On the existence of attracting domains for maps tangent to the identity

by

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Preface

Before discussing the specifics of this thesis, let's take a moment to explore the idea of a dynamical system. Suppose you have a pool table, a cue stick and a ball. Place the ball anywhere on the table, taking note of where you place it. Hit the ball with your cue stick at a particular angle, with a particular force. The ball will move around the table until it eventually stops; take note of the new position of the ball. Hit the ball with your cue stick *in the exact same manner* as before - with the same angle and force. The ball will move around the table until it eventually stops; take note of this new position. Continue to do this process indefinitely, each time adding onto the list of positions the ball attains. The list you create tells you about the behavior (or *dynamics*) of the system. Fix a manner in which to hit the ball; this is a function defined at all possible positions of the ball on the pool table.

By varying the initial position of the ball even slightly, our list of positions that the ball attains can change dramatically and may demonstrate very different behavior. For instance, it could happen that where you initially place the ball, after hitting the ball five times it returns to this initial position; in this case the behavior of the ball is *periodic*, repeating itself after 5 iterations. Or it could happen that you position the ball so that after hitting it once it ends in the exact same position; in this case the position of the ball is *fixed*. Or you could position the ball so that each time you hit it, the ball gets closer and closer to a particular position x on the table, but it never actually reaches that position x; in this case the position you chose is *attracted* to x. Or perhaps you place the ball somewhere on the table and when you hit it, the ball actually does what it is suppose to do and goes through one of the holes in the table; then the ball has left the system and we can no longer study how the cue stick acts on the position of the ball. In dynamics, we study the behavior of points under iteration by a function and characterize these possible behaviors (such as periodic, fixed, or attracted to a point).

One of the guiding questions behind the study of local (discrete) holomorphic dynamics is: given a germ f of a holomorphic self-map of $\mathbb{C}^m$ that fixes a point (say the origin), can f be expresed in a simpler form? If so, then the dynamical behavior of the map can be more easily understood.

In one dimension, f can be expressed near the origin as:

$$f(z) = \lambda z + a_k z^k + a_{k+1} z^{k+1} + \dots,$$

where $k \in \mathbb{N}$ and either both $k > 1$ and $a_k \neq 0$ or $f(z) = \lambda z$. The local dynamics of f is well-understood except when $|\lambda| = 1$, but λ is not a root of unity. The remaining scenarios for how f depends on λ can be divided into 3 cases: (1) If $|\lambda| \neq 0, 1$, then f is locally holomorphically conjugate to the linear map $z \mapsto \lambda z$. Hence, the local dynamics in this case is clear: when $|\lambda| < 1$ (or $|\lambda| > 1$), points tend towards the origin upon repeated application of f (or f^{-1}, respectively). (2) If $\lambda = 0$ and $a_k \neq 0$, then f is locally holomorphically conjugate to the map $z \mapsto z^k$. Hence, all points near the origin tend towards the origin upon repeated application of f. (3) If λ is a q-th root of unity, then we can better understand the local dynamics of f by understanding the local dynamics of f^q. In this case, assuming $f(z) \not\equiv \lambda z$, f is not holomorphically conjugate to a linear map in a full neighborhood of the origin, however, the dynamics of f in a neighborhood of the origin is still well-understood. We discuss this case further in §1.1. More information on the topic of local dynamics in one dimension can be found in [A3, CG, M].

In higher dimensions, f can be expressed near the origin as:

$$f(z) = P_1(z) + P_k(z) + P_{k+1}(z) + \dots,$$

where $k \in \mathbb{N}$, P_j is a homogeneous polynomial of degree $j \in \mathbb{N}$ and either both $k > 1$ and $P_k \not\equiv 0$ or $f(z) = P_1(z)$. The local dynamics of f is much more complicated and less well-understood in higher dimensions. In this paper, we only consider maps that are *tangent to the identity*, which means that $P_1 \equiv \mathrm{Id}$ and $f \not\equiv \mathrm{Id}$. Given additional assumptions on f, there are several results on the existence of curves, submanifolds, and domains that are invariant under f and whose points are attracted to the origin upon iteration by f. We discuss many of these results in §1.2 as well as more recent results in the following chapters and open questions in Chapters 3 and 4.

In this paper, we focus on the following question: How do different assumptions on f affect the existence of a domain that is invariant under f and whose points are attracted to the origin tangentially to a particular direction under iteration by f? In Chapter 1, we introduce definitions and previous results necessary for understanding the context and content of the subsequent chapters. In Chapter 2, we show the existence of an invariant attracting domain for a germ of a holomorphic self-map of $\mathbb{C}^2$ whose unique characteristic direction is non-degenerate. In Chapter 3, we restrict to maps in $\mathbb{C}^2$ and discuss results on the existence of invariant attracting domains. We also introduce examples of maps that do (do not) have an invariant attracting domain whose points converge tangentially to a direction, ask some (currently) open questions on the subject, and concisely summarize (in Table 3.1) results on the existence of invariant attracting domains in $\mathbb{C}^2$. In Chapter 4, we focus on the existence of invariant attracting domains in $\mathbb{C}^m$, for $m \geq 3$. We discuss what results are known, provide additional examples of maps that do (do not) have such invariant attracting domains, ask some (currently) open questions on the subject, and concisely summarize (in Table 4.1) results on the existence of invariant attracting domains in $\mathbb{C}^m$, for $m \geq 3$.

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Abstract

One of the guiding questions behind the study of local (discrete) holomorphic dynamics is: given f, a germ of a holomorphic self-map of $\mathbb{C}^m$ that fixes a point (say the origin), can f be expressed in a simpler form? If so, then the dynamical behavior of the map can be more easily understood. In general, we want to know how points near the origin behave under iteration by the map f. More specifically, we want to know when there exists a domain whose points are attracted to the origin under iteration by f and, if such a domain exists, when its points converge to the origin tangentially to a given direction. In dimension one, the Leau-Fatou Flower Theorem shows the existence of such domains. In higher dimensions, Hakim showed that given some assumptions on f and the direction v, a domain of attraction whose points converge to the origin tangentially to v does exist. In this thesis, we consider a collection of maps that do not satisfy the assumptions of Hakim's theorem. The main result we discuss is for maps in $\mathbb{C}^2$ that have a unique characteristic direction and this direction is non-degenerate. We show that there exists a domain of attraction whose points converge to the origin along the characteristic direction and on which the map is conjugate to translation: $(\tau, \omega) \mapsto (\tau + 1, \omega)$. When the map is a global automorphism, there exists such a domain of attraction that is also a Fatou-Bieberbach domain. In addition, we discuss other types of germs of holomorphic self-maps of $\mathbb{C}^2$ or $\mathbb{C}^3$ that fix the origin, in each case determining whether or not there exists an invariant attracting domain whose points converge to the origin tangentially to the same direction.

