The Pentagram Map: Combinatorial and Geometric Perspectives

by

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TABLE OF CONTENTS

LIST OF FIGURES

LIST OF TABLES

Table

CHAPTER I

Introduction

1.1 Overview

The pentagram map, introduced by R. Schwartz [Sch92], is a geometric construction which produces one polygon from another. Figure 1.1 illustrates three successive applications of this operation. Throughout, the pentagram map will be denoted T.

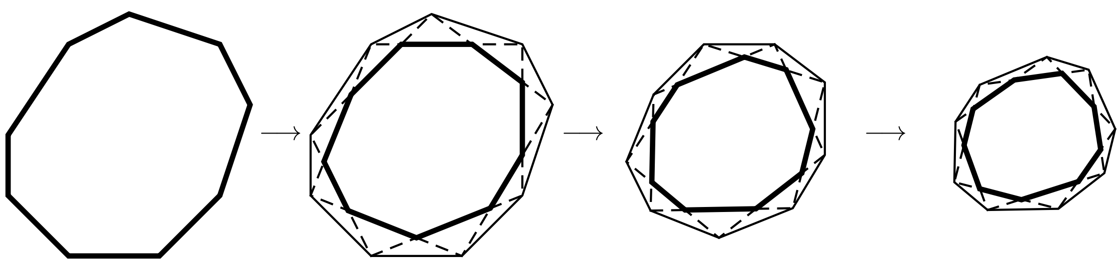

Figure 1.1: Three iterations of the pentagram map in the space of 9-gons

The pentagram map was revisited several times by Schwartz [Sch01a, Sch01b, Sch08], and has recently attracted a great deal of interest from a number of researchers, see e.g. [GSTV12, Gli11, Gli, MGOT, OST10, OST, Sol, ST10, ST11]. At present, the pentagram map is being approached within the theories of discrete integrable systems on the one hand, and cluster algebras on the other. These two areas, and the connections between them, are quite active areas of research.

Much of the current framework for the study of the map comes from [Sch08] in which Schwartz worked with certain generalized polygons called twisted polygons.

He introduced a set of coordinates $x_1, \ldots, x_{2n}$ on the configuration space of twisted n-gons and demonstrated that the pentagram map can be expressed in these coordinates by a simple rational function. He also found a collection of invariants of the map, which served as part of the proof by V. Ovsienko, Schwartz, and S. Tabachnikov [OST10] that the pentagram map is a discrete integrable system.

Let T^k denote the kth iterate of the pentagram map. Our first main result (Theorem I.2) is a nonrecursive description of T^k as a rational map. We work with a collection of quantities $y_1, \ldots, y_{2n}$ called the y-parameters, which are closely related to Schwartz's x-coordinates. In the y-parameters, the components of the map T^k factor into polynomials which we describe as generating functions over certain combinatorial objects (specifically, order ideals of the posets P_k of N. Elkies, G. Kuperberg, M. Larsen, and J. Propp [EKLP92]). The y-parameters themselves are not coordinates on the relevant space, as they satisfy the relation $y_1 \cdots y_{2n} = 1$. However, our formula in the y-parameters, together with other results about the map, leads to a similar formula (Theorem II.8) in the x-coordinates.

In principle, Theorem I.2 can be proven by induction, but it was discovered by relating the pentagram map to cluster algebras. Cluster algebras, introduced by S. Fomin and A. Zelevinsky [FZ02], are certain commutative algebras defined within rational function fields through a process known as seed mutation. Affiliated with each cluster algebra is a Y-pattern, defined by Fomin and Zelevinsky [FZ07] in terms of a separate, but related, mutation process. We show that the y-parameters transform under T according to the mutation dynamics of a certain Y-pattern. Our formulas then follow from results of [FZ07]. From this point of view, the polynomials which show up are certain F-polynomials $F_{j,k}$ of the corresponding cluster algebra. Most of the work goes into proving the generating function description of the $F_{j,k}$.

We next apply our formula to help explain some observations due to Schwartz regarding the long term behavior of the pentagram map. The first observation pertains to axis aligned polygons, or stated more naturally, polygons whose sides all pass through one of two points in an alternating fashion. Remarkably, after a predictable number of iterations depending on the number of sides, the original polygon transforms into one with the dual property, namely having its vertices alternate between two lines. We give proofs of this fact for both closed (Theorem I.3) and twisted (Theorem I.4) polygons, extending Theorem 3 of [Sch08].

The second application of our main formula is to the problem of understanding singularity confinement for the pentagram map. Singularity confinement, first studied by B. Grammaticos, A. Ramani, and V. Papageorgiou [GRP91], refers to a situation in which a rational map is singular for certain input, while some higher iterate of the map is well-defined. Singularity confinement is thought to be present in many discrete integrable systems. In the case of the pentagram map, Schwartz provided experimental evidence for singularity confinement several years before integrability was proven.

For us, a singular point of a rational map is an input at which one of the componenents of the map has a vanishing denominator. A singularity of an iterate of the pentagram map, then, can be described as a point at which one of the corresponding F-polynomials vanishes. Proving confinement amounts to proving that none of these polynomials vanish after a certain number of steps. We establish confinement (Theorems I.5 and I.6) for a large class of singular polygons. Moreover, we give an indication of how many steps such singularities persist depending on which collinearities exist among the vertices of the initial polygon.

Lastly, we provide an examination of singularity confinement from a geometric point of view. Consider some class X of polygons for which $T, T^2, \dots, T^{k-1}$ are singular, but T^k is generically defined. It seems natural to ask if there exists a geometric construction corresponding to the rational map $T^k|_X$. The usual construction will fail because of the singularities, but it is possible that some alternate approach circumvents the problem. We give such a construction (Algorithm 4) for the most simple singularity types and make some progress towards developing a procedure that could work in general.

This thesis is organized as follows. Section 1.2 introduces the main objects of study and includes a precise statement of our main results. Chapter II follows [Gli11] developing the connection of the pentagram map to Y-patterns and proving our main formulas. First, Section 2.1 gives the formula for T in the y-parameters. In Section 2.2 we review the definition of a Y-pattern and include the pentagram map within this framework. Section 2.3 then exploits the connection to Y-patterns to express the map T^k in terms of certain polynomials $F_{j,k}$. In Section 2.4 we provide background on alternating sign matrices which is used in Section 2.5 to give a combinatorial formula for $F_{j,k}$. Lastly, we apply our formulas to prove results pertaining to axis-aligned polygons in Section 2.6.

Chapter III gives a detailed look at the singularities of the pentagram map, following the paper [Gli]. Section 3.1 contains some technical results concerning when the F-polynomials do and do not vanish. Section 3.2 identifies a hierarchy of singularity types of the pentagram map and establishes that generic polygons of these types exhibit singularity confinement. The remainder of the chapter addresses the problem of moving past singularities by constructing $T^m(A)$ from A when A is a singular point of $T, T^2, \dots, T^{m-1}$. An approach which works for the simplest singularity type is given in Section 3.3 along with a discussion of its limitations in handling more severe singularities. Section 3.4 introduces decorated polygons which will serve as the underlying objects of the main construction. In Section 3.5 we develop the procedure which is iterated in our main construction. Section 3.6 states the main construction itself and discusses what is needed to prove its correctness for a given singularity type. All algorithms are stated explicitly, but some contain steps which are nontrivial to accomplish via a straightedge construction. In Section 3.7 we fill in the details for these steps.

Throughout , we adopt the convention that $\prod_{i=M}^{M-1} z_i = 1$ and $\prod_{i=M}^{N} z_i = \prod_{i=N+1}^{M-1}(1/z_i)$ for $N < M - 1$. With this convention, the property

$$\prod_{i=M}^{N} z_i \prod_{i=N+1}^{L} z_i = \prod_{i=M}^{L} z_i$$

holds for all $M, N, L \in \mathbb{Z}$. By way of notation, if $a, b \in \mathbb{R}$, $k \in \mathbb{Z}$, $k \geq 1$ and $(a-b)/k \in \mathbb{Z}_{\geq 0}$ then let $[a, b]_k$ denote the arithmetic progression

$$[a, b]_k = \{a, a+k, a+2k, \dots, b\} \tag{1.1}$$

1.2 Main definitions and results

Throughout this thesis, all geometric objects will live in the real projective plane. Points in the plane will be denoted by capital letters and lines by lower case letters. If A and B are distinct points then let $\overleftrightarrow{AB}$ denote the line passing through both of them. If a and b are distinct lines, then let $a \cap b$ denote their point of intersection.

The pentagram map is typically defined for objects called twisted polygons defined by Schwartz [Sch08]. A *twisted polygon* is a sequence $A = (A_i)_{i \in \mathbb{Z}}$ of points in the projective plane that is periodic modulo some projective transformation ϕ, i.e., $A_{i+n} = \phi(A_i)$ for all $i \in \mathbb{Z}$. We will place the additional restriction that every quadruple of consecutive points of A be in general position. Figure 1.2 depicts a

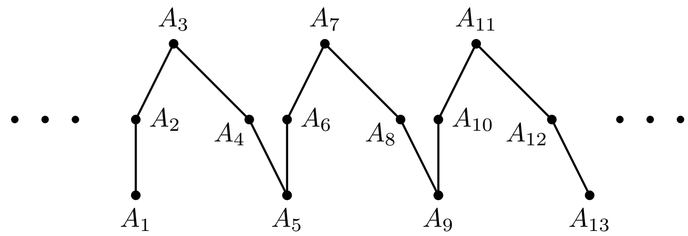

Figure 1.2: A twisted quadrilateral

twisted quadrilateral for which the map ϕ is a translation to the right. If ϕ is the identity, then the usual notion of a *closed* polygon is recovered.

Two twisted polygons A and B are said to be *projectively equivalent* if there exists a projective transformation ψ such that $\psi(A_i) = B_i$ for all i. Let $\mathcal{P}_n$ denote the space of twisted n-gons modulo projective equivalence. It is convenient to also allow twisted polygons to be indexed by $\frac{1}{2}+\mathbb{Z}$ instead of $\mathbb{Z}$. Let $\mathcal{P}_n^*$ denote the space of twisted n-gons indexed by $\frac{1}{2}+\mathbb{Z}$, modulo projective equivalence. If A is a twisted polygon with vertices A_i, then let a_j denote its side $a_j = \overleftrightarrow{A_{j-\frac{1}{2}}A_{j+\frac{1}{2}}}$. If the vertices of A are indexed by $\mathbb{Z}$ then the sides are indexed by $\frac{1}{2}+\mathbb{Z}$, and vice versa.

The *pentagram map*, denoted T, inputs a twisted polygon A and constructs a new twisted polygon B defined by $B_i = \overleftrightarrow{A_{i-\frac{3}{2}}A_{i+\frac{1}{2}}} \cap \overleftrightarrow{A_{i-\frac{1}{2}}A_{i+\frac{3}{2}}}$. Note that if A is indexed by $\mathbb{Z}$ then B is indexed by $\frac{1}{2}+\mathbb{Z}$ and vice versa. The pentagram map preserves projective equivalence, so it induces maps

$$\alpha_1 : \mathcal{P}_n^* \to \mathcal{P}_n$$

$$\alpha_2 : \mathcal{P}_n \to \mathcal{P}_n^*$$

We work with two collections of parameters defined for twisted polygons, both given by certain cross ratios. The *cross ratio* of four real numbers a, b, c, d is defined to be

$$\chi(a, b, c, d) = \frac{(a-b)(c-d)}{(a-c)(b-d)}.$$

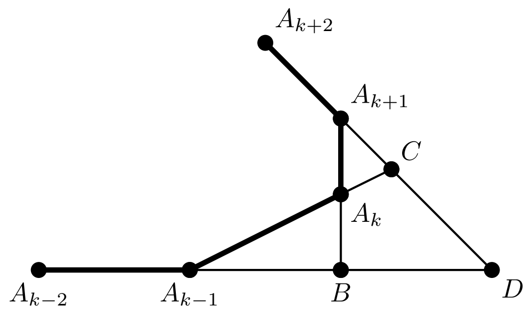

Figure 1.3: The x-coordinates of A. Here, $x_{2k}(A) = \chi(A_{k-2}, A_{k-1}, B, D)$ and $x_{2k+1}(A) = \chi(A_{k+2}, A_{k+1}, C, D)$.

This definition extends to the projective line, on which it gives a projective invariant of four points. We will be interested in taking the cross ratio of four collinear points in the projective plane, or dually, the cross ratio of four lines intersecting at a common point.

Let A be a twisted polygon with vertices indexed by either by $\mathbb{Z}$ or $\frac{1}{2} + \mathbb{Z}$. The *x-coordinates* $(x_j(A))_{j\in\mathbb{Z}}$ of A are defined by

$$x_{2k}(A) = \chi(A_{k-2}, A_{k-1}, \overleftrightarrow{A_kA_{k+1}} \cap \overleftrightarrow{A_{k-2}A_{k-1}}, \overleftrightarrow{A_{k+1}A_{k+2}} \cap \overleftrightarrow{A_{k-2}A_{k-1}})$$
$$x_{2k+1}(A) = \chi(A_{k+2}, A_{k+1}, \overleftrightarrow{A_kA_{k-1}} \cap \overleftrightarrow{A_{k+2}A_{k+1}}, \overleftrightarrow{A_{k-1}A_{k-2}} \cap \overleftrightarrow{A_{k+2}A_{k+1}}).$$

for each index k of A. This definition makes sense as all 4 points in the first cross ratio lie on the line $\overleftrightarrow{A_{k-2}A_{k-1}}$ and those in the second all lie on the line $\overleftrightarrow{A_{k+2}A_{k+1}}$ (see Figure 1.3).

The *y-parameters* of A are the real numbers $y_j(A)$ for $j \in \mathbb{Z}$ defined as follows. For each index k of A let

$$y_{2k}(A) = -\left(\chi(\overleftrightarrow{A_kA_{k-2}}, \overleftrightarrow{A_kA_{k-1}}, \overleftrightarrow{A_kA_{k+1}}, \overleftrightarrow{A_kA_{k+2}})\right)^{-1} \tag{1.2}$$

$$y_{2k+1}(A) = -\chi(\overleftrightarrow{A_{k-2}A_{k-1}} \cap l, A_k, A_{k+1}, \overleftrightarrow{A_{k+2}A_{k+3}} \cap l) \tag{1.3}$$

where $l = \overleftrightarrow{A_kA_{k+1}}$. Note that the 4 lines in (1.2) all pass through the point A_k, and the 4 points in (1.3) all lie on the line l. Therefore the cross ratios are defined. These

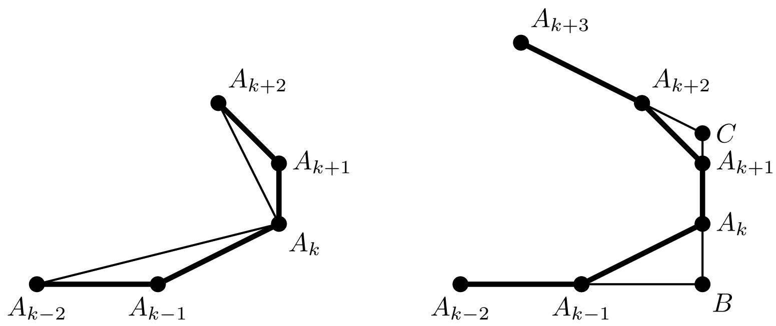

Figure 1.4: The cross ratios corresponding to the y-parameters. On the left, $-(y_{2k}(A))^{-1}$ is the cross ratio of the 4 lines through A_k. On the right, $y_{2k+1}(A) = -\chi(B, A_k, A_{k+1}, C)$.

cross ratios are illustrated in Figure 1.4.

The x_j and y_j are both projective invariants, so they are well-defined on $\mathcal{P}_n$ and $\mathcal{P}_n^*$. Both sequences are also $2n$ periodic on these spaces. Schwartz [Sch08] showed that $x_1, \ldots, x_{2n}$ give a set of coordinates on $\mathcal{P}_n$ and on $\mathcal{P}_n^*$ and expressed the pentagram map in these coordinates.

Proposition I.1 ([Sch08, (7)]). *Suppose that $(x_1, \ldots, x_{2n})$ are the x-coordinates of A. If $A \in \mathcal{P}_n^*$ then*

$$x_j(\alpha_1(A)) = \begin{cases} x_{j-1}\frac{1-x_{j-3}x_{j-2}}{1-x_{j+1}x_{j+2}}, & j \text{ even} \\ x_{j+1}\frac{1-x_{j+3}x_{j+2}}{1-x_{j-1}x_{j-2}}, & j \text{ odd} \end{cases} \tag{1.4}$$

Alternately, if $A \in \mathcal{P}_n$ then

$$x_j(\alpha_2(A)) = \begin{cases} x_{j+1}\frac{1-x_{j+3}x_{j+2}}{1-x_{j-1}x_{j-2}}, & j \text{ even} \\ x_{j-1}\frac{1-x_{j-3}x_{j-2}}{1-x_{j+1}x_{j+2}}, & j \text{ odd} \end{cases} \tag{1.5}$$

In contrast, we will show that $y_1 \cdots y_{2n} = 1$ proving that $y_1, \ldots, y_{2n}$ are not coordinates. However, our results are best stated in terms of the y-parameters, so we will generally prefer them. Typically, it is possible to reconstruct a polygon from its y-parameters together with a single extra quantity which is preserved by the map. As such, the interesting dynamics are captured in the y_j.

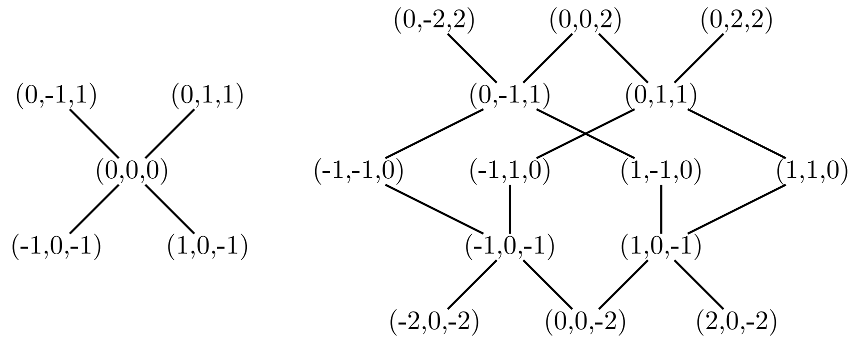

Figure 1.5: The poset P_k for $k = 2$ (left) and $k = 3$ (right)

We will be interested in T^k, the kth iterate of the pentagram map. Defined on $\mathcal{P}_n$ it takes the form $T^k = \underbrace{\cdots \circ \alpha_2 \circ \alpha_1 \circ \alpha_2}_{k}$ and has image in either $\mathcal{P}_n$ or $\mathcal{P}_n^*$ depending on the parity of k. By (1.4) and (1.5), T^k is a rational map.

We first provide a nonrecursive formula for T^k as it acts on the y-parameters. We will see that the transformation of the y_j is also rational. Our formulas for these maps involve the F-polynomials of a particular cluster algebra. These can in turn be expressed in terms of certain posets which we define now.

The original definition of the posets, given by Elkies, Kuperberg, Larsen, and Propp [EKLP92], involves height functions of domino tilings. Although we will use this characterization later, the following self-contained definition suffices for now. Let Q_k be the set of triples $(r, s, t) \in \mathbb{Z}^3$ such that $|r| + |s| \leq k - 2$, $r + s \equiv k \pmod 2$, and

$$t \in [2|s| - k + 2, k - 2 - 2|r|]_4$$

(recall the notation (1.1)). Let $P_k = Q_{k+1} \cup Q_k$. Define a partial order on P_k by saying that (r', s', t') covers (r, s, t) if and only if $t' = t + 1$ and $|r' - r| + |s' - s| = 1$. The Hasse diagrams of P_2 and P_3 are given in Figure 1.5.

We denote by $J(P_k)$ the set of *order ideals* in P_k, i.e., subsets $I \subseteq P_k$ such that $x \in I$ and $y < x$ implies $y \in I$.

Theorem I.2. *Let $A \in \mathcal{P}_n$ and let $y_j = y_j(A)$ for all $j \in \mathbb{Z}$. If $k \geq 1$ then the y-parameters of $T^k(A)$ are given by*

$$y_j(T^k(A)) = \begin{cases} \left(\prod_{i=-k}^{k} y_{j+3i}\right) \dfrac{F_{j-1,k}F_{j+1,k}}{F_{j-3,k}F_{j+3,k}}, & j+k \text{ even} \\ \left(\prod_{i=-k+1}^{k-1} y_{j+3i}\right)^{-1} \dfrac{F_{j-3,k-1}F_{j+3,k-1}}{F_{j-1,k-1}F_{j+1,k-1}}, & j+k \text{ odd} \end{cases} \tag{1.6}$$

where

$$F_{j,k} = \sum_{I \in J(P_k)} \prod_{(r,s,t) \in I} y_{3r+s+j}. \tag{1.7}$$

We next turn to applying our formula to explain some long term behavior of the pentagram map. Recall that our definition of a twisted polygon requires that any four consecutive vertices be in general position. A situation that is allowed is for the points A_{i-2}, A_i, and A_{i+2} to be collinear for some i since these vertices are far enough apart. We introduce some notation pertaining to such polygons, as well as polygons with the dual property.

For $i \in \mathbb{Z}$ let

$$X_i = \{A \in \mathcal{P}_n : A_{i-2}, A_i, A_{i+2} \text{ collinear}\}$$

For $j \in (\frac{1}{2} + \mathbb{Z})$ let

$$Y_j = \{A \in \mathcal{P}_n : a_{j-2}, a_j, a_{j+2} \text{ concurrent}\}$$

Define similarly $X_j \subseteq \mathcal{P}_n^*$ for $j \in (\frac{1}{2} + \mathbb{Z})$ and $Y_i \subseteq \mathcal{P}_n^*$ for $i \in \mathbb{Z}$. For $S \subseteq \mathbb{Z}$ or $S \subseteq (\frac{1}{2} + \mathbb{Z})$ let

$$X_S = \bigcap_{i \in S} X_i$$

$$Y_S = \bigcap_{i \in S} Y_i$$

For instance, $X_{\{3,5\}} = X_3 \cap X_5$ is the set of twisted polygons A for which A_1, A_3, A_5, A_7 are all collinear.

The X_S and Y_S play an important role in the singularity theory of the pentagram map. In particular, we will see that the X_i and Y_j are the components of the singular loci of T and T^{-1} respectively.

Taking $S = \{1/2, 3/2, \ldots, n - 1/2\}$, we get the most extreme instance of Y_S in which each side of the polygon contains one of two points P and Q. Applying a projective transformation, we can move P and Q to the line at infinity, one in the x direction and the other in the y direction. The sides of the polygon will then all be horizontal or vertical. As such we call a polygon $A \in Y_S$ *axis-aligned.*

In addition to being the most severe type of singularity of T^{-1}, axis-aligned polygons have interesting behavior under T itself. Schwartz showed [Sch08, Theorem 3] in certain cases that some T^k will transform an axis-aligned polygon to one with the dual property (namely, having its vertices confined to two lines). We extend this result to cover more cases.

