The F-Signature of Toric Varieties

by

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CHAPTER I

Introduction

Let R be a commutative Noetherian ring of characteristic $p > 0$. The *Frobenius map* $F : R \to R$, defined by $r \mapsto r^p$, is a ring homomorphism with no analog in characteristic zero. Its existence gives rise to many powerful tools in commutative algebra and algebraic geometry. For example, the Frobenius map may be used to define several numerical invariants which measure the singularity of a point on an algebraic variety. These include the F-threshold ([MTW05]), the Hilbert-Kunz multiplicity ([Mon83]), and the F-signature ([HL02]). We will study the F-signature in this thesis.

The *F-signature* of R is a real number between 0 and 1 which gives information about the singularity of R. It is equal to 1 precisely when R is regular, i.e., when R corresponds to a nonsingular point on an algebraic variety. It is positive if and only if R is *strongly F-regular* ([AL03]), a property arising in tight closure theory. The F-signature arose naturally in the study of differential operators in positive characteristic ([SVdB97]) and, starting with [HL02], has been studied as an object of independent interest. It is intimately related to the Hilbert-Kunz multiplicity; in fact, it has been characterized as a *minimal relative Hilbert-Kunz multiplicity* of R ([WY04]).

The thesis has two main goals. First, we consider the *local* setting: we compute the F-signature of a point on a toric variety. *Toric varieties* are mildly singular varieties, commonly studied in algebraic geometry, which may be described using combinatorial data. Our first main theorem (Theorem 3.2.3) gives a formula for the F-signature of the coordinate ring of an affine toric variety as the volume of an explicit polytope associated to this combinatorial data.

Second, we consider the *global* setting: we define a new F-signature function on the divisor class group of a projective variety (Definition 4.1.4). We compute this function in the case of a projective toric variety and show that it is a continuous and piecewise rational function on the interior of its domain of definition (Corollary 4.4.6).

1.1 Statement of Results and Motivation

1.1.1 Affine Toric F-Signature

There are many different characterizations of F-signature; we will describe several at the start of Chapter II. The simplest definition is the original one, which characterizes the F-signature of R, denoted $s(R)$, as a real number that determines the asymptotic growth rate of the rank of a maximal free R^{p^e}-module summand of R. The F-signature is generally difficult to compute. Rings for which it has previously been computed include Veronese subrings of polynomial rings and finite quotient singularities ([HL02], [Tuc11]), Segre products of polynomial rings ([Sin05]), and toric rings with no nontrivial units ([WY04]).

A *toric variety* is a normal variety which contains, as an open dense subset, an algebraic torus $T \simeq \operatorname{Spec} k[x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}]$, so that the group action of T on itself extends to an action of T on X. Toric varieties were first mentioned explicitly in [Dem70], but the concepts underlying their use arose informally much earlier. They are natural objects of study in large part because their geometric properties can be easily understood in terms of simple combinatorial data. In particular, their coordinate rings are monomial rings which may be studied using the perspective of convex polyhedral geometry ([Ful93], [CLS11]). In addition to providing geometers with a rich, easily understood class of algebraic varieties, toric varieties have been used to answer questions about polytopes. A notable result of this sort is Stanley's proof of McMullen's conjecture on the number of faces of a simplicial polytope ([Sta80]).

Our first main theorem is a formula for the F-signature of an affine toric variety. Recall that an affine toric variety is determined by a lattice N and a cone σ inside the real vector space $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$ (see, e.g., [Ful93]).

Theorem 3.2.3. *Let X_σ be an affine toric variety without torus factors. Let $\vec{v}_1, \dots, \vec{v}_r \in N$ be primitive generators for the cone σ. Define a polytope P_σ in the dual space $M_{\mathbb{R}} = N_{\mathbb{R}}^*$:*

$$P_\sigma := \{\vec{w} \in M_{\mathbb{R}} \mid \forall i, 0 \leq \vec{w} \cdot \vec{v}_i < 1\}.$$

Then $s(k[X_\sigma]) = \mathrm{Volume}(P_\sigma)$.

More generally, suppose $X = X' \times T$, where X' is a toric variety without torus factors and T is an algebraic torus. Let $N'_{\mathbb{R}} \subset N_{\mathbb{R}}$ be the vector subspace spanned by σ, and let σ' be σ viewed as a cone in $N'_{\mathbb{R}}$. Then $s(k[X]) = s(k[X']) = \mathrm{Volume}(P_{\sigma'})$.

We will devote much of this thesis to further generalizing Theorem 3.2.3, which is itself a slight generalization of a formula of Watanabe and Yoshida [WY04].

In Chapter III, we will compute the F-signature of toric *pairs* and *triples*. The notion of singularities of pairs is an important one in birational geometry. Instead of studying the singularities of a variety X, or the singularities of a $\mathbb{Q}$-divisor D on X, one studies the pair (X, D), for example by considering how D changes under various resolutions of X. More generally, it is often useful to study triples $(X, D, \mathfrak{a}^t)$, where D is a $\mathbb{Q}$-divisor, $\mathfrak{a}$ an ideal sheaf on X, and t a real parameter. This perspective first arose in the Minimal Model Program in algebraic geometry; see [Kol97] for an introduction to pairs in this setting.

Recently, commutative algebraists have begun to adapt the pairs and triples perspective for their own use. Working from the point of view that the F-signature can be understood via certain algebras of p^e-linear maps, Blickle, Schwede, and Tucker have defined the F-signature of pairs and triples ([BST11]). That is, they define the F-signature of a pair (R, D), where D is a $\mathbb{Q}$-divisor on $\operatorname{Spec} R$, and of a triple $(R, D, \mathfrak{a}^t)$, where $\mathfrak{a}$ is an ideal of R and $0 \leq t \in \mathbb{R}$. They show that the F-signature of a pair is nonzero precisely when the pair is strongly F-regular. Moreover, they use the F-signature of pairs to show the existence of the so-called *F-splitting ratio*, a generalization of the F-signature to rings which are not strongly F-regular ([BST11], Theorems 3.18 and 4.2).

1.1.2 A Projective F-Signature Function

We aim to use the F-signature to define a new function on the class group of a projective variety. A starting point for this definition is as follows:

Definition 4.1.1. Let X be a normal projective variety and D an ample Cartier divisor on X. We define the *F-signature of D*, denoted $s(X, D)$, to be the F-signature of the section ring $\operatorname{Sec}(X, D) = \oplus_{n \geq 0} \Gamma(X, O_X(nD))$.

In Chapter IV, we will prove Theorem 4.1.3, which allows us to extend the F-signature function of Definition 4.1.1 to a much broader subset of the space of $\mathbb{Q}$-divisors on X. In particular:

Definition 4.1.4. Let X be a normal projective variety over a field k of positive characteristic. Let D be a $\mathbb{Q}$-divisor on X, and suppose that $\mathrm{Sec}(X, D) = \bigoplus_n \Gamma(X, O_X(nD))$ is a finitely generated k-algebra of dimension at least two. Choose $c \in \mathbb{N}_{\geq 0}$ sufficiently divisible so that $\mathrm{Sec}(X, cD)$ is a normal section ring. We define the *F-signature* $s(X, D)$ to be $c \cdot s(X, cD)$.

This point of view—in which a numerical invariant of graded rings is used to construct a function on a subspace of the space of $\mathbb{Q}$-divisors on X—arose relatively recently but has been applied to other invariants with great success. One important invariant of this sort is the volume of a divisor (see [Laz04], 2.2C). Recall:

Definition 1.1.1. Let X be a projective variety of dimension d and D a $\mathbb{Q}$-divisor on X. The *volume* of D is $\frac{1}{d!} \lim_{n\to\infty} \frac{\dim \Gamma(X, O_X(nD))}{n^d}$.

A divisor is *big* if its volume is nonzero. The volume of divisors on X is invariant under numerical equivalence and gives rise to a function on the cone of big $\mathbb{Q}$-divisors on X which may be used to study the geometry of X. This function is in fact continuously differentiable ([LM09]) and scales according to $\mathrm{Volume}(nD) = n^{\dim X}\,\mathrm{Volume}(D)$. When X is a projective toric variety, the volume function is piecewise polynomial on the big cone of X ([Laz04]).

Theorem 3.2.3 will allow us to compute our new F-signature function on the space of $\mathbb{Q}$-divisors of a projective toric variety as the volume of an associated polytope. We will be able to show that the F-signature function shares analogous properties to the volume function in the toric setting, though the F-signature function is only piecewise *rational* rather than piecewise polynomial:

Corollary 4.4.6. *Let X be a projective toric variety over a field k of positive characteristic. Then the F-signature function $D \mapsto s(X, D)$ is a continuous, piecewise rational function of degree -1 on the set of big $\mathbb{Q}$-divisors of X.*

1.2 Outline

In Chapter II, we first define the F-signature and review some of its basic properties. Next, we prove a series of general facts about the F-signature, mostly in preparation for later chapters. Although the F-signature is well-defined over both local and graded rings, previous work on the F-signature has historically focused on the local case. We relate the local and graded settings by showing that when R is an appropriately graded ring with homogeneous maximal ideal m, $s(R) = s(R_m)$ (Theorem 2.2.1). We also give formulas for the F-signature of products (Theorems 2.3.1 and 2.3.4), which will later assist us in computing the F-signature of toric varieties which contain torus factors.

Several of our main results concern the F-signature of the section ring of a divisor on a projective variety. An essential ingredient of these results is the following formula for the F-signature of Veronese subrings of certain graded rings.

Theorem 2.6.2. *Let S be a normal section ring over a perfect field of positive characteristic, of dimension at least two. Let n be any positive integer, and let $S^{(n)}$ denote the nth Veronese subring of S. Then $s(S^{(n)}) = \frac{1}{n}s(S)$.*

The methods used to compute this formula generalize; we apply them to compute the F-signature of certain ring extensions which are étale in codimension one (Theorem 2.6.5). Finally, we define the F-signature of pairs and triples in preparation for computing formulas for these invariants on affine toric varieties in Chapter III (§2.7).

In Chapter III, we compute the F-signature of an affine toric variety X_σ, or more precisely of its coordinate ring $k[X_\sigma]$. We begin by reviewing the machinery of affine toric varieties, in particular the correspondence between an affine toric variety X and a convex polyhedral cone σ. This machinery enables us to compute the F-signature $s(k[X_\sigma])$ as the volume of a polytope P_σ (Theorem 3.2.3).

Previously, Watanabe and Yoshida computed the minimal relative Hilbert-Kunz multiplicity (which is equal to the F-signature) of a toric singularity in the case where X has no torus factors ([WY04]). We rederive their formula using different methods, as well as handling the case where X has torus factors. Furthermore, our methods allow us to give a formula for the F-signature of certain monomial rings which are not presented "torically" (Corollary 3.3.2). Armed with this formula, we rederive a formula of Singh for the F-signature of a normal monomial ring [Sin05].

We use the methods introduced in the proof of Theorem 3.2.3 to compute the F-signature of a toric pair $s(R, D)$ as the volume of a polytope P_σ^D, and we compute $s(R, D, \mathfrak{a}^t)$ as the volume of a polytope $P_\sigma^{D,\mathfrak{a}^t}$ (Theorems 3.4.10 and 3.4.13). When the pair (R, D) has $\mathbb{Q}$-Gorenstein singularities, we show that our F-signature formulas are especially simple (Corollary 3.4.17). Finally, we indicate how to generalize our computations to the case of imperfect residue field (Remark 3.5.3).

In Chapter IV, we investigate the F-signature $s(\mathrm{Sec}(X, D))$ of the section ring of a divisor D on a projective toric variety X. This F-signature will *not* be analogous to the F-signature of pairs $s(R, D)$ discussed above. The F-signature of pairs of [BST11] is a "local" invariant which measures the singularity of the pair. The new F-signature is a "global" invariant which contains data about the images of X under rational maps. We show that the F-signature function $D \mapsto s(\mathrm{Sec}(X, D))$, after a slight modification, may be extended to an F-signature function $s(X, D)$ on a larger subset of the group of $\mathbb{Q}$-divisors on X (Definition 4.1.4).

As a consequence of Theorem 3.2.3, we give a formula for the F-signature $s(X, D)$ of a $\mathbb{Q}$-divisor on a toric variety as the volume of a polytope. Our formula will be slightly simpler when D is a big divisor (Corollary 4.4.5), though we generalize our formula to apply to all divisors (Corollary 4.4.7). Our formula demonstrates that $s(X, D)$ is a piecewise rational function on the cone of big $\mathbb{Q}$-divisors of X (Corollary 4.4.6). Finally, we will compute a few examples, including a complete description of the F-signature function on $\mathbb{P}^1 \times \mathbb{P}^1$ (Example 4.4.10).

CHAPTER II

F-Signature

This chapter has two main goals. First, we introduce the F-signature; review some of its basic properties; and prove a few straightforward results about the F-signature of graded rings (§2.2) and products (§2.3). Second, we investigate the behavior of the F-signature under finite extensions. In particular, we give a formula for the F-signature of Veronese subrings of certain graded rings:

Theorem 2.6.2. *Let S be a normal section ring over a field of positive characteristic, of dimension at least two. Let n be any positive integer. Then $s(S^{(n)}) = \frac{1}{n}s(S)$.*

Theorem 2.6.2 will allow us to define an F-signature function for divisors on projective varieties in Chapter IV. The same methods that we use to prove this theorem also show that the F-signature behaves well under certain extensions which are *étale in codimension one*:

Theorem 2.6.5. *Let $(R, m_R, k) \to (S, m_S, k)$ be a module-finite inclusion of normal local rings, where the residue field k has positive characteristic. Suppose that $R \to S$ is split and étale in codimension one and that the maximal rank of a free R-submodule splitting from S is equal to one. Then $s(R) = \frac{1}{[\mathrm{Frac}(S):\mathrm{Frac}(R)]} \cdot s(S)$.*

At the end of the chapter, we will define the F-signature of pairs and triples in preparation for computing formulas for these invariants on toric varieties in Chapter III (§2.7).

2.1 Preliminaries

We begin by defining the F-signature. Let R be a ring containing a field k of characteristic $p > 0$. Let $F^e_* R$ be the R-module whose underlying abelian group is R and whose R-module structure is given by Frobenius: for $r \in R, s \in F^e_* R$, $r \cdot s = r^{p^e} s$. If R is reduced, it is easy to see that $F^e_* R$ is isomorphic to R^{1/p^e}, the R-module of p^eth roots of elements of R. (Incidentally, this also gives $F^e_* R$ a natural ring structure.) Recall that R is said to be *F-finite* if $F_* R$ is a finitely generated R-module. For example, every finitely generated algebra over a perfect field is F-finite.

Remark 2.1.1. (Conventions.) All rings are assumed to be Noetherian F-finite domains whose residue field k has prime characteristic $p > 0$. We assume that k is perfect unless stated otherwise, though by Remark 3.5.3, this assumption is mostly unnecessary. Moreover, all rings over which we compute F-signature will be either local with residue field k; $\mathbb{N}$-graded with zeroth graded piece ring-isomorphic to k; or $\mathbb{Z}^n$-graded for some $n \in \mathbb{N}$, with each nonzero graded piece a one-dimensional k-vector space.

Definition 2.1.2. Let R be a Noetherian local or graded ring as in Remark 2.1.1. Let M be a finitely generated R-module, which is assumed to be graded if R is graded, and consider a decomposition of M as a direct sum of indecomposable R-modules. The *free rank* of M as an R-module is the maximal rank of a free R-module appearing in this decomposition.

Remark 2.1.3. In general (if R is not local or $\mathbb{N}$-graded over a field), different decompositions may have free summands of different ranks. In the local or $\mathbb{N}$-graded setting, however, free rank is independent of decomposition (see, e.g., Remark 2.1.9 and Lemma 2.2.2).

Definition 2.1.4. [HL02] Let R be a ring (either local or graded, as in Remark 2.1.1) of dimension d. For each $e \in \mathbb{N}$, let a_e be the free rank of R^{1/p^e} as an R-module, so that $F^e_* R \simeq_{R-mod} R^{\oplus a_e} \oplus M_e$, where M_e has no free summands. We define the *F-signature* of R to be the limit

$$s(R) = \lim_{e \to \infty} \frac{a_e}{p^{ed}}.$$

Remark 2.1.5. Tucker [Tuc11] showed that the limit given in Definition 2.1.4 exists when R is a local ring. It follows from Theorem 2.2.1, below, that the F-signature is well-defined when R is $\mathbb{N}$-graded as in Remark 2.1.1.

More generally, we may define the F-signature of an module:

Definition 2.1.6. Let R be a ring (either local or graded, as in Remark 2.1.1) of dimension d. Let M be a finitely generated R-module. For each $e \in \mathbb{N}$, let $a_e^{M,R}$ be the free rank of $F^e_* M$ as an R-module, so that $F^e_* M \simeq_{R-mod} R^{\oplus a_e^{M,R}} \oplus M_e$, where M_e has no free summands. We define the *F-signature* of M to be the limit

$$s(M) = s_R(M) = \lim_{e \to \infty} \frac{a_e^{M,R}}{p^{ed}}.$$

In fact, the F-signature of an R-module depends only on the F-signature of R and the generic rank of the module:

Theorem 2.1.7 ([Tuc11], Theorem 4.11)**.** *Let R be a d-dimensional local domain, as in Remark 2.1.1, and let M be a finitely generated R-module. Then*

$$s_R(M) = \operatorname{rank}_R(M) \cdot s(R),$$

where $\operatorname{rank}_R(M) = \dim_{\operatorname{Frac}(R)} M \otimes_R \operatorname{Frac}(R)$ *is the generic rank of M.*

Theorem 2.1.7 is very useful for studying the behavior of the F-signature under module-finite ring maps. We will make use of it in section 2.6 to compute the F-signature of Veronese subrings.

Next, we consider several different characterizations of the F-signature.

Lemma 2.1.8 (cf. [Tuc11], Prop 4.5)**.** *Let (R, m) be a local or $\mathbb{N}$-graded ring as in Remark 2.1.1. Let M be a finitely generated local (or $\mathbb{N}$-graded) R-module. Let $Q \subset M$ be the subset $\{x \in M \mid \forall \phi \in \mathrm{Hom}_R(M, R), \phi(x) \in m\}$. Then Q is an R-submodule of M; in fact, if M is an S-module, where S is any R-algebra, then Q is an S-submodule of M. Moreover, the free rank of M as an R-module is equal to $l(M/Q)$.*

Proof. It is easily checked that if M is an S-module, then Q is an S-submodule of M. In particular, if $x \in Q$, $s \in S$, suppose for the sake of contradiction that $sx \notin Q$. Then there exists $\phi \in \mathrm{Hom}_R(M, R)$ such that $\phi(sx) = 1$, so $\phi(s \cdot -) : M \to R$ maps $x \mapsto 1$, a contradiction. It is also easy to see that if $M \simeq R^{\oplus a} \oplus N$, where N has no free R-module summands, then $Q \simeq m^{\oplus a} \oplus N$, so that $l(M/Q) = l((R/m)^{\oplus a}) = a$ is the free rank of M as an R-module. □

Remark 2.1.9. The F-signature of a local ring (R, m, k) may also be characterized as follows ([Tuc11], Prop 4.5): define $I_e \subset R$ to be the ideal

$$I_e = \{r \in R \mid \forall \phi \in \mathrm{Hom}_R(R^{1/p^e}, R), \phi(r^{1/p^e}) \in m\}.$$

In other words, I_e is the ideal of elements of R whose p^eth roots do not generate a free summand of R^{1/p^e}. Lemma 2.1.8 implies that $a_e = l(R/I_e)$, so

$$s(R) = \lim_{e \to \infty} \frac{l(R/I_e)}{p^{ed}}.$$

The F-signature of a finitely generated R-module may be characterized similarly. Define $N_e \subset M$ to be the submodule $\{x \in M \mid \forall \phi \in \operatorname{Hom}_R(F^e_* M, R), \phi(F^e_* x) \in m\}$. Then

$$s_R(M) = \lim_{e \to \infty} \frac{l(M/N_e)}{p^{ed}}.$$

Lemma 2.1.10. *Let (R, m) be a local or $\mathbb{N}$-graded ring as in Remark 2.1.1. Let M be a finitely generated R-module (which is graded if R is graded). Then the free rank of M may be characterized as the length $l(\operatorname{Hom}_R(M, R)/\operatorname{Hom}_R(M, m))$. It follows that in the notation of Definition 2.1.6,*

$$a_e^{M,R} = l(\operatorname{Hom}_R(F^e_* M, R)/\operatorname{Hom}_R(F^e_* M, m)).$$

Proof. We will prove this statement in the case that R is $\mathbb{N}$-graded over a field, referring the reader to ([Tuc11], Proposition 4.5) for the local case. Suppose that as a graded R-module, $M \simeq_{R-mod} R^{\oplus a} \oplus N$, where N has no graded free summands, hence no free summands as an ungraded R-module by Lemma 2.1.11 below. Then $\operatorname{Hom}_R(M, R) \simeq R^{\oplus a} \oplus \operatorname{Hom}_R(N, R)$. Since N has no free summands, for all $\phi \in \operatorname{Hom}_R(N, R)$, $\operatorname{im} \phi$ does not contain any nonzero elements of degree zero. It follows that $\operatorname{im} \phi \subset m$, so that $\operatorname{Hom}_R(N, R) = \operatorname{Hom}_R(N, m)$. Thus, $\operatorname{Hom}_R(M, m) \simeq m^{\oplus a} \oplus \operatorname{Hom}_R(N, R)$. Taking quotients, we find that $\operatorname{Hom}_R(M, R)/\operatorname{Hom}_R(M, m) \simeq (R/m)^{\oplus a}$, so that $l_R(\operatorname{Hom}_R(M, R)/\operatorname{Hom}_R(M, m))$ is the free rank of M as an R-module. The statement of the lemma follows. □

Lemma 2.1.11. *Let A be an abelian semigroup and R an A-graded ring whose zeroth graded piece is ring-isomorphic to a field k. Let M be a finitely generated graded R-module. Then M has a free summand as a graded R-module if and only if it has a free summand as an ungraded R-module.*

Proof. Suppose that M has a free summand as an ungraded R-module. Let $x \in M$ be a generator of this summand. Then there exists a surjective R-module homomorphism $\phi \in \mathrm{Hom}_R(M, R)$ such that $\phi(x) = 1$. Now we decompose ϕ and x as sums of homogeneous terms: $\phi = \sum_d \phi_d$, $x = \sum_e x_e$. Then $\sum_{d,e} \phi_d(x_e) = 1$. In particular, for some d and e, $\phi_d(x_e)$ is a nonzero degree zero element of R. Hence, $\phi_d(x_e)$ is a unit, so x_e generates an A-graded free summand of R. $\square$

2.2 F-Signature of Graded Rings

In this section, we compare the local and graded versions of the F-signature. In particular, we show that if R is $\mathbb{N}$-graded with homogeneous maximal ideal m, then the (graded) F-signature of R and the (local) F-signature of R_m are equal. Moreover, the ideal $I_e \subset R_m$ has a "graded counterpart" $I_e^{gr} \subset R$ which may be used to define F-signature in the graded category.