Chapter 1

Introduction to complex dynamics

In this paper, we study holomorphic self-maps of $\mathbb{C}^m$ that fix a point and, in particular, focus on the behavior of points near the fixed point under iteration. We begin by defining the main object of study.

Definition 1.1. A *(discrete) holomorphic local dynamical system* at $p \in \mathbb{C}^m$ is a holomorphic map $f : U \to \mathbb{C}^m$ such that $f(p) = p$, where $U \subseteq \mathbb{C}^m$ is an open neighborhood of p.

Notation. Let $\operatorname{End}(\mathbb{C}^m, p)$ denote the set of all (discrete) holomorphic local dynamical systems at a point $p \in \mathbb{C}^m$. Let $f, g \in \operatorname{End}(\mathbb{C}^m, p)$ be defined on open sets $U, V \subset \mathbb{C}^m$, respectively, that both contain the point p. We say that f and g are in the same equivalence class if and only if they agree on $U \cap V$ (i.e., $(f, U) \sim (g, V) \Leftrightarrow f|_{U\cap V} \equiv g|_{U\cap V}$).

In order to discuss the dynamics of $f \in \operatorname{End}(\mathbb{C}^m, p)$, we first need to define the iterates of f and determine where they are defined. Suppose f is defined on $U \subset \mathbb{C}^m$ with $p \in U$. Then the second iterate of f is defined on the open set $U \cap f^{-1}(U)$ containing p and additional iterates of f can be defined on open sets containing p in a similar fashion.

Definition 1.2. Let $f \in \operatorname{End}(\mathbb{C}^m, p)$ be defined on the open set $U \subset \mathbb{C}^m$ with $p \in U$. Denote the n-th iterate of f by f^n and the n-th iterate of a point $z \in U$ by z_n. In particular,

$$f^1 := f, \qquad f^2 := f \circ f, \qquad f^n := f \circ f^{n-1} = \underbrace{f \circ \cdots \circ f}_{n-times}, \qquad \text{and} \qquad z_n := f^n(z).$$

The set of points $\{z_n\}_{n=0}^{\infty}$, where $z_0 := z$, is called the *orbit* of z. In addition, the *stable set* K_f *of* f is:

$$K_f := \bigcap_{n=0}^{\infty} f^{-n}(U),$$

the set of all points on which f and all of its iterates are defined. A point (or its orbit) *escapes from* U if $z \in U \setminus K_f$.

In this paper, we study the set K_f and, in particular, when K_f is open we look for open subsets of K_f whose points converge to p under iteration. The set K_f is nonempty since $p \in K_f$; however, it can happen that K_f has empty interior and even that $K_f = \{p\}$.

Example 1.3. *Consider the germs $f(z) = 2z$ and $g(z) = \frac{1}{2}z$ both defined on $\mathbb{D}_r := \{z \in \mathbb{C}^m \mid ||z|| < r\}$ for some $0 < r < \infty$. Then $f^n(z) = 2^n z$ and $g^n(z) = 2^{-n}z$. Eventually for any point $z \in \mathbb{D}_r \setminus \{0\}$, $f^n(z) \notin \mathbb{D}_r$ and $g^n(z) \in \mathbb{D}_r$. Hence, $K_f = \{0\}$ and $K_g = \mathbb{D}_r$. In other words, the orbit of every point in $\mathbb{D}_r \setminus \{0\}$ under f escapes from $\mathbb{D}_r$, but the orbit of every point in $\mathbb{D}_r$ under g remains inside $\mathbb{D}_r$ and, in fact, converges to the origin.*

In order to simplify notation, we shall assume for the remainder of this paper that p is the origin, O. Near the origin, any $f \in \mathrm{End}(\mathbb{C}^m, \mathrm{O})$ can be expressed as:

$$f(z) = P_1(z) + P_2(z) + \cdots , \tag{1.1}$$

where $P_j = (P_j^1, \ldots, P_j^m)$ and each P_j^l is a homogeneous polynomial of degree j. In particular, P_1 is the differential of f at the origin, df_{O}, and f is locally invertible exactly when P_1 is invertible.

Definition 1.4. Let $f \in \mathrm{End}(\mathbb{C}^m, \mathrm{O})$. If all of the eigenvalues of df_{O}:

- have modulus strictly less than 1, then O is an *attracting* fixed point;
- have modulus strictly greater than 1, then O is a *repelling* fixed point;
- are roots of unity, then O is a *parabolic* fixed point.

Of course there are other types of fixed points, but these three, and primarily the last one, are the only types of fixed points we discuss in this paper. In Example 1.3, the origin is a repelling fixed point for f and $K_f = \{0\}$, while the origin is an attracting fixed point for g and $K_g = \mathbb{D}_r$, the open set containing the origin on which g is defined. In general, if the origin is a repelling fixed point for $f \in \mathrm{End}(\mathbb{C}^m, \mathrm{O})$, then $K_f = \{\mathrm{O}\}$. Conversely, if the origin is an attracting fixed point for $f \in \mathrm{End}(\mathbb{C}^m, \mathrm{O})$, then K_f is an open neighborhood of the origin. When the origin is a parabolic fixed point for f, it is much less clear what K_f, the stable set of f, will be and this is what we focus on in this paper. More specifically, we look at maps that are *tangent to the identity* at the origin, and, consequently, have the origin as a parabolic fixed point.

Definition 1.5. A germ $f \in \mathrm{End}(\mathbb{C}^m, \mathrm{O})$ is *tangent to the identity* if $P_1 = \mathrm{Id}$ in (1.1). In addition, if $P_k \not\equiv 0$ and $P_j \equiv 0$ for $1 < j < k$, then k is the *order* of f.

Let $f \in \mathrm{End}(\mathbb{C}^m, \mathrm{O})$ be tangent to the identity of order k defined on $U \subset \mathbb{C}^m$, an open neighborhood of the origin. Near the origin, f can be expressed as:

$$f(z) = z + P_k(z) + P_{k+1}(z) + \dots,$$

where $k > 1, P_k \not\equiv 0$ and $z \in \mathbb{C}^m$.

Definition 1.6. Let $f \in \mathrm{End}(\mathbb{C}^m, \mathrm{O})$ be tangent to the identity.

1. Suppose the orbit of a point $z \in \mathbb{C}^m$ converges to the origin ($z_n \to \mathrm{O}$), but does not reach the origin ($z_n \neq \mathrm{O}, \forall n$).
   - If $m = 1$, the orbit converges *tangentially* to a real direction v if:
   $$\frac{z_n}{|z_n|} \to v \in S^1 \subset \mathbb{C}.$$
   - If $m > 1$, the orbit converges *tangentially* to a complex direction $[v]$ if:
   $$[z_n] \to [v] \in \mathbb{P}^{m-1}(\mathbb{C}),$$
   where $[v]$ is the canonical projection of $v \in \mathbb{C}^m \setminus \{\mathrm{O}\}$ to $\mathbb{P}^{m-1}(\mathbb{C})$.
2. A domain $D \subset \mathbb{C}^m$ is *f-invariant* if $f(D) \subseteq D$.