Theorem I.3. *Let $n \geq 3$, $S = \{1/2, 3/2, \ldots, 2n - 1/2\}$, and $S' = \{1, 2, \ldots, 2n\}$. Suppose $A \in \mathcal{P}_{2n}$ is closed. If $A \in Y_S$ (i.e. A is axis-aligned) then $B = T^{n-2}(A)$ is in either $X_{S'}$ or X_S depending on whether n is even or odd respectively (i.e. the vertices of B lie alternately on two lines).*

Theorem I.4. *Let S and S' be as above and let $A \in Y_S \subseteq \mathcal{P}_{2n}$. Let P and Q denote the points of concurrency of the sides of A. Assume that A is not closed but twisted with $\phi(A_i) = A_{i+2n}$. Additionally assume that ϕ fixes every point on the line containing P and Q. Then $T^{n-1}(A)$ is in either X_S or $X_{S'}$ depending on whether n is even or odd respectively.*

As already mentioned, the map T is singular on X_i for each i. However, for generic singular points A of T, there exists a higher iterate T^k which is nonsingular at A. How many steps are needed to move past the singularity seems to depend in large part on which X_S the polygon A belongs to. As such, we call $A \in X_S$ a singular point of *type* S. The situation is simplest for S a finite arithmetic sequence whose terms differ by 2. Generically, the number of steps needed to bypass such a singularity equals $m+2$ where m is the length of the sequence.

Theorem I.5. *Let* $i, m \in \mathbb{Z}$ *with* $1 \leq m < n/3 - 1$. *Let*

$$S = [i-(m-1), i+(m-1)]_2$$
$$S' = \left[i - \frac{1}{2}(m-1), i + \frac{1}{2}(m-1)\right]_1$$

Then the map T^k *is singular on* $X_S \subseteq \mathcal{P}_n$ *for* $1 \leq k \leq m+1$, *but* T^{m+2} *is nonsingular at generic* $A \in X_S$. *Moreover,* $T^{m+2}(A) \in Y_{S'}$ *for such* A.

We will see that the roles of S and S' can be interchanged in this theorem. As such, we have a good understanding of singular behavior in X_S for S of the form $[a, b]_1$ or $[a, b]_2$. For general S, we do not know how many steps a generic singularity persists. We do however prove an upper bound for this number for all but the most severe singularity type of odd sided polygons.

Theorem I.6. *Suppose that* n *is odd and that* $S \subsetneq [1, n]_1$. *Then there exists a* $k \leq n+1$ *such that* T^k *is nonsingular for generic* $A \in X_S$.

We conjecture that the same bound holds for n even outside of some exceptional choices of S.

CHAPTER II

The pentagram map and Y-patterns

2.1 The transition equations

Let A be a twisted n-gon. In this section, we show that each y-parameter of $T(A)$ is a rational function of $y_1(A), \ldots, y_{2n}(A)$. First, we explain how to express the y-parameters of A in terms of its x-coordinates, and then we use (1.4) and (1.5) to derive our formula.

As observed by V. Ovsienko, R. Schwartz, and S. Tabachnikov in [OST10], the products $x_j x_{j+1}$ of consecutive x-coordinates are themselves cross ratios. In fact, $x_j x_{j+1}$ equals the cross ratios used in (1.2) or (1.3) (depending on the parity of j) to define y_j. Therefore

$$y_j = -(x_j x_{j+1})^{-1} \tag{2.1}$$

if $j/2$ is an index of A and

$$y_j = -(x_j x_{j+1}) \tag{2.2}$$

otherwise.

It follows that $y_1 y_2 y_3 \cdots y_{2n} = 1$ for any twisted polygon. Therefore the y-parameters do not "coordinatize" $\mathcal{P}_n$ or $\mathcal{P}_n^*$. However, the y_j together with Schwartz's pentagram invariants O_k and E_k [Sch08] can be used to determine the x-coordinates and hence the polygon (up to projective equivalence.) More precisely, the y-parameters determine the products $x_j x_{j+1}$ for $j = 1, \ldots, 2n$. These, together with $E_n = x_2 x_4 \cdots x_{2n}$, can be used to compute x_j^n for each j. If n is odd then this is all that is needed to find the x-coordinates. On the other hand, if n is even then $y_1, \ldots, y_{2n}$, and E_n only determine the x-coordinates up to a simultaneous change of sign. In this event, another pentagram invariant such as E_1 can be used to resolve the ambiguity.

The pentagram invariants are interchanged by the pentagram map: $O_k(T(A)) = E_k(A)$ and $E_k(T(A)) = O_k(A)$ for all twisted polygons A (Theorem 1.1 of [Sch08]). What remains, then, is to understand how the pentagram map and its iterates affect the y-parameters.

Proposition II.1. *Let $(y_1, \ldots, y_{2n})$ be the y-parameters of A. If $A \in \mathcal{P}_n^*$ then*

$$y_j' \stackrel{def}{=} y_j(\alpha_1(A)) = \begin{cases} y_{j-3} y_j y_{j+3} \frac{(1+y_{j-1})(1+y_{j+1})}{(1+y_{j-3})(1+y_{j+3})}, & j \text{ even} \\ y_j^{-1}, & j \text{ odd} \end{cases} \tag{2.3}$$

Alternately, if $A \in \mathcal{P}_n$ then

$$y_j'' \stackrel{def}{=} y_j(\alpha_2(A)) = \begin{cases} y_j^{-1}, & j \text{ even} \\ y_{j-3} y_j y_{j+3} \frac{(1+y_{j-1})(1+y_{j+1})}{(1+y_{j-3})(1+y_{j+3})}, & j \text{ odd} \end{cases} \tag{2.4}$$

Proof. We will prove the formula when $A \in \mathcal{P}_n^*$, i.e., when its vertices are indexed by $\frac{1}{2} + \mathbb{Z}$. If j is odd then $j/2$ is an index of A but not of $T(A)$, so $y_j = -(x_j x_{j+1})^{-1}$ and $y_j' = -x_j' x_{j+1}'$. Computing

$$\begin{aligned} y_j' &= -x_j' x_{j+1}' \\ &= -\left(x_{j+1} \frac{1 - x_{j+3} x_{j+2}}{1 - x_{j-1} x_{j-2}}\right) \left(x_j \frac{1 - x_{j-2} x_{j-1}}{1 - x_{j+2} x_{j+3}}\right) \qquad \text{(by (1.4))} \\ &= -x_j x_{j+1} \\ &= y_j^{-1}. \end{aligned}$$

On the other hand, if j is even then $y_j = -(x_j x_{j+1})$ and

$$\begin{aligned}
y'_j &= -(x'_j x'_{j+1})^{-1} \\
&= -\left(\left(x_{j-1}\frac{1-x_{j-3}x_{j-2}}{1-x_{j+1}x_{j+2}}\right)\left(x_{j+2}\frac{1-x_{j+4}x_{j+3}}{1-x_j x_{j-1}}\right)\right)^{-1} \\
&= -\frac{x_j x_{j+1}}{x_{j-1}x_j x_{j+1} x_{j+2}}\frac{(1-x_{j-1}x_j)(1-x_{j+1}x_{j+2})}{(1-x_{j-3}x_{j-2})(1-x_{j+3}x_{j+4})} \\
&= y_{j-1}y_j y_{j+1}\frac{(1+1/y_{j-1})(1+1/y_{j+1})}{(1+1/y_{j-3})(1+1/y_{j+3})} \\
&= y_{j-3}y_j y_{j+3}\frac{(1+y_{j-1})(1+y_{j+1})}{(1+y_{j-3})(1+y_{j+3})}
\end{aligned}$$

as desired. The case when $A \in \mathcal{P}_n$ is similar. □

Remark II.2. One can prove Proposition II.1 without using the x-coordinates at all. By definition, the y-parameters are certain negative cross ratios. It follows that the expressions $1+y_j$ and $1+1/y_j$ are also given by cross ratios. The equations (2.3)–(2.4) then become multiplicative cross ratio identities which can be proven geometrically.

Remark II.3. Section 12.2 of the survey [KNS11] provides formulas analogous to (2.3)–(2.4) in the setting of quadrilateral lattices in 3-space.

By a slight abuse of notation, write $(y'_1, \ldots, y'_{2n}) = \alpha_1(y_1, \ldots, y_{2n})$ and $(y''_1, \ldots, y''_{2n}) = \alpha_2(y_1, \ldots, y_{2n})$ for y'_j and y''_j defined as above.

2.2 The associated Y-pattern

The equations (2.3)–(2.4) can be viewed as transition equations of a certain Y-pattern. Y-patterns represent a part of cluster algebra dynamics; they were introduced by Fomin and Zelevinsky [FZ07]. A simplified (but sufficient for our current purposes) version of the relevant definitions is given below.

Definition II.4. A Y-seed is a pair $(\mathbf{y}, B)$ where $\mathbf{y} = (y_1, \ldots, y_n)$ is an n-tuple of rational functions and B is an $n \times n$ skew-symmetric, integer matrix. The integer

n is called the *rank* of the seed. Given a Y-seed $(\mathbf{y}, B)$ and some $k = 1, \ldots, n$, the *seed mutation* μ_k in direction k results in a new Y-seed $\mu_k(\mathbf{y}, B) = (\mathbf{y}', B')$ where

$$y'_j = \begin{cases} y_j^{-1}, & j = k \\ y_j y_k^{[b_{kj}]_+}(1 + y_k)^{-b_{kj}}, & j \neq k \end{cases}$$

and B' is the matrix with entries

$$b'_{ij} = \begin{cases} -b_{ij}, & i = k \text{ or } j = k \\ b_{ij} + \operatorname{sgn}(b_{ik})[b_{ik}b_{kj}]_+, & \text{otherwise} \end{cases}$$

In these formulas, $[x]_+$ is shorthand for $\max(x, 0)$.

The data of the exchange matrix B can alternately be represented by a *quiver*. This is a directed graph on vertex set $\{1, \ldots, n\}$. For each i and j, there are $|b_{ij}|$ arcs connecting vertex i and vertex j. Each such arc is oriented from i to j if $b_{ij} > 0$ and from j to i if $b_{ij} < 0$. In terms of quivers, the mutation μ_k consists of the three following steps

1. For every length 2 path $i \to k \to j$, add an arc from i to j.

2. Reverse the orientation of all arcs incident to k.

3. Remove all oriented 2-cycles.

Figure 2.1 illustrates some quiver mutations applied to the quiver associated with

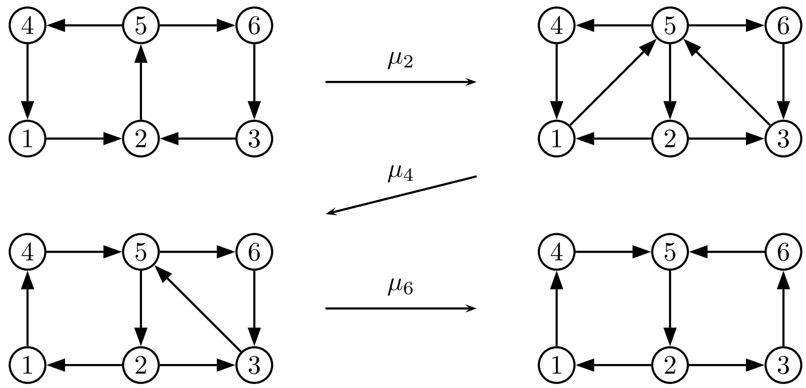

Figure 2.1: Some quiver mutations

the exchange matrix

$$\begin{bmatrix} 0 & 1 & 0 & -1 & 0 & 0 \\ -1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 & 0 \end{bmatrix}.$$

Note that in this example the mutated quiver is the same as the initial one except that all the arrows have been reversed. The is an instance of a more general phenomenon described by the following lemma.

Lemma II.5. *Suppose that* $(\mathbf{y}, B)$ *is a* Y*-seed of rank* $2n$ *such that* $b_{ij} = 0$ *whenever* i, j *have the same parity (so the associated quiver is bipartite). Assume also that for all* i *and* j *the number of length 2 paths in the quiver from* i *to* j *equals the number of length 2 paths from* j *to* i*. Then the* μ_i *for* i *odd pairwise commute as do the* μ_i *for* i *even. Moreover,* $\mu_{2n-1} \circ \cdots \circ \mu_3 \circ \mu_1(y, B) = (\mathbf{y}', -B)$ *and* $\mu_{2n} \circ \cdots \circ \mu_4 \circ \mu_2(\mathbf{y}, B) =$

$(\mathbf{y}'', -B)$ *where*

$$y'_j = \begin{cases} y_j \prod_k y_k^{[b_{kj}]_+}(1+y_k)^{-b_{kj}}, & j \text{ even} \\ y_j^{-1}, & j \text{ odd} \end{cases} \tag{2.5}$$

$$y''_j = \begin{cases} y_j^{-1}, & j \text{ even} \\ y_j \prod_k y_k^{[b_{kj}]_+}(1+y_k)^{-b_{kj}}, & j \text{ odd} \end{cases} \tag{2.6}$$

The proof of this lemma is a simple calculation using the description of quiver mutations above. Note that the term bipartite, as used in the statement of the lemma, simply means that each arc in the quiver connects an odd vertex and an even vertex. No condition on the orientation of the arcs is placed. A stronger notion would require that all arcs begin at an odd vertex and end at an even one. The discussion of bipartite belts in [FZ07] uses the stronger condition. As such, the results proven there do not apply to the current context. We will, however, use much of the same notation.

Let μ_{even} be the compound mutation $\mu_{\text{even}} = \mu_{2n} \circ \ldots \circ \mu_4 \circ \mu_2$ and let $\mu_{\text{odd}} = \mu_{2n-1} \circ \ldots \circ \mu_3 \circ \mu_1$. Equations (2.3)–(2.4) and (2.5)–(2.6) suggest that α_1 and α_2 are instances of μ_{odd} and μ_{even}, respectively. Indeed, let B_0 be the matrix with entries

$$b^0_{ij} = \begin{cases} (-1)^j, & i-j \equiv \pm 1 \pmod{2n} \\ (-1)^{j+1}, & i-j \equiv \pm 3 \pmod{2n} \\ 0, & \text{otherwise} \end{cases}$$

The corresponding quiver in the case $n = 8$ is shown in Figure 2.2.

Proposition II.6. *$\mu_{even}(\mathbf{y}, B_0) = (\alpha_2(\mathbf{y}), -B_0)$ and $\mu_{odd}(\mathbf{y}, -B_0) = (\alpha_1(\mathbf{y}), B_0)$.*

Proof. First of all, B_0 is skew-symmetric and $b^0_{i,j} = 0$ for i, j of equal parity. In the

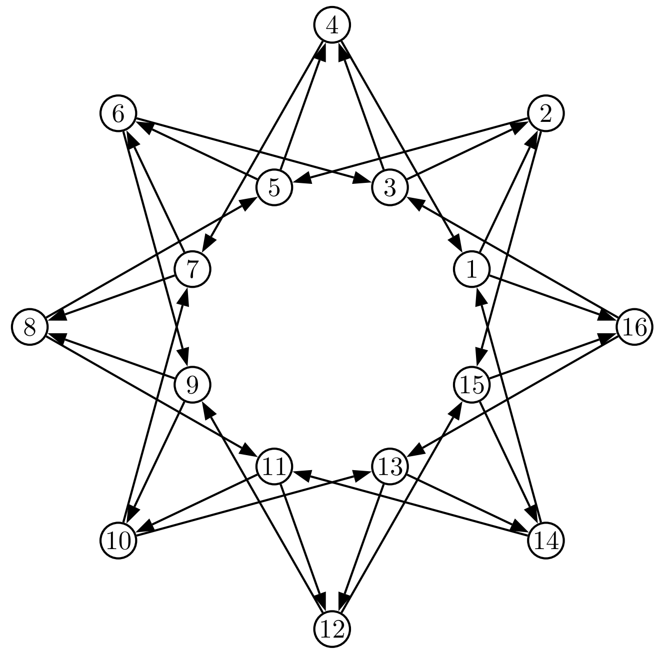

Figure 2.2: The quiver associated with the exchange matrix B_0 for $n = 8$

quiver associated to B_0, the number of length 2 paths from i to j is 1 if $|i-j| \in \{2, 4\}$ and 0 otherwise. Therefore, Lemma II.5 applies to B_0 and μ_{even} is given by (2.6).

Both α_2 and μ_{even} invert the y_j for j even. Now suppose j is odd. Then α_2 has the effect of multiplying y_j by

$$y_{j-3}y_{j+3}\frac{(1+y_{j-1})(1+y_{j+1})}{(1+y_{j-3})(1+y_{j+3})}$$

while μ_{even} multiplies y_j by

$$\prod_k y_k^{[b^0_{kj}]_+}(1+y_k)^{-b^0_{kj}}.$$

Since j is odd, we have $b^0_{j\pm 1,j} = -1$ and $b^0_{j\pm 3,j} = 1$. So these two factors agree. This shows that α_2 and μ_{even} have the same effect on the y-variables. That μ_{even} negates the exchange matrix B_0 also follows from Lemma II.5.

The proof that α_1 corresponds to the mutation μ_{odd}, applied with exchange matrix $-B_0$, is similar. $\square$

2.3 The formula for an iterate of the pentagram map

Let A be a twisted n-gon indexed by $\mathbb{Z}$, and let $\mathbf{y} = (y_1, \ldots, y_{2n})$ be its y-parameters. For $k \geq 0$ let $\mathbf{y}_k = (y_{1,k}, \ldots, y_{2n,k})$ be the y-parameters of $T^k(A)$. In other words, $\mathbf{y}_0 = \mathbf{y}$, $\mathbf{y}_{2m+1} = \alpha_2(\mathbf{y}_{2m})$, and $\mathbf{y}_{2m} = \alpha_1(\mathbf{y}_{2m-1})$. The results of the previous section show that the $\mathbf{y}_k$ are related by seed mutations:

$$(\mathbf{y}_0, B_0) \xrightarrow{\mu_{\text{even}}} (\mathbf{y}_1, -B_0) \xrightarrow{\mu_{\text{odd}}} (\mathbf{y}_2, B_0) \xrightarrow{\mu_{\text{even}}} (\mathbf{y}_3, -B_0) \xrightarrow{\mu_{\text{odd}}} \cdots$$

Note that each $y_{j,k}$ is a rational function of $y_1, \ldots, y_{2n}$. In the language of cluster algebras, this rational function is denoted $Y_{j,k} \in \mathbb{Q}(y_1, \ldots, y_{2n})$. Explicitly, $Y_{j,0} = y_j$ and by (2.3) and (2.4)

$$Y_{j,k+1} = \begin{cases} 1/Y_{j,k}, & j+k \text{ even} \\ Y_{j-3,k}Y_{j,k}Y_{j+3,k}\frac{(1+Y_{j-1,k})(1+Y_{j+1,k})}{(1+Y_{j-3,k})(1+Y_{j+3,k})}, & j+k \text{ odd} \end{cases}$$

To simplify formulas, it is easier to consider only the $Y_{j,k}$ for $j+k$ even. The recurrence satisfied by these rational functions is $Y_{j,-1} = 1/y_j$ for j odd, $Y_{j,0} = y_j$ for j even, and

$$Y_{j,k} = \frac{Y_{j-3,k-1}Y_{j+3,k-1}}{Y_{j,k-2}} \frac{(1+Y_{j-1,k-1})(1+Y_{j+1,k-1})}{(1+Y_{j-3,k-1})(1+Y_{j+3,k-1})}$$

for $j+k$ even and $k \geq 1$. From these, it is easy to compute the other $Y_{j,k}$ because if $j+k$ is odd, then $Y_{j,k} = 1/Y_{j,k-1}$.

Proposition 3.13 of [FZ07], specialized to the present context, says that if $j+k$ is even then $Y_{j,k}$ can be written in the form

$$Y_{j,k} = M_{j,k}\frac{F_{j-1,k}F_{j+1,k}}{F_{j-3,k}F_{j+3,k}}. \tag{2.7}$$

Here, $M_{j,k}$ is a Laurent monomial in $y_1, \ldots, y_{2n}$ and the $F_{i,k}$ are certain polynomials over $y_1, \ldots, y_{2n}$. A description of these component pieces follows.

The monomial $M_{j,k}$ is given by the evaluation of the rational expressions $Y_{j,k}$ in the tropical semifield $\mathbb{P} = Trop(y_1, \ldots, y_{2n})$. This is carried out as follows. First of all, $Y_{j,k}$ is expressed in such a manner that no minus signs appear (that this is possible is clear from transition equations of the Y-pattern). Next, each plus sign is replaced by the auxiliary addition $\oplus$ symbol. This is a binary operation on Laurent monomials defined by $\prod_i y_i^{a_i} \oplus \prod_i y_i^{a_i'} = \prod_i y_i^{\min(a_i, a_i')}$. Finally, this operation together with multiplication and division of monomials is used to compute a result. As an example,

$$Y_{3,1} = y_0 y_3 y_6 \frac{(1+y_2)(1+y_4)}{(1+y_0)(1+y_6)}$$

so

$$M_{3,1} = y_0 y_3 y_6 \frac{(1+y_2)(1+y_4)}{(1+y_0)(1+y_6)}\bigg|_{\mathbb{P}} = y_0 y_3 y_6 \frac{(1 \oplus y_2)(1 \oplus y_4)}{(1 \oplus y_0)(1 \oplus y_6)} = y_0 y_3 y_6.$$

Now $M_{j,-1} = Y_{j,-1} = 1/y_j$ for j odd and $M_{j,0} = Y_{j,0} = y_j$ for j even. The transition equation for the monomials is identical to the transition equation for the $Y_{j,k}$, except that $+$ is replaced throughout by $\oplus$. So, if $j+k$ is even and $k \geq 1$ then

$$M_{j,k} = \frac{M_{j-3,k-1} M_{j+3,k-1}}{M_{j,k-2}} \frac{(1 \oplus M_{j-1,k-1})(1 \oplus M_{j+1,k-1})}{(1 \oplus M_{j-3,k-1})(1 \oplus M_{j+3,k-1})}.$$

Proposition II.7. *The solution to this recurrence is given by*

$$M_{j,k} = \prod_{i=-k}^{k} y_{j+3i} \tag{2.8}$$

for $j+k$ even.

Proof. Clearly the initial conditions are satisfied. Suppose $j+k$ is even and $k \geq 1$.