Theorem 2.2.1. *Let R be an $\mathbb{N}$-graded ring over a field k as in Remark 2.1.1, with homogeneous maximal ideal m. Then $s(R) = s(R_m)$.*

Proof. We apply Lemma 2.2.2, which demonstrates the existence of a sequence of ideals I_e^{gr} which serve an analogous role to the ideals I_e in the local case. In particular, Lemma 2.2.2.5 shows that the sequences $a_e(R)$ and $a_e(R_m)$ are equal. It follows that the F-signatures $s(R)$ and $s(R_m)$ are equal. $\square$

Lemma 2.2.2. *Let R be an $\mathbb{N}$-graded ring over a field k as in Remark 2.1.1, with homogeneous maximal ideal m. Set $I_e^{gr} = \{r \in R \mid \forall \phi \in \mathrm{Hom}_R(R^{1/p^e}, R), \phi(r^{1/p^e}) \in m\}$. Let $I_e \subset R_m$ be the ideal defined in Remark 2.1.9, and $i : R \to R_m$ the natural localization map. Then:*

1. *$F_*^e I_e^{gr}$ is the kernel of the map $\psi : F_*^e R \to \mathrm{Hom}_R(\mathrm{Hom}_R(F_*^e R, R), R/mR)$ given by $x \mapsto [\phi \mapsto \phi(x) + mR]$.*

2. *I_e^{gr} is homogeneous.*

3. *$I_e^{gr} = i^{-1}I_e = I_e \cap R$.*

4. *If $a_e(R)$ is defined as in 2.1.4, then $a_e(R) = l(F_*^e R / F_*^e I_e^{gr})$.*

5. *For each e, $a_e(R) = a_e(R_m)$.*

Proof. 1. This follows immediately from the definition.

2. Since $F_*^e R$, R, and R/m are graded R-modules, so are the modules of homomorphisms between them. The map ψ from (1) is degree-preserving, that is, it is homogeneous of degree zero. It follows that $F_*^e I_e^{gr} = \ker \psi$ is a graded submodule. Hence, I_e^{gr} is homogeneous.

3. Clearly $F_*^e I_e = \ker(\psi \otimes_R R_m)$. It follows that $F_*^e I_e \simeq F_*^e I_e^{gr} \otimes_R R_m$ (by the flatness of localization). Thus, $I_e = I_e^{gr} R_m = i(I_e^{gr})$. Also, $i : R \to R_m$ is injective. (If $i(x) = 0$, then $wx = 0$ for $w \notin m$. If x_i is the lowest-degree term of x, and w_0 is the degree-zero term of w, then $w_0 x_i = 0$, but $0 \neq w_0 \in k$, so $x_i = 0$. Thus, $x = 0$.) We conclude that $I_e^{gr} = I_e \cap R$.

4. Suppose $F_*^e R = R^{\oplus a_e} \oplus M_e$, where the decomposition is graded and M_e has no graded free summands. We will show that $F_*^e I_e^{gr} = mR^{\oplus a_e} \oplus M_e$, from which the claim follows immediately. Since I_e^{gr} is homogeneous, to compute I_e^{gr} or $F_*^e I_e^{gr}$, we need only check which homogeneous elements they contain. Suppose $x \in F_*^e I_e^{gr}$ is homogeneous. Then $\phi(x) \notin m$ for some ϕ. Without loss of generality, we may assume that ϕ is homogeneous. Then $\phi(x)$ is homogeneous and not in m, hence $\phi(x) \in k$, so x generates a direct summand Rx of $F_*^e R$. This cannot occur if $x \in mR^{\oplus a_e} \oplus M_e$, so $mR^{\oplus a_e} \oplus M_e \subset F_*^e I_e^{gr}$. On the other hand, suppose x is homogeneous and $x \notin mR^{\oplus a_e} \oplus M_e$. This occurs precisely

when x is a generator of one of the copies of R in the decomposition. In that case, clearly $x \notin F^e_* I^{gr}_e$. We conclude that $F^e_* I^{gr}_e = mR^{\oplus a_e} \oplus M_e$.

5. It is easily checked that $m^{[p^e]} R_m \subset I_e$, so $m^{[p^e]} R \subset I^{gr}_e$. Thus, $\operatorname{Rad} I^{gr}_e = m$, so $R/I^{gr}_e \simeq R_m / I^{gr}_e R_m$ as k-vector spaces. We conclude that $l(R/I^{gr}_e) = l(R_m/I_e)$. It follows that $a_e(R) = l(F_* R/F_* I^{gr}_e) = l(F_* R_m / F_* I_e) = a_e(R_m)$, as we desired to show.

□

2.3 F-Signature of Products

Now we will show that the F-signature of a product of varieties (i.e., the F-signature a tensor product of rings over the appropriate field) is the product of the F-signatures. We will give proofs in both the local and graded cases.

Theorem 2.3.1. *Let A and B be abelian semigroups (e.g., $\mathbb{N}$ or $\mathbb{Z}^n$). Let R and S be A- and B-graded rings, respectively, over a field k, such that the zeroth graded piece of each is ring-isomorphic to k. Then $s(R \otimes_k S) = s(R) \cdot s(S)$.*

Proof. First, note that since k is perfect, $(R \otimes_k S)^{1/p^e} \simeq R^{1/p^e} \otimes_k S^{1/p^e}$ by Lemma 2.3.3. Suppose that as graded R- and S-modules, $R^{1/p^e} \simeq_{R-mod} R^{\oplus a_e} \oplus M_e$, and $S^{1/p^e} \simeq_{S-mod} S^{\oplus b_e} \oplus N_e$. As an $(A \oplus B)$-graded $(R \otimes_k S)$-module,

$$\begin{aligned}(R \otimes_k S)^{1/p^e} &\simeq (R^{\oplus a_e} \oplus M_e) \otimes_k (S^{\oplus b_e} \oplus N_e) \\ &\simeq (R \otimes_k S)^{\oplus a_e b_e} \oplus (R \otimes_k N_e)^{\oplus a_e} \oplus (M_e \otimes_k S)^{\oplus b_e} \oplus (M_e \otimes_k N_e).\end{aligned}$$

By Lemma 2.3.2, $(R \otimes_k N_e)$, $(M_e \otimes_k S)$, and $(M_e \otimes_k N_e)$ have no free summands as $(A \oplus B)$-graded $(R \otimes_k S)$-modules. It follows immediately that the free rank of $(R \otimes_k S)^{1/p^e}$ is $a_e b_e$. Since $\dim(R \otimes_k S) = \dim R + \dim S$, we conclude that $s(R \otimes_k S) = \lim_{e \to \infty} \frac{a_e b_e}{p^{e(\dim R + \dim S)}} = s(R) \cdot s(S)$. □

Lemma 2.3.2. *Let A, B be abelian semigroups, and let R and S be A- and B-graded rings, respectively, over a field k, such that the zeroth graded piece of each is ring-isomorphic to k. Let M and N be finitely generated graded R- and S-modules, respectively. Suppose $M \otimes_k N$ has a free summand as an $(A \oplus B)$-graded $R \otimes_k S$-module. Then both M and N have free summands as graded R- and S-modules.*

Proof. Suppose that $M \otimes_k N$ has a free summand as an $(A \oplus B)$-graded $R \otimes_k S$-module. Then we have a homogeneous map $\phi : M \otimes_k N \twoheadrightarrow R \otimes_k S$. In particular, $1 \in \operatorname{im} \phi$. Since $\operatorname{im} \phi$ is generated by the images of homogeneous simple tensors in $M \otimes_k N$, it follows that there is an homogeneous simple tensor $x \otimes y \in M \otimes_k N$ such that $\deg \phi(x \otimes y) = 0$ but $\phi(x \otimes y) \neq 0$. The degree zero part of $R \otimes_k S$ is isomorphic to k, so (after replacing ϕ by $\frac{1}{\phi(x \otimes y)}\phi$) we may assume that $\phi(x \otimes y) = 1 \in R \otimes_k S$.

Now, 1 generates a free summand $R \otimes 1$ of the free R-module $R \otimes_k S$, so there is an R-module map $\psi : R \otimes_k S \twoheadrightarrow R$ sending $\phi(x \otimes y) \mapsto 1$. Consider the map $\psi \circ \phi : M \to R \otimes_k S \to R$. This map sends $x \mapsto \phi(x \otimes y) \mapsto 1$. We conclude that M has a free summand as an R-module. By a symmetric argument, N has a free summand also. □

Lemma 2.3.3. *Let R and S be reduced k-algebras, where k is a perfect field. Then as $R \otimes_k S$-modules, $(R \otimes_k S)^{1/p^e} \simeq R^{1/p^e} \otimes_k S^{1/p^e}$.*

Proof. Since k is perfect, $R \otimes_k S$ is reduced (see, e.g., [Bou03], Chapter 5, Theorem 15.5.3). Consider the $R \otimes_k S$-module map $\phi : R^{1/p^e} \otimes_k S^{1/p^e} \to (R \otimes_k S)^{1/p^e}$ given by $a \otimes b \mapsto (a^{p^e} \otimes b^{p^e})^{1/p^e}$. Clearly ϕ is surjective. Since $R \otimes_k S$ is reduced, ϕ is injective as well. We conclude that ϕ is an isomorphism.

□

As for the local case:

Theorem 2.3.4. *Suppose (R, m_R, k) and (S, m_S, k) are local rings, each of which is a k-algebra. Let $m \subset R \otimes_k S$ be the maximal ideal $m_R \otimes_k S + R \otimes_k m_S$. Suppose that $(R \otimes_k S)_m$ is Noetherian. Then $s((R \otimes_k S)_m) = s(R) \cdot s(S)$.*

Proof. Since k is perfect, $(R \otimes_k S)^{1/p^e} \simeq R^{1/p^e} \otimes_k S^{1/p^e}$ by Lemma 2.3.3. Suppose that $R^{1/p^e} \simeq_{R-mod} R^{\oplus a_e} \oplus M_e$, and $S^{1/p^e} \simeq_{S-mod} S^{\oplus b_e} \oplus N_e$. As $(R \otimes_k S)_m$-modules,

$$\begin{aligned} (R \otimes_k S)_m^{1/p^e} &\simeq ((R^{\oplus a_e} \oplus M_e) \otimes_k (S^{\oplus b_e} \oplus N_e))_m \\ &\simeq (R \otimes_k S)_m^{\oplus a_e b_e} \oplus (R \otimes_k N_e)_m^{\oplus a_e} \oplus (M_e \otimes_k S)_m^{\oplus b_e}. \end{aligned}$$

By Lemma 2.3.5, $(R \otimes_k N_e)_m$ and $(M_e \otimes_k S)_m$ have no free summands. It follows immediately that the free rank of $(R \otimes_k S)_m^{1/p^e}$ is $a_e b_e$. Since $\dim(R \otimes_k S)_m = \dim R + \dim S$, we conclude that $s((R \otimes_k S)_q) = \lim_{e\to\infty} \frac{a_e b_e}{p^{e(\dim R + \dim S)}} = s(R) \cdot s(S)$. $\square$

Lemma 2.3.5. *Let (R, m_R, k) and (S, m_S, k) be local rings with residue field k, and suppose $(R \otimes_k S)_m$ is Noetherian. Let $m \subset R \otimes_k S$ be the maximal ideal $m_R \otimes_k S + R \otimes_k m_S$. Let M and N be finitely generated R- and S-modules, respectively. Suppose $(M \otimes_k N)_m$ has a free summand as an $(R \otimes_k S)_m$-module. Then both M and N have free summands as R- and S-modules.*

Proof. Suppose that $(M \otimes_k N)_m$ has a free summand. Then we have an $(R \otimes_k S)_m$-module map $\phi : (M \otimes_k N)_m \twoheadrightarrow (R \otimes_k S)_m$. Since $\phi \in \operatorname{Hom}_{((R\otimes_k S)_m)}((M \otimes_k N)_m, (R \otimes_k S)_m) \simeq (\operatorname{Hom}_{R\otimes_k S}(M \otimes_k N, R \otimes_k S))_m$, by clearing denominators we may assume that ϕ maps $M \otimes_k N \to R \otimes_k S$ so that $\operatorname{im}\phi \not\subset m(R \otimes_k S)$. It follows that there is a simple tensor $x \otimes y \in M \otimes_k N$ such that $\phi(x \otimes y) \notin m(R \otimes_k S)$. In particular, $\phi(x \otimes y) \notin m_R(R \otimes_k S)$. Recall that if A is a free module over a Noetherian local ring (R, m_R), then each element of $A \backslash m_R A$ generates a free summand of A. Thus, $\phi(x \otimes y)$ generates a free summand of the free R-module $R \otimes_k S$, and there is an R-module map $\psi : R \otimes_k S \twoheadrightarrow R$ sending $\phi(x \otimes y) \mapsto 1$.

Now consider the map $\psi \circ \phi : M \to R \otimes_k S \to R$. This map sends $x \mapsto \phi(x \otimes y) \mapsto 1$. We conclude that M has a free summand as an R-module. By a symmetric argument, N has a free summand also. $\square$

2.4 Background on Étale Maps and Reflexive Modules

This section, and the following one on section rings, will prepare us for our investigation into the behavior of the F-signature under finite extensions. A good introduction to étale morphisms and their properties and applications may be found in [Mil80].

Definition 2.4.1. Let $R \to S$ be an extension of rings. We say that $R \to S$ is *étale*, or equivalently that the map $\operatorname{Spec} S \to \operatorname{Spec} R$ is étale, if S is a finitely presented R-algebra and $R \to S$ is flat and unramified. Recall that $R \to S$ is *flat* if S is a flat R-module, and it is *unramified* if the module of relative differentials $\Omega_{S/R}$ is equal to 0.

Definition 2.4.2. Let $R \to S$ be a module-finite extension of rings. The *étale locus* of this extension is the set $U = \{P \in \operatorname{Spec} R \mid R_P \to S \otimes_R R_P \text{ is étale}\}$. We say that $R \to S$ is *étale in codimension one* if the closed set $\operatorname{Spec} R \backslash U$ is codimension two or greater.

We note that étaleness is an open property:

Proposition 2.4.3. *Let $R \to S$ be a module-finite ring extension. Then the étale locus of this extension is open, and the extension is étale if and only if the étale locus is all of* $\operatorname{Spec} R$.

Recall that when $K \hookrightarrow L$ is a finite field extension, the *field trace* $\operatorname{Tr}_{L/K} \in \operatorname{Hom}_K(L, K)$ is the map sending $\alpha \in L$ to the trace of the K-vector space automorphism of L given by multiplication by α.

Lemma 2.4.4. *([ST12], Prop. 7.4) Let $R \hookrightarrow S$ be a module-finite, étale in codimension one extension of normal local rings. Then $\mathrm{Hom}_R(S, R)$ is a free S-module of rank one, generated by the restriction to S of the trace map of function fields from* $\mathrm{Frac}(S) \to \mathrm{Frac}(R)$.

Proposition 2.4.5 ([HH90], 6.3)**.** *Suppose that $R \to S$ is an étale (or, more generally, smooth), module-finite ring map. Then the natural map $R^{1/q} \otimes_R S \to S^{1/q}$ is an isomorphism.*

We also recall some well-known facts about reflexive modules.

Definition 2.4.6. Let R be a ring and M an R-module. Then M is *reflexive* if the natural map $M \to \mathrm{Hom}_R(\mathrm{Hom}_R(M, R), R)$ is an isomorphism.

Proposition 2.4.7. *([Har94], Corollary 1.8) Let R be a domain and M a finitely generated R-module. Then $\mathrm{Hom}_R(M, R)$ is a reflexive R-module. More generally, if N is a reflexive R-module, then so is $\mathrm{Hom}_R(M, N)$.*

Theorem 2.4.8. *([Har94], Theorem 1.9) Let R be a normal ring and M a finitely generated R-module. Then M is reflexive if and only if it satisfies Serre's S_2 condition: that is, for each prime of R, depth $M_P \geq \min(2, ht\ P)$. In particular, if $R \to S$ is a module-finite inclusion of normal rings, then S is a reflexive R-module.*

Proposition 2.4.9. *([Har94], Proposition 1.11) Let X be a normal scheme, with $U \subset X$ an open subset whose complement is codimension at least two. Let $\mathfrak{F}$ be a reflexive sheaf on U. Then $\mathfrak{F}$ extends uniquely to a reflexive sheaf on X, namely $i_*\mathfrak{F}$. In particular, if $X = \mathrm{Spec}\, R$, M and N are reflexive R-modules, and $f : M \to N$ is an R-module homomorphism which is an isomorphism on an open set $U \subset \mathrm{Spec}\, R$ whose complement is codimension at least two, then f is an R-module isomorphism.*

Let X be a normal variety with function field $K(X)$ and D a Weil divisor on X. We denote by $O_X(D)$ the O_X-module given by $O_X(D)(U) = \{f \in K(X) : (\text{div } f + D)_U \geq 0\}$. Then:

Proposition 2.4.10 (see, e.g., [CLS11], Theorem 8.0.4)**.** *Let X be a normal variety and D a Weil divisor on X. Then $O_X(D)$ is a reflexive O_X-module.*

2.5 Background on Section Rings and Veronese Subrings

Excellent discussions of section rings and their properties may be found in [HS04], §2 and in [Smi97]. Recall:

Definition 2.5.1. A $\mathbb{Q}$-*divisor* on a variety X is a formal $\mathbb{Q}$-linear sum of prime Weil divisors. A $\mathbb{Q}$-divisor is $\mathbb{Q}$-*Cartier* if some integer multiple of D is an integral Weil divisor which is Cartier.

Definition 2.5.2. Let X be a normal algebraic variety. Let $\mathfrak{L}$ be an ample invertible sheaf on X. Then the *section ring* $\text{Sec}(X, \mathfrak{L})$ of $\mathfrak{L}$ is the finitely generated graded ring $\bigoplus_n H^0(X, \mathfrak{L}^n)$. (Here, multiplication is induced by the tensor product map $\mathfrak{L}^i \times \mathfrak{L}^j \to \mathfrak{L}^{i+j} : (s,t) \mapsto s \otimes t$.)

Definition 2.5.3. Let S be a finitely generated $\mathbb{N}$-graded ring. We will say that S *is a section ring* if $S \simeq \text{Sec}(X, \mathfrak{L})$ for some projective variety X and ample invertible sheaf $\mathfrak{L}$ on X.

Definition 2.5.4. Let X be a normal algebraic variety and D any $\mathbb{Q}$-divisor on X. The *generalized section ring* of D is taken to be $\bigoplus_n H^0(X, O_X(nD))$, where $O_X(nD)$ is the reflexive sheaf given by $O_X(nD)(U) = \{f \in K(X) : (\text{div } f + nD)|_U \geq 0\}$.

The following is well-known:

Proposition 2.5.5. *(cf. [HS04], §2) Let X be a projective variety and $\mathfrak{L}$ an ample invertible sheaf on X. Then* $\mathrm{Sec}(X, \mathfrak{L})$ *is normal if and only if X is normal.*

More generally, we have:

Proposition 2.5.6. *Let X be a normal variety and $\mathfrak{L}$ an integral Weil divisor on X. Then* $\mathrm{Sec}(X, D)$ *is normal.*

Proof. Let $X_{reg} \subset X$ be the regular locus of X. Since X is normal, X_{reg} is an open subset of X whose compliment has codimension ≥ 2. Since $O_X(nD)$ is reflexive by Proposition 2.4.10, it follows that $\Gamma(X, O_X(nD)) \simeq \Gamma(X_{reg}, O_X(nD))$, and $\mathrm{Sec}(X, D) \simeq \mathrm{Sec}(X_{reg}, D|_{X_{reg}})$. Without loss of generality, set $X = X_{reg}$ and $D = D|_{X_{reg}}$, so that D is a Cartier divisor on X.

Recall that the normalization of an $\mathbb{N}$-graded (not necessarily Noetherian) ring is also $\mathbb{N}$-graded ([HS06], Theorem 2.3.2). Thus, to prove that $\mathrm{Sec}(X, D)$ is normal, it suffices to show that if $f, g \in \mathrm{Sec}(X, D)$ are homogeneous elements with $\deg f \geq \deg g$, and $\frac{f}{g}$ is integral over $\mathrm{Sec}(X, D)$, then $\frac{f}{g} \in \mathrm{Sec}(X, D)$.

Suppose that $f \in \Gamma(X, O_X((m+d)D))$ and $g \in \Gamma(X, O_X(mD))$, where $d \geq 0$, and that $\frac{f}{g}$ is integral over $\mathrm{Sec}(X, D)$. Fix an open set $U \subset X$ on which $O_X(D)$ is trivial, say $O_X(D)|_U = O_X|_U \cdot \tau \subset K(X)$, and $O_X(rD)|_U = O_X|_U \cdot \tau^r$ for any $r \in \mathbb{Z}$. For some $a, b \in O_X(U)$, $f|_U = a\tau^{m+d}$ and $g|_U = b\tau^m$, so that $\frac{f}{g} = \frac{a}{b}\tau^d$. By hypothesis, $\frac{f}{g}$ satisfies a monic polynomial $F(x) = x^n + \sum_{n>i\geq 0} s_i x^i = 0$, where $s_i \in \mathrm{Sec}(X, D)$. Without loss of generality we may assume that each s_i is homogeneous of degree $d(n-i)$, so that for some $c_i \in O_X(U)$, $s_i|_U = c_i\tau^{d(n-i)}$. Then $\frac{a}{b}\tau^d$ satisfies $F(x)$, so

$$(\frac{a}{b})^n \tau^{dn} + \sum_{n>i\geq 0} c_i \tau^{d(n-i)} (\frac{a}{b})^i \tau^{di} = 0.$$

Dividing through by τ^{dn}, we see that $\frac{a}{b}$ is integral over $O_X(U)$. Since X is normal, we conclude that $\frac{a}{b} \in O_X(U)$, so that $\frac{f}{g} \in O_X(U) \cdot \tau^d = \Gamma(U, O_X(dD))$. Since D is

Cartier, this holds for open sets U covering X, so $\frac{f}{g} \in \Gamma(X, O_X(dD)) \subset \mathrm{Sec}(X, D)$, as we desired to show. $\square$

In subsequent sections we will take advantage of the special properties of section rings. In particular:

Proposition 2.5.7. *(cf. [HS04], Prop 2.1.4) Let X be a projective variety and $\mathfrak{L}$ an ample invertible sheaf on X. Set $S = \mathrm{Sec}(X, \mathfrak{L})$. Fix any homogeneous element $x \in S$ such that $\mathfrak{L}$ is free on $D_+(x)$, and any degree-one element $t \in S_x$. Then there is an isomorphism of $\mathbb{Z}$-graded rings $S_x \simeq (S_x)_0[t, t^{-1}]$, where $(S_x)_0$ denotes the subring of degree zero elements of S_x. The set of such homogeneous elements $x \in S$ generates an m_S-primary ideal of S.*

In §2.6, we will be interested in computing the F-signature of the *nth Veronese subring* of a section ring.

Notation 2.5.8. Let R be an $\mathbb{N}$-graded ring and $n \in \mathbb{N}$. We denote by $R^{(n)}$ the nth Veronese subring of R. That is, $R^{(n)}$ is the subring of R generated by elements of degree congruent to $0 \pmod{n}$.

Note that $\mathrm{Sec}(X, \mathfrak{L}^n) = \mathrm{Sec}(X, \mathfrak{L})^{(n)}$ as rings, though not as graded rings: the degree k component of $\mathrm{Sec}(X, \mathfrak{L}^n)$ is the degree kn component of $\mathrm{Sec}(X, \mathfrak{L})$.