In order to more easily study the dynamics of a map $f \in \mathrm{End}(\mathbb{C}^m, \mathrm{O})$, we would like f to be expressed in a simple form. Ideally we can conjugate f to a map of a simpler form whose dynamics can be more easily understood.

Definition 1.7. Let $f_1 \in \mathrm{End}(\mathbb{C}^m, p_1)$ and $f_2 \in \mathrm{End}(\mathbb{C}^m, p_2)$. Then f_1 and f_2 are *holomorphically* (or *topologically*) *locally conjugate* if there are open neighborhoods $U_1 \ni p_1$ and $U_2 \ni p_2$ in the domains of f_1 and f_2, respectively, and there is a biholomorphism (or a homeomorphism, respectively) $\phi : U_1 \to U_2$ with $\phi(p_1) = p_2$ such that:

$$f_1 = \phi^{-1} \circ f_2 \circ \phi \quad \text{on} \quad \phi^{-1}\left(U_2 \cap f_2^{-1}(U_2)\right) = U_1 \cap f_1^{-1}(U_1).$$

If two maps f_1, f_2 are locally conjugate, then the local dynamics of one map tells us about the local dynamics of the other one because:

$$f_1^k = \left(\phi^{-1} \circ f_2 \circ \phi\right)^k = \phi^{-1} \circ f_2^k \circ \phi,$$

for all $k \in \mathbb{N}$.

Notation. Let $||\cdot||$ denote the standard Euclidean norm so that for $z \in \mathbb{C}^p$,

$$||z|| = \left(\sum_{j=1}^{p} |z_j|^2 \right)^{\frac{1}{2}}.$$

Given $f, g_1, \ldots, g_s : \mathbb{C}^m \to \mathbb{C}^n$, we shall write $f = \mathrm{O}(g_1, \ldots, g_s)$ to mean $\exists C_1, \ldots, C_s > 0$ such that:

$$||f(z)|| \leq C_1 ||g_1(z)|| + \cdots + C_s ||g_s(z)||$$

and we shall write $f = o(g_1)$ to mean:

$$\lim_{||z|| \to 0} \frac{||f(z)||}{||g_1(z)||} = 0.$$

1.1. Complex dynamics in dimension one

We shall start by discussing holomorphic local dynamics at the origin in $\mathbb{C}$.

Definition 1.8. Let $f \in \mathrm{End}(\mathbb{C}, 0)$ be tangent to the identity of order k so that near the origin f has the form:

$$f(z) = z\left(1 + a_k z^{k-1} + \ldots\right), \text{ where } k > 1 \text{ and } a_k \neq 0.$$

1. An *attracting* (or *repelling*) *direction for* f at the origin is $v \in S^1 \subset \mathbb{C}$ such that $a_k v^{k-1}$ is real and negative (or positive, respectively).

2. The *basin* centered at an attracting direction v is:

$$B_{f,v} \equiv \left\{ z \in K_f \setminus \{0\} \,\middle|\, f^n(z) \to 0, \frac{f^n(z)}{|f^n(z)|} \to v \right\}.$$

3. An *attracting petal* centered at an attracting direction v is an open, simply-connected f-invariant set $P \subseteq K_f \setminus \{0\}$ such that a point $z \in B_{f,v}$ if and only if $\{f^n(z)\} \cap P \neq \emptyset$.

We can easily extend these definitions to repelling directions because a repelling direction direction for f is an attracting direction for f^{-1}. Hence, a *repelling petal* for f centered at a repelling direction v is an attracting petal for f^{-1} centered at v.

The following theorem, often called the Leau-Fatou Flower Theorem, completely describes the local dynamics of f near the origin. In addition, it can be extended to maps with a parabolic fixed point.[CG, M].

Theorem 1.9 (Leau-Fatou)**.** *Let* $f \in \mathrm{End}(\mathbb{C}, 0)$ *be tangent to the identity of order* k. *Then:*

(i) f *has* $k-1$ *attracting directions and* $k-1$ *repelling directions.*

(ii) For each attracting (or repelling) direction, there exists an attracting (or repelling) petal, so that the union of all of these $2(k-1)$ *petals together with the origin forms a neighborhood of the origin.*

(iii) $K_f \setminus \{0\}$ *is the (disjoint) union of the basins centered at the* $k-1$ *attracting directions.*

(iv) If B *is a basin centered at one of the attracting directions, then there exists a function* $\varphi : B \to \mathbb{C}$ *that conjugates* $f|_B$ *to translation, i.e.* $\varphi \circ f(z) = \varphi(z) + 1$ *for all* $z \in B$*. In addition, if* B *is a petal as in* (ii)*, then* $\varphi|_B$ *is a biholomorphism with an open subset of the complex plane containing a right half-plane.*

In Figure 1.1, we see a depiction of the local dynamics of $f(z) = z - z^3 = z(1 - z^2)$ as described by the Leau-Fatou Flower Theorem. The attracting (or repelling) directions of f are the positive and negative sides of the real (or imaginary) axis. Centered at each of the two attracting directions, there is an attracting petal; similarly, centered at each of the two repelling directions, there is a repelling petal. Points inside each attracting (or repelling) petal approach the origin along their central axis upon repeated application of f (or f^{-1}, respectively). Together with the origin, the four petals form a complete neighborhood of the origin.

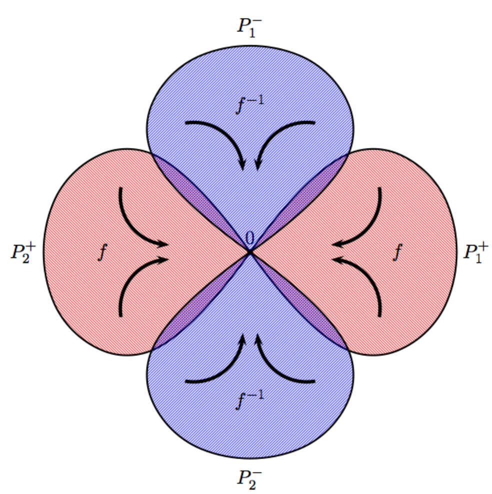

Figure 1.1: Leau-Fatou Flower for $f(z) = z - z^3$

Some of the techniques used to prove the Leau-Fatou Flower Theorem are also used to prove similar higher dimensional results; for instance, some of the techniques are used in the proofs of the main results of Chapter 2. We give an idea of the proof below, focusing on those relevant techniques. For a more detailed proof, see [A3, CG, M].