Then

$$\begin{aligned}
M_{j,k} &= \prod_{i=-k}^{k} y_{j+3i} \\
&= \frac{\prod_{i=-k+1}^{k-1} y_{j-3+3i} \prod_{i=-k+1}^{k-1} y_{j+3+3i}}{\prod_{i=-k+2}^{k-2} y_{j+3i}} \\
&= \frac{M_{j-3,k-1} M_{j+3,k-1}}{M_{j,k-2}} \\
&= \frac{M_{j-3,k-1} M_{j+3,k-1}}{M_{j,k-2}} \frac{(1 \oplus M_{j-1,k-1})(1 \oplus M_{j+1,k-1})}{(1 \oplus M_{j-3,k-1})(1 \oplus M_{j+3,k-1})}.
\end{aligned}$$

The last equality is justified because each $M_{j+i,k-1}$ for i odd is an actual monomial (as opposed to a Laurent monomial), so $1 \oplus M_{j+i,k} = 1$ for these i. $\square$

The $F_{j,k}$ for $j+k$ odd are defined recursively as follows. Put $F_{j,-1} = 1$ for j even, $F_{j,0} = 1$ for j odd, and

$$F_{j,k+1} = \frac{F_{j-3,k} F_{j+3,k} + M_{j,k} F_{j-1,k} F_{j+1,k}}{(1 \oplus M_{j,k}) F_{j,k-1}}$$

for $j+k$ even and $k \geq 0$. Recall $M_{j,k} = \prod_{i=-k}^{k} y_{j+3i}$ so this formula simplifies to

$$F_{j,k+1} = \frac{F_{j-3,k} F_{j+3,k} + (\prod_{i=-k}^{k} y_{j+3i}) F_{j-1,k} F_{j+1,k}}{F_{j,k-1}}. \tag{2.9}$$

For example, $F_{j,1} = 1 + y_j$ and

$$F_{j,2} = (1 + y_{j-3})(1 + y_{j+3}) + y_{j-3} y_j y_{j+3} (1 + y_{j-1})(1 + y_{j+1}). \tag{2.10}$$

Although it is not clear from this definition, the $F_{j,k}$ are indeed polynomials. This is a consequence of general cluster algebra theory.

Equations (2.7)–(2.8) and the fact that $Y_{j,k} = 1/Y_{j,k-1}$ for $j+k$ odd combine to prove that the formula given in Theorem I.2 is of the right form. What remains is to prove (1.7), which expresses the F-polynomials in terms of order ideals. This proof is developed in the next several sections. Before moving on we show how Theorem I.2

can be used to derive a similar formula expressing the iterates of the pentagram map in the x-coordinates.

Theorem II.8. *Let $A \in \mathcal{P}_n$, $x_j = x_j(A)$, and $y_j = y_j(A)$. Then $x_{j,k} = x_j(T^k(A))$ is given by*

$$x_{j,k} = \begin{cases} x_{j-3k} \left(\prod_{i=-k}^{k-1} y_{j+1+3i} \right) \dfrac{F_{j+2,k-1} F_{j-3,k}}{F_{j-2,k-1} F_{j+1,k}}, & j+k \text{ even} \\ x_{j+3k} \left(\prod_{i=-k}^{k-1} y_{j+1+3i} \right) \dfrac{F_{j-3,k-1} F_{j+2,k}}{F_{j+1,k-1} F_{j-2,k}}, & j+k \text{ odd} \end{cases} \tag{2.11}$$

Proof. Let $y_{j,k} = y_j(T^k(A))$. Based on the discussion in Section 2.1, the $x_{j,k}$ are uniquely determined by the identities $x_{j,k}x_{j+1,k} = -y_{j,k}^{-1}$ for $j+k$ even, $x_{j,k}x_{j+1,k} = -y_{j,k}$ for $j+k$ odd, and

$$x_{2,k}x_{4,k}\cdots x_{2n,k} = \begin{cases} x_2x_4\cdots x_{2n}, & k \text{ even} \\ x_1x_3\cdots x_{2n-1}, & k \text{ odd} \end{cases}$$

As such, it suffices to verify that these identities hold if the $x_{j,k}$ are given by (2.11).

If $j+k$ is even then

$$\begin{aligned} x_{j,k} &= x_{j-3k} \left(\prod_{i=-k}^{k-1} y_{j+1+3i} \right) \frac{F_{j+2,k-1} F_{j-3,k}}{F_{j-2,k-1} F_{j+1,k}} \\ x_{j+1,k} &= x_{j+3k+1} \left(\prod_{i=-k}^{k-1} y_{j+2+3i} \right) \frac{F_{j-2,k-1} F_{j+3,k}}{F_{j+2,k-1} F_{j-1,k}}. \end{aligned}$$

Therefore,

$$\begin{aligned} x_{j,k}x_{j+1,k} &= x_{j-3k} \left(\prod_{i=-k}^{k-1} y_{j+1+3i} y_{j+2+3i} \right) x_{j+3k+1} \frac{F_{j-3,k} F_{j+3,k}}{F_{j-1,k} F_{j+1,k}} \\ &= x_{j-3k} \left(\prod_{i=-3k}^{3k} y_{j+i} \right) x_{j+3k+1} \frac{F_{j-3,k} F_{j+3,k}}{M_{j,k} F_{j-1,k} F_{j+1,k}}. \end{aligned}$$

But

$$\prod_{i=-3k}^{3k} y_{j+i} = \frac{(-x_{j-3k+1}x_{j-3k+2})\cdots(-x_{j+3k-1}x_{j+3k})}{(-x_{j-3k}x_{j-3k+1})(-x_{j-3k+2}x_{j-3k+3})\cdots(-x_{j+3k}x_{j+3k+1})}$$
$$= -\frac{1}{x_{j-3k}x_{j+3k+1}}$$

by (2.1)–(2.2). Therefore,

$$x_{j,k}x_{j+1,k} = -\frac{F_{j-3,k}F_{j+3,k}}{M_{j,k}F_{j-1,k}F_{j+1,k}} = -y_{j,k}^{-1}$$

by (1.6). A similar calculation shows $x_{j,k}x_{j+1,k} = -y_{j,k}$ for $j+k$ odd.

Finally, in computing $x_{2,k}x_{4,k}\cdots x_{2n,k}$, all of the F-polynomials in (2.11) cancel out. Each of $y_1, \ldots, y_{2n}$ appear exactly k times in the product, but $y_1 \cdots y_{2n} = 1$ so the y-variables do not contribute either. All that remain are the x_{j-3k} or x_{j+3k} as appropriate. So the product equals $x_2x_4\cdots x_{2n}$ if k is even or $x_1x_3\cdots x_{2n-1}$ if k is odd. □

It will be convenient in the following sections to define $M_{j,k}$ and $F_{j,k}$ for all j, k (as opposed to just for $j+k$ even or, respectively, odd). More specifically, let $M_{j,k} = \prod_{i=-k}^{k} y_{j+3i}$ for all j and k, $F_{j,-1} = F_{j,0} = 1$ for all j, and

$$F_{j,k+1} = \frac{F_{j-3,k}F_{j+3,k} + M_{j,k}F_{j-1,k}F_{j+1,k}}{F_{j,k-1}} \tag{2.12}$$

for all j and k with $k \geq 0$.

2.4 Alternating sign matrix background

An *alternating sign matrix* is a square matrix of 1's, 0's, and -1's such that

- the non-zero entries of each row and column alternate in sign and
- the sum of the entries of each row and column is 1.

Let $ASM(k)$ denote the set of k by k alternating sign matrices. Alternating sign matrices are related to many other mathematical objects, including the posets P_k used in the formula for the F-polynomials. In this section, we explain the connection between these objects.

Recall that Q_k is defined to be the set of triples $(r, s, t) \in \mathbb{Z}^3$ such that $|r| + |s| \leq k - 2$, $r + s \equiv k \pmod 2$, and

$$t \in [2|s| - k + 2, k - 2 - 2|r|]_4$$

Also P_k is the poset $P_k = Q_{k+1} \cup Q_k$ ordered by saying that (r', s', t') covers (r, s, t) if and only if $t' = t + 1$ and $|r' - r| + |s' - s| = 1$. The partial order on P_k restricts to a partial order on Q_k.

A bijection is given by Elkies, Kuperberg, Larsen, and Propp [EKLP92] between $ASM(k)$ and $J(Q_k)$, the set of order ideals of Q_k. This bijection is defined in several steps. Given an order ideal I of Q_k, associate to I the height function H defined by

$$H(r, s) = \begin{cases} k + 2 + \min\{t : (r, s, t) \in I\}, & \text{if such a } t \text{ exists} \\ 2|s|, & \text{otherwise} \end{cases}$$

From H construct a matrix A^* with entries

$$a^*_{ij} = \frac{1}{2} H(-k + i + j, -i + j).$$

Finally, the alternating sign matrix A corresponding to I is defined to be the matrix with entries

$$a_{ij} = \frac{1}{2}(a^*_{i-1,j} + a^*_{i,j-1} - a^*_{i-1,j-1} - a^*_{i,j}).$$

As an example, the poset Q_3 (see Figure 2.3) has seven order ideals. Table 2.1 illustrates a couple instances of the bijection of $J(Q_3)$ with $ASM(3)$.

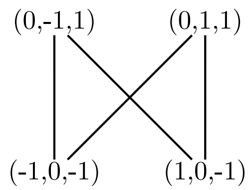

Figure 2.3: The poset Q_3

I	$\emptyset$	$\{(-1,0,-1),(1,0,-1)\}$
$H(r,s)$	$\begin{array}{ccccccc} & & & 6 & & & \\ & & 4 & & 4 & & \\ & 2 & & 2 & & 2 & \\ 0 & & 0 & & 0 & & 0 \\ & 2 & & 2 & & 2 & \\ & & 4 & & 4 & & \\ & & & 6 & & & \end{array}$	$\begin{array}{ccccccc} & & & 6 & & & \\ & & 4 & & 4 & & \\ & 2 & & 2 & & 2 & \\ 0 & & 4 & & 4 & & 0 \\ & 2 & & 2 & & 2 & \\ & & 4 & & 4 & & \\ & & & 6 & & & \end{array}$
A^*	$\begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 1 & 2 \\ 2 & 1 & 0 & 1 \\ 3 & 2 & 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 2 & 1 & 2 \\ 2 & 1 & 2 & 1 \\ 3 & 2 & 1 & 0 \end{bmatrix}$
A	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

Table 2.1: Two examples illustrating the bijection between $J(Q_3)$ and $ASM(3)$. The values of $H(r,s)$ are given for $r+s$ odd and $|r|+|s| \leq 3$ with r increasing from left to right and s from bottom to top.

Remark II.9. The intermediate objects H and A^* in the bijection are themselves of interest. The function H is the so-called height function of a domino tiling in [EKLP92]. In that paper, the posets P_k and Q_k are shifted upward, eliminating the need to add $k+2$ in the definition of H. The $(n+1)\times(n+1)$ matrix A^* (with row and column index starting at 0) is called the skew summation of A.

Call order ideals $I \subseteq Q_{k+1}$ and $J \subseteq Q_k$ compatible if $I \cup J$ is an order ideal of $P_k = Q_{k+1} \cup Q_k$. Call alternating sign matrices $A \in ASM(k+1)$ and $B \in ASM(k)$ compatible if they correspond under the above bijection to compatible order ideals.

This compatibility condition was introduced by D. Robbins and H. Rumsey in their study of a class of recurrences which includes the octahedron recurrence [RR86]. A three-dimensional array of quantities $f_{i,j,k}$ is said to satisfy the *octahedron recurrence* if

$$f_{i,j,k-1}f_{i,j,k+1} = f_{i-1,j,k}f_{i+1,j,k} + f_{i,j-1,k}f_{i,j+1,k}$$

for all $(i,j,k) \in \mathbb{Z}^3$.

Let $k \geq 1$ and consider the expression for $f_{0,0,k}$ in terms of the $(f_{i,j,-1})$ and $(f_{i,j,0})$. It is easy to check that $f_{0,0,k}$ only depends on

$$(f_{i,j,0} : |i|+|j| \leq k, i+j \equiv k \pmod 2)$$

and

$$(f_{i,j,-1} : |i|+|j| \leq k-1, i+j-1 \equiv k \pmod 2).$$

Rotating by 45 degrees, the relevant initial values can be stored in the matrices

$$U_{k+1} = \begin{bmatrix} f_{-k,0,0} & f_{-k+1,1,0} & \cdots & f_{0,k,0} \\ f_{-k+1,-1,0} & f_{-k+2,0,0} & \cdots & f_{1,k-1,0} \\ \vdots & \vdots & & \vdots \\ f_{0,-k,0} & f_{1,-k+1,0} & \cdots & f_{k,0,0} \end{bmatrix}$$

and

$$V_k = \begin{bmatrix} f_{-k+1,0,-1} & f_{-k+2,1,-1} & \cdots & f_{0,k-1,-1} \\ f_{-k+2,-1,-1} & f_{-k+3,0,-1} & \cdots & f_{1,k-2,-1} \\ \vdots & \vdots & & \vdots \\ f_{0,-k+1,-1} & f_{1,-k+2,-1} & \cdots & f_{k-1,0,-1} \end{bmatrix}.$$

In the following, the notation U^A, with U and A matrices of the same dimensions, represents the product $\prod_i \prod_j u_{ij}^{a_{ij}}$.

Proposition II.10 ([RR86, Theorem 1])**.** *Suppose $(f_{i,j,k})$ is a solution to the octahedron recurrence and let $k \geq 1$. Then*

$$f_{0,0,k} = \sum_{A,B} (U_{k+1})^A (V_k)^{-B}$$

where the sum is over all compatible pairs $A \in ASM(k+1)$, $B \in ASM(k)$.

2.5 Computation of the F-polynomials

This section proves the formula for the F-polynomials given in (1.7).

Define Laurent monomials $m_{i,j,k}$ for $k \geq -1$ recursively as follows. Let

$$m_{i,j,0} = \prod_{l=0}^{j-1} \prod_{m=0}^{l} y_{3i+j-4l+6m-1} \tag{2.13}$$

and $m_{i,j,-1} = 1/m_{i,j,0}$ for all $i, j \in \mathbb{Z}$. For $k \geq 1$, put

$$m_{i,j,k} = \frac{m_{i-1,j,k-1} m_{i+1,j,k-1}}{m_{i,j,k-2}}. \tag{2.14}$$

Note that in (2.13), if $j \leq 0$ the conventions for products mentioned in the introduction are needed. Applying these conventions and simplifying yields $m_{i,-1,0} = m_{i,0,0} = 1$ and

$$m_{i,j,0} = \prod_{l=j}^{-2} \prod_{m=l+1}^{-1} y_{3i+j-4l+6m-1}$$

for $j \leq -2$. A portion of the array $m_{i,j,0}$ is given in Figure 2.4.

$$\begin{array}{rccccc}
 & & \vdots & \vdots & \vdots & \\
(j=3) & \cdots & y_{-9}y_{-5}y_{-3}y_{-1}y_1y_3 & y_{-6}y_{-2}y_0y_2y_4y_6 & y_{-3}y_1y_3y_5y_7y_9 & \cdots \\
(j=2) & \cdots & y_{-6}y_{-2}y_0 & y_{-3}y_1y_3 & y_0y_4y_6 & \cdots \\
(j=1) & \cdots & y_{-3} & y_0 & y_3 & \cdots \\
(j=0) & \cdots & 1 & 1 & 1 & \cdots \\
(j=-1) & \cdots & 1 & 1 & 1 & \cdots \\
(j=-2) & \cdots & y_{-4} & y_{-1} & y_2 & \cdots \\
(j=-3) & \cdots & y_{-7}y_{-5}y_{-1} & y_{-4}y_{-2}y_2 & y_{-1}y_1y_5 & \cdots \\
 & & \vdots & \vdots & \vdots & \\
 & & (i=-1) & (i=0) & (i=1) &
\end{array}$$

Figure 2.4: The monomials $m_{i,j,0}$

Proposition II.11. *Let $f_{i,j,k} = m_{i,j,k}F_{3i+j,k}$ for all i, j, k with $k \geq -1$. Then $(f_{i,j,k})$ is a solution to the octahedron recurrence.*

Proof. Fix i, j, k with $k \geq 0$. Then

$$f_{i,j,k-1}f_{i,j,k+1} = (m_{i,j,k-1}m_{i,j,k+1})(F_{3i+j,k-1}F_{3i+j,k+1}).$$

By (2.14) and (2.12) we have

$$m_{i,j,k-1}m_{i,j,k+1} = m_{i-1,j,k}m_{i+1,j,k}$$

$$F_{3i+j,k-1}F_{3i+j,k+1} = F_{3i+j-3,k}F_{3i+j+3,k} + (M_{3i+j,k})F_{3i+j-1,k}F_{3i+j+1,k}.$$

Multiplying yields

$$f_{i,j,k-1}f_{i,j,k+1} = f_{i-1,j,k}f_{i+1,j,k} + (m_{i-1,j,k}m_{i+1,j,k}M_{3i+j,k})F_{3i+j-1,k}F_{3i+j+1,k}. \tag{2.15}$$

Lemma II.12. *For all i, j, k with $k \geq -1$,*

$$\frac{m_{i,j-1,k}m_{i,j+1,k}}{m_{i-1,j,k}m_{i+1,j,k}} = M_{3i+j,k}$$

for M as defined in Section 2.3.

Proof. Let $a_{i,j,k} = \frac{m_{i,j-1,k}m_{i,j+1,k}}{m_{i-1,j,k}m_{i+1,j,k}}$. Applying the recurrence (2.14) defining the $m_{i,j,k}$ to each of the four monomials on the right hand side shows that the $a_{i,j,k}$ satisfy the same recurrence:

$$a_{i,j,k} = \frac{a_{i-1,j,k-1}a_{i+1,j,k-1}}{a_{i,j,k-2}}$$

We want to show $a_{i,j,k} = M_{3i+j,k}$. The fact that

$$M_{3i+j,k} = \frac{M_{3i+j-3,k-1}M_{3i+j+3,k-1}}{M_{3i+j,k-2}}$$

was demonstrated in the proof of Proposition II.7. All that remains is to check the initial conditions $k = 0$ and $k = -1$.

From (2.13) it is easy to show that $m_{i,j+1,0}/m_{i-1,j,0} = \prod_{l=0}^{j} y_{3i+j+2l}$. Shifting yields $m_{i+1,j,0}/m_{i,j-1,0} = \prod_{l=0}^{j-1} y_{3i+j+2l+2}$. Therefore,

$$a_{i,j,0} = \frac{m_{i,j+1,0}/m_{i-1,j,0}}{m_{i+1,j,0}/m_{i,j-1,0}} = \frac{\prod_{l=0}^{j} y_{3i+j+2l}}{\prod_{l=0}^{j-1} y_{3i+j+2l+2}} = y_{3i+j} = M_{3i+j,0}.$$

Taking the reciprocal of both sides yields

$$a_{i,j,-1} = 1/a_{i,j,0} = 1/y_{3i+j} = M_{3i+j,-1}.$$

So, $a_{i,j,k} = M_{3i+j,k}$ for all $k \geq -1$, as desired. □

By Lemma II.12, $m_{i-1,j,k}m_{i+1,j,k}M_{3i+j,k} = m_{i,j-1,k}m_{i,j+1,k}$. Substituting this into (2.15) yields

$$\begin{aligned} f_{i,j,k-1}f_{i,j,k+1} &= f_{i-1,j,k}f_{i+1,j,k} + m_{i,j-1,k}m_{i,j+1,k}F_{3i+j-1,k}F_{3i+j+1,k} \\ &= f_{i-1,j,k}f_{i+1,j,k} + f_{i,j-1,k}f_{i,j+1,k} \end{aligned}$$

which completes the proof of Proposition II.11. □

The general solution to the octahedron recurrence given in Proposition II.10 in particular applies to this solution. Now $m_{i,0,k} = 1$ for all i and k, so $f_{0,0,k} = m_{0,0,k}F_{0,k} = F_{0,k}$. For all i and j, we have $f_{i,j,0} = m_{i,j,0}$ (since $F_{3i+j,0} = 1$), so the matrix U_{k+1} is given by

$$U_{k+1} = \begin{bmatrix} m_{-k,0,0} & m_{-k+1,1,0} & \cdots & m_{0,k,0} \\ m_{-k+1,-1,0} & m_{-k+2,0,0} & \cdots & m_{1,k-1,0} \\ \vdots & \vdots & & \vdots \\ m_{0,-k,0} & m_{1,-k+1,0} & \cdots & m_{k,0,0} \end{bmatrix}. \tag{2.16}$$

Finally $f_{i,j,-1} = m_{i,j,-1} = 1/m_{i,j,0}$ for all i and j. It follows that each entry of V_k is the reciprocal of the corresponding entry of U_k. So, $(V_k)^{-B} = U_k^B$ for any $B \in ASM(k-1)$. This proves the following result.

Proposition II.13. *Using the $m_{i,j,0}$ as the entries of U_{k+1} and U_k as in* (2.16)

$$F_{0,k} = \sum_{A,B} (U_{k+1})^A (U_k)^B \tag{2.17}$$

where the sum is over all compatible pairs $A \in ASM(k+1)$, $B \in ASM(k)$.

As an example, let $k = 2$. Then

$$U_2 = \begin{bmatrix} m_{-1,0,0} & m_{0,1,0} \\ m_{0,-1,0} & m_{1,0,0} \end{bmatrix} = \begin{bmatrix} 1 & y_0 \\ 1 & 1 \end{bmatrix}$$

and

$$U_3 = \begin{bmatrix} m_{-2,0,0} & m_{-1,1,0} & m_{0,2,0} \\ m_{-1,-1,0} & m_{0,0,0} & m_{1,1,0} \\ m_{0,-2,0} & m_{1,-1,0} & m_{2,0,0} \end{bmatrix} = \begin{bmatrix} 1 & y_{-3} & y_{-3}y_1y_3 \\ 1 & 1 & y_3 \\ y_{-1} & 1 & 1 \end{bmatrix}.$$

There are two elements $B \in ASM(2)$, namely the two permutation matrices, so U_2^B is either 1 or y_0. The seven alternating sign matrices $A \in ASM(3)$ are listed in Table 2.2. The associated monomials U_3^A are 1,y_{-3},y_3 (top row), $y_{-3}y_3$ (middle row), and $y_{-3}y_3y_{-1}, y_{-3}y_3y_1, y_{-3}y_3y_{-1}y_1$ (bottom row).

The compatibility condition for $ASM(3)$ and $ASM(2)$ is as follows. The three matrices in the top row of Table 2.2 are related only to the identity matrix. The

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$
	$\begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$	
$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

Table 2.2: The seven elements of $ASM(3)$

three matrices in the bottom row are related only to the other element of $ASM(2)$. Lastly, the middle matrix is related to both elements of $ASM(2)$. The resulting formula is

$$\begin{aligned} F_{0,2} &= (1 + y_{-3} + y_3 + y_{-3}y_3)1 \\ &\quad + (y_{-3}y_3 + y_{-3}y_3y_{-1} + y_{-3}y_3y_1 + y_{-3}y_3y_{-1}y_1)y_0 \\ &= (1 + y_{-3})(1 + y_3) + y_{-3}y_0y_3(1 + y_{-1})(1 + y_1) \end{aligned}$$

which matches (2.10).

