AN APPLICATION OF JORDAN ALGEBRAS. *Phys.Lett.*, B144:41, 1984.

[65] M. Gunaydin, G. Sierra, and P.K. Townsend. Gauging the d = 5 Maxwell-Einstein Supergravity Theories: More on Jordan Algebras. *Nucl.Phys.*, B253:573, 1985.

[66] Kentaro Hanaki, Masahiro Ibe, Yutaka Ookouchi, and Chang Soon Park. Constraints on Direct Gauge Mediation Models with Complex Representations. *JHEP*, 1108:044, 2011.

[67] Kentaro Hanaki and Hai Lin. M2-M5 Systems in N=6 Chern-Simons Theory. *JHEP*, 0809:067, 2008.

[68] Kentaro Hanaki, Keisuke Ohashi, and Yuji Tachikawa. Supersymmetric Completion of an R**2 term in Five-dimensional Supergravity. *Prog.Theor.Phys.*, 117:533, 2007.

[69] Kentaro Hanaki and Yutaka Ookouchi. Light Gauginos and Conformal Sequestering. *Phys.Rev.*, D83:125010, 2011.

[70] Kentaro Hanaki and Cheng Peng. Symmetries of Holographic Super-Minimal Models. 2012.

[71] S.W. Hawking and R. Penrose. The Singularities of gravitational collapse and cosmology. *Proc.Roy.Soc.Lond.*, A314:529–548, 1970.

[72] M. Henningson and K. Skenderis. The holographic Weyl anomaly. *JHEP*, 07:023, 1998.

[73] Gary T. Horowitz and Andrew Strominger. Black strings and P-branes. *Nucl. Phys.*, B360:197–209, 1991.

[74] C.M. Hull and P.K. Townsend. Unity of superstring dualities. *Nucl.Phys.*, B438:109–137, 1995.

[75] Vivek Iyer and Robert M. Wald. Some properties of Noether charge and a proposal for dynamical black hole entropy. *Phys.Rev.*, D50:846–864, 1994.

[76] Ted Jacobson, Gungwon Kang, and Robert C. Myers. On black hole entropy. *Phys.Rev.*, D49:6587–6598, 1994.

[77] Romuald A. Janik. Viscous plasma evolution from gravity using AdS/CFT. *Phys.Rev.Lett.*, 98:022302, 2007.

[78] Clifford V. Johnson. Heterotic Coset Models of Microscopic Strings and Black Holes. 2007.

[79] M. Kaku and P.K. Townsend. POINCARE SUPERGRAVITY AS BROKEN SUPERCONFORMAL GRAVITY. *Phys.Lett.*, B76:54, 1978.

[80] M. Kaku, P.K. Townsend, and P. van Nieuwenhuizen. Properties of Conformal Supergravity. *Phys.Rev.*, D17:3179, 1978.

[81] Yevgeny Kats and Pavel Petrov. Effect of curvature squared corrections in AdS on the viscosity of the dual gauge theory. *JHEP*, 0901:044, 2009.

[82] Igor R. Klebanov and Edward Witten. Superconformal field theory on threebranes at a Calabi-Yau singularity. *Nucl. Phys.*, B536:199–218, 1998.

[83] Pavel Kovtun, Dam T. Son, and Andrei O. Starinets. Holography and hydrodynamics: Diffusion on stretched horizons. *JHEP*, 0310:064, 2003.

[84] Per Kraus and Finn Larsen. Microscopic black hole entropy in theories with higher derivatives. *JHEP*, 0509:034, 2005.

[85] Per Kraus and Finn Larsen. Holographic gravitational anomalies. *JHEP*, 0601:022, 2006.

[86] Per Kraus, Finn Larsen, and Akhil Shah. Fundamental Strings, Holography, and Nonlinear Superconformal Algebras. *JHEP*, 0711:028, 2007.

[87] Taichiro Kugo and Keisuke Ohashi. Supergravity tensor calculus in 5-D from 6-D. *Prog.Theor.Phys.*, 104:835–865, 2000.

[88] Stanislav Kuperstein and Sameer Murthy. Stringy effects in black hole decay. *JHEP*, 1011:064, 2010.

[89] Joshua M. Lapan, Aaron Simons, and Andrew Strominger. Nearing the Horizon of a Heterotic String. 2007.

[90] James T. Liu and Phillip Szepietowski. Higher derivative corrections to R-charged AdS(5) black holes and field redefinitions. *Phys.Rev.*, D79:084042, 2009.

[91] Kengo Maeda, Makoto Natsuume, and Takashi Okamura. Viscosity of gauge theory plasma with a chemical potential from AdS/CFT. *Phys.Rev.*, D73:066013, 2006.