Proof. *(Sketch)* Assume f has order 2; this assumption simplifies some of the steps of the proof without significantly changing any of the techniques. In addition, assume that $a_2 = -1$, which is the case after a linear conjugation. Then

$$f(z) = z - z^2 + \mathrm{O}(z^3) = z(1 - z + \mathrm{O}(z^2))$$

has an attracting direction along the positive real axis and a repelling direction along the negative real axis. These directions along with the corresponding attracting and repelling petals are depicted in Figure 1.2.

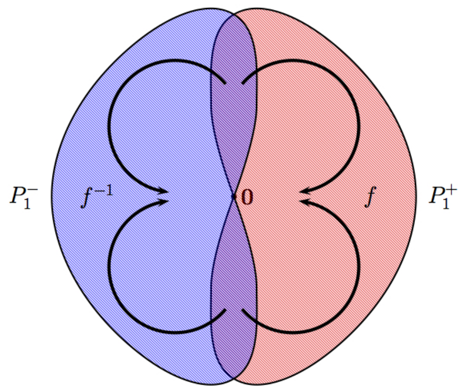

Figure 1.2: Leau-Fatou Flower for $f(z) = z\,(1 - z + \mathrm{O}(z^2))$

First of all, we want to find an attracting petal for f. For any $\delta > 0$, let P_δ be the disc centered at δ with the origin in its boundary, so that:

$$P_\delta := \left\{ z \in \mathbb{C} \;\middle|\; |z - \delta| < \delta \right\}.$$

It turns out that for δ small enough, P_δ is an attracting petal for f.

In our first coordinate change, we move the fixed point at the origin in $\mathbb{C}$ to infinity in $\mathbb{P}^1(\mathbb{C})$. Let $\Psi : \mathbb{C} \setminus \{0\} \to \mathbb{C} \setminus \{0\}$ be:

$$\Psi(z) := \frac{1}{z}.$$

Then Ψ is a biholomorphism, $\Psi^{-1}(w) = \frac{1}{w}$, and P_δ maps onto the right half-plane:

$$H_\delta := \Psi(P_\delta) = \left\{ w \in \mathbb{C} \;\middle|\; \mathrm{Re}(w) > \frac{1}{2\delta} \right\}.$$

We want to define f in our new coordinates, $w = \frac{1}{z}$. Since f is a germ defined in a neighborhood of the origin, for a small $\epsilon > 0$, f will be defined on the disc $\mathbb{D}(0, \epsilon)$ of radius ϵ centered at the origin. Hence, for $|w|$ large enough, f is defined at $\Psi^{-1}(w)$. Let $F := \Psi \circ f \circ \Psi^{-1}$, which is defined for $|w| > \epsilon^{-1}$. Then:

$$F(w) = \left(\frac{1}{w}\left(1 - \frac{1}{w} + \mathrm{O}\left(\frac{1}{w^2}\right)\right)\right)^{-1} = w + 1 + \frac{b}{w} + \mathrm{O}\left(\frac{1}{w^2}\right), \tag{1.2}$$

for some $b \in \mathbb{C}$. We can study the dynamics of F in a neighborhood of ∞ to learn about the dynamics of f in a neighborhood of the origin.

If $\delta > 0$ is small enough, then for any $w \in H_\delta$ and $n \geq 1$,

$$\mathrm{Re}(F(w)) > \mathrm{Re}(w) + \frac{1}{2} \qquad \Rightarrow \qquad \mathrm{Re}(F^n(w)) > \mathrm{Re}(w) + \frac{n}{2}$$

and

$$|w| + 2 > |F(w)| > \frac{1}{2} \qquad \Rightarrow \qquad |w| + 2n > |F^n(w)| > \frac{n}{2}.$$

Hence, H_δ is F-invariant and $F^n(w) \to \infty$ in H_δ as $n \to \infty$. Consequently, P_δ is f-invariant and $f^n(z) \to 0$ in P_δ as $n \to \infty$. Furthermore, we can show that the attracting direction for f is along the positive real axis by showing that $\frac{f^n(z)}{|f^n(z)|} \to 1$ in P_δ as $n \to \infty$ or, equivalently, that $\mathrm{Arg}(F^n(w)) \to 0$ as $n \to \infty$. After looking at (1.2), it should not be surprising that $\lim_{n\to\infty} \mathrm{Arg}(F^n(w)) = 0$, so we will not go through the details here.

Since P_δ is a simply connected domain centered along the positive real axis with the origin in its boundary, every orbit converging to the origin along the positive real axis must intersect P_δ. Hence, P_δ is an attracting petal. A repelling petal can be found in the same way as the attracting petal.

Now we have seen that f on P_δ is conjugate to $F(w) = w + 1 + \frac{b}{w} + \mathrm{O}\left(\frac{1}{w^2}\right)$ on H_δ and we want to finish by showing that both are conjugate to translation $\zeta \mapsto \zeta + 1$. Let

$$\phi_n(w) := F^n(w) - n - b \log n,$$

for any $w \in H_\delta$ and $n \geq 1$. We want to show that the sequence of univalent (i.e. injective and holomorphic) functions $\{\phi_n\}$ converges to a univalent function ϕ.

We can also express $\phi_n(w)$ as:

$$\phi_n(w) = w + (\phi_1(w) - w) + \sum_{k=1}^{n-1} (\phi_{k+1}(w) - \phi_k(w)), \tag{1.3}$$

for any $w \in H_\delta$ and $n > 1$. Then

$$\begin{aligned}
\phi_{k+1}(w) - \phi_k(w) &= F^{k+1}(w) - F^k(w) - 1 - b\left(\log(k+1) - \log k\right) \\
&= \frac{b}{F^k(w)} + \mathrm{O}\left(\frac{1}{F^k(w)^2}\right) - b \log\left(1 + \frac{1}{k}\right) \\
&= \frac{b}{kF^k(w)}\left[k - F^k(w)\right] + \mathrm{O}\left(\frac{1}{k^2}\right) = \mathrm{O}\left(\frac{1}{k}\right),
\end{aligned}$$

since $\frac{1}{F_k(w)} = \mathrm{O}\left(\frac{1}{k}\right)$, and

$$|\phi_n(w) - w| \leq |\phi_1(w) - w| + \sum_{k=1}^{n-1} |\phi_{k+1}(w) - \phi_k(w)| = \mathrm{O}(\log n)$$

for any $w \in H_\delta$ and $n > 1$. Combining these two results with (1.3), we get:

$$\phi_{k+1}(w) - \phi_k(w) = \frac{-b}{kF^k(w)} \left[\phi_k(w) + b \log k\right] + \mathrm{O}\left(\frac{1}{k^2}\right) = \mathrm{O}\left(\frac{\log k}{k^2}\right).$$