Using the bijection between alternating sign matrices and order ideals from the previous section, (2.17) can be expressed in terms of order ideals. Associate to each triple $(r, s, t) \in \mathbb{Z}^3$ the variable y_{3r+s}. Define the weight of a finite subset I of $\mathbb{Z}^3$ (particularly an order ideal of Q_k or P_k) to be the monomial

$$\mathrm{wt}(I) = \prod_{(r,s,t)\in I} y_{3r+s}.$$

Proposition II.14. *If $I \subseteq Q_k$ is an order ideal and $A \in ASM(k)$ is the associated alternating sign matrix then*

$$\mathrm{wt}(I) = U_k^A.$$

Proof. The proof is by induction on the number of elements of I. If I is empty then $\mathrm{wt}(I) = 1$. On the other hand, the height function corresponding to I is $H(r,s) = 2|s|$ so the matrix A^* has entries $a^*_{ij} = |-i+j|$. It follows that $a_{ij} = 1$ if $i = j$ and $a_{ij} = 0$ otherwise (i.e. A is the identity matrix). The diagonal entries of U_k are all 1, so $U_k^A = 1$.

Now suppose I is non-empty and that the proposition holds for all smaller order ideals. Then there is some $(r_0, s_0, t_0) \in I$ such that $I' = I - \{(r_0, s_0, t_0)\}$ is still an order ideal of Q_k. By the induction hypothesis, $\mathrm{wt}(I') = U_k^{A'}$ where A' is the alternating sign matrix corresponding to I'. Clearly, $\mathrm{wt}(I) = y_{3r_0+s_0}\,\mathrm{wt}(I')$. The addition of (r_0, s_0, t_0) to obtain I from I' propagates through the bijection as follows:

$$I = I' \cup \{(r_0, s_0, t_0)\}$$

$$H(r,s) = \begin{cases} H'(r,s) + 4, & (r,s) = (r_0, s_0) \\ H'(r,s), & \text{otherwise} \end{cases}$$

$$a^*_{ij} = \begin{cases} a^{*\prime}_{ij} + 2, & (i,j) = (i_0, j_0) \\ a^{*\prime}_{ij}, & \text{otherwise} \end{cases}$$

$$a_{ij} = \begin{cases} a'_{ij} - 1, & (i,j) \in \{(i_0, j_0), (i_0+1, j_0+1)\} \\ a'_{ij} + 1, & (i,j) \in \{(i_0+1, j_0), (i_0, j_0+1)\} \\ a'_{ij}, & \text{otherwise} \end{cases}$$

where i_0, j_0 are the integers satisfying $-k + i_0 + j_0 = r_0$, $-i_0 + j_0 = s_0$.

The (i_0, j_0) entry of U_k is $m_{-k-1+i_0+j_0, j_0-i_0, 0} = m_{r_0-1, s_0, 0}$. Similarly, the (i_0, j_0+1) entry is $m_{r_0, s_0+1, 0}$, the (i_0+1, j_0) entry is $m_{r_0, s_0-1, 0}$, and the (i_0+1, j_0+1) entry is

$m_{r_0+1,s_0,0}$. These are the only entries where A and A' differ, so

$$\begin{aligned} U_k^A &= \frac{m_{r_0,s_0+1,0}m_{r_0,s_0-1,0}}{m_{r_0-1,s_0,0}m_{r_0+1,s_0,0}}U_k^{A'} \\ &= y_{3r_0+s_0}U_k^{A'} \end{aligned}$$

by Lemma II.12. Therefore $U_k^A = y_{3r_0+s_0}\operatorname{wt}(I') = \operatorname{wt}(I)$ as desired. $\square$

Example II.15. Let A be the matrix in the middle row of Table 2.2. We have already seen that $U_3^A = y_{-3}y_3$. According to Table 2.1, the corresponding order ideal is $I = \{(-1,0,-1),(1,0,-1)\}$, so $\operatorname{wt}(I) = y_{-3}y_3$ as well.

Theorem II.16.

$$F_{j,k} = \sum_{I\in J(P_k)}\prod_{(r,s,t)\in I} y_{3r+s+j}$$

where $J(P_k)$ denotes the set of order ideals of P_k.

Proof. The effect on either side of this equation of changing j is to shift the index of each y-variable. As such, it suffices to verify the formula for $F_{0,k}$. By Proposition II.13,

$$F_{0,k} = \sum_{A,B} U_{k+1}^A U_k^B$$

where the sum is over compatible pairs $A \in ASM(k+1), B \in ASM(k)$. Such pairs are in bijection with compatible pairs of order ideals $I_1 \subseteq Q_{k+1}, I_2 \subseteq Q_k$, so by Proposition II.14

$$F_{0,k} = \sum_{I_1,I_2}\operatorname{wt}(I_1)\operatorname{wt}(I_2).$$

Every order ideal I of P_k is uniquely the union of such a compatible pair, namely $I_1 = I\cap Q_{k+1}$ and $I_2 = I \cap Q_k$. Since I is a disjoint union of I_1 and I_2, it follows that $\operatorname{wt}(I) = \operatorname{wt}(I_1)\operatorname{wt}(I_2)$. Therefore,

$$F_{0,k} = \sum_{I\in J(P_k)}\operatorname{wt}(I) = \sum_{I\in J(P_k)}\prod_{(r,s,t)\in I} y_{3r+s}.$$

$\square$

Remark II.17. Proposition II.13 and Theorem II.16 give two generating function descriptions of the polynomials $F_{j,k}$, one in terms of alternating sign matrices and the other in terms of order ideals. A third possible approach would be to use domino tilings of the Aztec diamond as the underlying combinatorial objects. It would almost certainly be possible to understand such a formula as a special case of results of D. Speyer [Spe07].

2.6 Axis-aligned polygons

We now turn to addressing our results on axis-aligned polygons, namely Theorems I.3 and I.4. First, we express the relevant geometric properties in terms of the y-parameters.

Lemma II.18. *Let A be a twisted polygon with vertices indexed either by $\mathbb{Z}$ or $\frac{1}{2}+\mathbb{Z}$. Let i be a vertex index of A and j a side index.*

1. *A_{i-2}, A_i, A_{i+2} are collinear if and only if $y_{2i}(A) = -1$.*

2. *a_{j-2}, a_j, a_{j+2} are concurrent if and only if $y_{2j}(A) = -1$.*

Proof. By (1.2), we have $y_{2i}(A) = -1$ if and only if

$$\chi(\overleftrightarrow{A_iA_{i-2}}, \overleftrightarrow{A_iA_{i-1}}, \overleftrightarrow{A_iA_{i+1}}, \overleftrightarrow{A_iA_{i+2}}) = 1.$$

A cross ratio $\chi(a, b, c, d)$ equals 1 if and only if $a = d$ or $b = c$. By assumption, A_{i-1}, A_i, A_{i+1} are not collinear so $\overleftrightarrow{A_iA_{i-1}} \neq \overleftrightarrow{A_iA_{i+1}}$. Therefore $y_{2i}(A) = -1$, if and only if $\overleftrightarrow{A_iA_{i-2}} = \overleftrightarrow{A_iA_{i+2}}$, i.e. A_{i-2}, A_i, A_{i+2} are collinear. The proof of the second statement is similar. $\square$

Recall if $S \subseteq \mathbb{Z}$ then $X_S \subseteq \mathcal{P}_n$ is defined as the collection of polygons with A_{i-2}, A_i, A_{i+2} collinear for all $i \in S$. So by Lemma II.18 we have that X_S is defined by the equations $y_{2i} = -1$ for all $i \in S$. Similarly, for $S \subseteq (\frac{1}{2} + \mathbb{Z})$, Y_S is defined in $\mathcal{P}_n$ by $y_{2j} = -1$ for all $j \in S$. The corresponding statements also hold for $\mathcal{P}_n^*$. Now an axis-aligned polygon A has a_{j-2}, a_j, a_{j+2} concurrent for all j. So axis-aligned polygons have half of their y-parameters equal to -1.

The polynomial $F_{0,k}$ takes an interesting form under the specialization $y_j = -1$ for all $j \equiv k \pmod 2$. Specifically, we will show that there is a $(k+1) \times (k+1)$ matrix whose entries are monomials in the other y_j and whose determinant equals $F_{0,k}$ in this case.

Let $(f_{i,j,k})$ be a solution to the octahedron recurrence and let $\tilde{f}_{i,j,k} = \sigma_j f_{i,j,k}$ where

$$\sigma_j = \begin{cases} 1, & j \equiv 0, 3 \pmod 4 \\ -1, & j \equiv 1, 2 \pmod 4 \end{cases}$$

The $\tilde{f}_{i,j,k}$ satisfy a slightly different recurrence. Indeed

$$\begin{aligned} \tilde{f}_{i,j,k+1}\tilde{f}_{i,j,k-1} &= f_{i,j,k+1}f_{i,j,k-1} \\ &= f_{i-1,j,k}f_{i+1,j,k} + f_{i,j-1,k}f_{i,j+1,k} \\ &= \tilde{f}_{i-1,j,k}\tilde{f}_{i+1,j,k} - \tilde{f}_{i,j-1,k}\tilde{f}_{i,j+1,k} \end{aligned}$$

since $\sigma_j^2 = 1$ and $\sigma_{j-1}\sigma_{j+1} = -1$ for all j.

This recurrence is the one used in Dodgson's method of computing determinants. More specifically, recall that $f_{0,0,k}$ can be expressed in terms of the initial conditions given in the matrices U_{k+1} and V_k. Similarly, $\tilde{f}_{0,0,k}$ can be expressed in terms of matrices $\tilde{U}_{k+1}$ and $\tilde{V}_k$ in which the f's have been replaced by $\tilde{f}$'s. According to Dodgson's condensation, if $\tilde{V}_k$ has all its entries equal to 1 then $\tilde{f}_{0,0,k} = \det(\tilde{U}_{k+1})$.

Proposition II.19. *If* $y_j = -1$ *for all* $j \equiv k \pmod 2$ *then*

$$F_{0,k} = \det \begin{bmatrix} \tilde{m}_{-k,0,0} & \tilde{m}_{-k+1,1,0} & \cdots & \tilde{m}_{0,k,0} \\ \tilde{m}_{-k+1,-1,0} & \tilde{m}_{-k+2,0,0} & \cdots & \tilde{m}_{1,k-1,0} \\ \vdots & \vdots & & \vdots \\ \tilde{m}_{0,-k,0} & \tilde{m}_{1,-k+1,0} & \cdots & \tilde{m}_{k,0,0} \end{bmatrix}$$

where $\tilde{m}_{i,j,0} = \sigma_j m_{i,j,0}$.

Proof. Consider the solution to the octahedron recurrence constructed in Section 2.5. The entries of V_k are of the form $f_{i,j,-1}$ for $i + j - 1 \equiv k \pmod 2$. Recall that $f_{i,j,-1} = m_{i,j,-1} = 1/m_{i,j,0}$. However, by (2.13) we have that $m_{i,j,0}$ is equal to a product of -1's since $3i + j - 4l + 6m - 1 \equiv k \pmod 2$. The number of terms in this product equals $\sum_{l=0}^{j-1}(l+1) = j(j+1)/2$. Therefore, $m_{i,j,0} = \sigma_j$, so $m_{i,j,-1} = \sigma_j$. All of the entries of V_k equal σ_j, so all the entries of $\tilde{V}_k$ equal $\sigma_j^2 = 1$.

It follows from Dodgson's condensation that $\tilde{f}_{0,0,k} = \det(\tilde{U}_{k+1})$. The matrix $\tilde{U}_{k+1}$ is exactly the one in the statement of this proposition, and $\tilde{f}_{0,0,k} = f_{0,0,k} = F_{0,k}$. $\square$

For $k = 1$, the proposition says

$$F_{0,1} = \det \begin{bmatrix} m_{-1,0,0} & -m_{0,1,0} \\ m_{0,-1,0} & m_{1,0,0} \end{bmatrix} = \det \begin{bmatrix} 1 & -y_0 \\ 1 & 1 \end{bmatrix}$$

which equals $1 + y_0$. For $k = 2$,

$$F_{0,2} = \det \begin{bmatrix} m_{-2,0,0} & -m_{-1,1,0} & -m_{0,2,0} \\ m_{-1,-1,0} & m_{0,0,0} & -m_{1,1,0} \\ -m_{0,-2,0} & m_{1,-1,0} & m_{2,0,0} \end{bmatrix}$$

$$= \det \begin{bmatrix} 1 & -y_{-3} & -y_{-3}y_1y_3 \\ 1 & 1 & -y_3 \\ -y_{-1} & 1 & 1 \end{bmatrix}.$$

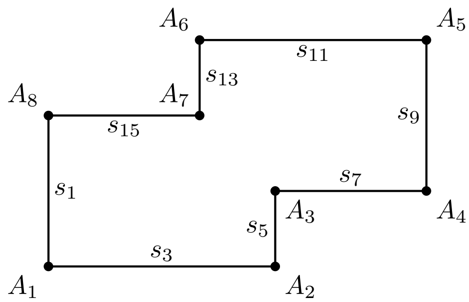

Figure 2.5: An axis-aligned octagon. The side lengths s_3, s_5, s_7, and s_9 are positive and the others are negative.

This determinant agrees with the result of substituting $y_0 = -1$ into (2.10).

Let $A \in \mathcal{P}_{2n}$ be an axis-aligned polygon, actually realized in the plane with its sides all vertical and horizontal. Suppose in addition that A is closed, i.e. $A_{i+2n} = A_i$ for all $i \in \mathbb{Z}$. Let s_{2j+1} denote the signed length of the side joining A_j and A_{j+1}, where the sign is taken to be positive if and only if A_{j+1} is to the right of or above A_j. An example of an axis-aligned octagon is given in Figure 2.5. It follows from the second statement in Lemma II.18 that $y_{2j+1}(A) = -1$ for all $j \in \mathbb{Z}$. On the other hand, the even y-parameters can be expressed directly in terms of the side lengths.

Lemma II.20. *For all* $j \in \mathbb{Z}$

$$y_{2j}(A) = -\frac{s_{2j-1}s_{2j+1}}{s_{2j-3}s_{2j+3}}.$$

Proof. By (1.2)

$$y_{2j}(A) = -\left(\chi(\overleftrightarrow{A_jA_{j-2}}, \overleftrightarrow{A_jA_{j-1}}, \overleftrightarrow{A_jA_{j+1}}, \overleftrightarrow{A_jA_{j+2}})\right)^{-1}.$$

A cross ratio of 4 lines can be calculated as the cross ratio of the 4 corresponding slopes. Suppose first that $\overleftrightarrow{A_jA_{j-1}}$ is vertical (slope $= \infty$) and $\overleftrightarrow{A_jA_{j+1}}$ is horizontal (slope $= 0$). Then the slope of $\overleftrightarrow{A_jA_{j-2}}$ is s_{2j-1}/s_{2j-3} and the slope of $\overleftrightarrow{A_jA_{j+2}}$ is

s_{2j+3}/s_{2j+1}. Therefore

$$\begin{aligned} y_{2j}(A) &= -\left(\chi\left(\frac{s_{2j-1}}{s_{2j-3}}, \infty, 0, \frac{s_{2j+3}}{s_{2j+1}}\right)\right)^{-1} \\ &= -\left(\frac{(-\infty)\left(-\frac{s_{2j+3}}{s_{2j+1}}\right)}{\left(\frac{s_{2j-1}}{s_{2j-3}}\right)(\infty)}\right)^{-1} \\ &= -\frac{s_{2j-1}s_{2j+1}}{s_{2j-3}s_{2j+3}} \end{aligned}$$

as desired. The calculation is similar if $\overleftrightarrow{A_jA_{j-1}}$ is horizontal and $\overleftrightarrow{A_jA_{j+1}}$ is vertical.

$\square$

Proposition II.21. *Let $A \in \mathcal{P}_{2n}$ be closed and axis-aligned, and let $y_j = y_j(A)$ for all j. If n is even then $F_{0,n-1}(y) = 0$.*

Proof. By Lemma II.18, $y_j = -1$ for all odd j, that is for all $j \equiv n-1 \pmod 2$. So by Proposition II.19 we have

$$F_{0,n-1} = \det \begin{bmatrix} \tilde{m}_{-n+1,0,0} & \tilde{m}_{-n+2,1,0} & \cdots & \tilde{m}_{0,n-1,0} \\ \tilde{m}_{-n+2,-1,0} & \tilde{m}_{-n+3,0,0} & \cdots & \tilde{m}_{1,n-2,0} \\ \vdots & \vdots & & \vdots \\ \tilde{m}_{0,-n+1,0} & \tilde{m}_{1,-n+2,0} & \cdots & \tilde{m}_{n-1,0,0} \end{bmatrix}.$$

We want to show that this matrix, call it X, is degenerate. Let x_{ij} be the i,j entry of X, that is $x_{ij} = \tilde{m}_{-n-1+i+j,j-i,0}$. Then

$$\begin{aligned} \frac{x_{i,j}x_{i+1,j+1}}{x_{i+1,j}x_{i,j+1}} &= \frac{\tilde{m}_{-n-1+i+j,j-i,0}\tilde{m}_{-n+1+i+j,j-i,0}}{\tilde{m}_{-n+i+j,j-i-1,0}\tilde{m}_{-n-1+i+j,j-i+1,0}} \\ &= -\frac{m_{-n-1+i+j,j-i,0}m_{-n+1+i+j,j-i,0}}{m_{-n+i+j,j-i-1,0}m_{-n-1+i+j,j-i+1,0}} \\ &= -\frac{1}{y_{3(-n+i+j)+j-i}} = -\frac{1}{y_{-3n+2i+4j}} \end{aligned}$$

by Lemma II.12. Since n is even, $-3n+2i+4j$ is even so

$$\begin{aligned}\frac{x_{i,j}x_{i+1,j+1}}{x_{i+1,j}x_{i,j+1}} &= \frac{s_{-3n+2i+4j-3}s_{-3n+2i+4j+3}}{s_{-3n+2i+4j-1}s_{-3n+2i+4j+1}} \\ &= \frac{z_{i,j}z_{i+1,j+1}}{z_{i+1,j}z_{i,j+1}}\end{aligned}$$

where $z_{ij} = s_{-3n+2i+4j-3}$. Therefore

$$\frac{x_{i,j}}{z_{i,j}}\frac{x_{i+1,j+1}}{z_{i+1,j+1}} - \frac{x_{i+1,j}}{z_{i+1,j}}\frac{x_{i,j+1}}{z_{i,j+1}} = 0.$$

The matrix with entries x_{ij}/z_{ij} has consecutive 2×2 minors equal to 0. The entries of this matrix are generically non-zero, so it follows that all of its 2×2 minors vanish. Therefore, the matrix has rank 1 and there exist non-zero scalars $\lambda_1, \ldots, \lambda_n$ and $\mu_1, \ldots, \mu_n$ such that $x_{ij}/z_{ij} = \lambda_i \mu_j$.

Now the matrix Z whose entries are z_{ij} is degenerate because its rows all have sum 0. Indeed for each fixed i

$$\sum_{j=1}^{n} z_{ij} = \sum_{j=1}^{n} s_{-3n+2i+4j-3} = 0$$

because A is closed so the sum of the lengths of its n horizontal (or vertical) sides must be 0. However, X can be obtained from Z by multiplying its rows by the λ_i and its columns by the μ_j. Therefore, X is degenerate as well. □

Corollary II.22. *Let $y_j = y_j(A)$ for A as above. Then $F_{j,n-1}(y) = 0$ for all $j \equiv n$ (mod 2).*

Proof. Suppose first that n is even. Then by Proposition II.21 we have $F_{0,n-1} = 0$. Cyclically permuting the vertex indexing has the effect of shifting the y-variables, and hence the F-polynomials, by an even offset. So $F_{j,n-1} = 0$ for all even j.

Now suppose n is odd. Shifting all of the y-variables up by 1 in the statement of Proposition II.19 yields that if $y_j = -1$ for all $j - 1 \equiv k \pmod 2$ then $F_{1,k}$ is given by the determinant of a matrix. Since A is axis-aligned, $y_j = -1$ for all odd j, that is for all $j-1 \equiv n-1 \pmod 2$. Therefore, $F_{1,n-1}$ is the determinant of some matrix X which is exactly like the matrix X in the proof of Proposition II.21, except that the y-variables have all been shifted by 1. The same proof shows that this matrix is degenerate so $F_{1,n-1} = 0$. Permuting the vertices yields that $F_{j,n-1} = 0$ for all odd j. □

Corollary II.22 is precisely the algebraic result needed to prove Theorem I.3

Proof (of Theorem I.3). Suppose $A \in \mathcal{P}_n$ is closed and axis-aligned. By Corollary II.22, $F_{j,n-1} = 0$ provided $j \equiv n \pmod 2$. Therefore,

$$\begin{aligned} 0 &= F_{j,n-1}F_{j,n-3} \\ &= F_{j-3,n-2}F_{j+3,n-2} + \left(\prod_{i=-n+2}^{n-2} y_{j+3i}\right) F_{j-1,n-2}F_{j+1,n-2}. \end{aligned}$$

Hence

$$\left(\prod_{i=-n+2}^{n-2} y_{j+3i}\right) \frac{F_{j-1,n-2}F_{j+1,n-2}}{F_{j-3,n-2}F_{j+3,n-2}} = -1.$$

By (1.6) the left hand side equals $y_{j,n-2}$, the jth y-parameter of $T^{n-2}(A)$. So $y_{j,n-2} = -1$ for all $j \equiv n \pmod 2$.

If n is even, then $T^{n-2}(A)$ is indexed by $\mathbb{Z}$ and $y_{j,n-2} = -1$ for all j even. On the other hand, if n is odd, then $T^{n-2}(A)$ is indexed by $\frac{1}{2} + \mathbb{Z}$ and $y_{j,n-2} = -1$ for all j odd. In either case, it follows from the first statement of Lemma II.18 that the odd vertices of $T^{n-2}(A)$ lie on one line and the even vertices lie on another. □

Remark II.23. Theorem I.3 is stated for all n in [Sch01a] and proven for n even (i.e. the number of sides of A divisible by 4) in [Sch08]. Schwartz's proof in [Sch08] also involves Dodgson's condensation, so it seems as though our proof must be related to his. However, our proof does handle the case of n odd as well as even.

Remark II.24. Theorem I.3 is only meant to hold for polygons A for which the map T^{n-2} is defined. The set of closed, axis-aligned $A \in \mathcal{P}_{2n}$ with this property is open, but it could, a priori, be empty. To rule out this possibility, it suffices to find a single example which works for each n. According to Schwartz, there is substantial experimental evidence to suggest that this is always possible [Sch01a].