Corollary 2.5.9. *Let X be a projective variety and $\mathfrak{L}$ an ample invertible sheaf on X. Set $S = \mathrm{Sec}(X, \mathfrak{L})$. Fix $n \in \mathbb{N}$. Choose $x \in S$ to which Proposition 2.5.7 applies. Then under the same notation as in Proposition 2.5.7, $S^{(n)}_{x^n} \simeq (S_x)_0[t^n, t^{-n}]$.*

Proof. We apply Proposition 2.5.7 to the sheaf $\mathfrak{L}^n$. Since $\mathfrak{L}$ is free on $D_+(x)$, so is $\mathfrak{L}^n$. Since t^n is a degree one element of $S^{(n)}$, we have $S^{(n)}_{x^n} \simeq (S^{(n)}_{x^n})_0[t^n, t^{-n}]$. Moreover, $(S_x)_0 \simeq (S^{(n)}_{x^n})_0 \simeq O_X(D_+(x^n))$. Thus, $S^{(n)}_{x^n} \simeq (S_x)_0[t^n, t^{-n}]$. $\square$

All normal graded rings are somewhat close to being section rings, up to taking Veronese subrings:

Proposition 2.5.10. *([Smi97], §1) Let S be any normal, $\mathbb{N}$-graded, finitely generated k-algebra. Choose n sufficiently large and divisible so that $S^{(n)}$ is finitely generated in degree one. Then $S^{(n)}$ is the section ring of the very ample invertible sheaf $O(1)$ on $\operatorname{Proj} S^{(n)} \simeq \operatorname{Proj} S$.*

2.6 Behavior of the F-Signature Under Finite Extensions

We next give a formula for the F-signature of a Veronese subring of a graded ring (Theorem 2.6.2). This formula will be useful in Chapter IV when we consider the F-signature of a line bundle on a projective variety. We also give a formula for the behavior of the F-signature under certain extensions which are finite and étale in codimension one (Theorem 2.6.5). Both formulas are consequences of the following:

Lemma 2.6.1. *Let $(R, m_R, k) \to (S, m_S, k)$ be a module-finite inclusion of local rings, where the residue field k has positive characteristic. Suppose that the free rank of S as an R-module is equal to one; that $\operatorname{Hom}_R(S, R)$ is a free rank-one S-module, generated by $\Phi \in \operatorname{Hom}_R(S, R)$; and that for all $e \in \mathbb{N}$, the natural S^{1/p^e}-module map*

$$\Phi^* : \operatorname{Hom}_S(S^{1/p^e}, S) \to \operatorname{Hom}_R(S^{1/p^e}, R),$$

defined by $\psi \mapsto \Phi \circ \psi$, is an isomorphism. Then $s(R) = \frac{1}{[\operatorname{Frac}(S):\operatorname{Frac}(R)]} \cdot s(S)$.

Proof. By Theorem 2.1.7, $s_R(S) = (\dim_{\operatorname{Frac}(R)} S \otimes_R \operatorname{Frac}(R)) s(R) = [\operatorname{Frac}(S) : \operatorname{Frac}(R)] s(R)$. Thus, to prove the lemma, it suffices to show that $s_R(S) = s_S(S)$. In particular, we will show that $a_e^{S,R} = a_e^S$. By Remark 2.1.10,

$$a_e^{S,R} = l_R(\operatorname{Hom}_R(S^{1/p^e}, R) / \operatorname{Hom}_R(S^{1/p^e}, m_R)),$$

and

$$a_e^S = l_S(\operatorname{Hom}_S(S^{1/p^e}, S)/\operatorname{Hom}_S(S^{1/p^e}, m_S)).$$

We wish to show that

$$\operatorname{Hom}_R(S^{1/p^e}, R)/\operatorname{Hom}_R(S^{1/p^e}, m_R) \simeq_{S^{1/p^e}-mod} \operatorname{Hom}_S(S^{1/p^e}, S)/\operatorname{Hom}_S(S^{1/p^e}, m_S),$$

so that these S^{1/p^e}-modules have the same length. If so, they will have the same length as S-modules. Since R and S have the same residue field, $R-$ and S-module length are equivalent, so it will follow that $a_e^{S,R} = a_e^S$.

By our hypotheses, $\operatorname{Hom}_R(S^{1/p^e}, R) \simeq_{S^{1/p^e}-mod} \operatorname{Hom}_S(S^{1/p^e}, S)$ via Φ^*. It remains only to check that Φ^* identifies $\operatorname{Hom}_R(S^{1/p^e}, m_R)$ with $\operatorname{Hom}_S(S^{1/p^e}, m_S)$. The preimage of $\operatorname{Hom}_R(S^{1/p^e}, m_R)$ under Φ^* is the S-module

$$\{\psi \in \operatorname{Hom}_S(S^{1/p^e}, S) : \Phi(\operatorname{im}\psi) \subset m_R\}.$$

For each such ψ, $\operatorname{im}\psi$ is an S-module. Thus, if $\Phi(\operatorname{im}\psi) \subset m_R$, then for all $s \in \operatorname{im}\psi, \Phi(sS) \subset m_R$. Let $J \subset S$ be the ideal $\{s \in S \mid \Phi(sS) \subset m_R\} = \{s \in S \forall \phi \in \operatorname{Hom}_R(S, R), \phi(s) \in m_R\}$. By Lemma 2.1.8, $J \subset S$ is an ideal, and $l_R(S/J)$ is the free rank of S as an R-module, which is one. Thus, $S/J \simeq_{R-mod} k$, so $J = m_S$. We conclude that $\operatorname{Hom}_R(S^{1/p^e}, m_R) \simeq \operatorname{Hom}_S(S^{1/p^e}, J) = \operatorname{Hom}_S(S^{1/p^e}, m_S)$, as we desired to show. □

Now we are ready to prove our two F-signature formulas.

Theorem 2.6.2. *Let S be a normal section ring over a field of positive characteristic, of dimension at least two. Let n be any positive integer. Then $s(S^{(n)}) = \frac{1}{n}s(S)$.*

Proof. Our goal is to apply Lemma 2.6.1. By Lemma 2.6.3 below, the free rank of S as an $S^{(n)}$-module is equal to one. By Lemma 2.5.7, the generic rank of S as an $S^{(n)}$-module is equal to n. Let $\Phi \in \operatorname{Hom}_{S^{(n)}}(S, S^{(n)})$ be the $S^{(n)}$-module homomorphism defined as follows: for $x \in S$ homogeneous of degree d,

$$\Phi(x) = \begin{cases} x & n | \deg x \\ 0 & n \nmid \deg x \end{cases}$$

Consider the S^{1/p^e}-module map $\Phi^* : \mathrm{Hom}_S(S^{1/p^e}, S) \to \mathrm{Hom}_{S^{(n)}}(S^{1/p^e}, S^{(n)})$, given by $\phi \mapsto \Phi \circ \phi$. In order to apply Lemma 2.6.1, it remains to show that Φ^* is an isomorphism for each $e \geq 0$.

The $S^{(n)}$-modules $\mathrm{Hom}_S(S^{1/p^e}, S)$ and $\mathrm{Hom}_{S^{(n)}}(S^{1/p^e}, S^{(n)})$ are reflexive by Proposition 2.4.7 and Theorem 2.4.8, so by Proposition 2.4.9, it suffices to show that Φ^* is an isomorphism in codimension one. Clearly, Φ^* is injective, since no ideal of S lies in the kernel of Φ, so we need only show that it is surjective in codimension one. Since $\dim S \geq 2$, it suffices to show surjectivity away from the homogeneous maximal ideal $m_S \in \mathrm{Spec}\, S^{(n)}$.

Fix $x \in S$ to which Lemma 2.5.7 applies, so that $S_x \simeq (S_x)_{[0]}[t, t^{-1}]$. Set $T = (S_x)_{[0]}$, so that $S_x \simeq T[t, t^{-1}]$. By Corollary 2.5.9, $S^{(n)}_{x^n} \simeq T[t^n, t^{-n}]$. Under these isomorphisms, we view Φ^* as a $T^{1/p^e}[x^{1/p^e}, \frac{1}{x^{1/p^e}}]$-module map,

$$\Phi^* : \mathrm{Hom}_{T[x, \frac{1}{x}]}(T^{1/p^e}[x^{1/p^e}, \frac{1}{x^{1/p^e}}], T[x, \frac{1}{x}]) \to \mathrm{Hom}_{T[x^p, \frac{1}{x^p}]}(T^{1/p^e}[x^{1/p^e}, \frac{1}{x^{1/p^e}}], T[x^p, \frac{1}{x^p}]).$$

The image of a homomorphism ϕ under this map is the homomorphism $\Phi \circ \phi$, which applies ϕ to an input, then kills all terms whose degree in x does not divide n. Given $\phi \in \mathrm{Hom}_{T[x^p, \frac{1}{x^p}]}(T^{1/p^e}[x^{1/p^e}, \frac{1}{x^{1/p^e}}], T[x^p, \frac{1}{x^p}])$, set $\overline{\phi} = \Sigma_{0 \leq i < p} x^i \phi(\frac{1}{x^i} \cdot -)$. We leave it to the reader to check that $\overline{\phi} \in \mathrm{Hom}_{T[x, \frac{1}{x}]}(T^{1/p^e}[x^{1/p^e}, \frac{1}{x^{1/p^e}}], T[x, \frac{1}{x}])$, and $\Phi \circ \overline{\phi} = \phi$. It follows that Φ^* is surjective on $D(x^n) \subset \mathrm{Spec}\, S^{(n)}$. By Lemma 2.5.7, these sets $D(x^n)$ cover $\mathrm{Spec}\, S^{(n)} \backslash \{m_{S^{(n)}}\}$. Thus, by our reflexivity argument, Φ^* is an isomorphism. $\square$

Lemma 2.6.3. *Let S be a normal section ring of dimension at least two. Let n be a positive integer. Then the free rank of S as an $S^{(n)}$-module is equal to one.*

Proof. For $0 \leq i < n$, let $S_{i \pmod n}$ be the $S^{(n)}$-submodule of S generated by homogeneous elements of degree congruent to $i \pmod n$. Clearly,

$$S \simeq_{S^{(n)}-mod} \bigoplus_{0 \leq i < n} S_{i \pmod n} = S^{(n)} \oplus \Big(\bigoplus_{0 < i < n} S_{i \pmod n} \Big).$$

It suffices to check that for $i > 0$, $S_{i \pmod n}$ has no free $S^{(n)}$-module summands.

Fix $0 < i < n$. Note first that for each i, $S_{i \pmod n}$ is an $S^{(n)}$-module of generic rank equal to one. To see this, fix any homogeneous $f \in S$ of degree congruent to i $\pmod n$. Then the map $S^{(n)}_{f^n} \to (S_{i \pmod n})_{f^n}$ given by $s \mapsto fs$ is an isomorphism. It follows that $S_{i \pmod n}$ has a free summand if and only if $S_{i \pmod n} \simeq S^{(n)}$ as a graded $S^{(n)}$-module.

Suppose for the sake of contradiction that $S_{i \pmod n} \simeq S^{(n)}$, generated by a homogeneous element $x \in S$ of degree i. Then $S_{i \pmod n}$ generates an ideal I of S which is contained in the principal ideal xS. It follows that I is of height one. On the other hand, $S_{i \pmod n}$ includes all elements of S of degree i. It follows that $m_S^i \subset I$, so I is of height equal to the dimension of S, contradicting our assumption that $\dim S \geq 2$. □

Remark 2.6.4. Theorem 2.6.2 does not hold in the one-dimensional case. If $S = k[x]$, then for any n, $S^{(n)} = k[x^n]$; the free rank of S as an $S^{(n)}$-module is equal to n; and $s(S^{(n)}) = s(S) = 1$. Likewise, if S is not a section ring, then the theorem may not hold. For example, if $S = k[x, y^2]$, then $S^{(2)} = k[x^2, y^2]$, and we see that $s(S) = s(S^{(2)}) = 1$.

Next, we prove a similar F-signature formula for certain ring extensions which are étale in codimension one.

Theorem 2.6.5. *Let $(R, m_R, k) \to (S, m_S, k)$ be a module-finite inclusion of normal local rings, where the residue field k has positive characteristic. Suppose that $R \to S$ is split and étale in codimension one, and the free rank of S as an R-module is equal to one. Then $s(R) = \frac{1}{[\mathrm{Frac}(S):\mathrm{Frac}(R)]} \cdot s(S)$.*

Proof. Our goal is to apply Lemma 2.6.1. By Lemma 2.4.4, $\mathrm{Hom}_R(S, R) \simeq_{S-mod} S$. Fix $e > 0$. Let Φ be an S-module generator for $\mathrm{Hom}_R(S, R)$. Consider the S^{1/p^e}-module map $\Phi^* : \mathrm{Hom}_S(S^{1/p^e}, S) \to \mathrm{Hom}_R(S^{1/p^e}, R)$ given by $\phi \mapsto \Phi \circ \phi$. In order to apply Lemma 2.6.1, we wish to show that Φ^* is an isomorphism for each $e \geq 0$. Since $R \hookrightarrow S$ splits, $\mathrm{Hom}_R(S, R)$ contains a surjective homomorphism. In particular, Φ is surjective, so Φ^* is injective.

To check surjectivity, we first give an S-module isomorphism from $\mathrm{Hom}_R(S^{1/p^e}, R)$ to $\mathrm{Hom}_R(R^{1/p^e} \otimes_R S, R)$. These R-modules are reflexive by Proposition 2.4.7 and Theorem 2.4.8. By Lemma 2.4.5, the S-module map $R^{1/p^e} \otimes_R S \to S^{1/p^e}$ induces an S-module isomorphism between them in codimension one. It follows from Proposition 2.4.9 that the induced map on Hom sets is an R-module isomorphism, hence an S-module isomorphism:

$$\mathrm{Hom}_R(S^{1/p^e}, R) \simeq_{S-mod} \mathrm{Hom}_R(R^{1/p^e} \otimes_R S, R).$$

By the adjointness of tensor and Hom, along with Lemma 2.4.4, we conclude that as R-modules,

$$\begin{aligned} \mathrm{Hom}_R(S^{1/p^e}, R) &\simeq \mathrm{Hom}_R(R^{1/p^e} \otimes_R S, R) \\ &\simeq \mathrm{Hom}_R(R^{1/p^e}, \mathrm{Hom}_R(S, R)) \\ &\simeq \mathrm{Hom}_R(R^{1/p^e}, S). \end{aligned}$$

Likewise, $\mathrm{Hom}_S(S^{1/p^e}, S) \simeq_{S-mod} \mathrm{Hom}_S(R^{1/p^e} \otimes_R S, S)$. Thus, it suffices to show that the induced R-module map $\mathrm{Hom}_S(R^{1/p^e} \otimes_R S, S) \to \mathrm{Hom}_R(R^{1/p^e}, S)$ is surjective. This map, given by $\phi \mapsto [x \mapsto \phi(x \otimes 1)]$, is surjective: a homomorphism $\phi \in \mathrm{Hom}_R(R^{1/p^e}, S)$ has preimage $\phi \otimes 1$. $\square$

Remark 2.6.6. When $R \to S$ is finite local, and f is the free rank of S as an R-module, then $s(R) \leq \frac{f}{[\mathrm{Frac}(S):\mathrm{Frac}(R)]} s(S)$ ([Tuc11]). Theorem 2.6.5 proves that this inequality is an equality when $R \to S$ is étale in codimension one, in the special case $f = 1$.

Remark 2.6.7. Theorem 2.6.2 is *almost* a special case of the graded version of Theorem 2.6.5. When $p \nmid n$, the inclusion $S^{(n)} \hookrightarrow S$ is étale in codimension one. However, when p divides n, this inclusion is not étale in codimension one, so our Veronese F-signature formula does not follow from Theorem 2.6.5.

2.7 F-Signature of Pairs and Triples

The F-signature of pairs and triples was recently defined in [BST11]. First, we recall:

Definition 2.7.1. Let R be a normal domain. Let D be a Weil divisor on $X = \operatorname{Spec} R$. We define $R(D)$ to be the module of global sections of $O_X(D)$. That is, $R(D) = \{f \in \mathrm{Frac}(R) \mid \mathrm{div}\ f + D \geq 0\}$.

Remark 2.7.2. Note that when D is effective, $R \subset R(D)$. moreover,if D is the principal divisor div g for some $g \in R$, then $R(D) = R \cdot \frac{1}{g}$ is the cyclic R-module generated by $\frac{1}{g}$.

Now we can define the F-signature of a pair or triple:

Definition 2.7.3. Let (R, m) be a normal local (or $\mathbb{N}$-graded) domain over k, of dimension d. Let $D = \sum_i a_i D_i$ be an effective $\mathbb{Q}$-divisor on $X = \operatorname{Spec} R$, $\mathfrak{a}$ an ideal of R, and $0 \leq t \in \mathbb{R}$. We define the F-signature of the triple $(R, D, \mathfrak{a}^t)$ as follows. For each

e, define $I_e^D \subset R$ to be the ideal $\{r \in R \,|\, \forall \phi \in \mathrm{Hom}_R(R(\lceil (p^e-1)D\rceil)^{1/p^e}, R), \phi(r^{1/p^e}) \in m\}$. Then the F-signature of the pair $s(R,D)$ is $\lim_{e\to\infty} \frac{l(R/I_e^D)}{p^{ed}}$.

Define $I_e^{\mathfrak{a}} \subset R$ to be the ideal $\{r \in R \,|\, \forall \phi \in \mathrm{Hom}_R(R(\lceil (p^e-1)D\rceil)^{1/p^e}, R), \forall a \in \mathfrak{a}^{\lceil (p^e-1)t\rceil}, \phi(a^{1/p^e} r^{1/p^e}) \in m\}$. The F-signature of the triple $s(R,D,\mathfrak{a}^t)$ is defined to be $\lim_{e\to\infty} \frac{l(R/I_e^{\mathfrak{a}})}{p^{ed}}$.

Remark 2.7.4. The limits given in Definition 2.7.3 have been shown to exist in [BST11], in the case of a local ring.

Remark 2.7.5. It is easily checked that the F-signature of the triple $(R, D, (1))$ (with $\mathfrak{a}$ as the unit ideal) is the F-signature of the pair (R, D). Likewise, the F-signature of the pair $(R, 0)$ (with D as the zero divisor) is the F-signature of R.

Just like the "usual" F-signature, the F-signature of pairs may be viewed as a measure of the number of splittings of the Frobenius map, or as a measure of the number of free summands splitting off from R^{1/p^e}, though the F-signature of pairs only counts certain summands:

Lemma 2.7.6. *([BST11], Proposition 3.5.) Suppose that we are in the setting of Definition 2.7.3. Set $a_e^D = l(F_*(R/I_e^D))$, so that $s(R,D) = \lim_{e\to\infty} \frac{a_e^D}{p^{e(d+\alpha)}}$. Then a_e^D is the maximum rank of a free summand of R^{1/p^e} that is simultaneously a free summand of $R(\lceil (p^e-1)D\rceil)^{1/p^e}$. Moreover, any k-vector space basis for $R^{1/p^e}/(I_e^D)^{1/p^e}$ lifts to a set of generators in R^{1/p^e} for such a free summand of maximum rank, and $(I_e^D)^{1/p^e}$ is the submodule of elements of R^{1/p^e} which do not generate such a free summand.*

Minor modifications can be made to the definition, without changing the F-signature ([BST11], Lemma 4.17). In particular:

Lemma 2.7.7. *([BST11], discussion following Lemma 4.17.) Let $R, D, \mathfrak{a}, t$ be as in Definition 2.7.3.*

1. *We may replace $(p^e-1)D$ by p^eD in the limits given in Definition 2.7.3 without changing the F-signature.*

2. *If we write $D = \sum_i a_i D_i$, then $s(R, D, \mathfrak{a}^t)$ is continuous in the a_i.*

3. *Likewise, we may replace $(p^e-1)t$ with p^et in the definition without changing $s(R, D, \mathfrak{a}^t)$. Moreover, $s(R, D, \mathfrak{a}^t)$ is continuous in t.*

4. *Replacing $\mathfrak{a}^{t'_e}$ with its integral closure $\overline{\mathfrak{a}^{t'_e}}$ does not change the F-signature.*

Lemma 2.7.7 will allow us to simplify our computations later by, for example, assuming that t and the coefficients a_i of D are rational numbers with denominator a power of p.

We also have, in the triples case:

Lemma 2.7.8. *Suppose that we are in the setting of Definition 2.7.3. Then $I_e^{\mathfrak{a}} = (I_e^D : \mathfrak{a}^{\lceil t(p^e-1)\rceil})$. Equivalently, set $a_e^{\mathfrak{a}} = l(F_*^eR/F_*^eI_e^{\mathfrak{a}})$, so $s(R, D, \mathfrak{a}^t) = \lim_{e\to\infty} \frac{a_e^{\mathfrak{a}}}{p^{e(d+\alpha)}}$. Then $I_e^{\mathfrak{a}} = (I_e^D :_R \mathfrak{a}^{\lceil t(p^e-1)\rceil})$, and*

$$a_e^{\mathfrak{a}} = l(F_*^eR/(F_*^eI_e^D :_{R^{1/p^e}} F_*^e\mathfrak{a}^{\lceil t(p^e-1)\rceil})).$$

Proof. Requiring that $r \in (I_e^D : \mathfrak{a}^{\lceil t(p^e-1)\rceil})$ is the same as requiring that multiplication by an element $a \in (\mathfrak{a}^{\lceil t(p^e-1)\rceil})^{1/p^e}$ sends r^{1/p^e} into $(I_e^D)^{1/p^e}$, so that for all $\phi \in \mathfrak{D}_e$, $\phi(a \cdot r^{1/p^e}) \in m$. Equivalently, for all $\phi \in \mathrm{Hom}_R(R(\lceil (p^e-1)D\rceil)^{1/p^e}, R), \forall a \in (\mathfrak{a}^{\lceil t(p^e-1)\rceil})^{1/p^e}, \phi(r^{1/p^e}) \in m$. We conclude that $r \in (I_e^D : \mathfrak{a}^{\lceil t(p^e-1)\rceil}) \iff r \in I_e^{\mathfrak{a}}$. We have proved our first claim; the second claim follows immediately. $\square$

CHAPTER III

The F-Signature of Affine Toric Singularities

An affine toric variety is determined by a lattice N and a cone σ inside the real vector space $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$. Its coordinate ring $k[X_\sigma]$ is a normal monomial ring. In this chapter, we will give a formula for the F-signature of this coordinate ring as the volume of a polytope P_σ which lies in $M_{\mathbb{R}}$, the dual vector space to $N_{\mathbb{R}}$.

Theorem 3.2.3. *(cf. [WY04], Theorem 5.1) Let X_σ be an affine toric variety without torus factors. Let $\vec{v}_1, \ldots, \vec{v}_r \in N$ be primitive generators for σ. Let $P_\sigma \subset M_{\mathbb{R}}$ be the polytope $\{\vec{w} \in M_{\mathbb{R}} \mid \forall i, 0 \leq \vec{w} \cdot \vec{v}_i < 1\}$. Then $s(k[X_\sigma]) = \mathrm{Volume}(P_\sigma)$.*

More generally, suppose $X = X' \times T$, where X' is a toric variety without torus factors and T is an algebraic torus. Let $N'_{\mathbb{R}} \subset N_{\mathbb{R}}$ be the vector subspace spanned by σ, and let σ' be σ viewed as a cone in $N'_{\mathbb{R}}$. Then $s(k[X]) = s(k[X']) = \mathrm{Volume}(P_{\sigma'})$.