Hence, $\sum_{k=1}^{\infty} |\phi_{k+1}(w) - \phi_k(w)| < \infty$. In addition, for any $w \in H_\delta$ and $n \geq 1$,

$$\phi_n \circ F(w) = F^{n+1}(w) - n - b \log n = \phi_{n+1}(w) + 1 + b \log\left(1 + \frac{1}{n}\right).$$

Therefore the univalent functions $\{\phi_n\}$ converge to a function Φ and their limit is non-constant since $\Phi \circ F(w) = \Phi(w) + 1$ for all $w \in H_\delta$, so Φ is also univalent. Thus, Φ conjugates F to translation, $\Phi \circ F \circ \Phi^{-1}(\zeta) = \zeta + 1$, and $\Phi \circ \Psi$ conjugates our original function f to translation:

$$(\Phi \circ \Psi) \circ f \circ (\Phi \circ \Psi)^{-1}(\zeta) = \zeta + 1.$$

In the following diagram, we summarize the coordinate changes we performed to show that f is conjugate to translation on P_δ. Recall that f is defined on $\mathbb{D}(0, \epsilon) \supset P_\delta$ and F is defined on $\mathbb{C} \setminus \mathbb{D}\left(0, \epsilon^{-1}\right) \supset H_\delta$. Let i be the inclusion map.

$$\begin{tikzcd}
\mathbb{D}(0,\epsilon) \setminus \{0\} \arrow[r, "\Psi", "z \mapsto \frac{1}{z}"'] & \mathbb{C} \setminus \mathbb{D}\left(0, \epsilon^{-1}\right) & \\
P_\delta = \{|z - \delta| < \delta\} \arrow[u, "i"] \arrow[r, "\Psi"] \arrow[d, "f"'] & H_\delta = \left\{\mathrm{Re}(w) > \frac{1}{2\delta}\right\} \arrow[u, "i"] \arrow[r, "\Phi"] \arrow[d, "F"'] & \Phi\left(H_\delta\right) \arrow[d, "\zeta \mapsto \zeta + 1"'] \\
P_\delta \arrow[r, "\Psi"] & H_\delta \arrow[r, "\Phi"] & \Phi\left(H_\delta\right)
\end{tikzcd}$$

Now we want to enlarge the size of our attracting petal P_δ and the corresponding repelling petal so that they together with the origin form a neighborhood of the origin on which f is defined. Since Ψ is already defined on $\mathbb{C} \setminus \{0\}$ and f is defined and expressed as above in a whole neighborhood of the origin, F can easily be extended to a larger domain. From (1.2), we know that there exist constants $R, C > 0$ such that

$$|F(w) - w - 1| \leq \frac{C}{|w|}$$

for any $|w| > R$. Given $0 < \alpha < 1$, choose $\delta > 0$ such that $|w| > \frac{1}{2\delta}$ implies:

$$|F(w) - w - 1| \leq \frac{\alpha}{2}.$$

Let $M_\alpha := \frac{\sqrt{1+\alpha^2}}{2\delta}$ and extend $H_\delta = \left\{ w \in \mathbb{C} \;\middle|\; \mathrm{Re}(w) > \frac{1}{2\delta} \right\}$ to:

$$H_\alpha := \{ w \in \mathbb{C} \mid \alpha |\operatorname{Im}(w)| > -\operatorname{Re}(w) + M_\alpha \} \cup H_\delta.$$

Then for any $w \in H_\alpha$ we have

$$\mathrm{Re}(F(w)) > \mathrm{Re}(w) + 1 - \frac{\alpha}{2} \qquad \text{and} \qquad |\operatorname{Im}(F(w)) - \operatorname{Im}(w)| < \frac{\alpha}{2}.$$

One can easily check that $F(H_\alpha) \subset H_\alpha$ and every orbit starting in H_α eventually enters H_δ. Then $P_\alpha := \Psi^{-1}(H_\alpha) \supset P_\delta$ is an enlarged attracting petal for f.

We can extend the domain of definition of Φ to any domain Ω on which F is defined and satisfies both:

$$F(\Omega) \subseteq \Omega \qquad \text{and} \qquad \mathrm{Re}(F^n(w)) \to +\infty, \quad \forall w \in \Omega,$$

since then we know that $F^n(w) \in H_\delta$ for some $n \in \mathbb{N}$. In particular, we can extend the domain of definition of Φ to H_α. We do so by rearranging and adjusting the equation $\Phi \circ F(w) = \Phi(w) + 1$ to get:

$$\Phi(w) := \Phi \circ F^n(w) - n$$

for any $w \in H_\alpha$, where $n \in \mathbb{N}$ such that $F^n(w) \in H_\delta$. □

Understanding the ideas and techniques used to prove the previous theorem is key to understanding the proofs of similar results in higher dimensions. In particular, we will be revisiting many of them to prove results in Chapter 2.

1.2. Complex dynamics in higher dimensions

Generalizing the concepts behind the Leau-Fatou Flower Theorem (Theorem 1.9) is a driving force behind the study of holomorphic self-maps of $\mathbb{C}^m$ that are tangent to the identity. There are two main objects that arise in the Leau-Fatou Flower Theorem: (1) the attracting directions, which correspond to real lines, and (2) the invariant attracting petals. The idea of attracting directions in $\mathbb{C}$ is generalized to characteristic directions in higher dimensions, which correspond to complex lines.

Definition 1.10. Let $f \in \mathrm{End}(\mathbb{C}^m, \mathrm{O})$ be tangent to the identity of order k.

1. If $v \in \mathbb{C}^m \setminus \{O\}$ is such that $P_k(v) = \lambda v$ for some $\lambda \in \mathbb{C}$, then $[v]$ is a *characteristic direction* of f, where $[v]$ is the canonical projection of v in $\mathbb{P}^{m-1}(\mathbb{C})$.

2. A characteristic direction $[v]$ is *degenerate* if $\lambda = 0$ and *non-degenerate* if $\lambda \neq 0$.

3. f is *dicritical* if all directions are characteristic, otherwise f is *non-dicritical.*

4. A *characteristic trajectory* for f is an orbit of a point z in the domain of f, such that $\{z_n\} := \{f^n(z)\}$ converges to the origin tangentially to the direction $[v] \in \mathbb{P}^{m-1}(\mathbb{C})$, that is:

$$\lim_{n\to\infty} z_n = 0 \qquad \text{and} \qquad \lim_{n\to\infty} [z_n] = [v].$$

The value of $\lambda \in \mathbb{C}$ is relevant only in that it is either zero or nonzero.

Remark 1.11. It is easy to check that f is dicritical if and only if $P_k \equiv \Lambda\,\mathrm{Id}$, where $\Lambda : \mathbb{C}^m \to \mathbb{C}$ is a homogeneous polynomial of degree $k-1$. In the generic case, f has finitely many characteristic directions; more precisely, in the generic case f has $\frac{k^m-1}{k-1}$ characteristic directions counted with multiplicity (see [AT]).