Suppose now that A is not closed, but twisted with $A_{i+2n} = \phi(A_i)$. Since A is axis-aligned, the projective transformation ϕ must send vertical lines to vertical lines and horizontal lines to horizontal lines. One can check that all such projective transformations are of the form

$$\phi(x, y) = (ax + b, cy + d)$$

for some reals a, b, c, d.

As before, let s_{2i+1} be the signed length of the side joining A_i to A_{i+1}. Since A is not closed, the side lengths are no longer periodic. More specifically, if s_j is the length of a horizontal edge then $s_{j+4n} = as_j$, while if it is the length of a vertical edge then $s_{j+4n} = cs_j$. In Theorem I.4, we assume that ϕ fixes pointwise the line containing the two points of concurrency, which we have fixed to be the line at infinity. Put another way, ϕ preserves the slopes of lines. This implies $a = c$ so $s_{j+4n}/s_j = a = c$ for all odd j.

Proof (of Theorem I.4). Following the proof of Theorem I.3, it suffices to show that $F_{j,n} = 0$ for all $j \equiv n - 1 \pmod 2$. By symmetry, it is enough to show that $F_{0,n} = 0$ if n is odd and $F_{1,n} = 0$ if n is even. Suppose n is odd. Since A is axis-aligned, Proposition II.19 applies and

$$F_{0,n} = \det \begin{bmatrix} \tilde{m}_{-n,0,0} & \tilde{m}_{-n+1,1,0} & \cdots & \tilde{m}_{0,n,0} \\ \tilde{m}_{-n+1,-1,0} & \tilde{m}_{-n+2,0,0} & \cdots & \tilde{m}_{1,n-1,0} \\ \vdots & \vdots & & \vdots \\ \tilde{m}_{0,-n,0} & \tilde{m}_{1,-n+1,0} & \cdots & \tilde{m}_{n,0,0} \end{bmatrix}.$$

Let X be this matrix, that is, $x_{ij} = \tilde{m}_{-n-2+i+j,j-i,0}$. As in the proof of Proposition II.21, we have

$$\frac{x_{i,j}x_{i+1,j+1}}{x_{i+1,j}x_{i,j+1}} = -\frac{1}{y_{-3n+2i+4j-3}}$$

for all i and j. Therefore,

$$\frac{x_{i,1}x_{i+1,n+1}}{x_{i+1,1}x_{i,n+1}} = -\frac{1}{y_{-3n+2i+1}y_{-3n+2i+5}\cdots y_{-3n+2i+4n-3}}.$$

Expressing the y-parameters in terms of the side lengths using Lemma II.20, the right hand side becomes a telescoping product leaving

$$\frac{x_{i,1}x_{i+1,n+1}}{x_{i+1,1}x_{i,n+1}} = \left(\frac{s_{-3n+2i-2}}{s_{-3n+2i}}\right)\left(\frac{s_{n+2i}}{s_{n+2i-2}}\right) = 1$$

since $s_{n+2i}/s_{-3n+2i} = s_{n+2i-2}/s_{-3n+2i-2}$ by the assumption on ϕ. Therefore $x_{i,1}/x_{i,n+1} = x_{i+1,1}/x_{i+1,n+1}$ for all $i = 1, \ldots, n$. The first and last columns of X are linearly dependent, so $F_{0,n} = \det(X) = 0$ as desired.

The proof that $F_{1,n} = 0$ for n even is similar. □

Remark II.25. It should be possible to deduce Theorem I.3 from Theorem I.4 along the following lines. Let A be a closed, axis-aligned $2n$-gon. If the vertices of a polygon lie alternately on 2 lines, then the pentagram map collapses it to a single point (the intersection of those 2 lines). As such, it suffices to show that $T^{n-1}(A)$ is a single point. Approximating A by twisted polygons with ϕ being smaller and smaller vertical translations, Theorem I.4 shows that the vertices of $T^{n-1}(A)$ lie on

2 lines, l_1 and l_2. In fact, it is easy to show that these lines must be parallel and, in this case, vertical. Similarly, approximating A by twisted polygons with smaller and smaller horizontal translations shows that the vertices of $T^{n-1}(A)$ lie on 2 horizontal lines m_1 and m_2. Combining these, the vertices of $T^{n-1}(A)$ alternate between the points $l_1 \cap m_1$ and $l_2 \cap m_2$. The pentagram map never collapses a polygon to a line segment of positive length, so it follows that these 2 points are equal.

CHAPTER III

On singularity confinement for the pentagram map

3.1 Vanishing of the F-polynomials

According to (1.4) and (1.5), the pentagram map has singularities for polygons with $x_j x_{j+1} = 1$, i.e., $y_j = -1$, for some j. According to (2.11), the iterate T^k has a singularity whenever $F_{j,k-1} = 0$ or $F_{j,k} = 0$ for some j. In this section we examine under which circumstances having $y_j = -1$ for certain j forces an F-polynomial to vanish. Results along these lines will indicate how many steps a given singularity persists.

For the purpose of this section, relax the assumptions $y_{i+2n} = y_i$ for all i and $y_1 y_2 \cdots y_{2n} = 1$. Instead consider the $F_{j,k}$ as polynomials in the countable collection of variables $\{y_i : i \in \mathbb{Z}\}$. By way of notation, if $S \subseteq \mathbb{Z}$ let $F_{j,k}|_S$ be the polynomial in $\{y_i : i \in \mathbb{Z} \setminus S\}$ obtained by substituting $y_i = -1$ for all $i \in S$ into $F_{j,k}$.

The following two lemmas contain more properties of alternating sign matrices and order ideals. Each of these statements can easily be deduced from results of [EKLP92].

Lemma III.1. *Let $A \in ASM(k)$. Suppose that A has m entries equal to 1, namely entries $(i_1, j_1), (i_2, j_2), \ldots, (i_m, j_m)$. Then there are 2^m alternating sign matrices $B \in ASM(k+1)$ compatible with A. Moreover, there exist an order ideal $J_0 \in J(Q_{k+1})$*

and elements $(r_1, s_1, t_1), \ldots, (r_m, s_m, t_m) \in Q_{k+1} \setminus J_0$ *such that*

- *The map* $(i, j) \mapsto (i+j-k-1, -i+j)$ *sends* $(i_1, j_1), \ldots, (i_m, j_m)$ *to* $(r_1, s_1), \ldots, (r_m, s_m)$.
- *An order ideal* $J \in J(Q_{k+1})$ *corresponds to a matrix* B *compatible with* A *if and only if*

$$J_0 \subseteq J \subseteq J_0 \cup \{(r_1, s_1, t_1), \ldots, (r_m, s_m, t_m)\}$$

By way of notation, let $B_0 = B_0(A) \in ASM(k+1)$ be the matrix corresponding to the order ideal J_0 from the lemma. As an example, take

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \in ASM(2)$$

As mentioned previously, A is compatible to four elements $B \in ASM(3)$, namely those in the bottom two rows of Table 2.2. The corresponding order ideals of Q_3 are those containing $J_0 = \{(-1, 0, -1), (1, 0, -1)\}$ (see Figure 2.3). From the last column of Table 2.1 we have

$$B_0(A) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Lemma III.2. *Let* $A \in ASM(k)$ *and suppose that* $a_{11} = a_{kk} = 1$. *This implies* $a_{1k} = a_{k1} = 0$. *Let* $A' \in ASM(k)$ *be identical to* A *except on the corners where* $a'_{11} = a'_{kk} = 0$ *and* $a'_{1k} = a'_{k1} = 1$. *Let* $B = B_0(A)$ *and* $B' = B_0(A')$.

- *Let* $I, I' \in J(Q_k)$ *be the order ideals corresponding to* A *and* A' *respectively. Then* $I \subseteq I'$, *and* $I' \setminus I$ *contains exactly one element* (r, s, t) *for each* r, s *with* $|r| + |s| \leq k-2$ *and* $r + s \equiv k \pmod 2$.

- *Let $J, J' \in J(Q_{k+1})$ be the order ideals corresponding to B and B' respectively. Then $J \subseteq J'$, and $J' \setminus J$ contains exactly one element (r,s,t) for each r,s with $|r|+|s| \leq k-1$, $r+s \equiv k+1 \pmod 2$, and $s \neq \pm(k-1)$.*

Define the weight of an alternating sign matrix A with respect to an integer j to be

$$\mathrm{wt}(A,j) = \prod_{(r,s,t)\in I} y_{3r+s+j}$$

where $I \in J(Q_k)$ is the order ideal corresponding to A. By (1.7) we have

$$F_{j,k} = \sum_{A,B} \mathrm{wt}(A,j)\,\mathrm{wt}(B,j)$$

where the sum is over all compatible pairs $A \in ASM(k)$, $B \in ASM(k+1)$. Recall that $r+s \equiv k \pmod 2$ for $(r,s,t) \in Q_k$. Therefore $\mathrm{wt}(A,j)$ is a monomial in $\{y_i : i \equiv j+k \pmod 2\}$ while $\mathrm{wt}(B,j)$ is a monomial in $\{y_i : i \equiv j+k+1 \pmod 2\}$.

Proposition III.3.

$$F_{j,k} = \sum_{A\in ASM(k)} \mathrm{wt}(A,j)\,\mathrm{wt}(B_0(A),j) \prod_{a_{il}=1} (1+y_{j+2i+4l-3k-3}) \tag{3.1}$$

Proof. We need to show for each fixed $A \in ASM(k)$ that

$$\sum_B wt(B,j) = \mathrm{wt}(B_0(A),j) \prod_{a_{il}=1} (1+y_{j+2i+4l-3k-3})$$

where the sum is over B compatible with A. By Lemma III.1, both sides have 2^m terms where m is the number of 1's in A. Moreover, the lowest degree term of both sides is $\mathrm{wt}(B_0(A),j)$. Let J_0 be the order ideal corresponding to $B_0(A)$. If $a_{il}=1$ then Lemma III.1 says that it is possible to add some (r,s,t) with $r = i+l-k-1$ and $s = l-i$ to J_0 to get a new order ideal J corresponding to a matrix B compatible with A. Computing:

$$\begin{aligned}\mathrm{wt}(B,j) &= \mathrm{wt}(B_0(A),j)y_{3r+s+j}\\ &= \mathrm{wt}(B_0(A),j)y_{j+2i+4l-3k-3}\end{aligned}$$

By Lemma III.1, this can be done for arbitrary subsets of the set of 1's of A, so the proposition follows. $\square$

Let S be a set of integers. Say that a matrix $A \in ASM(k)$ *avoids* (S,j) if $j+2i+4l-3k-3 \notin S$ for all (i,l) with $a_{il}=1$. As i and l range from 1 to k, the index $j+2i+4l-3k-3$ takes on values in the k by k array

$$\begin{pmatrix} j-3k+3 & j-3k+7 & \cdots & j+k-1 \\ j-3k+5 & j-3k+9 & \cdots & j+k+1 \\ \vdots & \vdots & \ddots & \vdots \\ j-k+1 & j-k+5 & \cdots & j+3k-3 \end{pmatrix} \tag{3.2}$$

Hence, A avoids (S,j) if and only if the entries of A equal to 1 avoid the entries of this array contained in S.

Let S be a set of integers. Then we can use (3.1) to compute $F_{j,k}|_S$ by substituting $y_i=-1$ for all $i\in S$. If some $A\in ASM(k)$ does not avoid (S,j) then the corresponding term of the sum will have a factor $1+y_i$ for some $i\in S$. Hence this whole term will vanish in $F_{j,k}|_S$.

Corollary III.4. *Fix j and k, and let $S\subseteq\{i : i\equiv j+k+1 \pmod 2\}$. Then*

$$F_{j,k}|_S = \sum_A \mathrm{wt}(A,j)(\mathrm{wt}(B_0(A),j))|_S \prod_{a_{il}=1}(1+y_{j+2i+4l-3k-3}) \tag{3.3}$$

where the sum is over those $A\in ASM(k)$ which avoid (S,j). In particular, if no such A exists then $F_{j,k}|_S\equiv 0$.

Unfortunately, if there do exist matrices $A \in ASM(k)$ avoiding (S, j), then it is not safe to conclude that $F_{j,k}|_S \not\equiv 0$. Indeed, recall that $\mathrm{wt}(B_0(A), j)$ is a monomial in $\{y_i : i \equiv j + k + 1 \pmod 2\}$. When we substitute $y_i = -1$ for $i \in S$, a sign is introduced. It is therefore possible that different terms of (3.3) cancel with each other. We can at least conclude that $F_{j,k}|_S \not\equiv 0$ if there exists a unique A avoiding (S, j).

Proposition III.5. *Fix $l \in [-(k-1), k-1]_2$. Let $S \subseteq \mathbb{Z}$ be either*

1. $[j + l - 2(k-1), j + l + 2(k-1)]_4$ *or*
2. $[j + 2l - (k-1), j + 2l + (k-1)]_2$.

Then $F_{j,k}|_S \equiv 0$.

Proof. Suppose $S = [j+l-2(k-1), j+l+2(k-1)]_4$. Then S consists precisely of the elements of some row of (3.2). Every $A \in ASM(k)$ must have at least one entry equal to 1 in this row by the definition of alternating sign matrices. So by Corollary III.4, $F_{j,k}|_S \equiv 0$. Similarly $S = [j + 2l - (k-1), j + 2l + (k-1)]_2$ corresponds to a column of (3.2), so the same result holds. □

Lemma III.6. *Fix $l \in [-(k+1), k+1]_2$ and let*

$$\sigma = \begin{cases} 0, & k \text{ odd} \\ -2, & k \text{ even and } l + k \equiv 1 \pmod 4 \\ 2, & k \text{ even and } l + k \equiv 3 \pmod 4 \end{cases}$$

Then the set of entries of (3.2) contained in $\{j+l-2k, j+l+\sigma, j+l+2k\}$ corresponds to the 1's of a permutation matrix.

Proof. Consider a "knight's path" in a k by k matrix which starts at some entry in the first column and moves one column right and two rows up each step. Suppose the rows are ordered cyclically so whenever the knight passes the top of the matrix it wraps around to the bottom. Continue until the knight reaches the last column placing 1's everywhere it visits. One possible resulting matrix for $k = 7$ is:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The wrap arounds divide the path into three segments (one of which might have length zero). The entries of (3.2) are constant along each segment, and differ by $2k$ between consecutive segments. If k is odd then the result is a permutation matrix, and the 1's correspond to entries of the array equal to $j + l - 2k$, $j + l$, or $j + l + 2k$ for some l. One can check that the set of values of l that arise in this manner is precisely $[-(k + 1), k + 1]_2$.

If k is even then the resulting matrix will not be a permutation matrix, but one can be obtained by shifting the 1's of the middle segment either up or down by one row. For instance, in the following example with $k = 6$ the middle segment is shifted down:

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The effect on the corresponding entry of the array is accounted for by adding σ. $\square$

Proposition III.7. *Fix $l \in [-(k+1), k+1]_2$ and let*

$$S = [j-3k+3, j+3k-3]_2 \setminus \{j+l-2k, j+l+\sigma, j+l+2k\}$$

for σ as in Lemma III.6. Let A be the permutation matrix from the conclusion of Lemma III.6. Then

$$F_{j,k}|_S = \mathrm{wt}(A,j)(\mathrm{wt}(B_0(A),j))|_S(1+y_{j+l-2k})^a(1+y_{j+l+\sigma})^b(1+y_{j+l+2k})^c$$

for some nonnegative integers a, b, c with $a+b+c=k$. In particular, $F_{j,k}|_S \not\equiv 0$.

Proof. By Lemma III.6, all entries of (3.2) are contained in S except those corresponding to the 1's of A. On the one hand, this means that A avoids (S, j). On the other hand, let $A' \in ASM(k)$ be any matrix avoiding (S, j). Each column of A' must have at least one entry equal to 1, and 1's can only occur away from elements of S. So each column of A' must have exactly one 1 and it must be in the same place as a 1 of A. As each column of A' has only a single 1, there cannot be any -1's, so in fact $A' = A$. Hence A is the only element of $ASM(k)$ avoiding (S, j). The proposition follows from Corollary III.4, where a, b, and c are the sizes of the three segments of the knight's path as defined in the proof of Lemma III.6. $\square$

Corollary III.8. *Fix $l \in [-(k+1), k+1]_2$ and let S be any of*

1. $S = [j + l - 2k + 4, j + l + 2k - 4]_4$,

2. $S = [j + l - 2k + 2, j + l + \sigma - 2]_2$, *or*

3. $S = [j + l + \sigma + 2, j + l + 2k - 2]_2$.

for σ as above. Then $F_{j,k}|_S \not\equiv 0$.

Proof. Each S is contained in the corresponding one from Proposition III.7. As fewer substitutions are made, $F_{j,k}|_S$ remains nonzero. □

Corollary III.9. *Let S be a finite arithmetic sequence such that*

- $|S| < k$,

- *consecutive terms of S differ by 4 or 2, and*

- *the elements of S have the same parity as $j + k + 1$.*

Then $F_{j,k}|_S \not\equiv 0$.

Proof. We may assume without loss of generality that $S \subseteq [j-3(k-1), j+3(k-1)]_1$, as those are the indices of the only y-variables that $F_{j,k}$ depends on. In this case, S is contained in some S from Corollary III.8 so we still have $F_{j,k}|_S \not\equiv 0$. □

Proposition III.5 and Corollary III.9 give a complete picture as to when $F_{j,k}|_S \equiv 0$ for S of the form $[a, b]_4$ or $[a, b]_2$. We will use these results to prove confinement for certain singularity types in the next section. There, we will assume that n is large relative to $|S|$ so that the relations among the y-variables do not enter into play. In contrast, the following proposition pertains to a more severe singularity type, so we will reintroduce those relations at this point.

Proposition III.10. *Suppose that n is odd and $S = [2, 2n-2]_2$. Assume that $y_{i+2n} = y_i$ for all $i \in \mathbb{Z}$, $y_i = -1$ for all $i \in S$, and $y_1 \cdots y_{2n} = 1$. Let $j, k \in \mathbb{Z}$ with $j+k$ odd and $k \in \{n, n+1\}$. Then evaluated at this input, $F_{j,k} \neq 0$ provided $y_0 \neq -1$ and $y_i \neq 0$ for all i.*

Proof. First suppose that $k = n$, which implies that k is odd and j is even. Then in Proposition III.7, $\sigma = 0$. Moreover, we can choose $l \in [-(k+1), (k+1)]_2$ such that $l \equiv -j \pmod{2n}$. So

$$j + l - 2k \equiv j + l + \sigma \equiv j + l + 2k \equiv 0 \pmod{2n}$$

By Proposition III.7 we have

$$F_{j,k} = M(1 + y_0)^n$$

for some monomial M in $y_1, \ldots, y_{2n}$. Since $y_0 \neq -1$ and no y_i is zero, we have that $F_{j,k} \neq 0$.

Now let $k = n+1$, in which case k is even and j is odd. Since $y_{i+2n} = y_i$ for all i, we can consider the entries of (3.2) modulo $2n$. For example, if $n = 7$, $k = 8$, and $j = 3$ the result is

$$\begin{pmatrix}
10 & 0 & 4 & 8 & 12 & 2 & 6 & 10 \\
12 & 2 & 6 & 10 & 0 & 4 & 8 & 12 \\
0 & 4 & 8 & 12 & 2 & 6 & 10 & 0 \\
2 & 6 & 10 & 0 & 4 & 8 & 12 & 2 \\
4 & 8 & 12 & 2 & 6 & 10 & 0 & 4 \\
6 & 10 & 0 & 4 & 8 & 12 & 2 & 6 \\
8 & 12 & 2 & 6 & 10 & 0 & 4 & 8 \\
10 & 0 & 4 & 8 & 12 & 2 & 6 & 10
\end{pmatrix}$$

Since n is odd, each row and each column contains each of $0, 2, 4, \ldots, 2n-2$ with a single repeat, namely the first and last entry. We are interested in matrices $A \in ASM(k)$ whose 1's avoid S and hence all correspond to elements of the array equal to 0. There are two cases. If $j \neq n$, then the corner entries of the array are nonzero. This is the situation in the above example. There will always be a single row and column in the interior of the array that each start and end with 0. Every other row and column will have a single 0. There is a unique alternating sign matrix A avoiding (S, j) in this case. It has 1's everywhere there is a 0 in the array, and a single -1 where needed. The matrix A corresponding to the above example is

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

This matrix will always have $n+2$ entries equal to 1, so by Corollary III.4 we have

$$F_{j,k} = M(1+y_0)^{n+2} \neq 0$$

where M is a monomial.

Lastly, if $j = n$ (still assuming $k = n+1$) then the corner entries of the array all equal 0. In the case $n = 7$ this looks like

$$\begin{pmatrix} 0 & 4 & 8 & 12 & 2 & 6 & 10 & 0 \\ 2 & 6 & 10 & 0 & 4 & 8 & 12 & 2 \\ 4 & 8 & 12 & 2 & 6 & 10 & 0 & 4 \\ 6 & 10 & 0 & 4 & 8 & 12 & 2 & 6 \\ 8 & 12 & 2 & 6 & 10 & 0 & 4 & 8 \\ 10 & 0 & 4 & 8 & 12 & 2 & 6 & 10 \\ 12 & 2 & 6 & 10 & 0 & 4 & 8 & 12 \\ 0 & 4 & 8 & 12 & 2 & 6 & 10 & 0 \end{pmatrix}$$

In general, there will be two alternating sign matrices avoiding (S, j), both permutation matrices. For $n = 7$ they are

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Call these A and A' respectively and let $B = B_0(A)$, $B' = B_0(A')$. By Corollary III.4 we have

$$F_{j,k} = (\mathrm{wt}(A, j)\,\mathrm{wt}(B, j) + \mathrm{wt}(A', j)\,\mathrm{wt}(B', j))(1 + y_0)^{n+1}$$

Now, Lemma III.2 determines how the order ideals corresponding to A, A', B, and B' relate to each other. Using this result, one can check that the weights are related by

$$\mathrm{wt}(A', j) = (y_1 y_3 \cdots y_{2n-1})^n \, \mathrm{wt}(A, j)$$

$$\mathrm{wt}(B', j) = y_0^{n+1} (y_2 y_4 \cdots y_{2n-2})^{n+2} \, \mathrm{wt}(B, j)$$

Consequently,

$$F_{j,k} = M(1 + (y_0 y_1 \cdots y_{2n-1})^n y_0 (y_2 y_4 \cdots y_{2n-2})^2)(1 + y_0)^{n+1}$$

where $M = \mathrm{wt}(A, j)\,\mathrm{wt}(B, j)$. Now, $y_0 y_1 y_2 \cdots y_{2n-1} = 1$ and $y_2 = y_4 = \ldots = y_{2n-2} = -1$ so

$$F_{j,k} = M(1 + y_0)(1 + y_0)^{n+1} = M(1 + y_0)^{n+2} \neq 0$$

□

3.2 Singularity patterns

We are now ready to prove our main results on singularity confinement. Fix n to be the number of sides of the polygons being considered. Beginning with Theorem I.5, suppose that $1 \leq m < n/3 - 1$ and

$$S = [i - (m-1), i + (m-1)]_2$$

$$S' = \left[i - \frac{1}{2}(m-1), i + \frac{1}{2}(m-1)\right]_1$$

Lemma III.11. *$F_{j,m}|_{2S} \equiv 0$ for all $j \in 2S'$.*

Proof. Suppose $j \in 2S'$. Then $j = 2i - l$ for some $l \in [-(m-1), m-1]_2$. Hence $2S = [j + l - 2(m-1), j + l + 2(m-1)]_4$. So by Proposition III.5 we have that $F_{j,m}|_{2S} \equiv 0$. □

Lemma III.12. *$F_{j,k}|_{2S} \not\equiv 0$ for all $j, k \in \mathbb{Z}$ with $k \in \{m+1, m+2\}$ and $j + k$ odd.*

Proof. This would seem to follow immediately from Corollary III.9. The only difficulty is that the assumption $y_{i+2n} = y_i$ was relaxed in that section. Recall that $F_{j,k}$ depends only on those y_i for $i \in [j-3(k-1), j+3(k-1)]_1$, a total of $6k-5$ consecutive variables. We are assuming that $k \leq m+2 < n/3+1$ so $6k-5 \leq 2n$. It follows that assuming $y_{i+2n} = y_i$ has no effect on the y-variables appearing in $F_{j,k}$. □

Proof of Theorem I.5. By Lemma III.11 and (2.11) we have that T^{m+1} and T^m are singular on X_S. This same lemma applied to smaller m implies that all T^k with $k < m$ are also singular on X_S.