Here, all notation is standard as in Fulton's book [Ful93]; we will review this notation in the next section.

Our formula is equivalent to the one given in [WY04] in the case where X has no torus factors; when X does have torus factors, our formula corrects the one given in [WY04], which does not hold in that case. Our method of proof uses the usual machinery of toric geometry and in this respect is closer in spirit to the computation of toric Hilbert-Kunz multiplicities in [Wat99]. Singh gives a "non-toric" formula for the F-signature of a normal monomial ring in [Sin05]. The methods used in this chapter allow us to give an easy proof of Singh's result (Theorem 3.3.6).

We also compute the F-signature of pairs and triples in the toric case:

Theorem 3.4.10. *Let R be the coordinate ring of an affine toric variety, with conventions as in Remark 3.2.1. Let D be a torus-invariant $\mathbb{Q}$-divisor, with associated polytope P_σ^D as in Definition 3.4.9. Then* $s(R) = \mathrm{Volume}(P_\sigma^D)$.

Theorem 3.4.13. *Let R be the coordinate ring of an affine toric variety, with conventions as in Remark 3.2.1. Let D be a torus-invariant $\mathbb{Q}$-divisor as in Definition 3.4.9. Let $\mathfrak{a} \subset R$ be a monomial ideal, with associated polytope $\mathrm{P}_\sigma^{\mathrm{D},\mathfrak{a}^{\mathrm{t}}}$ as in Definition 3.4.12. Then* $s(R, D, \mathfrak{a}^t) = \mathrm{Volume}(\mathrm{P}_\sigma^{\mathrm{D},\mathfrak{a}^{\mathrm{t}}})$.

Corollary 3.4.17. *Let R be the coordinate ring of an affine toric variety, D a $\mathbb{Q}$-divisor on* $\operatorname{Spec} R$, *and $\mathfrak{a}$ a monomial ideal, presented as in Theorem 3.4.13. Let* $\mathrm{Newt}(\mathfrak{a})$ *denote the Newton polyhedron of $\mathfrak{a}$ as in Definition 3.4.4. Suppose that the pair (R, D) is $\mathbb{Q}$-Gorenstein. Then* $s(R, D, \mathfrak{a}^t) = \mathrm{Volume}(P_\sigma^D \cap t \cdot \mathrm{Newt}(\mathfrak{a}))$.

3.1 Affine Toric Varieties

Here, we present enough background on toric varieties to prove Theorem 3.2.3. Almost all notation is standard as in Fulton's book [Ful93], which the reader may consult for further details.

A toric variety X may be defined as a normal variety which contains an algebraic torus $T = \operatorname{Spec} k[x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}]$ as an open dense subset, so that the action of T on itself extends to an action of T on X. Toric varieties can be presented in terms of simple combinatorial data, making algebro-geometric computations easier on toric varieties.

Let N be a free abelian group of rank n. Let $M = N^* = \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$ the dual group to N. Consider M as a lattice, called the *character lattice*, in the $\mathbb{R}$-vector space $M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{R}$. Let $k[M]$ be the semigroup ring on M, so that up to non-canonical isomorphism, $k[M] \simeq k[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ is the coordinate ring of an "algebraic torus." Elements of the semigroup M are called *characters* but may also be thought of as exponents; the injective group homomorphism $\chi : M^+ \hookrightarrow (k[M])^\times$ is called the *exponential map* and is written $m \mapsto \chi^m$. Elements $\chi^m \in k[M]$ are called *monomials*. A *monomial ring* R is a k-subalgebra of $k[M]$, finitely generated by monomials: $R = k[S]$, where χ^S is the set of monomials in R. Of course, the set of monomials in R forms a semigroup under multiplication which is naturally isomorphic to S. We denote by $L = \mathrm{Lattice}(S)$ the (additive) subgroup of M generated by S, which is isomorphic under the exponential map to the (multiplicative) group of monomials in $\mathrm{Frac}(R)$.

In what follows, let $\sigma \subset N_{\mathbb{R}}$ be a *strongly convex rational polyhedral cone.* By *rational polyhedral cone* we mean that σ is the cone of vectors $\{\sum_i a_i \vec{v}_i \,|\, 0 \leq a_i \in \mathbb{R}\}$, where $\vec{v}_i \in N$ are a collection of finitely many *generators* for σ. Moreover, we require that σ be *strongly convex*: that is, if $0 \neq \vec{v} \in \sigma$ then $-\vec{v} \notin \sigma$.

A minimal set of generators of a cone is uniquely determined up to rescaling. (For each i, $\mathbb{R}_{\geq 0} \cdot \vec{v}_i$ is a ray which forms one edge of the cone $\sigma \subset N_{\mathbb{R}}$.) It is often useful to take the vectors $\vec{v}_i$ to be *primitive generators*: that is, we replace each $\vec{v}_i$ with the shortest vector in N that lies on the same ray. The primitive generators of σ are themselves uniquely determined.

A *face* of σ is $F = \sigma \cap H$, where $H \subset \mathbb{R}^n$ is a hyperplane that only intersects σ on its boundary $\partial\sigma$. (Equivalently, $H = \vec{w}^{\perp}$, where $\vec{w} \cdot \vec{v} \geq 0$ for all $\vec{v} \in \sigma$. Such H is called a *supporting hyperplane.*) A codimension-one face is called a *facet.* As is (hopefully) intuitively clear, one can show that the union of the facets of σ is equal to the boundary of the cone, $\partial\sigma$. Every face of σ is itself a strongly convex rational polyhedral cone, whose generators are a subset of the generators of σ.

A strongly convex rational polyhedral cone σ in the vector space V has a *dual cone*, $\sigma^\vee = \{\vec{u} \in V^* \,|\, \vec{u} \cdot \vec{v} \geq 0 \forall \vec{v} \in \sigma\}$. It is a basic fact of convex geometry that $\sigma^\vee$ is also a rational polyhedral cone, and that $\sigma = (\sigma^\vee)^\vee$. Moreover, σ is full-dimensional if and only if its dual cone $\sigma^\vee$ is strongly convex ([Ful93], §1.2).

Remark 3.1.1. Let σ be a polyhedral cone. Then there is a one-to-one order-reversing correspondence between faces of σ and faces of $\sigma^\vee$, given by $\tau \mapsto \tau^\vee \cap \sigma^\vee$.

Now we define affine toric varieties in the language of cones. Every affine toric variety may be presented in the following form:

Definition 3.1.2. Let N be an n-dimensional lattice, $N \subset N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$. Let $M = N^*$, and $M_{\mathbb{R}} = M \otimes \mathbb{R}$. Let $\sigma \subset N_{\mathbb{R}}$ be a strongly convex rational polyhedral cone, and $S = \sigma^\vee \cap M$, where $\sigma^\vee \subset M_{\mathbb{R}}$ is the dual cone to σ. Let $R = k[S]$. The affine toric variety corresponding to σ is defined to be $X = \operatorname{Spec} R$.

Remark 3.1.3. Only normal monomial rings arise as the coordinate rings of toric varieties. Since strongly F-regular rings are normal, there will be no loss of generality in restricting our F-signature computations to only those monomial rings arising from toric varieties. (If a monomial ring does not arise in this fashion, it is not normal, hence not strongly F-regular, so we already know that its F-signature is zero.)

The following fact will be useful later. It says that the group $\text{Lattice}(S)$ generated by the semigroup S is equal to the character lattice M.

Lemma 3.1.4. *([Ful93], §1.3) Let N be a rank-n lattice, $M = N^*$, $\sigma \subset N_{\mathbb{R}}$ a strongly convex rational polyhedral cone, and $S = \sigma^\vee \cap M$ (so that* $\operatorname{Spec} k[S]$ *is the affine toric* *variety corresponding to* σ*). Then* $\mathrm{Lattice}(S) = M$*. More generally, if* $L' \subset M_{\mathbb{R}}$ *is any* n*-dimensional lattice,* $\sigma^{\vee} \subset M_{\mathbb{R}}$ *any* n*-dimensional cone, and* $S = \sigma^{\vee} \cap L'$*, then* $\mathrm{Lattice}(S) = L'$*, that is,* L' *is the group generated by the semigroup* S*.*

It will be convenient, during our F-signature computations, to temporarily assume that the cone σ defining our toric variety X_{σ} is full-dimensional. Equivalently, we assume that our toric variety contains no *torus factors*, i.e, is not the product of two lower-dimensional toric varieties, one of which is a torus. The following (easily checked) facts about products of cones will allow us to reduce to the case of a toric variety with no torus factors:

Lemma 3.1.5. *([CLS11], Proposition 3.3.9) Let* $X = \operatorname{Spec} R$ *be the affine toric variety corresponding to the cone* σ*, so that* $R = k[\sigma^{\vee} \cap M]$*. Let* $N'_{\mathbb{R}} \subset N_{\mathbb{R}}$ *be the vector subspace spanned by* σ*. Let* $N' = N'_{\mathbb{R}} \cap N$*. Let* σ' *be* σ*, viewed as a full-dimensional cone in* $N'_{\mathbb{R}}$*. Let* $N'' = N/N'$*. Then* $X \simeq X' \times T_{N''}$*, where* X' *is the affine toric variety (with no torus factors) corresponding to* σ' *and* $T_{N''} = \operatorname{Spec} k[M'']$ *is an algebraic torus.*

Finally, we recall the definition of a polytope.

Definition 3.1.6. A *polytope* in $\mathbb{R}^n$ is the convex hull of a finite set of points, which we will call extremal points. Equivalently, a polytope is a bounded set given as the intersection of finitely many closed half-spaces $H = \{\vec{v} \,|\, \vec{v} \cdot \vec{u} \geq 0\}$ (see, e.g., [CLS11], §2.2), or a bounded set defined by finitely many linear inequalities.

Remark 3.1.7. We will abuse notation by allowing polytopes to be intersections of half-spaces which are either open ($H = \{\vec{v} \,|\, \vec{v} \cdot \vec{u} > 0\}$) or closed. (We will compute F-signatures to be the volumes of various polytopes. Since the volume of an intersection of half-spaces is the same whether the half-spaces are open or closed, this technicality will not affect our arguments.)

3.2 Toric F-Signature Computation

3.2.1 The Formula, and an Example

Remark 3.2.1. (Conventions.) For the remainder of this section, N is a lattice; $M = N^*$ is the dual lattice; $\sigma \subset N_{\mathbb{R}}$ is a strongly convex rational polyhedral cone; $S = M \cap \sigma^\vee$, so that $k[S]$ is the coordinate ring of an affine toric variety in the notation of [Ful93], and $\vec{v}_1, \ldots, \vec{v}_r$ are primitive generators for σ.

Definition 3.2.2. Let σ be a cone as in Remark 3.2.1, with primitive generators $\vec{v}_1, \ldots, \vec{v}_r$. We define $P_\sigma \subset \sigma^\vee$ to be the polytope $\{\vec{w} \in M_{\mathbb{R}} \mid \forall i, 0 \leq \vec{w} \cdot \vec{v}_i < 1\}$.

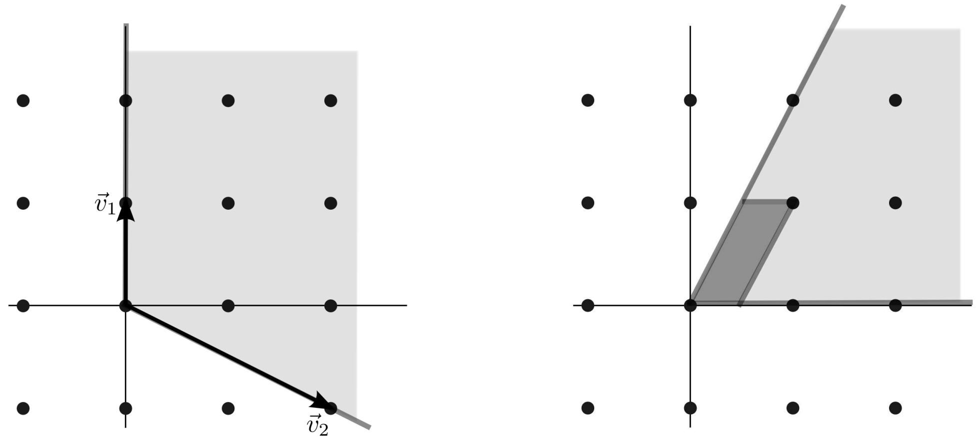
Theorem 3.2.3. *Let R be the coordinate ring of an affine toric variety X with no torus factors, with the conventions of Remark 3.2.1. Then $s(R)$ is the volume of P_σ. More generally, suppose $X = X' \times T$, where X' is a toric variety without torus factors and T is an algebraic torus. Let $N'_{\mathbb{R}} \subset N_{\mathbb{R}}$ be the vector subspace spanned by σ, and let σ' be σ viewed as a cone in $N'_{\mathbb{R}}$. Then $s(k[X]) = s(k[X']) = \mathrm{Volume}(P_{\sigma'})$.*

We will prove this theorem in Section 3.2.3. For now, we provide an example computation:

Example 3.2.4. Figure 3.1(a) shows the cone σ corresponding to a plane quadric $\mathbb{V}(xy - z^2)$, with primitive generators $\vec{v}_1, \vec{v}_2$. Figure 3.1(b) shows the dual cone $\sigma^\vee$. The coordinate ring $k[\sigma^\vee \cap M]$ is $k[x, xy, xy^2]$. In this case, P_σ, shaded in the figure, is the parallelogram $\{\langle x, y\rangle \mid 0 \leq y < 1, 0 \leq 2x - y < 1\}$. The F-signature is $s(R) = \mathrm{Volume}(P_\sigma) = \frac{1}{2}$.

3.2.2 R-module Decomposition of $R^{1/q}$

The main supporting result proved in this section is Lemma 3.2.8, which gives a formula for the free rank of $R^{1/q}$ as an R-module in terms of the number of monomials



(a) The cone $\sigma \subset N_{\mathbb{R}}$.

(b) The dual cone $\sigma^\vee \subset M_{\mathbb{R}}$.

Figure 3.1: Computing the F-signature of the coordinate ring $k[x, xy, xy^2]$ of a quadric cone.

in $R^{1/q}$ having a certain property. That lemma will be integral to our proof of the main theorem. Lemma 3.2.8 follows immediately from Lemma 3.2.7, which describes how $R^{1/q}$ decomposes as a direct sum of indecomposable R-modules.

We will be able to compute the F-signature of a monomial ring R because the R-module $R^{1/q}$ has an especially nice graded structure. In particular:

Lemma 3.2.5. *Let R be a monomial ring, with $q = p^e$, and with character lattice $M \simeq \mathbb{Z}^n$. Then:*

1. *$R^{1/q}$ is finitely generated, as an R-module, by q^{th} roots of monomials in R of bounded degree.*

2. *$R^{1/q}$ admits a natural $\frac{1}{q}M$-grading which respects the M-grading on $R \subset R^{1/q}$. Each graded piece of $R^{1/q}$ a one-dimensional k-vector space.*

Proof. 1. If R is generated by a finite set of monomials τ_i, we can pick a minimal set of t generators from among $\{\prod_i \tau_i^{a_i/q} \mid 0 \leq a_i < q\}$.

2. We consider R as a graded subring of the M-graded ring $k[M]$. The grading

on $R^{1/q}$ is inherited in the obvious way: $\deg(\chi^m)^{1/q} = \frac{1}{q}\deg\chi^m = \frac{1}{q}m$. We conclude that $R^{1/q}$ has a natural $\frac{1}{q}M$-grading. Each graded piece consists of the set of k-multiples of a single monomial $\chi^{m/q}$.

$\square$

It is well-known that relations on graded modules over monomial rings are generated by so-called "binomial" relations. We supply a proof here, for lack of a better reference:

Lemma 3.2.6. *Let W be a G-graded R-module, G an abelian group, with each nonzero graded piece a one-dimensional k-vector space. (For example, when R is a monomial subring of $k[x_1, \ldots, x_n]$, $W = R^{1/q}$ is $(\mathbb{Z}/q)^n$-graded.)*

1. *We can write W as a quotient of a free module so that the relations are generated by "binomial" relations, of the form $r \cdot \rho = s \cdot \mu$, for r, s nonzero homogeneous elements of R and ρ, μ homogeneous elements of W such that $\deg r + \deg \rho = \deg s + \deg \mu$.*

2. *We will say that two monomials $\rho, \mu \in W$ are* related *if they satisfy a binomial relation. Then being related is an equivalence relation.*

Proof. 1. Let μ_i be graded generators for W. Let $\sum_i r_i\mu_i = 0$ be a relation. Since W is graded, $\sum_i r_i\mu_i$ may be written as a sum of graded pieces, each of which is itself equal to 0. In other words, the relations on W are generated by relations with the property that $r_i\mu_i$ has the same degree for each i. In that case, since each graded piece of W is a one-dimensional k-vector space, we have that for each i, and each j for which $r_i\mu_i \neq 0$, $r_i\mu_i = c_{ij}r_j\mu_j$ for some $c_{ij} \in k$. This is a binomial relation on μ_i and μ_j, and binomial relations of this form generate the original relation $\sum_i r_i\mu_i = 0$.

2. If $r\rho = s\mu$, and $s'\mu = t\tau$, then $rs'\rho = st\tau$, so $\rho \sim \tau$.

$\square$

The following lemma essentially indicates how to decompose $R^{1/q}$ as a direct sum of R-submodules generated by monomials. It also gives a condition describing which monomials generate free summands of $R^{1/q}$.

Lemma 3.2.7. *Let W be a finitely generated G-graded R-module, G an abelian group, with each nonzero graded piece a one-dimensional k-vector space. (For example, $W = R^{1/q}$, $G = \mathbb{Z}^n$.) Let H be a set of homogeneous generators for W. Let $A_1, \ldots, A_k \subset H$ be the distinct equivalence classes of elements of H which are related (in the sense of Lemma 3.2.6). Then:*

1. *$W \simeq \bigoplus_i R \cdot A_i$ is a direct sum of submodules generated by the sets A_i.*

2. *Each of the submodules $R \cdot A_i$ is rank one (hence indecomposable even as an ungraded R-module).*

3. *Finally, a homogeneous element $\mu \in W$ generates a free summand of W if and only if the only homogeneous elements of W that are related to μ are R-multiples of μ.*

Proof. 1. Suppose that $A, B \subset H$, and $C = A \coprod B$. Since the corresponding submodules $R \cdot A$ and $R \cdot B$ are graded, their intersection must also be graded. In particular, in order for these modules to have nonempty intersection (equivalently, for the sum $R \cdot A + R \cdot B = R \cdot C$ to fail to be direct), we should have a binomial relation $r\mu = s\tau$ for some $\mu \in A, \tau \in B$, and $r, s \in R$, by Lemma 3.2.6. However, we constructed the sets A_i so that no binomial relations exist between them. We conclude that the sum is direct, and $W = R \cdot H \simeq \sum_i R \cdot A_i$.

2. Suppose that we have a subset $A \subset H$ of homogeneous elements which are are all related to one another (for example, $A = A_i$ for some i). Then pick a homogeneous $\mu \in A$. For any $\tau \in A$, we have that $r\mu = s\tau$ for $r, s \in R$; equivalently, $\tau = \frac{r}{s}\mu \in \mathrm{Frac}(R) \cdot \mu$. We conclude that $R \cdot A$ has rank 1, that is, $R \cdot A \otimes_R \mathrm{Frac}(R) \simeq \mathrm{Frac}(R)$.

3. Fix $\mu \in W$. Let $A_1, \ldots, A_k$ be the equivalence classes of related monomials in W. Without loss of generality, we may assume that $\mu \in A_1$. Then $R \cdot A_1$ is free of rank one if and only if it is generated by a single monomial. Thus, $R \cdot \mu$ is a free summand of W if and only if μ generates $R \cdot A_1$, that is, if and only if μ divides every homogeneous element of W that is related to μ.

□

The following lemma will be essential in the next section when we compute the free rank a_e of $R^{1/q}$ as an R-module. (We will also use this lemma in Section 2.7 when computing the F-signature of pairs and triples.)

Lemma 3.2.8. *Let $R = k[S]$ be a monomial ring, S a semigroup, and let $L =$ $\mathrm{Lattice}(S)$ be the group generated by S. Fix $q = p^e$. Let $H \subset \frac{1}{q}L$ be a finitely generated S-module, so that $k[H] \subset k[\frac{1}{q}L]$ is an R-module finitely generated by monomials. Let a_e be the free rank of $k[H]$ as an R-module. Then the set of monomials in H which generate a free summand of $k[H]$ is $\{\chi^{\vec{v}} \mid \vec{v} \in H$, and $\forall \vec{k} \in L \backslash S, \vec{v} + \vec{k} \notin H\}$. Moreover, if 0 is the only unit in H, then a_e is the size of this generating set.*

Proof. By Lemma 3.2.7, a monomial $\mu \in k[H]$ generates a free summand of $k[H]$ if and only if it is unrelated to all monomials in $k[H]$ that are not R-multiples of itself. We may characterize τ being related to μ (but not a multiple of it) by $\tau = \frac{r}{s}\mu$, with $\frac{r}{s} \in (\mathrm{Frac}\, R) \backslash R$. Thus, the set of monomial generators for $k[H]$ which generate a free summand of $k[H]$ is $\{\mu \in \chi^H \mid$ for all monomials $\frac{r}{s} \in (\operatorname{Frac} R)\backslash R, \frac{r}{s}\mu \notin k[H]\}$. We may rewrite this set as $\{\chi^{\vec{v}} \,|\, \vec{v} \in H$, and $\forall \vec{k} \in L\backslash S, \vec{v}+\vec{k} \notin H\}$, which is precisely the set described in the statement of the lemma. If 0 is the only unit in H, then there is a one-to-one-correspondence between monomials in the generating set and free summands of W. In that case, a_e is the size of the generating set. □

Remark 3.2.9. As we will see shortly, when we apply Lemma 3.2.8 to the case of the R-module $R^{1/q}$, the technical requirement that $0 \in H$ be the only unit corresponds to the cone σ being full-dimensional.

3.2.3 Derivation of the Formula

Remark 3.2.10. (An aside on computing volumes.) Consider $M \subset M_{\mathbb{R}}$, a lattice abstractly isomorphic to $\mathbb{Z}^n$ contained in a vector space abstractly isomorphic to $\mathbb{R}^n$. Choosing a basis for M gives us an identification of $M_{\mathbb{R}}$ with $\mathbb{R}^n$, hence a way to measure volume on $M_{\mathbb{R}}$. It is easily checked that this volume measure depends only on M and not on our choice of basis for M. (Such a measure is uniquely determined by the fact that with respect to it, the measure of a fundamental parallelepiped for M, also called the *covolume* of M, is 1.) Thus, it makes sense to talk about measuring volume "relative to the lattice M," denoted Volume_M, or simply Volume when there is no risk of ambiguity.

Now we are ready to prove our main result.