The idea of invariant attracting petals in $\mathbb{C}$ is generalized to $\mathbb{C}^m$ in two main ways: parabolic curves and invariant attracting domains. This paper will focus on the latter, but will address the former first. Characteristic directions play a prominent role in both cases.

Proposition 1.12 (Hakim, [H1])**.** *Let $f \in \mathrm{End}(\mathbb{C}^m, \mathrm{O})$ be tangent to the identity, and let $\{z_n\}$ be a characteristic trajectory tangent to the direction $[v] \in \mathbb{P}^{m-1}(\mathbb{C})$ at the origin. Then $[v]$ is a characteristic direction of f.*

We can also think about characteristic directions in another way, by considering the lift $\tilde{f}$ of f to the blowup of the origin in $\mathbb{C}^m$. Let $f \in \mathrm{End}(\mathbb{C}^m, \mathrm{O})$ be tangent to the identity of order k with characteristic direction $[v]$ and assume, without loss of generality, that $[v] = [1 : u_o]$. Let $\pi : M \to \mathbb{C}^m$ be the blowup of the origin, and let $E = \pi^{-1}(\mathrm{O})$ be the exceptional divisor of the blowup, which is canonically biholomorphic to $\mathbb{P}(T_{\mathrm{O}}\mathbb{C}^m)$, or $\mathbb{P}^{m-1}(\mathbb{C})$. Then characteristic directions of f are points in the blowup. Since f is tangent to the identity, the holomorphic lift $\tilde{f} = \pi^{-1} \circ f \circ \pi$ of f to M restricts to the identity map on E (i.e. $\tilde{f}|_E = \mathrm{Id} : E \to E$) and $\pi|_{M\setminus E} : M \setminus E \to \mathbb{C}^m \setminus \{0\}$ is a biholomorphism.

To better understand the dynamics of f, we will study the dynamics of the lift $\tilde{f}$. Recall that, near the origin, $f(z) = z + P_k(z) + P_{k+1}(z) + \ldots$, where $P_k \not\equiv 0$ and $P_l := (p_l, q_l) : \mathbb{C}^m \to \mathbb{C} \times \mathbb{C}^{m-1}$ are homogeneous polynomials of degree l. Choose local coordinates (x, u) on $M \setminus E$ so that

$$\pi(x, u) = (x, ux) := (x, y) \in \mathbb{C} \times \mathbb{C}^{m-1}.$$

In these coordinates, the lift $\tilde{f}$ can be expressed as:

$$\begin{aligned} x_1 &= x\left(1 + p_k(1,u)x^{k-1} + \mathrm{O}(x^k, ||u||x^k)\right) \\ u_1 &= u + r(u)x^{k-1} + \mathrm{O}(x^k, ||u||x^k), \end{aligned} \tag{1.4}$$

where

$$r(u) := q_k(1, u) - p_k(1, u)u$$

and u_1 was simplified from:

$$u_1 = \frac{y_1}{x_1} = \frac{u + x^{k-1}(q_k(1,u) + \mathrm{O}(x, ||u||x))}{1 + x^{k-1}(p_k(1,u) + \mathrm{O}(x, ||u||x))} = u + (q_k(1,u) - p_k(1,u)u)x^{k-1} + \mathrm{O}(x^k, ||u||x^k).$$

The characteristic directions of f of the form $[v] = [1 : u_o]$ correspond to points in E and are precisely the zeros of r:

$$[1 : u] \text{ characteristic direction } \Leftrightarrow P_k(1, u) = \lambda(1, u) \Leftrightarrow q_k(1, u) = up_k(1, u) \Leftrightarrow r(u) = 0,$$

which gives us another way of viewing characteristic directions.

Now assume that the characteristic direction $[v] = [1 : u_o]$ is non-degenerate, so $p_k(1, u_o) \neq 0$. Let

$$A(v) := \frac{1}{(k-1)p_k(1, u_o)} r'(u_o)$$

be a matrix associated to the characteristic direction $[v]$ of f. The class of similarity of $A(v)$ is invariant under a change of coordinates of f by formal power series (see Prop. 2.4 of [H2] or Prop. 4.7 of [AR]). Hence, the eigenvalues of $A(v)$ are holomorphic (and formal) invariants associated to $[v]$.

In addition, assume, without loss of generality, that $u_o = 0$. We can further simplify our expression for f by using Lemma 4.4 in [AR] (also in [H2]), which tells us that there is a polynomial change of coordinates holomorphically conjugating f to a germ $F := (F_1, \Psi)$ of the form:

$$\begin{aligned} x_1 &= F_1(x, u) = x\left(1 - \frac{1}{k-1}x^{k-1} + \mathrm{O}\left(||u||x^{k-1}, x^{2k-2}\right)\right) \\ u_1 &= \Psi(x, u) = u - x^{k-1}\frac{1}{(k-1)p_k(1, 0)} r(u) + \mathrm{O}(x^k, ||u||x^k), \end{aligned} \tag{1.5}$$

where we relabeled our new coordinates (x, u). By using the Taylor series expansion for r about 0, we can rewrite u_1 as:

$$u_1 = \Psi(x, u) = u - x^{k-1}\frac{1}{(k-1)p_k(1, 0)} r'(0)u + \mathrm{O}\left(||u||^2 x^{k-1}, ||u||x^k, x^k\right),$$

where $r(0) = 0$ since $[1 : 0]$ is a characteristic direction of f. Then F is of the form:

$$\begin{aligned} x_1 &= F_1(x, u) = x\left(1 - \frac{1}{k-1}x^{k-1} + \mathrm{O}\left(||u||x^{k-1}, x^{2k-2}\right)\right) \\ u_1 &= \Psi(x, u) = (I - x^{k-1}A)u + \mathrm{O}\left(||u||^2 x^{k-1}, ||u||x^k\right) + x^k \psi_1(x), \end{aligned} \tag{1.6}$$

where I is the identity matrix, $A := A(v)$, and ψ_1 is holomorphic. Since A is invariant under a formal change of coordinates, we can assume that A is in Jordan normal form. Now that we have expressed f in a simpler form, we return to the topic of generalizing invariant attracting petals in $\mathbb{C}$ to parabolic curves and invariant attracting domains in higher dimensions.

Definition 1.13. Let $f \in \mathrm{End}(\mathbb{C}^m, \mathrm{O})$ be tangent to the identity. A *parabolic curve* for f is an injective holomorphic map $\phi : \Delta \to \mathbb{C}^m \setminus \{\mathrm{O}\}$ such that:

1. Δ is a simply connected domain in $\mathbb{C}$ with $0 \in \partial\Delta$,
2. ϕ is continuous at the origin and $\phi(0) = \mathrm{O}$,
3. $\phi(\Delta)$ is f-invariant and $\left(f|_{\phi(\Delta)}\right)^n \to \mathrm{O}$ uniformly on compact subsets as $n \to \infty$.