Now let $k = m+2$. Then Lemma III.12 shows that none of the factors in the expression (2.11) for T^k are identically zero. Hence (2.11) defines T^{m+2} generically on X_S. It remains to show that the image is in $Y_{S'}$. Let $A \in X_S$ be such that $B = T^{m+2}(A)$ is defined and let $y_j = y_j(A)$ for all $j \in \mathbb{Z}$. Let $j \in S'$ be given. By Lemma III.11, $F_{2j,m} = 0$. Therefore by (2.12)

$$0 = F_{2j-3,m+1}F_{2j+3,m+1} + M_{2j,m+1}F_{2j-1,m+1}F_{2j+1,m+1}$$

Rearranging and using (1.6)

$$-1 = (M_{2j,m+1})^{-1}\frac{F_{2j-3,m+1}F_{2j+3,m+1}}{F_{2j-1,m+1}F_{2j+1,m+1}} = y_{2j}(B)$$

This is justified because $T^{m+2}(A)$ is defined so the factors being divided by are nonzero. We have $y_{2j}(B) = -1$ for all $j \in S'$ so $B \in Y_{S'}$ as desired. □

The roles of S and S' can be interchanged in Theorem I.5. This is apparently an instance of projective duality.

Theorem III.13. *Let S and S' be as in Theorem I.5. Then the map T^k is singular on $X_{S'}$ for $1 \leq k \leq m+1$, but T^{m+2} is nonsingular at generic $A \in X_{S'}$. Moreover,*

$T^{m+2}(A) \in Y_S$ *for such* A.

Proof. The proof is essentially obtained by switching S and S' throughout in the proofs of Theorem I.5 and its lemmas. The only difference is that the proof now utilizes different cases of Proposition III.5 and Corollary III.9, namely the cases involving arithmetic sequences with common difference 2. $\square$

We now have singularity confinement on X_S for S an arithmetic sequence whose terms differ by 1 or 2. Generally, if S is a disjoint union of such sequences which are far apart from each other, then the corresponding singularities do not affect each other. Hence singularity confinement holds and the number of steps needed to get past the singularity is dictated by the length of the largest of the disjoint sequences.

Not all singularity types are of this form. For instance, consider $S = \{3, 4, 7, 8\}$. By Theorem III.13, T^4 is defined generically on both $X_{\{3,4\}}$ and $X_{\{7,8\}}$. One can check that T^4 is singular on X_S, although another step does suffice to move past the singularity. For general types S, it is difficult to predict how many steps the corresponding singularities last. However, it would seem that singularity confinement does hold outside of some exceptional cases.

If n is odd, the only exceptional type is $S = [1, n]_1$. Moreover, for any other S and generic $A \in X_S$ the corresponding singularity lasts at most n steps. We establish this by considering the worst case where $|S| = n - 1$.

Proposition III.14. *Suppose* n *is odd and let* $S = [1, n]_1 \setminus \{i\}$ *for some* $i \in [1, n]_1$. *Then* T^{n+1} *is nonsingular at generic* $A \in X_S$.

Proof. Suppose without loss of generality that $i = n$. Then $y_2 = y_4 = \ldots = y_{2n-2} = -1$. As always, we have $y_{j+2n} = y_j$ for all j and $y_1 y_2 \cdots y_{2n} = 1$. So Proposition III.10 implies that, generically, $F_{j,k} \neq 0$ for $j + k$ odd and $k \in \{n, n+1\}$. Therefore,

T^{n+1} is generically defined by (2.11). $\square$

Remark III.15. In fact, Proposition III.10 says more, namely that the relevant F-polynomials never vanish unless $y_0 = -1$ or some $y_i = 0$. The assumption that quadruples of consecutive vertices be in general position forces all of the y_i to be nonzero. As such, we have that the only singularities of T^{n+1} on X_S occur when $y_{2n} = y_0 = -1$. Hence, T^{n+1} restricts to a regular map on $X_S \setminus X_{[1,n]_1}$.

Proof of Theorem I.6. Suppose n is odd and $S \subsetneq [1,n]_1$. Then there exists some S' such that $S \subseteq S' \subsetneq [1,n]_1$ and $|S'| = n-1$. Now T^{n+1} is nonsingular at generic $A \in X_{S'}$ by Proposition III.14. In particular, the map is defined at some such A, which is necessarily also in X_S. It follows that T^{n+1} is defined generically on X_S. Letting k be minimal such that T^k is defined for generic $A \in X_S$, we get that $k \leq n+1$. $\square$

Conjecture III.16. *Suppose that n is even.*

- *Singularity confinement holds generically on X_S unless $[1, n-1]_2 \subseteq S$ or $[2,n]_2 \subseteq S$.*
- *Whenever singularity confinement holds for a type, there exists an $m \leq n$ such that generic singularities of that type last m steps (i.e. T^m is singular but T^{m+1} is not).*

Remark III.17. The cases where singularity confinement fails to hold are quite extreme. If $S = [1,n]_1$ then $A \in X_S$ has all its vertices lying on two lines. It follows that all the vertices of $T(A)$ are equal. If n is even and say S contains $[1, n-1]_2$ then half the vertices of $A \in X_S$ are collinear and $T(A)$ will be contained in the common line. Remarkably, experiments suggest that if S is an exceptional type then there exist $k > 0$ and S' such that $(T^{-1})^k$ maps X_S to $Y_{S'}$. Theorems I.3 and I.4 are two partial results to this effect.

3.3 Straightedge constructions: a first attempt

Let $A \in \mathcal{P}_n$ be a singular point of T^k for $1 \leq k < m$ but not of T^m. The remainder of this chapter focuses on the problem of constructing $B = T^m(A)$.

One possible approach would be to compute the x-coordinates of A, plug into (2.11) and (1.7) to find the x-coordinates of B, and then use these to construct B itself. This process would be computationally expensive as the number of terms of $F_{j,k}$ grows superexponentially with k. More to the point, this approach has the drawback that it ignores the geometry of the pentagram map.

As an alternative, we could choose a one-parameter family $A(t)$ of twisted polygons varying continuously with t such that

1. $A(0) = A$ and
2. $A(t)$ is a regular point of T^k for all $t \neq 0$ and $k \leq m$.

For small $t \neq 0$, we can obtain $B(t) = T^m(A(t))$ by iterating the geometric construction defining T. By continuity, B is given by $\lim_{t \to 0} B(t)$ which can be found numerically. This method is perhaps more feasible, but it involves a limiting procedure. More satisfying would be a finite construction, preferably one which can be carried out with a straightedge alone, as is the case with the pentagram map.

In this section we introduce an iterative approach to finding such a straightedge construction, which works in simple situations. The idea is to attempt to make sense of the polygon $T^k(A)$ for $k < m$ despite the presence of the singularity. Let $A(t)$ be as above, and fixing $k < m$, let $C(t) = T^k(A(t))$. For each appropriate index i, let

$$C_i = \lim_{t \to 0}(C_i(t))$$

We say that C_i is well-defined if this limit always exists and is independent of the choice of the curve $A(t)$ through A. We can define sides c_j of $T^k(A)$ in the same way. In fact it is possible that each of the C_i and c_j are well-defined, despite the singularity. This would simply indicate that the resulting polygon C fails to satisfy the property that quadruples of consecutive vertices be in general position, which is needed for all the x-coordinates to be defined.

As before, suppose $A \in \mathcal{P}_n$ is a singular point of T^k for $1 \leq k < m$ but not of T^m. In addition, assume that all of the vertices and sides of $T^k(A)$ for $1 \leq k < m$ are well-defined. Then it should be possible to construct the components of these intermediate polygons successively. Ideally, each individual side or vertex can be constructed by a simple procedure depending only on nearby objects.

The most basic of these local rules is the usual definition of the pentagram map, namely, if $B = T(A)$ then

$$b_i = \overleftrightarrow{A_{i-1}A_{i+1}}$$

for each index i of A and

$$B_j = b_{j-\frac{1}{2}} \cap b_{j+\frac{1}{2}}$$

for each index j of B. These rules only work when $A_{i-1} \neq A_{i+1}$ and $b_{j-\frac{1}{2}} \neq b_{j+\frac{1}{2}}$ respectively. Other rules are needed to handle other cases. The next simplest rule involves triple ratios which are a six point analogue of cross ratios.

Definition III.18. Let A, B, C, D, E, F be points in the plane with A, B, C collinear, C, D, E collinear, and E, F, A collinear. The *triple ratio* of these points is defined to be

$$[A, B, C, D, E, F] = \frac{AB}{BC}\frac{CD}{DE}\frac{EF}{FA}$$

where for instance, $\frac{AB}{BC}$ refers to the ratio of these two lengths, taken to be positive if B lies between A and C and negative otherwise.

It will be convenient to have an analogous definition and notation for cross ratios, so define

$$[A, B, C, D] = \frac{AB}{BC}\frac{CD}{DA}$$

where A, B, C, D are collinear points in the plane. This agrees with the previous definition of cross ratio taken in a different order:

$$[A, B, C, D] = \chi(B, A, C, D)$$

We will need to following properties of triple ratios, which can be found for instance in [RG11].

Proposition III.19.

- *Triple ratios are invariant under projective transformations.*
- *(Ceva's theorem) If the lines* $\overleftrightarrow{AD}$, $\overleftrightarrow{CF}$, *and* $\overleftrightarrow{EB}$ *are concurrent then*

$$[A, B, C, D, E, F] = 1$$

- *(Menelaus' theorem) If* B, D, *and* F, *are collinear then*

$$[A, B, C, D, E, F] = -1$$

Proposition III.20. *Suppose* $A \in \mathcal{P}_n$ *is a regular point of* T *and* T^2, *and let* $B = T(A)$, $C = T^2(A)$. *Then for all* $i \in \mathbb{Z}$

$$[B_{i-\frac{3}{2}}, B_{i-\frac{1}{2}}, A_i, B_{i+\frac{3}{2}}, B_{i+\frac{1}{2}}, C_i] = -1$$

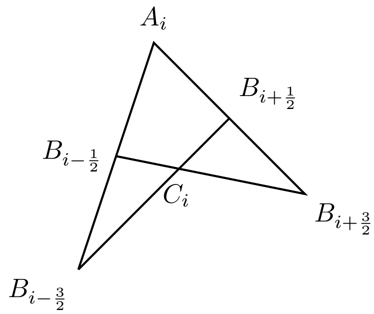

Figure 3.1: By Menelaus' theorem, $[B_{i-\frac{3}{2}}, B_{i-\frac{1}{2}}, A_i, B_{i+\frac{3}{2}}, B_{i+\frac{1}{2}}, C_i] = -1$ for any index i of A.

Proof. That the triple ratio makes sense, and that it satisfies the condition of Menelaus' theorem, are both clear from Figure 3.1. □

Now imagine continuously deforming the polygons until all six of these points are collinear. This relation holds as the polygons are being deformed, so it continues to hold in the limit. In particular, if five of these points are well-defined and collinear, then generically the sixth is also well-defined and is the unique point on the line for which the relation holds. Stated as a rule, if $B = T(A)$, $C = T(B)$, and if $A_i, B_{i-\frac{3}{2}}, B_{i-\frac{1}{2}}, B_{i+\frac{1}{2}}, B_{i+\frac{3}{2}}$ are collinear for some index i of A then

$$C_i = \texttt{TripleConjugate}(B_{i-\frac{3}{2}}, B_{i-\frac{1}{2}}, A_i, B_{i+\frac{3}{2}}, B_{i+\frac{1}{2}}) \tag{3.4}$$

Here, `TripleConjugate` is a function that inputs five collinear points P_1, P_2, P_3, P_4, P_5 and outputs the unique point P_6 on the common line such that $[P_1, P_2, P_3, P_4, P_5, P_6] = -1$.

The ordered sextuple of sides $a_j, b_{j-\frac{3}{2}}, b_{j+\frac{1}{2}}, b_{j-\frac{1}{2}}, b_{j+\frac{3}{2}}, c_j$ satisfy the same incidences as the vertices $A_i, B_{i-\frac{3}{2}}, B_{i-\frac{1}{2}}, B_{i+\frac{1}{2}}, B_{i+\frac{3}{2}}, C_i$ (see Figure 3.2). So we get the analogous rule, that if for some index j of B the sides $a_j, b_{j-\frac{3}{2}}, b_{j+\frac{1}{2}}, b_{j-\frac{1}{2}}, b_{j+\frac{3}{2}}$ are all concurrent then

$$c_j = \texttt{TripleConjugate}(b_{j-\frac{3}{2}}, b_{j+\frac{1}{2}}, a_j, b_{j+\frac{3}{2}}, b_{j-\frac{1}{2}}) \tag{3.5}$$

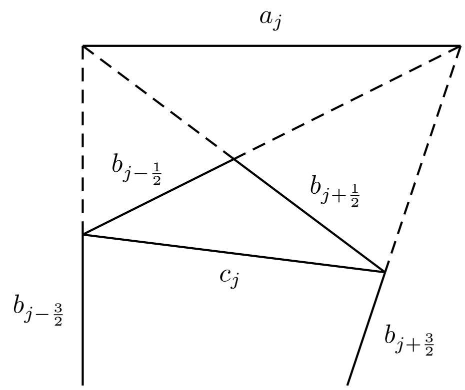

Figure 3.2: A configuration of lines dual to the configuration of points in Figure 3.1
.

These rules are already enough to handle singularities of the simplest type. The full construction is described in the next subsection. The following subsection explains the difficulty in handling more complicated singularities.

3.3.1 The map $T^3 : X_3 \to Y_3$

The case $m = 1$ of Theorem I.5 says that T^3 restricts to a rational map from X_i to Y_i. Assume without loss of generality that $i = 3$. Throughout this subsection, assume $A \in X_3$ and let $B = T(A)$, $C = T^2(A)$, and $D = T^3(A)$. Since $A \in X_3$ we have that A_1, A_3, A_5 are collinear, so let l denote the line containing them. In the following, i and j will denote elements of $\mathbb{Z}$ and $\frac{1}{2} + \mathbb{Z}$ respectively.

We start by constructing B, which can be done via the usual pentagram map. So construct $b_i = \overleftrightarrow{A_{i-1}A_{i+1}}$ for all i and $B_j = b_{j-\frac{1}{2}} \cap b_{j+\frac{1}{2}}$ for all j. Note that $b_2 = b_4 = l$. Therefore, $B_{1.5}, B_{2.5}, B_{3.5}, B_{4.5}$ all lie on l and moreover

$$B_{2.5} = B_{3.5} = l \cap b_3$$

Let P be this common point. The construction of B is shown in Figure 3.3.

Generically, $B_{j-1} \neq B_{j+1}$ for all j, so the sides of C can all be constructed as $c_j = \overleftrightarrow{B_{j-1}B_{j+1}}$. Note that $c_{2.5} = c_{3.5} = l$. Hence, we cannot use $c_{2.5} \cap c_{3.5}$ to

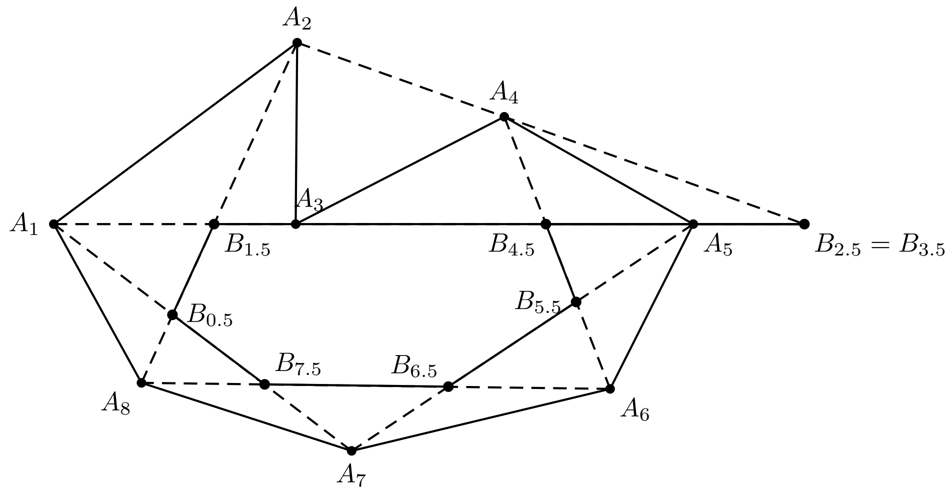

Figure 3.3: The construction of $B = T(A)$ from A for $A \in X_3$

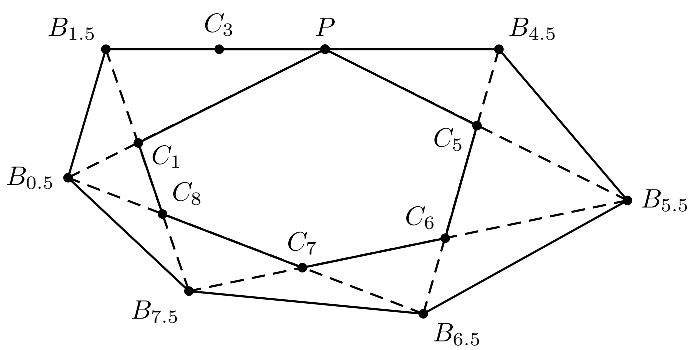

Figure 3.4: The construction of $C = T^2(A)$ from $B = T(A)$ for $A \in X_3$. Here, $P = B_{2.5} = B_{3.5} = C_2 = C_4$.

construct C_3. However, $A_3, B_{1.5}, B_{2.5}, B_{3,5}, B_{4.5}$ all lie on l so the rule (3.4) applies:

$$C_3 = \texttt{TripleConjugate}(B_{1.5}, B_{2.5}, A_3, B_{4.5}, B_{3.5})$$

As usual, $C_i = \overleftrightarrow{c_{i-\frac{1}{2}} c_{i+\frac{1}{2}}}$ for all $i \neq 3$. In particular we have

$$C_2 = c_{1.5} \cap c_{2.5} = \overleftrightarrow{B_{0.5}B_{2.5}} \cap l = B_{2.5} = P$$

$$C_4 = c_{3.5} \cap c_{4.5} = l \cap \overleftrightarrow{B_{3.5}B_{5.5}} = B_{3.5} = P$$

Figure 3.4 shows the construction of C from B.

The last difficulty is in constructing the side d_3 since $C_2 = C_4$. However, $b_3, c_{1.5}, c_{2.5}, c_{3.5}, c_{4.5}$

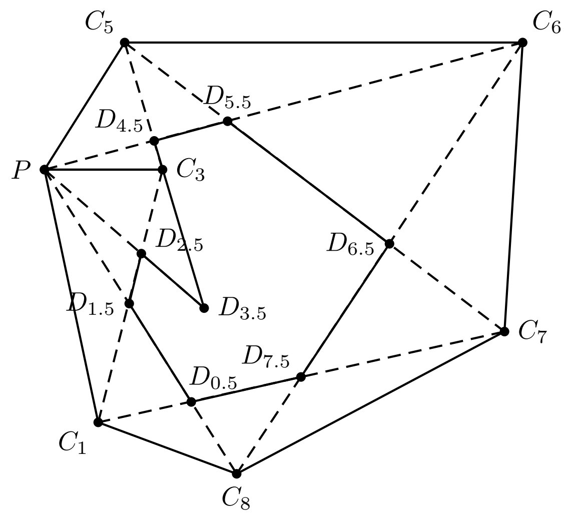

Figure 3.5: The construction of $D = T^3(A)$ from $C = T^2(A)$ for $A \in X_3$. Here, $P = C_2 = C_4$.

all pass through P so by (3.5)

$$d_3 = \texttt{TripleConjugate}(c_{1.5}, c_{3.5}, b_3, c_{4.5}, c_{2.5})$$

In particular, d_3 contains P. Letting $d_i = C_{i-1} \cap C_{i+1}$ for all $i \neq 3$, we have that d_1 contains $C_2 = P$ and d_5 contains $C_4 = P$ as well. This verifies that $D \in Y_3$. Finally, the vertices of D are constructed as $D_j = d_{j-\frac{1}{2}} \cap d_{j+\frac{1}{2}}$ for all j. The construction of D from C is given in Figure 3.5. The full construction of D from A is summarized in Algorithm 1.