Proof of Theorem 3.2.3. Suppose first that X has no torus factors. We apply Lemma 3.2.8, with $H = \sigma^\vee \cap \frac{1}{p^e}M$, $k[H] = R^{1/p^e}$. Since σ is full-dimensional, $\sigma^\vee$ is strongly convex, so H contains no nontrivial units. Then

$$a_e = \#\{\vec{v} \in (\sigma^\vee \cap \frac{1}{p^e}M) \,|\, \forall \vec{k} \in M\backslash\sigma^\vee, \vec{v} + \vec{k} \notin \sigma^\vee\}.$$

Let P'_σ be the set $\{\vec{v} \in \sigma^\vee \,|\, \forall \vec{k} \in M \backslash \sigma^\vee, \vec{v} + \vec{k} \notin \sigma^\vee\}$. Then $a_e = \#\{\vec{v} \in P'_\sigma \cap \frac{1}{p^e} M\}$. By Lemma 3.2.11, $P'_\sigma = P_\sigma$. Set $q = p^e$. Then $s(R)$, defined to be $\lim_{e\to\infty} \frac{a_e}{p^{en}}$, is equal to $\lim_{q\to\infty} \frac{\#(P_\sigma \cap \frac{1}{q} M)}{q^n}$. We apply Lemma 3.2.12 to conclude that $s(R) = \text{Volume}(P_\sigma)$.

Suppose now that X has torus factors. By Lemma 3.1.5, $X \simeq X' \times T_{N''}$, where $X' = \operatorname{Spec} k[\sigma' \cap M']$ and $T_{N''}$ is the algebraic torus $\operatorname{Spec} k[M'']$. In particular, $R \simeq k[X'] \otimes_k k[M'']$. We apply Theorem 2.3.1 on the F-signature of products to see that $s(R) = s(k[X']) \cdot 1 = s(k[X']) = \text{Volume}(P_{\sigma'})$. (It is easy to check directly that $s(k[M'']) = 1$: writing $M'' \simeq \mathbb{Z}^{d''}$, we see that $k[\mathbb{Z}^{d''}]^{1/q}$ is a free $k[\mathbb{Z}^{d''}]$-module of rank $q^{d''}$.) $\square$

It remains to prove the two lemmas referenced in the proof of Theorem 3.2.3.

Lemma 3.2.11. *Suppose that we are in the situation of Remark 3.2.1. Then*

$$P'_\sigma := \{\vec{v} \in \sigma^\vee \,|\, \forall \vec{k} \in M \backslash \sigma^\vee, \vec{v} + \vec{k} \notin \sigma^\vee\} = \{\vec{v} \in M_\mathbb{R} \,|\, \forall i, 0 \le \vec{v} \cdot \vec{v}_i < 1\} =: P_\sigma.$$

Proof. Recall that $\sigma^\vee = \{\vec{u} \,|\, \vec{u} \cdot \vec{v}_i \ge 0 \text{ for all } i\}$.

Suppose $\vec{v} \in P_\sigma$, so that for each i, $0 \le \vec{v} \cdot \vec{v}_i < 1$. Fix $\vec{k} \in M \backslash \sigma^\vee$. Since $\vec{k} \notin \sigma^\vee$, we know that $\vec{k} \cdot \vec{v}_j < 0$ for some j. For such j, since $\vec{k} \cdot \vec{v}_j \in \mathbb{Z}$, we know that $\vec{k} \cdot \vec{v}_j \le -1$. It follows that $(\vec{v} + \vec{k}) \cdot \vec{v}_j < 0$. Thus, $\vec{v} + \vec{k} \notin \sigma^\vee$. We conclude that $\vec{v} \in P'_\sigma$. Hence, $P_\sigma \subset P'_\sigma$.

Conversely, suppose that $\vec{v} \notin P_\sigma$, so that for some j, $\vec{v} \cdot \vec{v}_j \ge 1$. Set $\vec{k}_0$ to be any vector in M such that $\vec{k} \cdot \vec{v}_j = -1$. (Such $\vec{k}_0$ exists since by Lemma 3.1.4, $\text{Lattice}(S) = M$.) Choose $\vec{k}_1$ to be any vector in M that also lies in the interior of the facet $F_j = \vec{v}_j^\perp \cap \sigma^\vee$ of $\sigma^\vee$. Then $\vec{k}_1 \cdot \vec{v}_i > 0$ for each $i \neq j$. Thus, for sufficiently large m, $(\vec{k}_0 + m\vec{k}_1) \cdot \vec{v}_i \ge 0$ for $i \neq j$, while $(\vec{k}_0 + m\vec{k}_1) \cdot \vec{v}_j = 0$. Set $\vec{k} = \vec{k}_0 + m\vec{k}_1$. Then $\vec{k} \in M$, but $\vec{v} \notin \backslash \sigma^\vee$, since $\vec{k} \cdot \vec{v}_j = -1 < 0$. On the other hand, $\vec{v} + \vec{k} \in \sigma^\vee$, since $(\vec{v} + \vec{k}) \cdot \vec{v}_i \ge 0$ for each i. We conclude that $\vec{v} \notin P'_\sigma$.

Hence, $P'_\sigma \subset P_\sigma$. We conclude that $P_\sigma = P'_\sigma$, as we desired to show. $\square$

Lemma 3.2.12. *Let M be a lattice and $P \subset M \otimes_{\mathbb{Z}} \mathbb{R}$ a polytope (or, more generally, any set whose boundary has measure zero). Then* $\lim_{q\to\infty} \frac{\#\{P \cap \frac{1}{q}M\}}{q^n} = \mathrm{Volume}(P)$.

Proof. In fact, when P is a polytope, it can be shown that the quantity $\#\{P \cap \frac{1}{q}M\}$ is polynomial in q of degree n, known as the *Ehrhart polynomial* of P, and that its leading coefficient $\lim_{q\to\infty} \frac{\#\{P \cap \frac{1}{q}M\}}{q^n}$ is $\mathrm{Volume}(P)$ ([MS05], Thm 12.2). Even without this fact, however, it is easy to sketch a proof of the special case that we require: the quantity $\lim_{q\to\infty} \frac{\#\{P \cap \frac{1}{q}M\}}{q^n}$ is a limit of Riemann sums measuring the volume of P with respect to the lattice M. (See, for example, [Fol99], Theorem 2.28.) $\square$

3.3 Alternative Monomial Ring Presentations

3.3.1 A Slightly More General F-Signature Formula

Theorem 3.2.3 can be made to apply to alternative presentations of monomial rings. In particular, suppose $R = k[S]$, where $S = L \cap \sigma^\vee$ for *any* lattice L, not just $L = M$. We may apply a slightly modified version of Theorem 3.2.3 to compute $s(R)$.

Definition 3.3.1. Let σ be a cone as in Remark 3.2.1, with primitive generators $\vec{v}_1, \dots, \vec{v}_r$. Let L be a lattice. For each i, let $c_i = \min_{\vec{v}\in L} |\vec{v} \cdot \vec{v}_i|$. We define $P^L_\sigma \subset \sigma$ to be the polytope $\{\vec{w} \in M_{\mathbb{R}} \mid \forall i, 0 \le \vec{w} \cdot \vec{v}_i < c_i\}$. (Note that if $L = M$, then $P^L_\sigma = P_\sigma$, as $c_i = 1$ for each i.)

Corollary 3.3.2. *(We use the conventions of Remark 3.2.1.) Let $L \subset M$ be a sublattice, and set $S = \sigma^\vee \cap L$. (By Remark 3.1.4, $L = \mathrm{Lattice}(S)$.) If σ is a full-dimensional cone, then $s(R) = \mathrm{Volume}(P^L_\sigma)$, with the volume measured with respect to the lattice L. Moreover, for each e, $a_e = \#(P^L_\sigma \cap \frac{1}{p^e}L)$.*

Proof. The proof is essentially the same as that of Theorem 3.2.3 with M replaced by L, except that in the supporting Lemma 3.2.11, for each i, we replace the condition $0 \leq \vec{v}_i < 1$ with $0 \leq \vec{v}_i < c_i$. (In the original proof, we made use of the fact that $c_i = 1$ for $L = M$. It is easily checked that the proof holds in this more general case if we just replace each 1 with c_i as necessary.) □

Example 3.3.3. Proposition 3.3.2 may be used to recover the F-signature of a Veronese subring $R^{(n)}$ of a polynomial ring $R = k[x_1, \ldots, x_n]$. (This computation has already been performed in [HL02] and [Sin05]; it also follows from Theorem 2.6.2.) In particular, $R^{(n)} = k[\sigma^\vee \cap L]$, where σ is the first orthant and $L \subset M = \mathbb{Z}^n$ is the lattice of vectors whose coordinates sum to a multiple of n. It is easily checked that for such L and σ, $c_i = 1$ for all i, so that $P_\sigma^L = P_\sigma$. Moreover, $\#(M/L) = n$, so $\text{Volume}_L = \frac{1}{n} \cdot \text{Volume}_M$. We conclude that $s(R^{(n)}) = \text{Volume}_L(P_\sigma) = \frac{1}{n}\text{Volume}_M(P_\sigma) = \frac{1}{n}s(R)$.

3.3.2 A New Proof of an Old F-Signature Formula

Now we can provide an elementary proof of the F-signature formula given by Singh. First, we will need to discuss a few relevant properties of monomial rings.

Definition 3.3.4. Let S be a semigroup of monomials contained in the semigroup T generated by monomials $x_1, \ldots, x_n$. (So $k[A]$ is the polynomial ring $k[x_1, \ldots, x_n]$.) Then S is *full* if $\operatorname{Frac} k[S] \cap k[x_1, \ldots, x_n] = k[S]$. Equivalently, $\text{Lattice}(S) \cap T = S$.

Definition 3.3.5. Let S be a semigroup of monomials contained in the semigroup T generated by monomials $x_1, \ldots, x_n$. Then we say that S satisfies property $(*)$ if the following holds: consider any variable $x_i \in T$. Then there exist monomials $\zeta, \eta \in k[S]$ such that $\frac{\zeta}{\eta}$, as a fraction in $\operatorname{Frac} k[T]$ in lowest terms, can be written as $\frac{\tau}{x_i}$ (where τ is a monomial in S but not necessarily in T). Equivalently, the lattice $L \subset \mathbb{Z}^n$ generated by S should contain, for each i, an element with i^{th} coordinate equal to -1.

Theorem 3.3.6 ([Sin05]). *Let $R \subset A = k[x_1, \ldots, x_n]$ be a subring generated by finitely many monomials, $R = k[S]$, where S is a finitely generated semigroup. Let m_A be the homogeneous maximal ideal of A. Assuming that R is presented so that S is full and satisfies property $(*)$, the F-signature of R is $s(R) = \lim_{e\to\infty} \frac{l(R/(m_A^{[p^e]} \cap R))}{p^{ed}}$. In particular, $a_e = l(R/(m_A^{[p^e]} \cap R))$.*

Proof. We are given that $S = \text{Lattice}(S) \cap T = \text{Lattice}(S) \cap \sigma^\vee$, where σ is the first orthant, with primitive generators $\vec{v}_i$ equal to the unit vectors in $\mathbb{R}^n$. Thus, we may apply Proposition 3.3.2 to the cone σ and the lattice $L = \text{Lattice}(S)$. It remains only to show that $l(R/(m_A^{[p^e]} \cap R)) = \#(P_\sigma^L \cap \frac{1}{p^e} L)$. Since $c_i = 1$ for each i, the right-hand side is $\#\{\vec{v} \in \frac{1}{p^e} L \,|\, 0 \leq \vec{v} \cdot \vec{v}_i < 1\}$. The left-hand side is equal to the number of $\vec{v} \in L$ whose coordinates are all less than p^e, which is $\#\{\vec{v} \in L \,|\, 0 \leq \vec{v} \cdot \vec{v}_i < p^e\}$. Dividing all elements of the left-hand side by q, we see that the left-hand side and right-hand side are equal. Thus, $s(R) = \lim_{e\to\infty} \frac{a_e}{p^{ed}} = \lim_{e\to\infty} \frac{l(R/(m_A^{[p^e]} \cap R))}{p^{ed}}$. □

3.4 Toric F-Signature of Pairs and Triples

3.4.1 Toric Preliminaries

For our pair and triple computations, we will require some understanding of divisors on toric varieties. (Unless stated otherwise, proofs of these results may be found in [Ful93].)

Definition 3.4.1. A prime Weil divisor D on a toric variety X is *torus-invariant* if it is invariant under the action of the embedded torus on X. More generally, a $\mathbb{Q}$-divisor D is torus-invariant if $D = \sum_i a_i D_i$, where D_i are the torus-invariant prime divisors of X.

The torus-invariant prime divisors of X are in bijective correspondence with primitive generators $\vec{v}_i$ of the cone $\sigma \subset N_{\mathbb{R}}$ of X. (In particular, the prime divisor corresponding to $\vec{v}_i$ is $D_i = \mathbb{V}(I_i)$, where I_i is the ideal generated by monomials $\vec{u}$ such that $\vec{u} \cdot \vec{v}_i \neq 0$.) It can be shown that $\nu_i : \mathrm{Frac}(R) \to \mathbb{Z}$, the discrete valuation corresponding to D_i, is given by $\nu_i(\vec{u}) = \vec{u} \cdot \vec{v}_i$. From this, it follows that:

Lemma 3.4.2. *Let $X = \operatorname{Spec} R$ be an affine toric variety, $R = k[S]$, $S = \sigma^\vee \cap M \subset M_{\mathbb{R}}$. Let $D = \sum_i a_i D_i$ be a torus-invariant $\mathbb{Q}$-divisor on X, where each D_i corresponds to a primitive generator $\vec{v}_i$ of σ. Then $R(D) = \sum_{\vec{u}} R \cdot x^{\vec{u}}$, where the sum is taken over all $\vec{u} \in S$ such that $\vec{u} \cdot \vec{v}_i \geq -a_i$.*

For our F-signature of triples computation, we will require the concept of the Newton polyhedron of a monomial ideal.

Definition 3.4.3. A *polyhedron* is a possibly unbounded intersection of finitely many half-spaces in $\mathbb{R}^n$.

Definition 3.4.4. Let $R = k[S]$ be a monomial ring, with $S \subset M_{\mathbb{R}}$, as above. Let $\mathfrak{a} \subset R$ be a monomial ideal (i.e., an ideal generated by monomials). The *Newton polyhedron* of $\mathfrak{a}$, denoted $\mathrm{Newt}(\mathfrak{a})$, is the polyhedron in $M_{\mathbb{R}}$ which is the convex hull of the set of monomials in $\mathfrak{a}$.

The Newton polyhedron is closely related to the integral closure of monomial ideals:

Lemma 3.4.5. *(see, e.g., [Vil01], Proposition 7.3.4) Let $R = k[S]$ be a monomial ring as above. Let $\mathfrak{a} \subset R$ be a monomial ideal. Then the integral closure $\overline{\mathfrak{a}}$ of $\mathfrak{a}$ in R is a monomial ideal generated by those monomials in the set* $\mathrm{Newt}(\mathfrak{a}) \cap M$.

Definition 3.4.6. Let Q_1, Q_2 be subsets of $\mathbb{R}^n$. The *Minkowski sum* of Q_1 and Q_2, denoted $Q_1 + Q_2$, is the set $\{\vec{u}_1 + \vec{u}_2 \,|\, \vec{u}_1 \in Q_1, \vec{u}_2 \in Q_2\}$. We denote by $Q_1 - Q_2$ the set $Q_1 + (-Q_2) = \{\vec{u}_1 - \vec{u}_2 \,|\, \vec{u}_1 \in Q_1, \vec{u}_2 \in Q_2\}$.

It is easy to see that the Minkowski sum of two polyhedrons is itself a polyhedron, and that the sum of two polytopes is a polytope. (See, for example, [Grü03], §15.1.)

As Corollary 3.4.17 is a statement about $\mathbb{Q}$-Gorenstein pairs, we recall the definition of the $\mathbb{Q}$-Gorenstein condition.

Definition 3.4.7. If D is an effective $\mathbb{Q}$-divisor on $X = \operatorname{Spec} R$, the pair (R, D) is $\mathbb{Q}$-*Gorenstein* if, fixing a canonical divisor K_X on X, the divisor $K_X + D$ is $\mathbb{Q}$-Cartier; that is, some integer multiple of $K_X + D$ is Cartier.

It happens that on a toric variety, a canonical divisor may be given by $K_X = -\sum_i D_i$, where the sum is taken over all torus-invariant prime divisors on X. It is also a fact that Cartier divisors on an affine toric variety are principal. We conclude:

Lemma 3.4.8. *Let $X = \operatorname{Spec} k[S]$ be an affine toric variety with a corresponding strongly convex polyhedral cone σ. Let $D = \sum_i a_i D_i$ be a $\mathbb{Q}$-divisor. Then (R, D) is $\mathbb{Q}$-Gorenstein if and only if for some $\vec{w} \in M \otimes \mathbb{Q}$, $\vec{w} \cdot \vec{v}_i = -1 + a_i$ for each i.*

Proof. Given $\vec{u} \in M$, $\operatorname{div} x^{\vec{u}} = \sum_i (\vec{u} \cdot \vec{v}_i) D_i$. This operation extends linearly to $\mathbb{Q}$-divisors, so that for $\vec{u} \in M \otimes_{\mathbb{Z}} \mathbb{Q}$, $\operatorname{div} x^{n\vec{u}} = n(\sum_i c_i D_i)$ if and only if $\vec{u} \cdot \vec{v}_i = c_i$ for each i. Thus, $K_X + D$ is $\mathbb{Q}$-Gorenstein if and only if for some $\vec{w} \in M \otimes \mathbb{Q}$, $\vec{w} \cdot \vec{v}_i = -1 + a_i$ for each i. ☐

3.4.2 Pairs Computation

Now we will compute the F-signature of pairs and triples. We begin with the pairs case, in which our proof requires little modification from that of Theorem 3.2.3.

Definition 3.4.9. Let σ be a cone as in Remark 3.2.1, with primitive generators $\vec{v}_1, \ldots, \vec{v}_r$. Let $D = \sum_i a_i D_i$ be a torus-invariant $\mathbb{Q}$-divisor on $\operatorname{Spec} R$, with D_i

the prime divisor corresponding to $\vec{v}_i$. We define $P_\sigma^D \subset \sigma^\vee$ to be the polytope $\{\vec{v} \in M_\mathbb{R} \mid \forall i, 0 \leq \vec{v} \cdot \vec{v}_i < 1 - a_i\}$.

Theorem 3.4.10. *Let R be the coordinate ring of an affine toric variety, with conventions as in Remark 3.2.1. Let D be a torus-invariant $\mathbb{Q}$-divisor, with associated polytope P_σ^D as in Definition 3.4.9. Then $F_*^e I_e^D$ is generated by the monomials in the set $(\sigma \backslash P_\sigma^D) \cap \frac{1}{p^e} M$, and $s(R) = \mathrm{Volume}(P_\sigma^D)$.*

Proof. First, we apply Lemma 2.7.7 to replace $(p^e - 1)D$ by $p^e D$ without changing the F-signature. By the same lemma, $s(R, D)$ is continuous as a function of the a_i, so we may assume that $a_i \in \frac{1}{p^e}\mathbb{Z}$. (Proving the claim on that dense subset will prove it for all divisors by continuity.) We also assume that e is sufficiently large, so that $p^e D$ is an integral divisor, and $\lceil p^e D \rceil = p^e D$. As a result of all this simplification, we may ignore the rounding-up operation.

By Lemma 3.4.2, $R(p^e D)$ is an $\mathbb{N}^n$-graded R-module, generated by $\{\chi^{\vec{v}} \mid \vec{v} \in M$, and $\vec{v} \cdot \vec{v}_i \geq -p^e a_i\}$. It follows that $R(p^e D)^{1/p^e}$ is $\mathbb{N}^n/q$-graded: it is generated by $\{\chi^{\vec{v}} \mid \vec{v} \in \frac{1}{p^e} M$, and $\vec{v} \cdot \vec{v}_i \geq -a_i\}$.

Thus, we may apply Lemma 3.2.7. The graded module $R(p^e D)^{1/p^e}$ decomposes as a direct sum of graded submodules, where each submodule is generated by related monomials. Each submodule splits off from $R(p^e D)^{1/p^e}$ if and only if it is generated by a single monomial; likewise, each submodule generated by monomials in R^{1/p^e} splits off from R^{1/p^e} if and only if it is generated by a single monomial.

Set $\sigma' = \{\vec{v} \in M_\mathbb{R} \mid \forall i, \vec{v} \cdot \vec{v}_i \geq -a_i\}$, so that if $S' := M \cap \sigma'$, then $\chi^{S'}$ is the set of generators for $R(D)$. By Lemma 3.2.8, $a_e = \#\{\vec{v} \in \frac{1}{p^e} S \mid \forall \vec{k} \in M \backslash S, \vec{v} + \vec{k} \notin \frac{1}{p^e} S'\}$. Note that $F_*^e I_e^D$ is generated by those monomials in R^{1/p^e} whose corresponding characters are *not* in this set.

Following the proof of Theorem 3.2.3, we find that $a_e = \#\{\vec{v} \in \frac{1}{p^e} S \mid \forall \vec{k} \in M \backslash S, \vec{v} +$

$\vec{k} \notin \frac{1}{p^e}S'\}$. Equivalently, $a_e = \#\{\vec{v} \in \sigma^\vee \cap \frac{1}{p^e}M \mid \forall \vec{k} \in M \backslash \sigma^\vee, \vec{v} + \vec{k} \notin \sigma'\}$. That is,

$$a_e = \#\{\frac{1}{p^e}M \cap P'\},$$

where $P' = \{\vec{v} \mid \vec{v} \cdot \vec{v}_i \geq 0$, and $\forall \vec{k} \in M \backslash \sigma^\vee, (\vec{v} + \vec{k}) \cdot \vec{v}_i < -a_i$ for some $i\}$. (By the same argument, $F^e_* I_e$ is generated by the monomials whose characters lie in $\sigma^\vee \backslash P'$.) By Lemma 3.4.11 (the pairs analogue to Lemma 3.2.11), $P' = P^D_\sigma$. Our claim then follows from our lemma on volumes of polytopes, Lemma 3.2.12, just as in our original proof of Theorem 3.2.3. □

Lemma 3.4.11. *Suppose that we are in the situation of Lemma 3.4.10, and that* $P' = \{\vec{v} \mid \vec{v} \cdot \vec{v}_i \geq 0$, *and* $\forall \vec{k} \in M \backslash \sigma^\vee, (\vec{v} + \vec{k}) \cdot \vec{v}_i < -a_i$ *for some* $i\}$. *Then* $P' = P^D_\sigma = \{\vec{v} \in M_\mathbb{R} \mid \forall i, 0 \leq \vec{v} \cdot \vec{v}_i < 1 - a_i\}$.

Proof. We follow the proof of Lemma 3.2.11. Suppose that $\vec{v} \in P^D_\sigma$, $\vec{k} \in M$, and $\vec{k} \cdot \vec{v}_i < 0$. Then $(\vec{v} + \vec{k}) \cdot \vec{v}_i < (1 - a_i) + (-1) = -a_i$, so $\vec{v} + \vec{k} \notin \sigma'$. It follows that $P^D_\sigma \subset P'$. On the other hand, suppose $\vec{v} \notin P^D_\sigma$. Either $\vec{v} \cdot \vec{v}_i < 0$ for some i, in which case $\vec{v} \notin P'$, or $\vec{v} \cdot \vec{v}_i \geq 1 - a_i$ for some i. In the latter case, we may, as in Lemma 3.2.11, choose $\vec{k} \in M$ such that $\vec{k} \cdot \vec{v}_i = -1$ and $\vec{k} \cdot \vec{v}_j \geq 0$ for all $j \neq i$. Then $\vec{k} \notin \sigma^\vee$, but $(\vec{v} + \vec{k}) \cdot \vec{v}_j \geq -a_j$ for all j. It follows that $\vec{v} \notin P'$.