[92] Juan Martin Maldacena. The Large N limit of superconformal field theories and supergravity. *Adv.Theor.Math.Phys.*, 2:231–252, 1998.

[93] J. Mas. Shear viscosity from R-charged AdS black holes. *JHEP*, 0603:016, 2006.

[94] David Mateos, Robert C. Myers, and Rowan M. Thomson. Holographic viscosity of fundamental matter. *Phys.Rev.Lett.*, 98:101601, 2007.

[95] Harvey B. Meyer. A Calculation of the shear viscosity in SU(3) gluodynamics. *Phys.Rev.*, D76:101701, 2007.

[96] Thomas Mohaupt. Black hole entropy, special geometry and strings. *Fortsch.Phys.*, 49:3–161, 2001.

[97] Wolfgang Mueck and Alessio Celi. Private communications.

[98] Robert C. Myers, Miguel F. Paulos, and Aninda Sinha. Holographic Hydrodynamics with a Chemical Potential. *JHEP*, 0906:006, 2009.

[99] Robert C. Myers and Oyvind Tafjord. Superstars and giant gravitons. *JHEP*, 0111:009, 2001.

[100] Shin'ichi Nojiri and Sergei D. Odintsov. On the conformal anomaly from higher derivative gravity in AdS / CFT correspondence. *Int.J.Mod.Phys.*, A15:413–428, 2000.

[101] Shesansu Sekhar Pal. eta/s at finite coupling. *Phys.Rev.*, D81:045005, 2010.

[102] Joseph Polchinski. Dirichlet Branes and Ramond-Ramond charges. *Phys.Rev.Lett.*, 75:4724–4727, 1995.

[103] G. Policastro, D.T. Son, and A.O. Starinets. The Shear viscosity of strongly coupled N=4 supersymmetric Yang-Mills plasma. *Phys.Rev.Lett.*, 87:081601, 2001.

[104] Giuseppe Policastro, Dam T. Son, and Andrei O. Starinets. From AdS / CFT correspondence to hydrodynamics. *JHEP*, 0209:043, 2002.

[105] Joel Scherk and John H. Schwarz. Dual Models for Nonhadrons. *Nucl.Phys.*, B81:118–144, 1974.

[106] Martin F. Sohnius and Peter C. West. An Alternative Minimal Off-Shell Version of N=1 Supergravity. *Phys.Lett.*, B105:353, 1981.

[107] Dam T. Son and Andrei O. Starinets. Minkowski space correlators in AdS / CFT correspondence: Recipe and applications. *JHEP*, 0209:042, 2002.

[108] Dam T. Son and Andrei O. Starinets. Hydrodynamics of r-charged black holes. *JHEP*, 0603:052, 2006.

[109] Huichao Song, Steffen A. Bass, Ulrich Heinz, Tetsufumi Hirano, and Chun Shen. 200 A GeV Au+Au collisions serve a nearly perfect quark-gluon liquid. *Phys.Rev.Lett.*, 106:192301, 2011. 4 pages, 2 figures, accepted by PRL.

[110] K.S. Stelle and Peter C. West. Minimal Auxiliary Fields for Supergravity. *Phys.Lett.*, B74:330, 1978.

[111] Andrew Strominger and Cumrun Vafa. Microscopic origin of the Bekenstein-Hawking entropy. *Phys.Lett.*, B379:99–104, 1996.

[112] Leonard Susskind. The World as a hologram. *J.Math.Phys.*, 36:6377–6396, 1995.

[113] Gerard 't Hooft. Dimensional reduction in quantum gravity. 1993. Essay dedicated to Abdus Salam.

[114] Yuji Tachikawa. Black hole entropy in the presence of Chern-Simons terms. *Class.Quant.Grav.*, 24:737–744, 2007.

[115] Antoine Van Proeyen. N = 2 supergravity in d = 4, 5, 6 and its matter couplings. http://itf.fys.kuleuven.be/ toine/LectParis.pdf.

[116] Robert M. Wald. Black hole entropy is the Noether charge. *Phys.Rev.*, D48:3427–3431, 1993.

[117] Edward Witten. String theory dynamics in various dimensions. *Nucl. Phys.*, B443:85–126, 1995.

[118] Edward Witten. Anti-de Sitter space and holography. *Adv.Theor.Math.Phys.*, 2:253–291, 1998.

[119] T. Yoneya. Quantum gravity and the zero slope limit of the generalized Virasoro model. *Lett.Nuovo Cim.*, 8:951–955, 1973.

[120] Max Zucker. Gauged N=2 off-shell supergravity in five-dimensions. *JHEP*, 0008:016, 2000.

[121] Max Zucker. Minimal off-shell supergravity in five-dimensions. *Nucl.Phys.*, B570:267–283, 2000.