Remark III.21. All algorithms in this paper can be carried out as straightedge constructions. When this is not completely apparent from the algorithm itself, more details are provided in the surrounding text and/or in the appendix. For example, Algorithm 1 uses the function `TripleConjugate`, a construction for which is given in Algorithm 5 of the appendix. In addition to the usual operations of projective geometry (finding a line through two points or the intersection point of two lines) we assume as a primitive a function `RandomPoint`() which returns the next in an arbitrarily long sequence $P_1, P_2, \ldots$ of points in the plane. All algorithms are only

Algorithm 1 $T^3(A)$

Require: A_1, A_3, A_5 collinear
  for all i **do**
    $b_i := \overleftrightarrow{A_{i-1}A_{i+1}}$
  end for
  for all j **do**
    $B_j := b_{j-\frac{1}{2}} \cap b_{j+\frac{1}{2}}$
  end for
  for all j **do**
    $c_j := \overleftrightarrow{B_{j-1}B_{j+1}}$
  end for
  for all $i \neq 3$ **do**
    $C_i := c_{i-\frac{1}{2}} \cap c_{i+\frac{1}{2}}$
  end for
  $C_3 :=$ `TripleConjugate`$(B_{1.5}, B_{2.5}, A_3, B_{4.5}, B_{3.5})$
  for all $i \neq 3$ **do**
    $d_i := \overleftrightarrow{C_{i-1}C_{i+1}}$
  end for
  $d_3 :=$ `TripleConjugate`$(c_{1.5}, c_{3.5}, b_3, c_{4.5}, c_{2.5})$
  for all j **do**
    $D_j := d_{j-\frac{1}{2}} \cap d_{j+\frac{1}{2}}$
  end for
  return D

claimed to behave correctly for generic choices of these points. For convenience, define

- `RandomLine()` $:= \overleftrightarrow{\texttt{RandomPoint()RandomPoint()}}$
- `RandomPointOn`(l) $:=$ `RandomLine()` $\cap\, l$
- `RandomLineThrough`(P) $:= \overleftrightarrow{\texttt{RandomPoint()}P}$

3.3.2 The map $T^4 : X_{\{3,5\}} \to Y_{\{3.5,4.5\}}$

The next simplest case, $m = 2$, of Theorem I.5 concerns a singularity which disappears after four steps. Specifically, taking $i = 4$ there is a map $T^4 : X_{\{3,5\}} \to Y_{\{3.5,4.5\}}$. Suppose $A \in X_{\{3,5\}}$ which means that A_1, A_3, A_5, and A_7 are collinear. Let l be their common line. Then $E = T^4(A) \in Y_{\{3.5,4.5\}}$, i.e., $e_{1.5}, e_{3.5}, e_{5.5}$ are concurrent and $e_{2.5}, e_{4.5}, e_{6.5}$ are also concurrent. As before, we will attempt to successively construct the intermediate polygons, namely $B = T(A)$, $C = T^2(A)$, and $D = T^3(A)$.

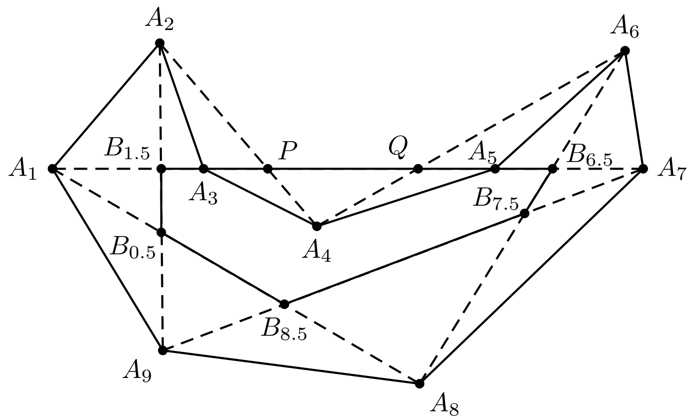

Figure 3.6: The construction of $B = T(A)$ from A for $A \in X_{\{3,5\}}$. Here, $P = B_{2.5} = B_{3.5}$ and $Q = B_{4.5} = B_{5.5}$.

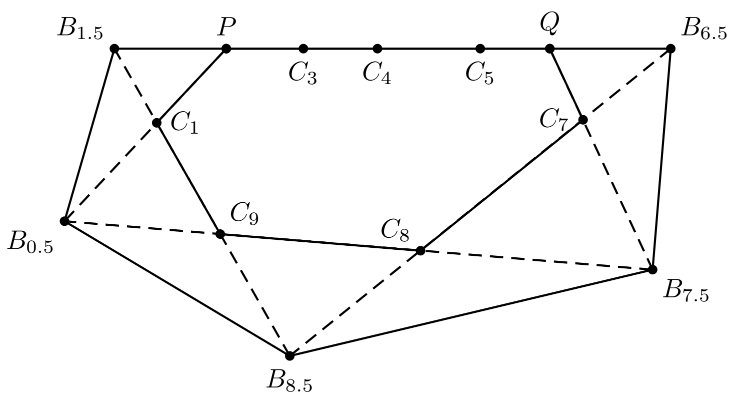

Figure 3.7: The construction of $C = T^2(A)$ from $B = T(A)$ for $A \in X_{\{3,5\}}$. Here, $P = B_{2.5} = B_{3.5} = C_2$ and $Q = B_{4.5} = B_{5.5} = C_6$.

However, in this case not all of these polygons will be completely well-defined. Again, let i and j range over $\mathbb{Z}$ and $\frac{1}{2} + \mathbb{Z}$ respectively. The constructions that follow are illustrated in Figures 3.6–3.9.

As before, B can be constructed using the standard pentagram map. In this case, it will have three sides equal to l, namely b_2, b_4, and b_6. As a result, the six consecutive vertices $B_{1.5}, B_{2.5}, \ldots, B_{6.5}$ will all lie on l. Moreover, we have

$$B_{2.5} = B_{3.5} = l \cap b_3$$

$$B_{4.5} = B_{5.5} = l \cap b_5$$

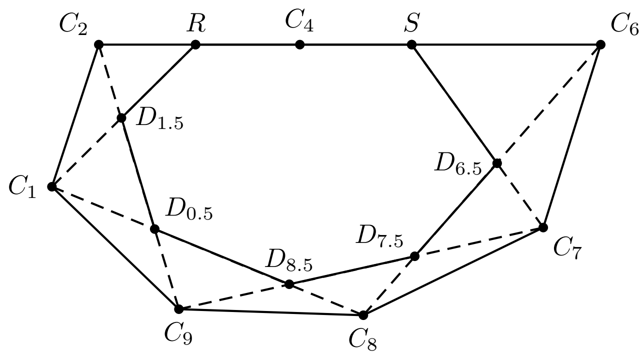

Figure 3.8: The construction of $D = T^3(A)$ from $C = T^2(A)$ for $A \in X_{\{3,5\}}$. Here, $R = C_3 = D_{2.5} = D_{4.5}$ and $S = C_5 = D_{3.5} = D_{5.5}$.

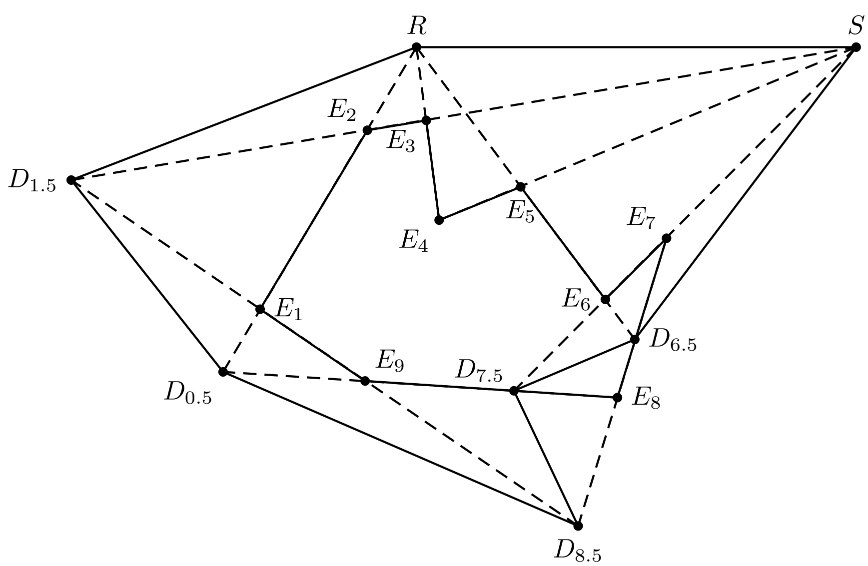

Figure 3.9: The construction of $E = T^4(A)$ from $D = T^3(A)$ for $A \in X_{\{3,5\}}$. Here, $R = D_{2.5} = D_{4.5}$ and $S = D_{3.5} = D_{5.5}$.

The sides $c_j = \overleftrightarrow{B_{j-1}B_{j+1}}$ of C are all defined, although $c_{2.5}, c_{3.5}, c_{4.5}, c_{5.5}$ all equal l. So the first problems arise in constructing the vertices C_3, C_4, and C_5. In general, C_3 only depends on vertices A_0 through A_6 of the original polygon, so the assumption $A_7 \in l$ is irrelevant for its construction. This puts us in the context of the previous subsection, so as there we have

$$C_3 = \texttt{TripleConjugate}(B_{1.5}, B_{2.5}, A_3, B_{4.5}, B_{3.5})$$

By symmetry there is a similar construction of C_5:

$$C_5 = \texttt{TripleConjugate}(B_{3.5}, B_{4.5}, A_5, B_{6.5}, B_{5.5})$$

The situation with C_4 is more complicated. Ordinarily, we would use the fact

$$[B_{2.5}, B_{3.5}, A_4, B_{5.5}, B_{4.5}, C_4] = -1$$

and solve for C_4. However, $B_{2.5} = B_{3.5}$ and $B_{5.5} = B_{4.5}$ so the triple ratio comes out to $0/0$. As such, we can not use this method to construct C_4. In fact, it turns out that C_4 is simply not well-defined.

The fact that an intermediate vertex is not well-defined causes great difficulty in the current approach to devising straightedge constructions. In the following sections, we demonstrate how enriching the input A with first-order data counteracts this difficulty and leads to a general algorithm. Before moving on, we finish describing a construction particular to the present context which works around the matter of C_4.

Recall that sides $c_{2.5}$ through $c_{5.5}$ of C all equal l, so its vertices C_2 through C_6 all lie on this line. As such, we know $d_3 = \overleftrightarrow{C_2C_4} = l$ and $d_5 = \overleftrightarrow{C_4C_6} = l$ even though C_4 is itself not well-defined. The rest of the sides of D are constructed similarly, and of note d_4 also equals l. The other sides are all generic so we can construct $D_j = d_{j-\frac{1}{2}} \cap d_{j+\frac{1}{2}}$ for all j other than 3.5 and 4.5. For these two vertices, we work backwards. We know ultimately that $e_{1.5}, e_{3.5}, e_{5.5}$ will be concurrent. But $e_{1.5} \cap e_{3.5} = D_{2.5}$ and $e_{3.5} \cap e_{5.5} = D_{4.5}$ so $D_{4.5} = D_{2.5}$. Similarly, the fact that $e_{2.5}, e_{4.5}, e_{6.5}$ are concurrent implies that $D_{3.5} = D_{5.5}$.

For the final step, let $e_j = \overleftrightarrow{D_{j-1}D_{j+1}}$ for all j besides 3.5 and 4.5. The usual construction fails for $e_{3.5}$ because $D_{2.5} = D_{4.5}$ and the construction involving triple conjugates also fails because $d_3 = d_4 = d_5$. However, $e_{3.5}$ certainly is well-defined as it is a side of $E = T^4(A) \in \mathcal{P}_n$. Through trial and error we discovered the following construction for $e_{3.5}$, and by symmetry, one for $e_{4.5}$.

Proposition III.22. *Under the assumptions of this subsection*

$$e_{3.5} = \overleftrightarrow{(\overleftrightarrow{(b_5 \cap d_2)C_5} \cap c_{6.5})C_3}$$
$$e_{4.5} = \overleftrightarrow{(\overleftrightarrow{(b_3 \cap d_6)C_3} \cap c_{1.5})C_5}$$

Remark III.23. In principle, results like Proposition III.22 can be proven computationally. In instances for which we are unaware of a more illuminating proof, we will simply fall back on this sort of reasoning. The computations required are generally quite tedious, so we will tend to omit them.

3.4 Decorated polygons

Let A be a twisted polygon which is a singular point of T^k. As explained in the previous section, we can attempt to define $T^k(A)$ as a limit of $T^k(A(t))$ where $A(t)$ is a curve in the space of polygons passing through $A = A(0)$. As we saw in Section 3.3.2, the result sometimes depends on the choice of the curve. This suggests a different approach to constructing the first nonsingular iterate $T^m(A)$. Start by fixing arbitrarily the one-parameter family $A(t)$. With respect to this choice the intermediate polygons $T^k(A)$ are well-defined. Constructing them in turn we eventually get $T^m(A)$. Since A is not a singular point of T^m, the final result will not depend on the choice of $A(t)$.

Working with actual curves would be difficult. However, all that will actually matter will be the first order behavior of the curve near $t = 0$. This information can be encoded using geometric data which we call decorations.

Let A be a point in the projective plane, and let γ be a smooth curve with $\gamma(0) = A$. Define the associated *decoration* of A, denoted A^*, to be the tangent line of γ at A:

$$A^* = \lim_{t\to 0} \overleftrightarrow{A\gamma(t)}$$

When defined, A^* is a line passing through A.

By the same token, if a is a line in the projective plane then a can be thought of as a point in the dual plane. Given a curve γ through that point we can define the *decoration* a^* as

$$a^* = \lim_{t\to 0} a \cap \gamma(t)$$

When defined, a^* is a point lying on a.

Finally, let A be a twisted polygon and γ a curve in the space of twisted polygons with $\gamma(0) = A$. Then γ determines a curve in the plane through each vertex of A and a curve in the dual plane through each side of A. By the above, we can define decorations on each of these individual objects.

Definition III.24. A *decorated polygon* is a twisted polygon A together with the decorations of each of its vertices and sides induced by some curve γ in the space of twisted polygons with $\gamma(0) = A$.

Decorated polygons will be denoted by the appropriate script letter. For instance if the underlying polygon is A then the decorated polygon will be called $\mathcal{A}$. It is

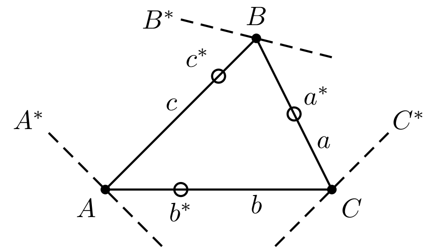

Figure 3.10: A decorated triangle

possible for different curves to give rise to the same decorated polygon $\mathcal{A}$. As such, $\mathcal{A}$ corresponds to a whole class of curves. We will call any curve γ in this class a *representative* of $\mathcal{A}$.

Remark III.25. Definition III.24 refers to the space of twisted polygons. This should not be confused with $\mathcal{P}_n$ or $\mathcal{P}_n^*$ which are spaces of projective equivalence classes of twisted polygons. In general, we will be working with actual polygons instead of equivalence classes of polygons for the remainder of the paper.

Given a collection of geometric objects which satisfy certain incidences (e.g. the vertices and sides of a polygon), a consistent choice of decorations of these objects may have to satisfy some relations. The simplest example of such a relation occurs for a closed triangle.

Proposition III.26. *In $\triangle ABC$ let $a = \overleftrightarrow{BC}$, $b = \overleftrightarrow{AC}$, and $c = \overleftrightarrow{AB}$ (see Figure 3.10). Let $A(t)$, $B(t)$, and $C(t)$ be curves through the vertices and use them to define curves corresponding to the sides (e.g. $a(t) = \overleftrightarrow{B(t)C(t)}$). Then the corresponding decorations satisfy*

$$[A, c^*, B, a^*, C, b^*] = [a, C^*, b, A^*, c, B^*]^{-1} \tag{3.6}$$

Lemma III.27. *Let $\triangle A'B'C'$ be another triangle with $a' = \overleftrightarrow{B'C'}$, $b' = \overleftrightarrow{A'C'}$, and*

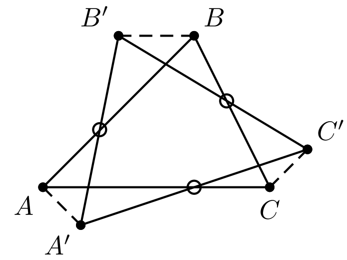

Figure 3.11: The points and lines involved in Lemma III.27

$c' = \overleftrightarrow{A'B'}$ (see Figure 3.11). Then

$$[A, c \cap c', B, a \cap a', C, b \cap b'] = [a', \overleftrightarrow{CC'}, b', \overleftrightarrow{AA'}, c', \overleftrightarrow{BB'}]^{-1} \tag{3.7}$$

Proof. See Remark III.23. $\square$

Proof of Proposition III.26. In Lemma III.27, take $A' = A(t)$, $B' = B(t)$, and $C' = C(t)$ and consider the limit as t goes to 0. In the limit, the vertices and sides of $\triangle A'B'C'$ approach their counterparts in $\triangle ABC$. By definition, $\overleftrightarrow{AA'}$ approaches A^* and similarly for the other vertices and sides. So the limit of (3.7) is precisely (3.6). $\square$

Remark III.28. For each n, there is a relation similar to (3.6) among the decorations of a closed n-gon. Moreover, this is the only relation that holds. Hence, one can pick n vertex decorations and $n-1$ sides decorations (or the other way around) independently, and the last decoration is then determined. The space of decorations of a fixed polygon A, then, has dimension $2n-1$. One can check that this space naturally corresponds to the projectivized tangent space of A within the space of closed n-gons. A similar statement is probably true for twisted polygons, but we do not understand what the relations are among the individual decorations.

The next proposition shows that repeated applications of (3.6) can be used to extend the pentagram map to decorated polygons.

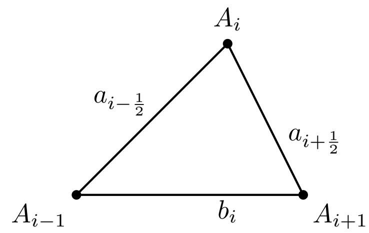

Figure 3.12: The triangle used to compute b_i^*

Proposition III.29. *Let $A(t)$ be a curve in the space of polygons and let $B(t) = T(A(t))$ for all t. Let $\mathcal{A}$ and $\mathcal{B}$ be the corresponding decorations of $A = A(0)$ and $B = B(0)$ respectively. Then $\mathcal{B}$ is uniquely determined by $\mathcal{A}$.*

Proof. Given an index i of A, consider the triangle with vertices A_{i-1}, A_i, and A_{i+1} (see Figure 3.12). It has all three vertices and two of its sides coming from A. The last side is $\overleftrightarrow{A_{i-1}A_{i+1}} = b_i$. Applying Proposition III.26 to this triangle, then, expresses b_i^* in terms of $\mathcal{A}$.

Next, consider the triangle with vertices $A_{j-\frac{1}{2}}$, B_j and $A_{j+\frac{1}{2}}$ for some index j of B (see Figure 3.13). Two of its vertices and one of its sides belong to A. The other two sides are in fact sides of B, namely

$$\overleftrightarrow{A_{j-\frac{1}{2}}B_j} = b_{j+\frac{1}{2}}$$
$$\overleftrightarrow{A_{j+\frac{1}{2}}B_j} = b_{j-\frac{1}{2}}$$

These were both decorated in the previous step. Another application of Proposition III.26, then, determines B_j^*.

□

The procedure above to construct $\mathcal{B}$ from $\mathcal{A}$ should be thought of as a lift of the pentagram map to the space of decorated polygons. To distinguish this operation from the original map, write $\mathcal{B} = \tilde{T}(\mathcal{A})$. The construction defining $\tilde{T}$ is given in Algorithm 2.

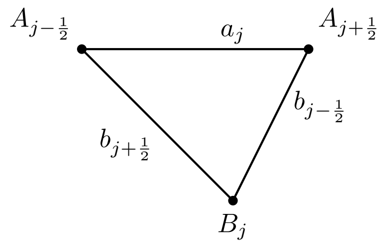

Figure 3.13: The triangle used to compute B_j^*

Algorithm 2 $\tilde{T}(\mathcal{A})$

for all i **do**
  $b_i := \overleftrightarrow{A_{i-1}A_{i+1}}$
  $b_i^* :=$ `DecorateSide`$(\mathcal{A}, b_i, i)$
end for
for all j **do**
  $B_j := b_{j-\frac{1}{2}} \cap b_{j+\frac{1}{2}}$
  $B_j^* :=$ `DecorateVertex`$(\mathcal{A}, (b_i), (b_i^*), B_j, j)$
end for
return $\mathcal{B}$

The subroutines `DecorateSide` and `DecorateVertex` build the triangles in Figures 3.12 and 3.13 respectively, and use (3.6) to compute the desired decoration.

Remark III.30. We will only be using decorated polygons and the map $\tilde{T}$ as tools in our straightedge constructions. However, these are likely interesting objects to study in their own right. Some immediate questions come to mind such as

- What would be a good set of coordinates on the space of decorated polygons?
- In such coordinates, does the map $\tilde{T}$ take a nice form?
- Does $\tilde{T}$ define a discrete integrable system?

3.5 Degenerations

We saw in the previous section that (3.6) is the only identity needed to apply the pentagram map to a generic decorated polygon. However, the motivation for introducing decorations is to handle degenerate cases. In this section, we introduce methods which will eventually be used to apply the pentagram map to a large class of degenerate polygons. Everything will be expressed in terms of local rules involving triangles and complete quadrilaterals.

3.5.1 Triangles

Let $A(t)$, $B(t)$, $C(t)$ be curves in the plane passing through points A, B, and C at time $t = 0$. Assume that for all $t \neq 0$, the points $A(t)$, $B(t)$, and $C(t)$ are in general position, and let $a(t)$, $b(t)$, and $c(t)$ denote the sides of the triangle they form. We allow A, B, and C to be collinear, or even equal to each other. However, assume that the limits

$$a = \lim_{t\to 0} a(t)$$

$$b = \lim_{t\to 0} b(t)$$

$$c = \lim_{t\to 0} c(t)$$

all exist. Assume that the limits defining the decorations $A^*, B^*, C^*, a^*, b^*, c^*$ all exist as well.

Now, Proposition III.26 did not allow for A, B and C to be collinear. However, by continuity (3.6) still holds in the present context.

Proposition III.31. *Assume that the decorations of the triangle are generic (i.e. distinct from each other and from the vertices and sides of the triangle.) Then the vertex B is uniquely determined by the sides a, b, c, the vertices A, C, and all the corresponding decorations.*

Proof. If $a \neq c$, then $B = a \cap c$, so assume $a = c$. There are two cases depending on if b equals the other sides.

If $b \neq a = c$ then $A = b \cap c = b \cap a = C$. In general, if $P_1 = P_5$ then

$$\begin{aligned}[P_1, P_2, P_3, P_4, P_5, P_6] &= \frac{P_1P_2}{P_2P_3}\frac{P_3P_4}{P_4P_5}\frac{P_5P_6}{P_6P_1} \\ &= -\frac{P_1P_2}{P_2P_3}\frac{P_3P_4}{P_4P_1} \\ &= -[P_1, P_2, P_3, P_4]\end{aligned}$$

so (3.6) simplifies to

$$[A, c^*, B, a^*] = [a, C^*, b, A^*]^{-1} \tag{3.8}$$

Note that the dependence on B^* has disappeared, so this identity determines B from the given geometric data.