In addition, if $[\phi(\zeta)] \to [v]$ in $\mathbb{P}^{m-1}(\mathbb{C})$ as $\zeta \to 0$ in Δ, then ϕ is *tangent* to the direction $[v]$.

Theorem 1.14 (Ecalle, [E]; Hakim, [H2])**.** *Let $f \in \mathrm{End}(\mathbb{C}^m, \mathrm{O})$ be tangent to the identity of order k. Then for any non-degenerate characteristic direction $[v] \in \mathbb{P}^{m-1}(\mathbb{C})$ there exist (at least) $k-1$ parabolic curves for f tangent to $[v]$.*

Abate in [A1] extended this result to maps with an isolated fixed point at the origin.

Theorem 1.15 (Abate, [A1])**.** *Let $f \in \mathrm{End}(\mathbb{C}^m, \mathrm{O})$ be tangent to the identity of order k and such that the origin is an isolated fixed point. Then there exist (at least) $k-1$ parabolic curves for f at the origin (tangent to some singular direction).*

We will not define a singular direction here, but it is helpful to know that for a map $f \in \mathrm{End}(\mathbb{C}^m, \mathrm{O})$ that is tangent to the identity, non-degenerate characteristic directions of f must be singular directions of f which, in turn, must be characteristic directions of f. Furthermore, singular directions are particularly interesting types of characteristic directions because if $f \in \mathrm{End}(\mathbb{C}^m, \mathrm{O})$ has a non-trivial orbit converging to the origin tangentially to a direction $[v] \in \mathbb{P}^{m-1}(\mathbb{C})$, then $[v]$ must be singular [A3]. This is an extension of Proposition 1.12.

We give an outline of the main ideas of the proof of Theorem 1.14 below. For a detailed proof, see [H2, AR].

Proof. (Sketch) Assume that $[v] = [1 : \mathrm{O}]$ and $k = 2$ to simplify the discussion. Conjugate f to F as in (1.6). We want to find a parabolic curve for F that is tangent to the direction $u = \mathrm{O}$. In particular, we want to find a holomorphic function μ defined on an open set $U \subset \mathbb{C}$ with $0 \in \partial U$ such that:

$$\mu : U \to \mathbb{C}^{m-1}, \qquad \mu(0) = \mathrm{O}, \mu'(0) = \mathrm{O}, \qquad \text{and} \qquad \mu(F_1(x, \mu(x))) = \Psi(x, \mu(x)).$$

If we find such a function, then $\phi(x) := (x, \mu(x))$ will be an F-invariant holomorphic curve:

$$F \circ \phi(x) = (F_1(x, \mu(x)), \Psi(x, \mu(x))) = \phi\left(F_1(x, \mu(x))\right).$$

There are three key steps to finding μ as above. The first step is to find a change of coordinates that simplifies the expression for u_1 so that the pure x term is of arbitrarily high order. It is possible to do so in a domain with the origin in its boundary, but we will not go into the details here. This coordinate change is performed and the coordinates are relabeled (x, u).

The second step is to prove the existence of a parabolic curve for f by finding a fixed point of a suitable operator between Banach spaces. This involves performing the following coordinate change:

$$u := x^A w = \exp(A \log x) w,$$

which is defined for $\mathrm{Re}(x) > 0$ and the new coordinates are (x, w). In addition, we define:

$$H(x, u) := x^A (w - w_1) = u - x^A x_1^{-A} u_1$$

so that:

$$w_1 = w - x^{-A} H(x, u).$$

Given a map $\mu(\cdot) := x^2 l(\cdot)$, where l is defined on a particular domain in $\mathbb{C}$, the iterates $\{x_n\}$ defined by:

$$x_{j+1} := f_\mu(x_j) := F_1(x_j, \mu(x_j))$$

are well-defined on that domain. With this μ, the operator:

$$T\mu(x) := x^A \sum_{n=0}^{\infty} x_n^{-A} H(x_n, \mu(x_n))$$

is well-defined. Restricting the domain of definition of T, we obtain that T is a continuous contraction, thus admitting a unique fixed point, μ. The third (and final) step is proving that the fixed point μ is a solution to:

$$\mu(F_1(x, \mu(x)) = \Psi(x, \mu(x)) \qquad \text{and} \qquad \lim_{x \to 0} \mu(x) = \lim_{x \to 0} \mu'(x) = \mathrm{O}\,.$$

Therefore $\phi = (\mathrm{Id}, \mu)$ is a parabolic curve for f tangent to $[v] = [1 : \mathrm{O}]$. □

Given f as in (1.6), we mentioned in the previous proof that we can perform a coordinate change in a domain with the origin in its boundary to make the pure x term in u_1 of arbitrarily high order (see [H2, AR]). Then, by using this coupled with the existence of parabolic curves for f from Theorem 1.14, we can perform a change of coordinates to remove the pure x term in our expression for u_1 on a reduced domain, $S^j_{r,c}$ (defined below). After performing this coordinate change, we rename F to be the new map and (x, u) to be the new coordinates. Then $(x_1, u_1) := F(x, u)$ can be expressed as:

$$x_1 = F_1(x, u) = x \left(1 - \frac{1}{k-1} x^{k-1} + \mathrm{O}\left(||u|| x^{k-1}, x^{2k-2} \log x\right)\right) \tag{1.7}$$
$$u_1 = \Psi(x, u) = \left(I - x^{k-1} A\right) u + \mathrm{O}\left(||u||^2 x^{k-1}, ||u|| x^k \log x\right),$$

where

$$(x, u) \in S^j_{r,c} = \left\{(x, u) \in \mathbb{C} \times \mathbb{C}^{m-1} \;\middle|\; x \in \Pi^j_r, ||u|| \le c\right\}$$

for $r, c > 0$, $j \in \{1, \ldots, k-1\}$, and Π^j_r a connected component of $\mathbb{D}_r := \left\{\left|x^{k-1} - r\right| < r\right\}$. Now that we have discussed the existence of parabolic curves for f and used such curves to simplify our expression for f, we turn to the main topic of this paper, the existence of invariant attracting domains for f.

Given a map $f \in \mathrm{End}(\mathbb{C}^m, \mathrm{O})$ that is tangent to the identity and of order $k \ge 2$, the existence of invariant attracting domains for f depends significantly on characteristic directions of f and, for non-degenerate characteristic directions, their directors. Directors are defined for non-degenerate characteristic directions, so for now assume that $[v] = [1 : u_o]$ is a non-degenerate characteristic direction of f. We provide two equivalent definitions of directors corresponding to the direction $[v]$, the second of which requires the following set-up.