We conclude that $P' = P^D_\sigma$. □

3.4.3 Triples Computation

Definition 3.4.12. Let σ be a cone as in Remark 3.2.1 and D a torus-invariant $\mathbb{Q}$-divisor, with corresponding polytope P^D_σ as in Definition 3.4.9. Let $\mathfrak{a} \subset R$ be a monomial ideal, and $0 \leq t \in \mathbb{R}$. We define $\mathrm{P}^{\mathrm{D},\mathfrak{a}^\mathrm{t}}_\sigma$ to be the polytope $(P^D_\sigma - t \cdot \mathrm{Newt}(\mathfrak{a})) \cap \sigma^\vee$.

Theorem 3.4.13. *Let R be the coordinate ring of an affine toric variety, with conventions as in Remark 3.2.1. Let D be a torus-invariant $\mathbb{Q}$-divisor as in Definition*

3.4.9. Let $\mathfrak{a} \subset R$ be a monomial ideal, with associated polytope $\mathrm{P}_\sigma^{\mathrm{D},\mathfrak{a}^\mathrm{t}}$ as in Definition 3.4.12. Then $s(R, D, \mathfrak{a}^t) = \mathrm{Volume}(\mathrm{P}_\sigma^{\mathrm{D},\mathfrak{a}^\mathrm{t}})$.

Proof of Theorem 3.4.13. As in our proof of Lemma 3.4.10, we apply Lemma 2.7.7 to replace $(p^e - 1)D$ by $p^e D$, and to assume that $a_i \in \frac{1}{p^e}\mathbb{Z}$. Likewise, we assume that $t \in \frac{1}{p^e}\mathbb{Z}$, and we replace $(p^e - 1)t$ with $p^e t$, so that for sufficiently large e, $\lceil p^e t \rceil = p^e t$, and $\lceil p^e D \rceil = p^e D$. We also replace $\mathfrak{a}^{p^e t}$ with its integral closure $\overline{\mathfrak{a}^{p^e t}}$, which is generated by monomials in the set $p^e t \cdot \mathrm{Newt}(\mathfrak{a})$.

We will use the characterization of F-signature of triples given in Lemma 2.7.8. Thus, we study $(I_e^{\mathfrak{a}})^{1/p^e} = ((I_e^D)^{1/p^e} : (\overline{\mathfrak{a}^{p^e t}})^{1/p^e})$. By Lemma 3.4.10, $(I_e^D)^{1/p^e}$ is generated by the monomials whose characters lie in $(\sigma^\vee \backslash P_\sigma^D) \cap \frac{1}{p^e}M$. The set of characters $\vec{v}$ with $\chi^{\vec{v}} \in (\overline{\mathfrak{a}^{tp^e}})^{1/p^e}$ is $(t \cdot \mathrm{Newt}(\mathfrak{a})) \cap \frac{1}{p^e}M$. Thus, the monomials in $R^{1/p^e} \backslash (I_e^{\mathfrak{a}})^{1/p^e}$ are those $\chi^{\vec{v}}, \vec{v} \in \frac{1}{p^e}M \cap \sigma^\vee$, such that for some $\vec{w} \in \frac{1}{p^e}M \cap t \cdot \mathrm{Newt}(\mathfrak{a}), \vec{v} + \vec{w} \in P_\sigma^D$. This set of characters can be written as a Minkowski sum, so that the size $a_e^{\mathfrak{a}}$ of the set is:

$$a_e^{\mathfrak{a}} = \#(((P_\sigma^D \cap \frac{1}{p^e}M) - ((t \cdot \mathrm{Newt}(\mathfrak{a})) \cap \frac{1}{p^e}M)) \cap \sigma^\vee).$$

We obtain a slightly larger (but easier-to-count) set if we intersect with the lattice $\frac{1}{p^e}M$ only *after* taking the Minkowski sum. In particular, set

$$a'_e := \#((P_\sigma^D - t \cdot \mathrm{Newt}(\mathfrak{a})) \cap \sigma^\vee \cap \frac{1}{p^e}M).$$

Note that $a'_e = \#(\mathrm{P}_\sigma^{\mathrm{D},\mathfrak{a}^\mathrm{t}} \cap \frac{1}{p^e}M)$. Now, a'_e may be larger than $a_e^{\mathfrak{a}}$. However, we apply Lemma 3.4.15 applied to the polytopes $P = P_\sigma^D$ and $Q = -t \cdot \mathrm{Newt}(\mathfrak{a})$ to conclude that $\lim_{e\to\infty} \frac{a_e^{\mathfrak{a}}}{p^{ed}} = \lim_{e\to\infty} \frac{a'_e}{p^{ed}}$.

Thus, $s(R, D, \mathfrak{a}^t) = \lim_{e\to\infty} \frac{a'_e}{p^{ed}}$. We can apply Lemma 3.2.12 (with $M = \mathbb{Z}^n$, $P = \mathrm{P}_\sigma^{\mathrm{D},\mathfrak{a}^\mathrm{t}}$, and $a'_e = \#(P \cap \frac{1}{p^e}M)$) to conclude that the F-signature of triples is the volume of the polytope $\mathrm{P}_\sigma^{\mathrm{D},\mathfrak{a}^\mathrm{t}}$. $\square$

All that remains is to prove Lemma 3.4.15, which implies that in the proof of Theorem 3.4.13, the quantities $a_e^{\mathfrak{a}}$ and a_e' are "close enough" that either one may be used to compute F-signature. These two quantities are obtained similarly: to compute $a_e^{\mathfrak{a}}$, we start with the polytopes P_σ^D and $-t \cdot \mathrm{Newt}(\mathfrak{a})$; intersect each with the lattice $\frac{1}{p^e}M$; then take the Minkowski sum of these two sets. To compute a_e', we take the Minkowski sum of the two polytopes, then intersect with the lattice $\frac{1}{p^e}M$.

Our plan will be to approximate the polytopes in our problem as unions of small cubes. To that end, we introduce some notation:

Notation 3.4.14. Let M be a lattice of rank n. Fix an isomorphism $M \simeq \mathbb{Z}^n$. Every vector $\vec{v} \in M_{\mathbb{R}}$ is contained in a cube of side length 1 whose vertices are those lattice points obtained by rounding each of the coordinates $v_1, \ldots, v_n$ of $\vec{v}$ either up or down. We will call this cube the *M-unit cube* containing $\vec{v}$.

Lemma 3.4.15. *Let M be a lattice. Let P and Q be polytopes in $M_{\mathbb{R}}$. Let S_e be the set of lattice points $\{((P+Q)\cap\frac{1}{p^e}M)\setminus((P\cap\frac{1}{p^e}M)+(Q\cap\frac{1}{p^e}M)\}$. Then* $\lim_{e\to\infty}\frac{\#S_e}{p^{ed}} = 0$.

Proof. Fix an isomorphism $M \simeq \mathbb{Z}^n$. For each e, let $P_e \subset P$ be the union of all $\frac{1}{p^e}M$-unit cubes which are entirely contained within P. Similarly, let $Q_e \subset Q$ be the union of all $\frac{1}{p^e}M$-unit cubes which are entirely contained within Q. By Lemma 3.4.16, $(P_e + Q_e) \cap M = (P_e \cap M) + (Q_e \cap M)$. Thus, $S_e \subset (P + Q)\setminus(P_e + Q_e)$, so $\frac{\#S_e}{p^{ed}} <$ Volume$((P + Q)\setminus(P_e + Q_e))$. It remains to the show that the limit of these volumes is zero. But every interior point of P lies in some P_e, and likewise for Q and Q_e. Thus, every interior point of $P + Q$ lies in some $P_e + Q_e$. We conclude that $\lim_{e\to\infty}$ Volume$((P + Q)\setminus(P_e + Q_e)) = 0$, and the claim is proved. ∎

Lemma 3.4.16. *Let M be a lattice of rank n. Fix a basis for M, so that without loss of generality $M \simeq \mathbb{Z}^n$. Let each of $A, B \subset M_{\mathbb{R}}$ be a union of M-unit cubes. Then*

$(A+B)\cap M = (A\cap M)+(B\cap M)$.

Proof. Say $\vec{v}\in(A+B)\cap M$, so that $\vec{v}=\vec{a}+\vec{b}$ with $\vec{a}\in A, \vec{b}\in B$. Let $\vec{a}'$ be the vector obtained by rounding up the coordinates of $\vec{a}$; let $\vec{b}'$ be the vector obtained by rounding down the coordinates of $\vec{b}$. Since A and B are unions of M-unit cubes, $\vec{a}'\in A$ and $\vec{b}'\in B$. Since $\vec{a}+\vec{b}\in M$, it is easily checked that $\vec{a}'+\vec{b}'=\vec{a}+\vec{b}\in M$. $\square$

3.4.4 $\mathbb{Q}$-Gorenstein Triples

Finally, we prove Corollary 3.4.17, which gives a particularly nice characterization of $\mathrm{P}_\sigma^{\mathrm{D},\mathfrak{a}^{\mathrm{t}}}$ when (R,D) is a $\mathbb{Q}$-Gorenstein pair.

Corollary 3.4.17. *Let R be the coordinate ring of an affine toric variety, D a $\mathbb{Q}$-divisor on* $\operatorname{Spec} R$, *and $\mathfrak{a}$ a monomial ideal, presented as in Theorem 3.4.13. Suppose that the pair (R,D) is $\mathbb{Q}$-Gorenstein. Then* $s(R,D,\mathfrak{a}^t)=\mathrm{Volume}(P_\sigma^D\cap t\cdot\mathrm{Newt}(\mathfrak{a}))$.

Proof of Corollary 3.4.17. Since the pair (R,D) is $\mathbb{Q}$-Gorenstein, for some $\vec{w}\in M\otimes\mathbb{Q}$, $\vec{w}\cdot\vec{v}_i=1-a_i$ for each i. (Just let $\vec{w}$ be the negative of the vector given by Lemma 3.4.8.) Set ϕ to be the map $\vec{u}\mapsto\vec{w}-\vec{u}$. We claim that ϕ is a volume-preserving bijection from $\mathrm{P}_\sigma^{\mathrm{D},\mathfrak{a}^{\mathrm{t}}}$ to $(t\cdot\mathrm{Newt}(\mathfrak{a}))\cap P_\sigma^D$. The corollary will follow immediately.

Before we prove the claim, we first check that ϕ maps P_σ^D to itself. Suppose $\vec{z}\in P_\sigma^D$. Then $0\le\vec{z}\cdot\vec{v}_i$, so $(\vec{w}-\vec{z})\cdot\vec{v}_i=(1-a_i)-(\vec{z}\cdot\vec{v}_i)\le 1-a_i$. Similarly, $0\le(\vec{w}-\vec{z})\cdot\vec{v}_i$. We conclude that $\phi(\vec{z})\in P_\sigma^D$.

Returning to our claim: the map ϕ is clearly linear, volume-preserving, and self-inverse, so it suffices to show that $\phi(\mathrm{P}_\sigma^{\mathrm{D},\mathfrak{a}^{\mathrm{t}}})=(t\cdot\mathrm{Newt}(\mathfrak{a}))\cap P_\sigma^D$. Suppose $\vec{u}\in\mathrm{P}_\sigma^{\mathrm{D},\mathfrak{a}^{\mathrm{t}}}$. In particular, $\vec{u}\in P_\sigma^D$, so (as we just showed) $\phi(\vec{u})\in P_\sigma^D$.

Since $\vec{u}\in\mathrm{P}_\sigma^{\mathrm{D},\mathfrak{a}^{\mathrm{t}}}$, we may write $\vec{u}=\vec{x}-\vec{y}$, with $\vec{x}\in P_\sigma^D, \vec{y}\in(t\cdot\mathrm{Newt}(\mathfrak{a}))\cap P_\sigma^D$. Then $\vec{w}-\vec{u}=\vec{y}+(\vec{w}-\vec{x})$. Since $(\vec{w}-\vec{x})\cdot\vec{v}_i\ge(1-a_i)-(1-a_i)=0$, we conclude that $\vec{w}-\vec{x}\in\sigma^\vee$. Since $t\cdot\mathrm{Newt}(\mathfrak{a})$ is closed under addition by vectors in $\sigma^\vee$, we conclude that $\phi(\vec{u}) = \vec{y} + (\vec{w} - \vec{x}) \in t \cdot \mathrm{Newt}(\mathfrak{a})$.

So far, we have shown that $\phi(\mathrm{P}_\sigma^{\mathrm{D},\mathfrak{a}^t}) \subset P_\sigma^D \cap t \cdot \mathrm{Newt}(\mathfrak{a})$. On the other hand, suppose that $\vec{y} \in P_\sigma^D \cap t \cdot \mathrm{Newt}(\mathfrak{a})$. We wish to show that $\vec{w} - \vec{y} \in \mathrm{P}_\sigma^{\mathrm{D},\mathfrak{a}^t}$. Since $\vec{w} \in P_\sigma^D$, $\vec{w} - \vec{y} \in P_\sigma^D - ((t \cdot \mathrm{Newt}(\mathfrak{a})) \cap P_\sigma^D)$. Moreover, since $\vec{y} \in P_\sigma^D$, we have that $\phi(\vec{y}) \in P_\sigma^D$. We conclude that $\phi(\vec{y}) \in (P_\sigma^D - ((t \cdot \mathrm{Newt}(\mathfrak{a})) \cap P_\sigma^D)) \cap P_\sigma^D = \mathrm{P}_\sigma^{\mathrm{D},\mathfrak{a}^t}$.

It follows that ϕ is a volume-preserving bijection. Thus,

$$s(R, D, \mathfrak{a}^t) = \mathrm{Volume}(\mathrm{P}_\sigma^{\mathrm{D},\mathfrak{a}^t}) = \mathrm{Volume}(P_\sigma^D \cap t \cdot \mathrm{Newt}(\mathfrak{a})),$$

as we desired to show. □

3.5 The Case of Imperfect Residue Field

Thus far we have assumed that we are working over a perfect field. In fact, F-signature may be defined over an field which is imperfect but still F-finite, and our F-signature computations generalize to that setting.

Let R be a ring, either local or graded, as in Remark 2.1.1, but over a possibly imperfect field k. Set $d = \dim R$, and $\alpha = \log_p [k^p : k] < \infty$.

Definition 3.5.1. For each $e \in \mathbb{N}$, let a_e be the free rank of R^{1/p^e} as an R-module. We define the *F-signature* of R to be the limit

$$s(R) = \lim_{e \to \infty} \frac{a_e}{p^{e(d+\alpha)}}.$$

Our definition of the F-signature of an R-module generalizes similarly.

Remark 3.5.2. ([Tuc11], Prop 4.5) As in Remark 2.1.9, define $I_e \subset R$ to be the ideal

$$I_e = \{r \in R \,|\, \forall \phi \in \mathrm{Hom}_R(R^{1/p^e}, R), \phi(r^{1/p^e}) \in m\}.$$

Then $a_e = l(F_*^e(R/I_e))$, so

$$s(R) = \lim_{e \to \infty} \frac{l(F_*^e(R/I_e))}{p^{e(d+\alpha)}}.$$

Since $l(F^e_* M) = [k^p : k]^e l(M)$, we arrive at the following definition of F-signature, which does not depend on α:

$$s(R) = \lim_{e\to\infty} \frac{l(R/I_e)}{p^{ed}}.$$

Remark 3.5.3. We wish to extend Theorem 3.2.3 to the case of an imperfect (but still F-finite) residue field. One can show using [Yao06] (Remark 2.3) that F-signature is in a precise sense residue field independent. We give an less general but more concrete argument. Suppose k is not perfect. The arguments of Theorem 3.2.3 still compute the asymptotic growth rate of the number of splittings of $k[\frac{1}{p^e}S]$:

$$\lim_{e\to\infty} \frac{\text{free rank}(k[\frac{1}{p^e}S])}{p^{ed}} = \text{Volume}(P_\sigma).$$

But for imperfect k, $R^{1/p^e} = k^{1/p^e}[\frac{1}{p^e}S] \simeq k^{1/p^e} \otimes_k k[\frac{1}{p^e}S]$. In particular, R^{1/p^e} is a free $k[\frac{1}{p^e}S]$-module of rank $[k^{1/p^e} : k] = p^{e\alpha}$. It follows that the free rank of R^{1/p^e} is $p^{e\alpha}$ times the free rank of $k[\frac{1}{p^e}S]$. Thus, by Definition 2.1.4, as well as the above formula, we see immediately that $s(R) = \text{Volume}(P_\sigma)$.

This approach generalizes to the case of pairs and triples. In the pairs case, let Σ^D_e denote the set of monomials in $R(p^e D)^{1/p^e}$. Regardless of whether k is perfect, the arguments of Theorem 3.4.10 still compute the asymptotic growth rate of the number of splittings of $k[\Sigma^D_e]$ that also split from $k[\frac{1}{p^e}S]$ to be Volume(P^D_σ). However, $R(p^e D)^{1/p^e} \simeq k^{1/p^e} \otimes_k k[\Sigma^D_e]$, so the number of splittings of $R(p^e D)^{1/p^e}$ that also split from R^{1/p^e} is $p^{e(d+\alpha)} \cdot \text{Volume}(P^D_\sigma)$, and $s(R, D) = \text{Volume}(P^D_\sigma)$, as we desired to show.

In the triples case, let $\Sigma^{D,\mathfrak{a}}_e$ denote the set of monomials in $((I^D_e)^{1/p^e} : (\overline{\mathfrak{a}^{p^e t}})^{1/p^e})$. Regardless of whether k is perfect, the arguments of Theorem 3.4.13 still compute the asymptotic growth rate of $k[\frac{1}{p^e}S]/(\Sigma^{D,\mathfrak{a}}_e)$ to be $\text{Volume}(\mathrm{P}^{\mathrm{D},\mathfrak{a}^{\mathrm{t}}}_\sigma)$. Now consider Lemma 2.7.8. We see that $F^e_* R/((I^D_e)^{1/p^e} : (\overline{\mathfrak{a}^{p^e t}})^{1/p^e}) \simeq k^{1/p^e} \otimes_k k[\frac{1}{p^e}S]/(\Sigma^{D,\mathfrak{a}}_e)$.

Thus, $a_e^{\mathfrak{a}} = p^{e(d+\alpha)} \cdot \mathrm{Volume}(\mathrm{P}_\sigma^{\mathrm{D},\mathfrak{a}^{\mathrm{t}}})$, and $s(R, D, \mathfrak{a}) = \mathrm{Volume}(\mathrm{P}_\sigma^{\mathrm{D},\mathfrak{a}^{\mathrm{t}}})$, as we desired to show.

CHAPTER IV

An Interpretation of F-Signature for Projective Varieties

In this chapter, we propose a definition for the F-signature of a $\mathbb{Q}$-divisor on a projective variety:

Definition 4.1.4. Let X be a normal projective variety over a field k of positive characteristic. Let D be a $\mathbb{Q}$-divisor on X, and suppose that $\mathrm{Sec}(X, D) = \bigoplus_n \Gamma(X, O_X(nD))$ is a finitely generated k-algebra of dimension at least two. Choose $c \in \mathbb{N}_{\geq 0}$ sufficiently divisible so that $\mathrm{Sec}(X, cD)$ is a normal section ring. We define the *F-signature* $s(X, D)$ to be $c \cdot s(X, cD)$.

We will show that this F-signature is a well-defined function on the set of $\mathbb{Q}$-divisors of X with finitely generated section rings, which is invariant under $\mathbb{Q}$-linear equivalence (Theorem 4.1.3).

Using Theorem 3.2.3, we will give a formula for the F-signature of a $\mathbb{Q}$-divisor on a projective toric variety (Corollary 4.4.5). As a consequence of this formula, we show:

Corollary 4.4.6. *Let X be a projective toric variety over a field k of positive characteristic. Then the F-signature function $D \mapsto s(X, D)$ is a continuous, piecewise rational function of degree -1 on the set of big $\mathbb{Q}$-divisors of X.*

We illustrate our formula by computing several examples; this includes a complete description of the F-signature function on the class group of $\mathbb{P}^1 \times \mathbb{P}^1$ (Proposition 4.4.10).

4.1 The F-Signature of a Projective Variety

The F-signature may be used to construct a function on a subset of the divisor class group of a projective variety. We begin by proposing the following definition:

Definition 4.1.1. Let X be a normal projective variety and D an ample Cartier divisor on X. We define the *F-signature* of D, denoted $s(X, D)$, to be the F-signature of the section ring $\operatorname{Sec}(X, D) = \bigoplus_n \Gamma(X, O_X(nD))$.

Remark 4.1.2. We will shortly generalize this definition to apply to $\mathbb{Q}$-divisors on X. We *could* set $s(X, D) = s(\operatorname{Sec}(X, D))$ for *all* (not necessarily ample Cartier) divisors D. However, we will see shortly that a slightly modified function has more desirable properties.

In any case, we suspect that the F-signature $s(\operatorname{Sec}(X, D))$ may be of greatest interest when D is ample, as in that case $s(X, D)$ contains information about the geometry of projective embeddings of X.

A natural question is: how does the F-signature behave as the divisor D varies? We have already shown in Theorem 2.6.2 that when R is a normal section ring (of dimension at least two), the F-signature of R scales predictably when we take Veronese subrings: $s(R^{(n)}) = \frac{1}{n}s(R)$. Thus we obtain the following theorem, which will allow us to generalize Definition 4.1.1:

Theorem 4.1.3. *Let X be a normal projective variety over a field k of positive characteristic. Let D be a $\mathbb{Q}$-divisor on X; that is, D is a formal $\mathbb{Q}$-linear combination of prime Weil divisors on X. Suppose that* $\operatorname{Sec}(X, D)$ *is a finitely generated k-algebra* *of dimension at least two. Choose* c, c' *sufficiently divisible so that* $\mathrm{Sec}(X, cD)$ *is a normal section ring in the sense of Definition 2.5.3. Then* $c \cdot s(X, cD) = c' \cdot s(X, c'D)$.

Proof. Since $\mathrm{Sec}(X, D)$ is finitely generated, so is the Veronese subring $\mathrm{Sec}(X, cD)$ for each $c \in \mathbb{N}$. By Proposition 2.5.6, for any c such that cD is an integral Weil divisor, $\mathrm{Sec}(X, cD)$ is normal. Moreover, since $\mathrm{Sec}(X, cD)$ is finitely generated, by Lemma 2.5.10, there exists c such that $\mathrm{Sec}(X, cD)$ is a section ring. Suppose that $cD, c'D$ both satisfy the conditions given above. We wish to check that $c \cdot s(X, cD) = c' \cdot s(X, c'D)$, so that our definition is choice-independent. By Theorem 2.6.2, $\frac{1}{c'} s(\mathrm{Sec}(X, cD)) = \frac{1}{c} s(\mathrm{Sec}(X, c'D)) = s(\mathrm{Sec}(X, cc'D))$. Multiplying by cc', we obtain the desired equality. $\square$

As an immediate consequence, we obtain the following generalization of Definition 4.1.1:

Definition 4.1.4. Let X be a normal projective variety over a field k of positive characteristic. Let D be a $\mathbb{Q}$-divisor on X, and suppose that $\mathrm{Sec}(X, D)$ is a finitely generated k-algebra of dimension at least two. Choose $c \in \mathbb{N}_{\geq 0}$ sufficiently divisible so that $\mathrm{Sec}(X, cD)$ is a normal section ring in the sense of Definition 2.5.3. We define the *F-signature* $s(X, D)$ to be $c \cdot s(X, cD)$.