Alternately, suppose $b = a = c$. Whenever $P_1 = P_3 = P_5$ we have

$$\begin{aligned}[P_1, P_2, P_3, P_4, P_5, P_6] &= \frac{P_1P_2}{P_2P_3}\frac{P_3P_4}{P_4P_5}\frac{P_5P_6}{P_6P_1} \\ &= \frac{P_1P_2}{P_2P_1}\frac{P_3P_4}{P_4P_3}\frac{P_5P_6}{P_6P_5} \\ &= -1\end{aligned}$$

Consequently, (3.6) becomes

$$[A, c^*, B, a^*, C, b^*] = -1$$

so again we can construct B. □

To sum up, if $a = c$ in a triangle then (3.6) can be used to construct the vertex B. The downside is that this identity can no longer be used to determine the decoration B^*. In fact, B^* is independent from the rest of the triangle when $a = c$. As such, we will need more data to construct vertex decorations when degeneracies occur.

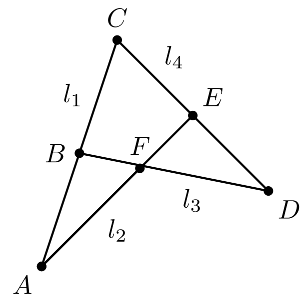

Figure 3.14: A complete quadrilateral

3.5.2 Complete quadrilaterals

A *complete quadrilateral* is a projective configuration consisting of four lines (called sides) in general position together with the six points (called vertices) at which they intersect. Call the sides l_1, l_2, l_3, l_4 and call the vertices A, B, C, D, E, F as in Figure 3.14.

As with triangles, we define degenerate complete quadrilaterals to be configurations that can be obtained as a limit of ordinary complete quadrilaterals. More precisely, let $l_1(t), \dots, l_4(t)$ and $A(t), \dots, F(t)$ be smooth curves which define a complete quadrilateral at each time $t \neq 0$. Let $l_1 = l_1(0), l_2 = l_2(0), \dots, F = F(0)$. Then some or all of the sides l_i could be equal. Assume that the decorations $l_1^*, l_2^*, \dots, F^*$ are all defined.

Proposition III.32. *Consider a degenerate complete quadrilateral with $l_1 = l_2 = l_3 = l_4$. Then the vertex decoration F^* is uniquely determined by the other vertices, sides, and decorations.*

Proof. By Menelaus' theorem, $[A, B, C, D, E, F] = -1$. According to the appendix, F can be constructed from the other vertices using a construction as in Figure 3.19. The idea of the present construction is to build a configuration as in that figure at each time t.

Choose generically in the plane a point c^* and lines P^* and $(C')^*$ (the reason for these names will be clear shortly.) For each t, define

$$c(t) = \overleftrightarrow{C(t)c^*}$$

$$P(t) = c(t) \cap P^*$$

$$C'(t) = c(t) \cap (C')^*$$

Note that

$$\lim_{t \to 0} c(t) \cap c(0) = \lim_{t \to 0} \overleftrightarrow{C(t)c^*} \cap \overleftrightarrow{Cc^*} = c^*$$

which justifies the notation c^*. Similar remarks hold for P^* and C'^*. Defining

$$B'(t) = \overleftrightarrow{A(t)C'(t)} \cap \overleftrightarrow{B(t)P(t)}$$
$$D'(t) = \overleftrightarrow{E(t)C'(t)} \cap \overleftrightarrow{D(t)P(t)}$$

we get at time $t = 0$ a configuration as in Figure 3.19. In particular B', D', and F are collinear. The proof of this fact given in the appendix generalizes to show that $B'(t)$, $D'(t)$, and $F(t)$ are collinear for all t. Let $l'_3(t)$ denote the common line. By way of notation, let $l'_1(t) = \overleftrightarrow{A(t)C'(t)}$, $l'_4(t) = \overleftrightarrow{E(t)C'(t)}$, $b(t) = \overleftrightarrow{B(t)P(t)}$, and $d(t) = \overleftrightarrow{D(t)P(t)}$.

We are given decorations of $A, B, \ldots, E$ and l_1, l_2, l_3, l_4 from the outset. We chose arbitrarily decorations of P, C' and c. In a generic triangle, knowing five of the six decorations determines the sixth by Proposition III.26. This fact can be used to find all missing decorations in our configuration. To start, use the first triangle in Figure 3.15 to find $(l'_1)^*$ and the second triangle to find b^*. Once these decorations are found, the third triangle in the figure can be used to determine $(B')^*$. A similar method is used to find $(l'_4)^*$, d^*, and then $(D')^*$. Finally, use the first triangle in Figure 3.16 to find $(l'_3)^*$ and the other triangle to find F^*. $\square$

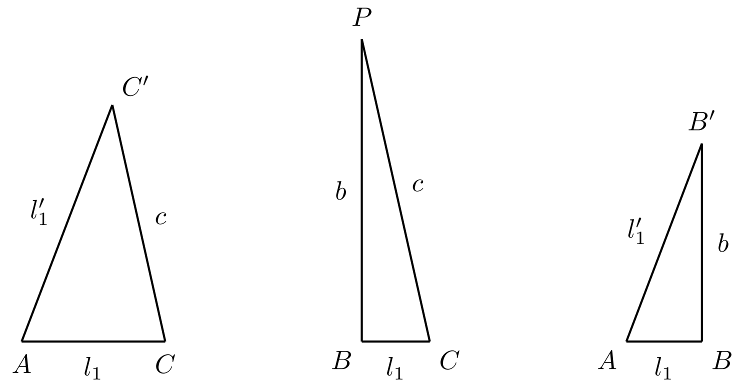

Figure 3.15: The triangles used to find $(l_1')^*$, b^*, and finally $(B')^*$

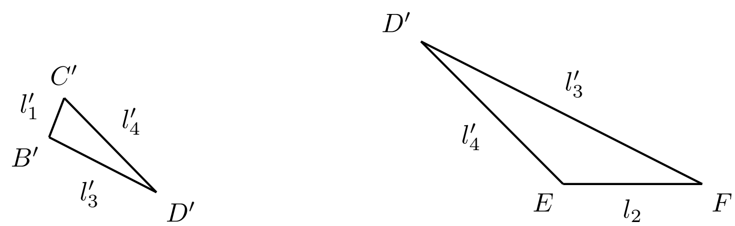

Figure 3.16: The triangles used to find $(l_3')^*$ and F^*

3.5.3 Degenerate polygons

A *degenerate polygon* is a sequences of points and an interlacing sequence of lines, which occur as the limits of the vertices and sides, respectively, of some twisted polygons. More precisely, if $A(t)$ is a twisted polygon for $t \neq 0$ and the appropriate limits are defined, then we get a degenerate polygon A with vertices $A_i = \lim_{t\to 0} A_i(t)$ and sides $a_j = \lim_{t\to 0} a_j(t)$. Fixing such a curve $A(t)$, we get a decoration for A as before.

Using our understanding of degenerate triangles and complete quadrilaterals, we are ready to state a version of $\tilde{T}$ which works for degenerate polygons. As will be explained, the algorithm requires two consecutive iterates of the pentagram map, $\mathcal{A}$ and $\mathcal{B}$, as input. The output is the iterate $\mathcal{C}$ which follows these two. The new procedure is called $\tilde{T}_2$ and is given in Algorithm 3. In the algorithm, j ranges over the

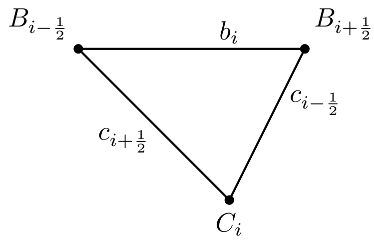

Figure 3.17: The triangle used in `ConstructVertex2`

index set of B and i ranges over the index set of C. A description of the subroutines appearing in the algorithm will follow.

Algorithm 3 $\tilde{T}_2(\mathcal{A}, \mathcal{B})$

for all j **do**
    $c_j :=$ `ConstructSide2`$(\mathcal{B}, j)$
end for
for all j **do**
    $c_j^* :=$ `DecorateSide2`$(\mathcal{A}, \mathcal{B}, c_j, j)$
end for
for all i **do**
    $C_i :=$ `ConstructVertex2`$(\mathcal{B}, (c_j), (c_j^*), i)$
end for
for all i **do**
    $C_i^* :=$ `DecorateVertex2`$(\mathcal{A}, \mathcal{B}, (c_j), (c_j^*), C_i, i)$
end for
return $\mathcal{C}$

The outline of this algorithm is the same as that of the original version of $\tilde{T}$. First the sides of $\mathcal{C}$ are constructed, then the side decorations, then the vertices, and finally the vertex decorations. Each individual step, though, is made more complicated by the possibility of degeneracies.

The subroutine `ConstructVertex2` works with the triangle pictured in Figure 3.17. All of the components of the triangle besides C_i have already been constructed. Hence by Proposition III.31, it is always possible to construct C_i.

The subroutine `DecorateVertex2` begins from this same triangle. If $c_{i-\frac{1}{2}} \neq c_{i+\frac{1}{2}}$ then (3.6) is used to compute C_i^*. Otherwise, more data is needed. Consider the complete quadrilateral in Figure 3.1. We are assuming that two of its sides, namely

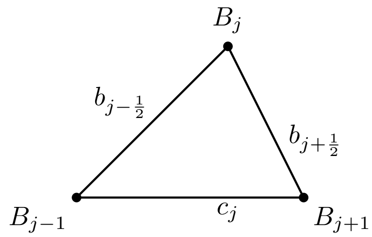

Figure 3.18: The triangle used in `ConstructSide2`

$c_{i-\frac{1}{2}}$ and $c_{i+\frac{1}{2}}$, are equal. This forces the five vertices other than A_i to lie on the common line. Generically, these five vertices are distinct forcing the two remaining sides (namely b_{i-1} and b_{i+1}) of the complete quadrilateral to be equal to each other and to $c_{i-\frac{1}{2}} = c_{i+\frac{1}{2}}$. This puts us in the situation of Proposition III.32. All vertices besides C_i and all sides have been decorated already, so the procedure can determine the decoration on C_i.

We used above the fact that if two sides of a complete quadrilateral are equal, then generically, they all must be equal. However, non-generic example where this fails will arise in practice. For instance, in Figures 3.6 and 3.7 we have $c_{3.5} = c_{4.5}$ but b_3 and b_5 are different. In these situations, `DecorateVertex2` will simply return a random decoration, i.e., a random line passing through C_i.

The subroutines `ConstructSide2` and `DecorateSide2` behave like `ConstructVertex2` and `DecorateVertex2` respectively. The difference is that they operate with configurations which are projectively dual to the ones in Figures 3.17 and 3.1. First, Figure 3.18 shows the triangle used by `ConstructSide2`. This procedure constructs c_j in the dual manner to how `ConstructVertex2` finds C_i in Figure 3.17.

Similarly, Figure 3.2 contains a configuration that is projectively dual to the complete quadrilateral in Figure 3.1. As such, `DecorateSide2` can find c_j^* via the projective dual of the construction used by `DecorateVertex2`.

The case where `DecorateVertex2` outputs a random line, and the analogous case of `DecorateSide2`, are not currently justified. However, all other cases are covered by Propositions III.31 and III.32. Hence we have the following correctness property of Algorithm 3.

Proposition III.33. *Let $A(t)$ be a curve in the space of twisted polygons that is generic away from $t = 0$. Let $B(t) = T(A(t))$ and $C(t) = T(B(t))$ for $t \neq 0$. Let $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ be the decorated polygons associated to these curves. Suppose that*

$$(B_{j-1} = B_{j+1} \Longrightarrow A_{j-\frac{1}{2}} = A_{j+\frac{1}{2}} = B_{j-1}) \text{ for all } j \tag{3.9}$$

$$(c_{i-\frac{1}{2}} = c_{i+\frac{1}{2}} \Longrightarrow b_{i-1} = b_{i+1} = c_{i-\frac{1}{2}}) \text{ for all } i \tag{3.10}$$

where j and i run over the vertex indices of B and C respectively. Then $\mathcal{C} = \tilde{T}_2(\mathcal{A}, \mathcal{B})$.

More specifically, if (3.9) fails for some j, then `DecorateSide2` chooses a random decoration for c_j. If (3.10) fails for some i, then `DecorateVertex2` chooses a random decoration for C_i. Otherwise, the algorithm $\tilde{T}_2$ behaves deterministically and correctly.

3.6 The main algorithm

The goal of our main algorithm is to construct $B = T^m(A)$ from A when the usual construction fails, i.e. when A is a singular point of various T^k for $k < m$. According to the previous section, it is typically possible to construct and decorate $T^2(A)$ given A, $T(A)$, and the corresponding decorations, even when singularities arise. The main construction, given in Algorithm 4, simply iterates this procedure.

Given $S \subseteq \{1, 2, \ldots, n\}$ such that singularity confinement holds on X_S, let m be the smallest positive integer such that T^m is generically defined on X_S. We want to say for generic $A \in X_S$ that the main algorithm, given A and m as input, produces $T^m(A)$. For the simplest singularity types, $S = \{i\}$, this result follows from

Algorithm 4 `main`$(A,\, m)$

$\mathcal{A}$:= `DecorateRandomly`(A)
Iterates[0] := $\mathcal{A}$
Iterates[1] := $\tilde{T}(A)$
for $k := 2$ **to** m **do**
  Iterates[k] := $\tilde{T}_2$(Iterates[$k-2$], Iterates[$k-1$])
end for
$\mathcal{B}$:= Iterates[m]
return B

Propositions III.29 and III.33.

For more complicated S, a difficulty arises because the assumptions (3.9) and (3.10) in Proposition III.33 will not hold at every step. Hence, some applications of $\tilde{T}_2$ in the main algorithm will produce random decorations. To prove correctness of the algorithm for such S, it is necessary to determine at which steps this occurs and to demonstrate that the outcome is independent of the random choices.

We will focus our attention on the types covered by Theorem I.5. Taking $i = 0$ for convenience in the Theorem, let $S = [-(m-1), (m-1)]_2$. Suppose $A \in X_S$ is generic. Tracing through the beginning of the main algorithm, let $\mathcal{A}$ be some decoration of A, let $\mathcal{B} = \tilde{T}(\mathcal{A})$, and let $\mathcal{C} = \tilde{T}_2(\mathcal{A}, \mathcal{B})$.

Since $A \in X_S$, the A_i for $i \in [-m-1, m+1]_2$ all lie on a common line, say l. It follows (see e.g. Figures 3.3, 3.4, 3.6, and 3.7) that $b_i = l$ for $i \in [-m, m]_2$ and $c_j = l$ for $j \in [-m+1/2, m-1/2]_1$. Consequently, condition (3.10) holds for $i \in [-m+1, m-1]_2$ but fails for $i \in [-m+2, m-2]_2$ (assuming $m \geq 2$). As such the corresponding C_i are decorated randomly.

To establish that `main`$(A,\, m+2) = T^{m+2}(A)$ for A as above, we need to prove two facts. The first is that the output of the algorithm does not depend on the decorations of the C_i that are chosen randomly. For given m, we can check this computationally by showing that any such choice of decorations is possible for an appropriate choice of representative $A(t)$ of $\mathcal{A}$. The second fact is that no other violations of (3.9) or (3.10) occur until computing Iterates$[m+2]$ in the last step. For given m, it suffices to check that this fact is true for a single $\mathcal{A}$ as it then follows for generic examples.

We have no general proof for the necessary facts, but we have verified that they hold for the first several values of m. Assuming them, we can repeatedly apply Proposition III.33 to conclude that Iterates$[k]$ is the decoration corresponding to the curve $T^k(A(t))$ for some curve $A(t)$ through A and all $k < m+2$. At the last step condition (3.9) will fail in some places, so Iterates$[k+2]$ will have some randomly decorated sides. However, the sides of Iterates$[k+2]$ themselves will be correct proving that its underlying polygon is in fact $T^{m+2}(A)$. Hence we get that the main algorithm works correctly for polygons of type $[-m+1, m-1]_2$ when m is small. We expect that this result holds for all m.

Remark III.34. The main algorithm is stated without regard to a particular singularity type. Hence it has the potential to work in greater generality than is discussed above. Experiments indicate that the algorithm does work for many, but not all, other singularity types. The simplest types for which it fails are $S = \{3, 4, 6\}$ and similar.

3.7 Appendix: Some basic constructions

This appendix states and proves straightedge constructions for the primitives used in the algorithms throughout the paper. The first, namely `TripleConjugate`, is given in Algorithm 5. This construction was shown to me by Pavlo Pylyavskyy.

This algorithm constructs points B', C', D' and F such that (A, B', C', D', E, F) is a Menelaus configuration (see Figure 3.19). Therefore $[A, B', C', D', E, F] = -1$. Applying a projective transformation we may assume that P is a point at infinity.

Algorithm 5 `TripleConjugate`(A, B, C, D, E)

$P :=$ `RandomPoint`$()$
$C' :=$ `RandomPointOn`$(\overleftrightarrow{CP})$
$B' := \overleftrightarrow{BP} \cap \overleftrightarrow{AC'}$
$D' := \overleftrightarrow{DP} \cap \overleftrightarrow{C'E}$
$F := \overleftrightarrow{B'D'} \cap \overleftrightarrow{AE}$
return F

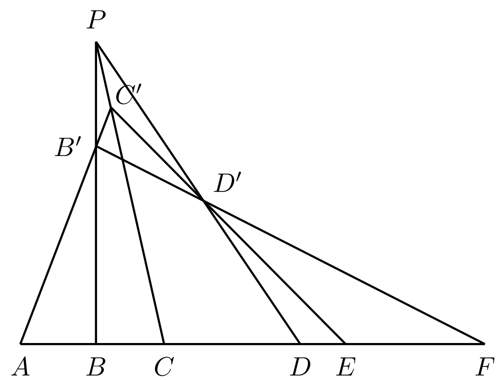

Figure 3.19: The construction of `TripleConjugate`

The lines $\overleftrightarrow{BB'}$, $\overleftrightarrow{CC'}$, and $\overleftrightarrow{DD'}$ all pass through this point, so they must be parallel. Therefore $\triangle ABB'$ is similar to $\triangle ACC'$, so $\frac{AB}{BC} = \frac{AB'}{B'C'}$. Also, $\triangle EDD'$ is similar to $\triangle ECC'$, so $\frac{CD}{DE} = \frac{C'D'}{D'E}$. It follows that

$$[A, B, C, D, E, F] = [A, B', C', D', E, F] = -1$$

as desired.

Next, Algorithm 6 inputs four points A, B, C, D, on one line, and three points A', B', C' on another. There exists a unique projective transformation from the first line to the second taking A to A', B to B', and C to C'. The algorithm returns the result of applying this projective transformation to D.

The algorithm selects a line l and a point P such that projection through P onto l sends A to A'. The images of B, C, are called B'' and C'' respectively. Then Q is constructed so that projection through Q onto the target line sends B'' to B' and C'' to C', while necessarily fixing A' (see Figure 3.20). Hence, the composition of these

Algorithm 6 `ProjectiveTransformation`(A, B, C, D, A', B', C')

$l :=$ `RandomLineThrough`(A')
$P :=$ `RandomPointOn`$(\overleftrightarrow{AA'})$
$B'' := l \cap \overleftrightarrow{BP}$
$C'' := l \cap \overleftrightarrow{CP}$
$Q := \overleftrightarrow{B''B'} \cap \overleftrightarrow{C''C'}$
$D'' := l \cap \overleftrightarrow{DP}$
$D' := \overleftrightarrow{D''Q} \cap \overleftrightarrow{A'B'}$
return D'

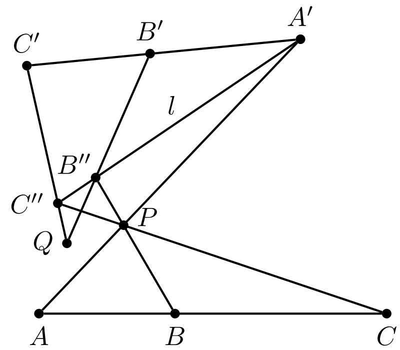

Figure 3.20: Part of the construction used in `ProjectiveTransformation`

two projections is the desired projective transformation. Applying it to D gives the output D'.

Cross ratios are invariant under projective transformation. Hence $[A', B', C', D'] = [A, B, C, D]$ and D' is the unique point on the line containing A', B', C' with this property. As such, we use this construction to find a point B satisfying (3.8). This appears to be a more complicated situation because one of the cross ratios is inverted, and also because both points and lines are involved. The identity can be expressed in terms of points alone using the fact that $[l_1, l_2, l_3, l_4] = [l_1 \cap l, l_2 \cap l, l_3 \cap l, l_4 \cap l]$ for any other line l. The reciprocal can be eliminated by reordering via the property $[P_1, P_2, P_3, P_4]^{-1} = [P_1, P_4, P_3, P_2]$.

Another component of several of our algorithms involves finding one point or line from (3.6) in terms of the others. By similar remarks to before, it is possible to cast this as a problem involving points alone, namely to construct F' from the other

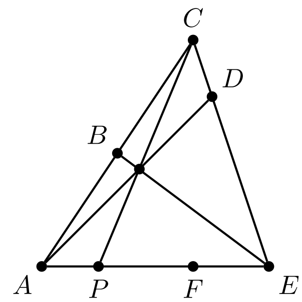

Figure 3.21: The construction of a point P satisfying $[A,B,C,D,E,F] = [A,P,E,F]$

points assuming

$$[A,B,C,D,E,F] = [A',B',C',D',E',F']$$

Here, not all points are assumed to be collinear, only those triples required by the definition of triple ratios.

Lemma III.35. $[A,B,C,D,E,F] = [A,P,E,F]$ *where*

$$P = \overleftrightarrow{C(\overleftrightarrow{AD} \cap \overleftrightarrow{BE})} \cap \overleftrightarrow{AE}$$

(see Figure 3.21).

Proof. For any point P on $A \cap E$, we have

$$\frac{[A,B,C,D,E,F]}{[A,B,C,D,E,P]} = [A,P,E,F]$$

For the particular P chosen, Ceva's theorem guarantees that $[A,B,C,D,E,P] = 1$. □

In light of this lemma, it is easy to construct the point F' above.

Algorithm 7 `ProjectiveTransformation2`$(A,B,C,D,E,F,A',B',C',D',E')$

$P := \overleftrightarrow{C(\overleftrightarrow{AD} \cap \overleftrightarrow{BE})} \cap \overleftrightarrow{AE}$
$P' := \overleftrightarrow{C'(\overleftrightarrow{A'D'} \cap \overleftrightarrow{B'E'})} \cap \overleftrightarrow{A'E'}$
$F' :=$ `ProjectiveTransformation`(A,P,E,F,A',P',E')
return F'

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