Remark 4.1.5. The function $D \mapsto s(X, D)$ is clearly invariant under linear equivalence of integral Weil divisors, since linearly equivalent divisors have isomorphic section rings. By construction, it is also invariant under $\mathbb{Q}$-linear equivalence. Indeed, suppose that D and D' are linearly equivalent $\mathbb{Q}$-divisors, so that for some $a \in \mathbb{N}$, aD and aD' are linearly equivalent integral Weil divisors. Then for sufficiently divisible c,

$$s(X, acD) = s(\mathrm{Sec}(X, acD)) = s(\mathrm{Sec}(X, acD')) = s(X, acD').$$

Thus, $s(X, D) = \frac{1}{ac} s(X, acD) = s(X, D')$.

Remark 4.1.6. When $\mathrm{Sec}(X, D)$ is not a section ring, Definition 4.1.4 does not necessarily define the F-signature $s(X, D)$ as being equal to $s(\mathrm{Sec}(X, D))$. Thus, under this definition, we might have $s(X, D) \neq s(\mathrm{Sec}(X, D))$. We will have to take this failure of equality into account in Section 4.4 when we compute the F-signature function on projective toric varieties. It is the price that we pay for extending our F-signature function to $\mathbb{Q}$-divisors. Of course, whenever D is an ample Cartier divisor, the two notions of F-signature agree.

Remark 4.1.7. We are not missing much by restricting to the dimension ≥ 2 case: when $\dim \mathrm{Sec}(X, D) = 1$, $\mathrm{Sec}(X, D)$ is isomorphic to a polynomial ring in one variable, and $s(\mathrm{Sec}(X, D)) = 1$.

Remark 4.1.8. Recall that a Cartier divisor D is *semiample* if some integer multiple is basepoint-free. If D is a semiample Cartier divisor on a projective variety X, its generalized section ring is finitely generated ([Laz04], 2.1.30). Theorem 4.1.3 demonstrates that we may view the F-signature as a function on the cone of semiample $\mathbb{Q}$-Cartier divisors on X, or more generally on the set of $\mathbb{Q}$-divisors on X with finitely generated generalized section rings.

There are as yet many questions still to be answered regarding the function $D \mapsto s(X, D)$. Motivated by the promise of the volume situation described in the introduction, we ask:

Question 4.1.9. *What can be said about the F-signature function on the set of $\mathbb{Q}$-divisors on X with finitely generated section rings? Is this function continuous, so that it extends to a function defined on a subset of the space of $\mathbb{R}$-divisors on X? Is it invariant under numerical equivalence—that is, does F-signature, like volume,* *induce a function on the Neron-Severi space of X?*

Remark 4.1.10. Given a projective variety X over a field of positive characteristic, it seems reasonable that we might be especially interested in the behavior of the F-signature function on the ample cone of X. In fact, if the F-signature of any ample divisor on X is nonzero, then X is *globally F-regular*, hence has *log Fano* singularities ([SS10]. Thm 1.1). Recall that a variety X is $\mathbb{Q}$*-factorial* if every Weil divisor on X is $\mathbb{Q}$-Cartier. In characteristic zero, is is known that every $\mathbb{Q}$-factorial log Fano projective variety is a so-called *Mori Dream Space* ([BCHM10]). Mori Dream Spaces have many good properties; in particular, the generalized section ring of any divisor on a Mori Dream Space is finitely generated. (For an introduction to Mori Dream Spaces, see [McK10] or [LV09].)

The question of whether globally F-regular varieties are Mori Dream Spaces is open in positive characteristic. Still, we observe:

Proposition 4.1.11 ([SS10], Theorem 1.1; [BCHM10], Corollary 1.3.1)**.** *Let X be a globally F-regular Mori Dream Space over a field k of positive characteristic. Then the generalized section ring of any divisor on X is a finitely generated k-algebra. Consequently, the F-signature function given in Definition 4.1.4 is a well-defined function on the cone of effective $\mathbb{Q}$-divisors on X.*

4.2 Projective Toric Varieties

Here we present enough background on projective toric varieties to compute the F-signature of a divisor on a projective toric variety. As in §3.1, almost all of the notation is standard as in [Ful93], which the reader may consult for further details.

Definition 4.2.1. A *fan* Σ is a finite set of (strongly convex rational polyhedral) cones such that if $\sigma, \tau \in \Sigma$ then $\sigma \cap \tau$ is a face of each of σ and τ; and if $\sigma \in \Sigma$, then all faces of σ are cones in Σ.

As before, let N be a free abelian group of rank n. Its dual lattice $M = N^*$ will be the character lattice of our (not necessarily affine) toric variety X. In what follows, let $\Sigma \subset N_{\mathbb{R}}$ be a fan.

The intersection of any two cones $\sigma, \sigma' \in \Sigma$ is a cone $\tau = \sigma \cap \sigma' \in \Sigma$, which is itself a face of both σ and σ'. Moreover, it may be shown that faces of a cone σ correspond to open affine subsets of X_σ. In particular:

Lemma 4.2.2 ([Ful93],§1.3)**.** *Let $\tau \subset \sigma \subset N_{\mathbb{R}}$ be (strongly convex, rational polyhedral) cones such that τ is a face of σ. Then the inclusion of τ into σ induces an inclusion of X_τ into X_σ as an open affine subset. These inclusions induce an order-preserving bijection between faces of σ and torus-invariant open affine subsets of X_σ.*

As a consequence, the fan Σ provides gluing data for a abstract toric variety.

Definition 4.2.3. Let N be an n-dimensional lattice, $M = N^*$. Let Σ be a fan of cones in $N_{\mathbb{R}}$. Then we denote by X_Σ the n-dimensional toric variety obtained by gluing together the affine toric varieties $\{X_\sigma : \sigma \in \Sigma\}$ in the manner suggested by Lemma 4.2.2. That is, open affine sets X_σ, X'_σ are glued along the open affine subsets $X_{\sigma\cap\sigma'}$.

The fan of a toric variety encodes global data about the variety. For example, let Σ be a fan in $N_{\mathbb{R}}$. Then the toric variety X_Σ is complete if and only if every point of $N_{\mathbb{R}}$ lies in a cone of Σ. We call such a fan *complete*. In particular, the fan of a projective toric variety is complete ([Ful93], §2.4).

As in the affine case, the torus-invariant prime Weil divisors of X_Σ are in one-to-one correspondence with the rays ρ_i of Σ, or equivalently with the primitive generators $\vec{v}_i$ of these rays. Given a torus-invariant divisor D on X_Σ, we may compute sections of the sheaf $O_X(D)$ on any toric affine open subset of X using Lemma 3.4.2. From these computations, it is not difficult to determine the global sections of $O_X(D)$:

Lemma 4.2.4. *[[Ful93], §3.4] Let $\Sigma \subset N_\mathbb{R}$ be a fan with primitive generators $\vec{v}_i$. Let $D = \Sigma_i a_i D_i$ be a torus-invariant Weil divisor on X_Σ. Then $\Gamma(X, O_X(D))$ is generated by the monomials $\{x^{\vec{u}} \in M \mid \vec{u} \cdot \vec{v}_i \geq -a_i\}$.*

When Σ is complete, it may be shown that the set $\{x^{\vec{u}} \in M_\mathbb{R} \mid \vec{u} \cdot \vec{v}_i \geq -a_i\}$ is a polytope in $M_\mathbb{R}$. Lemma 4.2.4 motivates the following definition:

Definition 4.2.5. Let $\Sigma \subset N_\mathbb{R}$ be a complete fan with primitive generators $\vec{v}_i$. Let $D = \Sigma_i a_i D_i$ be a torus-invariant divisor on X_Σ. We define the *polytope associated to the divisor* D, denoted $P_D \subset M_\mathbb{R}$, to be the polytope $\{\vec{u} \in M_\mathbb{R} \mid \vec{u} \cdot \vec{v}_i \geq -a_i\}$. Thus, $\Gamma(X, O_X(D))$ is generated by monomials $\chi^{\vec{u}}$ such that $\vec{u} \in P_D \cap M$.

Remark 4.2.6. Note that P_D may not be a lattice polytope; that is, its vertices may not lie in the lattice M. Any polytope P_D corresponding to a globally generated Cartier divisor is a lattice polytope ([CLS11], Theorem 6.1.7). In any case, since the vertices of P_D are certainly M-rational, it is clear that $kP_D = P_{kD}$ is a lattice polytope for sufficently divisible $k \in \mathbb{N}$.

Notation 4.2.7. The polytope P_D may not be full-dimensional. In general, assuming without loss of generality that P_D contains the origin, it spans a vector subspace $P_D \cdot \mathbb{R}$ of $M_\mathbb{R}$. We denote by M_D the lattice $M \cap (P_D \cdot \mathbb{R})$. Equivalently, for sufficiently large $k \in \mathbb{N}$, M_D is the lattice spanned by $M \cap kP_D$. We denote by N_D the lattice M_D^* dual to M_D. Of course, if P_D is full-dimensional, then $N_D = N$ and $M_D = M$.

Remark 4.2.8. The split inclusion of lattices $M_D \hookrightarrow M$ induces a surjective lattice homomorphism $N \twoheadrightarrow N_D$. That is, elements of N pair naturally with elements of M_D.

Lemma 4.2.4 allows us to easily compute the generalized section ring of a torus-invariant divisor:

Definition 4.2.9. Let V be an $\mathbb{R}$-vector space and $U \subset V$ any set. The *cone over* U is the set $\{t\vec{v} \mid \vec{v} \in U, 0 \le t \in \mathbb{R}\}$.

Lemma 4.2.10. *Let N be a lattice of rank n. Let $\Sigma \subset N_{\mathbb{R}}$ be a complete fan with primitive generators $\vec{v}_i$. Let $D = \Sigma_i a_i D_i$ be a torus-invariant effective divisor on X_Σ. Then the generalized section ring* $\mathrm{Sec}(X, D)$ *is the coordinate ring of the affine toric variety X_σ, where $\sigma^\vee$ is the cone over $P_D \times \{1\}$ lying in the* $(\dim P_D + 1)$*-dimensional space $(M_D \times \mathbb{Z})_{\mathbb{R}}$. This cone is*

$$\sigma^\vee = \{(\vec{v}, t) \mid (\vec{v}, t) \cdot (\vec{v}_i, a_i) \ge 0\}.$$

Proof. The cone $\sigma^\vee$ over $P_D \times \{1\}$ is easily seen to be $\{(\vec{v}, t) \mid \vec{v} \cdot \vec{v}_i \ge -ta_i, t \ge 0\}$, which we may rewrite as $\{(\vec{v}, t) \mid (\vec{v}, t) \cdot (\vec{v}_i, a_i) \ge 0, t \ge 0\}$.

The degree k part of $\mathrm{Sec}(X, D)$ is $\Gamma(X, O_X(kD))$, which is generated by the monomials $\chi^{\vec{u}}$ with exponents such that $\vec{u} \in kP_D \cap M = kP_D \cap M_D$. These are precisely those exponents $(\vec{u}, k) \in \sigma^\vee \cap (M_D \times \mathbb{Z})$ such that $t = k$. We conclude that $\mathrm{Sec}(X, D) = k[\sigma^\vee \cap M_D]$.

All that remains to be proven is our characterization of $\sigma^\vee$. In particular, we show that the inequality $t \ge 0$ is superfluous in defining $\sigma^\vee$. Since $a_i \ge 0$ for all i, for fixed $t < 0$, the set $\{\vec{v} \mid \vec{v} \cdot \vec{v}_i \ge -ta_i\}$ is contained in the set $\{\vec{v} \mid \vec{v} \cdot \vec{v}_i \ge 0\}$, which is equal to $\{\vec{0}\}$. To see this, note that for fixed nonzero $\vec{v}$, each inequality $\vec{v} \cdot \vec{v}_i \ge 0$ determines that $\vec{v}_i$ must lie in a particular half-space in $N_{\mathbb{R}}$. However, since Σ is complete, the vectors $\vec{v}_i$ are not confined to any half-space. We conclude that $\vec{v}$ must equal $\vec{0}$.

It follows that $\sigma^\vee = \{(\vec{v}, t) \mid (\vec{v}, t) \cdot (\vec{v}_i, a_i) \geq 0\}$, as we desired to show. $\square$

Now that we have characterized $\sigma^\vee$, we characterize the cone σ in terms of the fan Σ and the divisor $D = \Sigma_i a_i D_i$.

Lemma 4.2.11. *Let $\sigma \subset N_D \times \mathbb{Z}$ be the cone described in Lemma 4.2.10, so that X_σ is the affine toric variety with coordinate ring $\mathrm{Sec}(X, D)$. If P_D is full-dimensional, then the primitive generators for σ are those $(\vec{v}_j, a_j)$ such that the hyperplane $\{\vec{u} \mid \vec{u} \cdot \vec{v}_j = -a_j\}$ determines a facet of P_D. If P_D is not full-dimensional, primitive rays for σ are generated by the images, under the projection $N \to N_D$, of those same $(\vec{v}_j, a_j)$.*

Proof. The primitive generators of σ correspond to normal vectors of facets of $\sigma^\vee$ ([Ful93], 1.2.10). By Lemma 4.2.10, $\sigma^\vee = \{(\vec{v}, t) \mid (\vec{v}, t) \cdot (\vec{v}_i, a_i) \geq 0\}$. Each facet of $\sigma^\vee$ corresponds to an inequality from this list which actually determines a half-space bounding $\sigma^\vee \subset (M_D)_\mathbb{R}$ along a facet. For fixed j, a half-space $H_j = \{(\vec{v}, t) \mid (\vec{v}, t) \cdot (\vec{v}_j, a_j) \geq 0\}$ bounds $\sigma^\vee$ along a facet precisely when $H_j \cap \{t = 1\}$ bounds $\sigma^\vee \cap \{t = 1\}$ along a facet. This occurs precisely when $\{\vec{v} \in P_D \mid \vec{v} \cdot \vec{v}_j \geq -a_j\}$ is a facet of P_D. The claim follows. $\square$

4.3 Polytope Volumes are Piecewise Rational Functions of Facet Data

Consider a polytope in $\mathbb{R}^n$ cut out by linear inequalities of the form $\vec{v} \cdot \vec{v}_i \geq c_i$. The volume of this polytope is clearly a continuous function of the parameters $\vec{v}_i, c_i$. In preparation for Corollary 4.4.6, on the rationality of the F-signature function of a projective toric variety, we will show that the volume of such a polytope is in fact a *piecewise rational* function of these parameters.

Definition 4.3.1. Let $P \subset \mathbb{R}^n$ be a polytope. A *triangulation* T of P is a finite set of n-simplices with pairwise disjoint interiors whose union is P. The *vertices* of T are the vertices of simplices in T.

Definition 4.3.2. Let $P, P' \subset \mathbb{R}^n$ be polytopes.We say that P, P' are *combinatorially equivalent* if there is an inclusion-preserving bijection between the faces of P and those of P'.

Remark 4.3.3. Let X be a set, and let $\{P(x) \mid x \in X\}$ be a family of polytopes in $\mathbb{R}^n$ which are combinatorially equivalent. Suppose that we fix $x_0 \in X$ as well as a face $F(x_0) \in P(x_0)$. This face, along with the combinatorial equivalence, determines a unique face $F(x)$ for each $x \in X$. In this way, we may refer to a *face* $F(x)$ of the family, by which we mean a choice of equivalent faces, one for each $x \in X$.

Intuitively, two polytopes are combinatorially equivalent if they are "the same shape." For a more careful (if slightly more technical) development of this notion of combinatorial equivalence, see [KR12].

Proposition 4.3.4. *Let $X \subset (\mathbb{R}^{n+1})^r$ be the space of size-r families of linear inequalities in $\mathbb{R}^n$ whose solution set is a polytope. In particular, given $\vec{v}_1, \ldots, \vec{v}_r \in \mathbb{R}^n$ and $c_1, \ldots, c_r \in \mathbb{R}$, set $x = ((\vec{v}_1, c_1), \ldots, (\vec{v}_r, c_r))$, corresponding to the system of linear inequalities $\vec{v} \cdot \vec{v}_i \geq c_i, 1 \leq i \leq r$. Then $x \in X$ if these linear inequalities define a bounded polytope, in which case we denote that polytope by $P(x)$. Let $V : X \to \mathbb{R}$ be the function $V(x) = \mathrm{Volume}(P(x))$. Then V is a continuous,* piecewise rational *function on X, in the sense that X is a finite union of cells such that on each cell, V restricts to a quotient of polynomial functions with coefficients in $\mathbb{Q}$.*

Proof. Continuity is obvious; it remains to check piecewise rationality. Given $x, x' \in X$, there is a natural correspondence between the linear equations defining $P(x)$ and those defining $P(x')$, given by $(\vec{v}_i, c_i) \mapsto (\vec{v}'_i, c'_i)$. Fix a cell $U \subset X$ on which, with respect to this correspondence, the polytopes $\{P(x) \mid x \in U\}$ are combinatorially equivalent. Note that there are only finitely many such cells.

Let $\vec{u}(x)$ be a vertex of $P(x)$. Then for some $i_1, \ldots, i_n \in \{1, \ldots, r\}$, $\vec{u}(x)$ is the intersection of the i_1st,$\ldots, i_n$th facets of $P(x)$. We claim that the coordinates of $\vec{u}(x)$ are rational functions of x with coefficients in $\mathbb{Q}$. In particular, if $M(x)$ is the invertible matrix whose jth row is $\vec{v}_{i_j}(x)$, then $\vec{u} = M^{-1}\langle c_1, \ldots, c_r\rangle$. By Cramer's rule, the entries of M^{-1} are rational functions of the entries of M. It follows that the entries of $\vec{u}$ are rational functions of x.

Next, we triangulate each $P(x)$ simultaneously, according to the method in Lemma 4.3.5 below. Let $\mathcal{S}$ be the set of simplices in our triangulation. Then $V(x) = \sum_{S(x)\in\mathcal{S}} \mathrm{Volume}(S(x))$. By Lemma 4.3.6, for each $S \in \mathcal{S}$, Volume$(S(x))$ is a polynomial function with coefficients in $\mathbb{Q}$ of the vertices $\vec{u}_0(x), \ldots, \vec{u}_n(x)$ of S. By Lemma 4.3.5, the vertices of S are themselves rational functions of x, so $V(x)$ is a rational function of x, with coefficients in $\mathbb{Q}$. $\square$

Lemma 4.3.5 (cf. [Grü03], Theorem 6.2.4)**.** *Let X be a set, and let $\{P(x) \mid x \in X\}$ be a family of polytopes in $\mathbb{R}^n$ which are combinatorially equivalent. Then we may triangulate each $P(x)$ simultaneously, as a finite union of simplices $\{S_i(x) \mid i \in I\}$, compatibly with the combinatorial equivalence on $P(x)$. Moreover, for each i, the vertices of $S_i(x)$ are each given by a linear function in the vertices of $P(x)$, with coefficients in $\mathbb{Q}$.*

Proof. Let $\vec{v}_1(x), \ldots, \vec{v}_l(x)$ be the vertices of $P(x)$. We will inductively triangulate each of the d-dimensional faces of $P(x)$. When $d = 0$, the zero-dimensional faces of $P(x)$ are the points $\vec{v}_1(x), \ldots, \vec{v}_l(x)$; triangulating these zero-dimensional faces is trivial. Suppose we have triangulated the $(d-1)$-dimensional faces of $P(x)$, and let F be a d-dimensional face with k vertices. Set $p(x) = \sum_{\vec{v}_i(x)\in F(x)} \frac{1}{k}\vec{v}_i(x)$. (The precise coefficients $\frac{1}{k}$ are not important, merely the fact that all coefficients are nonzero and lie in $\mathbb{Q}$.) Then $p(x)$ is in the *relative interior* of $F(x)$; that is, it does not lie on a proper face of $F(x)$. Let $G(x)$ be any proper face of $F(x)$; by our inductive hypothesis, we have already triangulated $G(x)$ as a union of simplices. For each such simplex $S_i^G(x) \subset G(x)$, let $S_i^{G,p}(x)$ be the convex hull of $S_i^G(x)$ with $p(x)$. It is easily checked that the simplices $S_i^{G,p}(x)$ together form a triangulation of F, as desired. Moreover, by construction, each vertex of each simplex is a linear function of $\vec{v}_1(x), \ldots, \vec{v}_l(x)$ with coefficients in $\mathbb{Q}$. □

Lemma 4.3.6. *Let $\vec{u}_0, \ldots, \vec{u}_n \in \mathbb{R}^n$ be the vertices of an n-simplex S. Then the volume of S is a polynomial function of the coordinates of $\vec{u}_0, \ldots, \vec{u}_n$, with coefficients in $\mathbb{Q}$.*

Proof. We leave it to the reader to check (say, by basic multivariable calculus) that Volume(S) is equal to $\frac{1}{n!}$ times the determinant of the matrix whose ith row is $\vec{u}_i - \vec{u}_0$. □

4.4 The F-Signature Function of a Projective Toric Variety

4.4.1 A Projective Toric F-Signature Formula

Armed with Theorem 3.2.3 and Lemma 4.2.11, we may easily compute the F-signature function on a projective toric variety.

Remark 4.4.1. (Conventions.) For the remainder of this section, N is a lattice of dimension d; $M = N^*$ is the dual lattice; and $\Sigma \subset N_{\mathbb{R}}$ is the fan associated to the d-dimensional projective toric variety $X = X_\Sigma$, with primitive generators $\vec{v}_1, \ldots, \vec{v}_r$.

Remark 4.4.2. Let $D = \sum_i a_i D_i$ be a $\mathbb{Q}$-divisor on X_Σ. As we have previously discussed, each primitive generator $\vec{v}_i$ of Σ, along with the coefficient $a_i \in \mathbb{Q}$, determines a linear equation of the form $\vec{u} \cdot \vec{v}_i = -a_i$ which determines a hyperplane in $N_{\mathbb{R}}$. The facets of P_D are determined by these hyperplanes; however, not all hyperplanes determine facets. They may intersect P_D in a lower-dimensional face, or not at all.

We will need to pay attention to this technicality when we compute our F-signature formula.

Definition 4.4.3. Let Σ be a fan as in Remark 3.2.1. Let $D = \sum_i a_i D_i$ be a torus-invariant $\mathbb{Q}$-divisor on X. Let $M_D \subset M$ be as defined in §4.2. We define $P_{\Sigma,D}$ to be the polytope $\{(\vec{v}, t) \in ((M_D) \times \mathbb{Z})_{\mathbb{R}} \mid \forall i$ such that the hyperplane $\{\vec{u} \mid \vec{u} \cdot \vec{v}_i = -a_i\}$ determines a facet of P_D, we have $0 \leq (\vec{v}, t) \cdot (\vec{v}_i, a_i) < 1\}$.

We begin by computing the F-signature $s(\mathrm{Sec}(X, D))$ of the section ring of a torus-invariant divisor on a projective toric variety.

Theorem 4.4.4. *Let Σ be the fan of a projective toric variety $X = X_\Sigma$ of dimension d over a field k of positive characteristic. Let $\vec{v}_1, \ldots, \vec{v}_r$ be the primitive generators of Σ. Let $D = \Sigma_{i=1}^r a_i D_i$ be an effective torus-invariant Weil divisor on X_Σ. Let $P_D \subset M_{\mathbb{R}}$ be the polytope corresponding to D. Suppose that P_D is a full-dimensional lattice polytope. Then $s(\mathrm{Sec}(X, D)) = \mathrm{Volume}(P_{\Sigma,D})$.*

Proof. Let $I = \{i \in \mathbb{N} \mid$ the hyperplane $\{\vec{u} \mid \vec{u} \cdot \vec{v}_i = -a_i\}$ determines a facet of $P_D\}$. By Lemma 4.2.11, $\mathrm{Sec}(X, D) = k[\sigma^\vee \cap M]$, where $\sigma \subset (N \times \mathbb{Z})^*_{\mathbb{R}}$ has rays which are generated by the vectors $\{(\vec{v}_i, a_i) \mid i \in I\}$. Note that since $\vec{v}_i$ is N-primitive, $(\vec{v}_i, a_i)$ is an $(N \times \mathbb{Z})$-primitive generator for the ray on which it lies. Since $\sigma^\vee \subset (M \times \mathbb{Z})_{\mathbb{R}}$ is full-dimensional, σ is strongly convex. Thus, we may apply Theorem 3.2.3. We conclude that $s(\mathrm{Sec}(X, D)) = \mathrm{Volume}(P_\sigma) = \mathrm{Volume}(P_{\Sigma,D})$. □

Recall that by our definition, $s(X, D)$ need not equal $s(\mathrm{Sec}(X, D))$ if $\mathrm{Sec}(X, D)$ is not a section ring. However:

Corollary 4.4.5. *Let Σ be a the fan of a projective toric variety $X = X_\Sigma$ of dimension d over a field k of positive characteristic. Let D be any torus-invariant $\mathbb{Q}$-divisor* *on X such that P_D is full-dimensional (equivalently, D is big). Let $s(X, D)$ be defined as in Definition 4.1.4. Let $P_{\Sigma,D}$ be defined as in the statement of Theorem 4.4.4. Then*

$$s(X, D) = \text{Volume}(P_{\Sigma,D}).$$

Proof. Fix $D = \sum_i a_i D_i$. For sufficiently divisible $n \in \mathbb{N}$, $\text{Sec}(X, nD)$ is a section ring, by Lemma 2.5.10, and P_D is a lattice polytope, by Remark 4.2.6. In that case, $s(X, nD) = \text{Volume}(P_{\Sigma,nD})$. By our definition, for any $\mathbb{Q}$-divisor D, $s(X, D) = n \cdot s(X, nD)$. It remains to show that $\text{Volume}(P_{\Sigma,D}) = n \cdot \text{Volume}(P_{\Sigma,nD})$.

Let $I = \{i \in \mathbb{N} \,|\, \text{the hyperplane } \{\vec{u} \,|\, \vec{u} \cdot \vec{v}_i = -a_i\} \text{ determines a facet of } P_D\}$. Then

$$P_{\Sigma,D} = \{(\vec{v}, t) \in ((M_D) \times \mathbb{Z})_{\mathbb{R}} \,|\, \forall i \in I, 0 \leq (\vec{v}, t) \cdot (\vec{v}_i, a_i) < 1\},$$

and

$$P_{\Sigma,nD} = \{(\vec{v}, t) \in ((M_D) \times \mathbb{Z})_{\mathbb{R}} \,|\, \forall i \in I, 0 \leq (\vec{v}, t) \cdot (\vec{v}_i, n \cdot a_i) < 1\}.$$

Observe that the linear transformation $(\vec{v}, t) \mapsto (\vec{v}, nt)$ transforms $P_{\Sigma,nD}$ into $P_{\Sigma,D}$. It follows that

$$\text{Volume}(P_{\Sigma,D}) = n \cdot \text{Volume}(P_{\Sigma,nD}),$$

as we desired to show. $\square$

Recall that a $\mathbb{Q}$-divisor on a variety X of dimension d is *big* if its generalized section ring has dimension $d+1$. It is easy to see in the toric case that a big divisor D is one such that P_D is full-dimensional, and the set of big $\mathbb{Q}$-divisors of X is the interior of the set of effective $\mathbb{Q}$-divisors on X.

Corollary 4.4.6. *Let Σ be the fan of a projective toric variety $X = X_\Sigma$ of dimension $d \geq 2$ over a field k of positive characteristic. Let $\vec{v}_1, \ldots, \vec{v}_r$ be the primitive generators of Σ. Then the F-signature function $\langle a_1, \ldots, a_r \rangle \mapsto s(X, \sum_i a_i D_i)$ is a* *continuous, piecewise rational function, of degree -1, with coefficients in $\mathbb{Q}$, on the set of big $\mathbb{Q}$-divisors of X.*

Proof. We apply Proposition 4.3.4 to the polytope

$$P_{\Sigma,D} = \{(\vec{v},t) \in \mathbb{A}^{d+1}_{\mathbb{R}} \mid (\vec{v},t)\cdot(\vec{v}_i,a_i) \geq 0, (\vec{v},t)\cdot(-\vec{v}_i,-a_i) \geq -1\},$$

concluding that $\mathrm{Volume}(P_{\Sigma,D})$ is a continuous, piecewise rational function of the a_i, with coefficients in $\mathbb{Q}$.

Over an infinite field k, it is easily checked that the homogeneous rational functions of degree d on $\mathbb{A}^n_k$ are characterized by the property that for each $c \in k$, $f(c\underline{x}) = c^d f(\underline{x})$. In our case, $s(X, \sum_i (na_i)D_i) = \frac{1}{n}s(X, \sum_i a_i D_i)$ for each $n \in \mathbb{Z}$, hence the same holds for each $c \in \mathbb{Q}$. We conclude that $D \mapsto s(X,D)$ is homogeneous of degree -1. □

If P_D is not full-dimensional, it is possible that for some vertex $\vec{v}_i$ of P_D, the projection of $(\vec{v}_i, a_i)$ to N_D may not be primitive. In that case, the formula of Corollary 4.4.5 need not compute $s(\mathrm{Sec}(X,D))$. The formula is easily corrected, however:

Corollary 4.4.7. *Suppose that we are in the situation of Theorem 4.4.4, except that P_D is not necessarily a full-dimensional lattice polytope. For each i, let $c_i \in \mathbb{Q}$ be such that c_i times the projection of $\vec{v}_i \in N$ to N_D is a primitive generator for its ray. Let $P'_{\Sigma,D}$ be the polytope $\{(\vec{v},t) \in ((M_D)\times\mathbb{Z})_{\mathbb{R}} \mid \forall i$ such that the hyperplane $\{\vec{u} \mid \vec{u}\cdot\vec{v}_i = -a_i\} \subset (M_D)_{\mathbb{R}}$ determines a facet of $P_D, 0 \leq (\vec{v},t)\cdot c_i(\vec{v}_i,a_i) < 1\}$. Then*

$$s(X,D) = \mathrm{Volume}_{M_D}(P'_{\Sigma,D}).$$

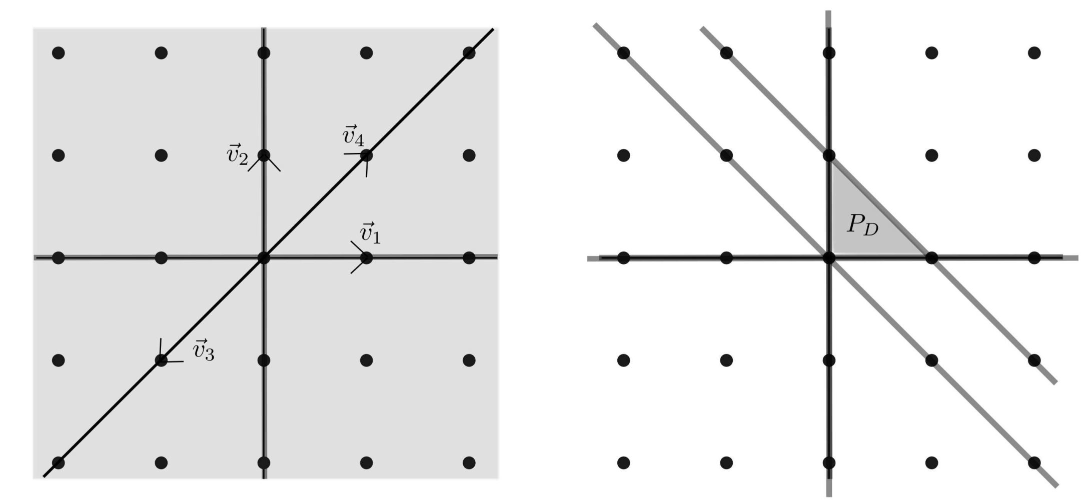
Proof. The argument here is identical to the proof of Theorem 4.4.4 and Corollary 4.4.5, except that we first rescale the vectors $(\vec{v}_i, a_i)$ by c_i. When the divisor D

is sufficiently divisible, the vectors $c_i \cdot (\vec{v}_i, a_i)$ are primitive generators for σ, and we conclude that $s(X, D) = s(\mathrm{Sec}(X, D)) = \mathrm{Volume}(P_{\Sigma,D})'$. As in Corollary 4.4.5, once we have proved that the formula holds for sufficiently divisible divisors D, it immediately generalizes to hold for all divisors. $\square$

4.4.2 Example F-Signature Computations

Now we consider a few examples which illustrate the subtleties arising in our toric F-signature formulas.

Example 4.4.8 (The blowup of $\mathbb{P}^2$ at a point). Let X be the blowup of the toric variety $\mathbb{P}^2$ at a torus-fixed point. The fan of X has four primitive generators $\vec{v}_1, \ldots, \vec{v}_4$ (Figure 4.1(a)). Set $D = D_3$. Then P_D is a simplex with height and width 1 (Figure 4.1(b)). From the figure, we easily compute the set $I = \{i \in \mathbb{N} \mid$ the hyperplane $\{\vec{u} \mid \vec{u} \cdot \vec{v}_i = -a_i\}$ determines a facet of $P_D\}$. Notice that the primitive generator $\vec{v}_4$ gives rise to the equation $\vec{v} \cdot \langle 1, 1\rangle \geq 0$, which does not determine a facet of P_D. Thus, $I = \{1, 2, 3\}$.



(a) The fan $\Sigma \subset N_{\mathbb{R}}$ of the blowup of $\mathbb{P}^2$ at a point. (b) The polytope $P_{D_3} \subset M_{\mathbb{R}}$.

Figure 4.1: The blowup of $\mathbb{P}^2$ at a torus-fixed point.

It follows from Theorem 4.4.4 that $s(X, D)$ is the volume of the polytope

$$\begin{aligned} P_\sigma &= \{(x,y,t) \in \mathbb{R}^3 \,|\, 0 \leq (x,y,t) \cdot (\vec{v}_i, a_i) < 1, 1 \leq i \leq 3\} \\ &= \{(x,y,t) \in \mathbb{R}^3 \,|\, 0 \leq x < 1, 0 \leq y < 1, 0 \leq -x - y + t < 1\}. \end{aligned}$$

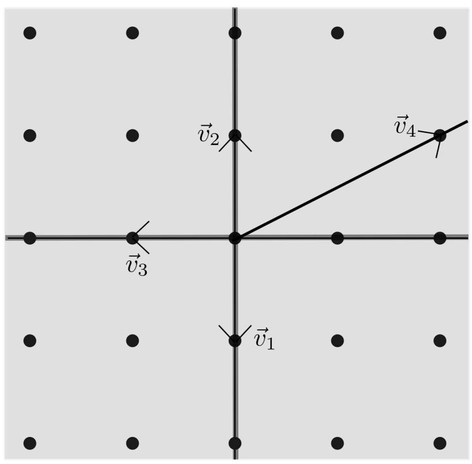
We leave it to the reader to check that the volume of this polytope is equal to 1. This is not surprising, as the divisor D_3 gives rise to the blowup map $X \to \mathbb{P}^2$. The generalized section ring of D_3 is the polynomial ring $k[x, y, z]$ in three variables, which has F-signature equal to 1.

This example illustrates the necessity, in defining $P_{\Sigma,D}$, of keeping track of the set I of primitive generators which actually determine facets of P_D. The equation that we omitted when defining $P_{\Sigma,D}$, corresponding to $i = 4$, was $0 \leq x + y \leq 1$. Note that had we (incorrectly) included this equation when applying Theorem 4.4.4, the resulting polytope would have been strictly smaller than $P_{\Sigma,D}$, and in computing its volume we would have obtained the incorrect F-signature.

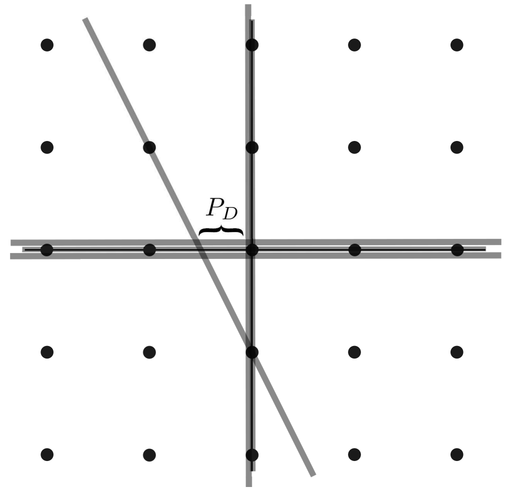
Example 4.4.9 (The F-signature of a non-full-dimensional polytope). Let $\Sigma \subset \mathbb{R}^2$ be the fan given in Figure 4.2(a). The fan Σ has four primitive generators $\vec{v}_1, \ldots, \vec{v}_4$, corresponding to torus-invariant divisors $D_1, \ldots, D_4$. We will compute the F-signature of the divisor D_4 on X_Σ. It is easily checked that P_{D_4} is the line segment $\{(x, y) \,|\, y = 0, -\frac{1}{2} \leq x \leq 0\}$ (Figure 4.2(b)).

We adopt the notation of Corollary 4.4.7. The primitive generators $\vec{v}_1$ and $\vec{v}_2$ do not determine facets of P_{D_4}, but the vectors $\vec{v}_3$ and $\vec{v}_4$ do. Note that $M_D \subset M$ is the one-dimensional sublattice $\mathbb{Z} \times \{0\}$, with basis $\{\langle 1, 0\rangle\}$. Let $\pi : N \twoheadrightarrow N_D$ be the natural projection map, so that a dual basis for N_D is $\{\pi(\langle 1, 0\rangle)\}$.

Since $\vec{v}_3 \cdot \langle 1, 0\rangle = -1$, we see that $\pi(\vec{v}_3)$ generates N_D, hence $\pi(\vec{v}_3)$ is N_D-primitive, and $c_3 = 1$. On the other hand, $\vec{v}_4 \cdot \langle 1, 0\rangle = 2$. Consequently, $\pi(\vec{v}_4)$ is not primitive,



(a) The fan $\Sigma \subset N_{\mathbb{R}}$.



(b) The polytope $P_{D_4} \subset M_{\mathbb{R}}$ is the line segment $\{(x,y) \mid y = 0, -\frac{1}{2} \leq x \leq 0\}$.

Figure 4.2: The F-signature of a polytope which is not full-dimensional.

and $c_4 = \frac{1}{2}$. We leave it to the reader to check that Volume$(P_{\Sigma,D_4}) = 1$, while Volume$(P'_{\Sigma,D_4}) = 2$. Thus, $s(X, D_4) = 2$.

We conclude that the F-signature $s(X, D)$ of an integral Weil divisor need not lie between 0 and 1. Moreover, the F-signature of a projective toric variety need not be a continuous function on its domain. In our case, for big divisors D approaching D_4, $s(X, D)$ approaches Volume$(P_{\Sigma,D_4}) = 1$, while $s(X, D_4) = 2$.

Example 4.4.10 (The F-signature function on $\mathbb{P}^1 \times \mathbb{P}^1$)*.* Now we compute a more complicated example. Consider the projective toric variety $X = \mathbb{P}^1 \times \mathbb{P}^1$. Then $H = \mathbb{P}^1 \times \{pt\}$ and $H' = \{pt\} \times \mathbb{P}^1$ are divisors on X which generate the class group of X. Denote by $O_X(a, b)$ the invertible sheaf $O_X(aH + bH')$. To compute the F-signature function on the class group of X, it suffices to compute $s(aH + bH') = s(X, O_X(a, b))$ for $a, b \in \mathbb{N}$.

Proposition 4.4.11. *Let $X = \mathbb{P}^1 \times \mathbb{P}^1$. Fix $0 < a \leq b \in \mathbb{N}$. Then*

$$s(X, O_X(a,b)) = \begin{cases} \frac{a}{b^2} & b \geq 2a \\ \frac{4}{b} + \frac{b}{3a^2} - \frac{5a}{3b^2} - \frac{2}{a} & a \leq b \leq 2a \end{cases}$$

Moreover, for $0 < b \in \mathbb{N}$, $s(X, O_X(0,b)) = \frac{1}{b}$.

Proof. It is easily checked that $\mathrm{Sec}(X, O_X(0,1)) \simeq k[x,y]$ is the coordinate ring of $\mathbb{P}^1$, which has F-signature equal to 1. Thus, for $b \neq 0$, $\mathrm{Sec}(X, O_X(0,b)) = \frac{1}{b}$ by Theorem 2.6.2. It remains to compute the F-signature when a and b are both positive.

The fan Σ of $\mathbb{P}^1 \times \mathbb{P}^1$ has four primitive generators $\vec{v}_1, \ldots, \vec{v}_4$ (Figure 4.3(a)). Set $D = aD_3 + bD_4$, so that $O_X(D) \simeq O_X(a,b)$. Then P_D is a rectangle with height a and width b (Figure 4.3(b)). In particular, every primitive generator corresponds to a facet of P_D. It follows from Theorem 4.4.4 that $s(X, D)$ is the volume of the polytope

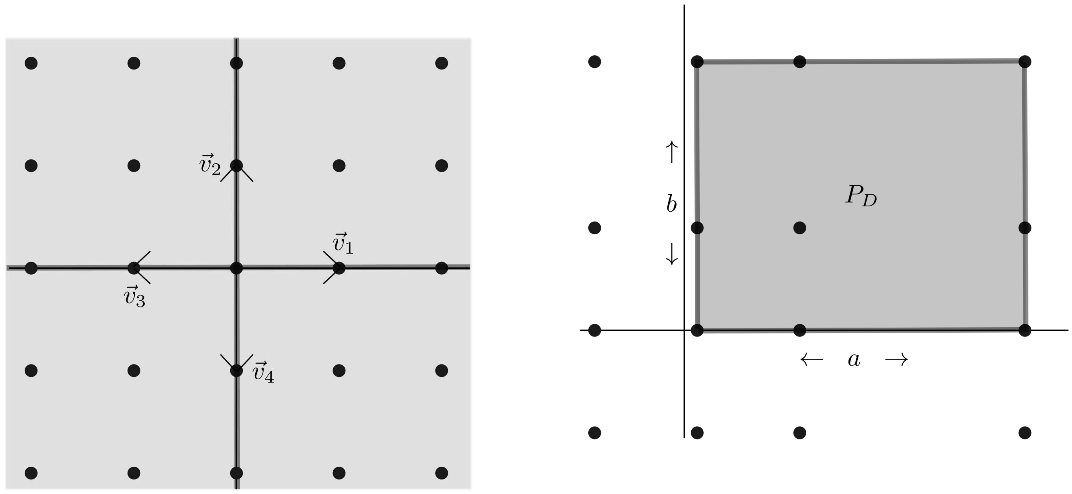
$$\begin{aligned} P_\sigma &= \{(x,y,t) \in \mathbb{R}^3 \mid 0 \leq (x,y,t) \cdot (\vec{v}_i, a_i) < 1, 1 \leq i \leq 4\} \\ &= \{(x,y,t) \in \mathbb{R}^3 \mid 0 \leq x < 1, 0 \leq y < 1, 0 \leq -x + at < 1, 0 \leq -y + bt < 1\} \\ &= \{(x,y,t) \in \mathbb{R}^3 \mid 0 \leq x < 1, 0 \leq y < 1, at - 1 \leq x < at, bt - 1 \leq y < bt\}. \end{aligned}$$

Now we compute the volume of P_σ via a series of integrals. Note that the volume remains unchanged if we replace P_σ with its closure (replacing all instances of $<$ with $\leq$).

For $0 \leq t \leq \frac{1}{b}$, so that $at - 1 \leq bt - 1 \leq 0$, we have $0 \leq x \leq at, 0 \leq y \leq bt$. The volume of this subset $P_\sigma \cap \{0 \leq t \leq \frac{1}{b}\}$ is $\int_{t=0}^{1/b} \int_{x=0}^{at} \int_{y=0}^{bt} dV = \int_{t=0}^{1/b} abt^2 dt = \frac{a}{3b^2}$.

Since $bt - 1 \leq y \leq 1$, we must have $t \leq \frac{2}{b}$. Similarly, $t \leq \frac{2}{a}$. Now we have two cases. First, suppose $b \geq 2a$, so that $\frac{1}{b} \leq t \leq \frac{2}{b} \leq \frac{1}{a} \leq \frac{2}{a}$. Then the volume of the subset $P_\sigma \cap \{\frac{1}{b} \leq t \leq \frac{2}{b}\}$ is

$$\int_{t=1/b}^{2/b} \int_{x=0}^{at} \int_{y=bt-1}^{1} dV = \frac{2a}{3b^2}.$$



(a) The fan $\Sigma \subset N_{\mathbb{R}}$ of $\mathbb{P}^1 \times \mathbb{P}^1$.

(b) The polytope $P_{aD_3+bD_4} \subset M_{\mathbb{R}}$.

Figure 4.3: The F-signature of a divisor on $\mathbb{P}^1 \times \mathbb{P}^1$.

We conclude that for $b \geq 2a$, $\text{Volume}(P_\sigma) = \frac{a}{3b^2} + \frac{2a}{3b^2} = \frac{a}{b^2}$.

Now suppose that $a \leq b \leq 2a$, so that $\frac{1}{b} \leq t \leq \frac{1}{a} \leq \frac{2}{b} \leq \frac{2}{a}$. Then the volume of the subset $P_\sigma \cap \{\frac{1}{b} \leq t \leq \frac{1}{a}\}$ is

$$\int_{t=1/b}^{1/a} \int_{x=0}^{at} \int_{y=bt-1}^{1} dV = \frac{1}{a} - \frac{2a}{3b^2} - \frac{b}{3a^2}.$$

And the volume of the subset $P_\sigma \cap \{\frac{1}{a} \leq t \leq \frac{2}{b}\}$ is

$$\int_{t=1/a}^{2/b} \int_{x=at-1}^{1} \int_{y=bt-1}^{1} dV = \frac{4}{b} - \frac{3}{a} - \frac{4a}{3b^2} + \frac{2b}{3a^2}.$$

We conclude that for $a \leq b \leq 2a$,

$$\begin{aligned}
\text{Volume}(P_\sigma) &= \frac{a}{3b^2} + \left(\frac{1}{a} - \frac{2a}{3b^2} - \frac{b}{3a^2}\right) + \left(\frac{4}{b} - \frac{3}{a} - \frac{4a}{3b^2} + \frac{2b}{3a^2}\right) \\
&= \frac{4}{b} + \frac{b}{3a^2} - \frac{5a}{3b^2} - \frac{2}{a}.
\end{aligned}$$

□

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