Assuming $||u||$ is small and remains small after iteration, we know from (1.6) or (1.7) that:

$$x_1 \approx x \left(1 - \frac{1}{k-1} x^{k-1}\right).$$

This expression only involves one variable, so we can use the one-dimensional results from the previous section (in particular, Theorem 1.9) to understand the behavior of the iterates of x. If $x \approx \mathrm{Re}(x)$ and $0 < \mathrm{Re}(x) \ll 1$, then $|x_1| < |x|$ and for $j > 0$ the same expressions hold, so $x_j \approx \mathrm{Re}(x_j)$ and $0 < \mathrm{Re}(x_j) \ll 1$. In addition, as we saw in the proof of Theorem 1.9, $\lim_{n\to\infty} x_n \to 0$. The expression for u_1 is more complicated, but from (1.6) or (1.7) one would expect $A = A(v)$ to play a significant role in the behavior of the iterates of u. To better understand how, view u_1 as:

$$u_1 \approx (I - x^{k-1} A) u.$$

Again assume that $x \approx \mathrm{Re}(x)$ and $0 < \mathrm{Re}(x) \ll 1$. From this simplified model, we can get an idea of how the eigenvalues of A would affect the size of $||u_1||$. If the eigenvalues of A have positive real parts, it seems likely that $||u_1|| < ||u||$; if the eigenvalues of A have negative real parts, it seems likely that $||u_1|| > ||u||$. However, if some or all of the eigenvalues of A

have real parts equal to zero, then we cannot use this simplified expression to get an idea of how the iterates of u behave; for the majority of this paper, we focus on this situation. Now we have sufficient background and motivation to define the directors of a non-degenerate characteristic direction.

Definition 1.16. Let $f \in \mathrm{End}(\mathbb{C}^m, \mathrm{O})$ be tangent to the identity of order k with non-degenerate characteristic direction $[v]$. The eigenvalues of the linear operator:

$$\frac{1}{k-1}\left(D(P_k)_{[v]} - \mathrm{Id}\right) : T_{[v]}\mathbb{P}^{m-1}(\mathbb{C}) \to T_{[v]}\mathbb{P}^{m-1}(\mathbb{C})$$

are the *directors* of $[v]$. Equivalently, the eigenvalues of $A = A(v)$ associated to $[v]$ are the *directors* of $[v]$. The direction $[v]$ is called *attracting* if all the real parts of its directors are strictly positive.

In general, for dimension $m > 1$, results on the existence of invariant attracting domains for f along a particular direction have mostly depended on the directors corresponding to that direction. The following theorem is a major result in that area:

Theorem 1.17 (Hakim, [H1]). *Let $f \in \mathrm{End}(\mathbb{C}^m, \mathrm{O})$ be tangent to the identity of order $k \geq 2$, and let $[v]$ be a non-degenerate attracting characteristic direction. Then there exists an invariant attracting domain D, with the origin in its boundary, in which every point is attracted to the origin tangentially to the direction $[v]$, and such that the restriction of f to D is conjugate to translation: $(\tau, \omega) \mapsto (\tau, \omega + 1)$.*

Given the same assumptions on f as in the previous theorem, Hakim also proved the existence of $k-1$ invariant attracting domains associated to $[v]$. For a detailed proof, see [AR, Theorem 1.2].

Theorem 1.18 (Hakim, [H2]; Arizzi-Raissy, [AR]). *Let $f \in \mathrm{End}(\mathbb{C}^m, \mathrm{O})$ be tangent to the identity of order $k \geq 2$, and let $[v]$ be a non-degenerate attracting characteristic direction. Then there exist $k-1$ invariant attracting domains, each with the origin in its boundary, in which every point is attracted to the origin tangentially to the direction $[v]$.*

The previous two theorems showed the existence of invariant attracting domains for non-degenerate *attracting* characteristic directions. The following corollary [AR, Corollary 8.11] dictates what the directors must be in order for an invariant attracting domain to possibly exist along a non-degenerate characteristic direction:

Corollary 1.19. *Let $f \in \mathrm{End}(\mathbb{C}^m, \mathrm{O})$ be tangent to the identity of order $k \geq 2$, and let $[v]$ be a non-degenerate characteristic direction. If there exists an invariant attracting domain* *where all the orbits converge to the tangentially to the direction $[v]$, then all of the directors of $[v]$ must have non-negative real parts.*

The following theorem lies in the middle ground between Theorem 1.14, which dictates the existence of invariant curves, and Theorem 1.17, which dictates the existence of invariant attracting domains when all of the directors have strictly positive real parts. In particular, it discusses the existence of invariant manifolds in the situation where the characteristic direction might not be attracting.

Theorem 1.20 (Hakim, [H1])**.** *Let $f \in \mathrm{End}(\mathbb{C}^m, \mathrm{O})$ be tangent to the identity of order $k \geq 2$, and let $[v]$ be a non-degenerate characteristic direction. Assume that the directors associated to $[v]$ are divided into two sets $\{\lambda_j\}_{1\leq j\leq q}$ and $\{\mu_k\}_{1\leq k\leq l}$ in such a way that for some $\alpha > 0$, we have:*

$$\mathrm{Re}\,\lambda_j > \alpha > \mathrm{Re}\,\mu_k \text{ for all } j, k.$$

Let d_j be the multiplicity of λ_j, and let $d := d_1 + \ldots + d_q$. Let E be the sum of the generalized eigenvector space associated to the λ_js. Then there exists an invariant piece of analytic manifold of dimension $d + 1$, with the origin in its boundary, tangent to $\mathbb{C}V + E$ at O, and in which every point is attracted to the origin tangentially to the direction $[v]$.

Brochero Martínez show that when the origin is dicritical (i.e., all directions are characteristic directions) there is an invariant attracting domain whose points are attracted to the origin, but such a domain might not be tangential to any characteristic direction.

Theorem 1.21 (Brochero Martínez, [Bro2])**.** *Let $f \in \mathrm{End}(\mathbb{C}^m, \mathrm{O})$ be tangent to the identity, and let the origin be dicritical. Let $\pi : (M, E) \to (\mathbb{C}^m, \mathrm{O})$ be the blowup of the origin in $\mathbb{C}^m$ with $E = \pi^{-1}(\mathrm{O})$, and let $\tilde{f} \in \mathrm{End}(M, E)$ be the lift of f to the blowup. Then there exists a finite number of points $p_1, \ldots, p_l \in E$ and open sets $U^+, U^- \subset M$ such that:*

- $\overline{U^+ \cup U^-} \supset E \setminus \{p_1, \ldots, p_l\}$,
- *$\tilde{f}(U^+) \subset U^+$ and for all $q \in U^+$, $\lim_{n\to+\infty} \tilde{f}^n(q)$ exists and is a point of E, and*
- *$\tilde{f}^{-1}(U^-) \subset U^-$ and for all $q \in U^-$, $\lim_{n\to+\infty} \tilde{f}^{-n}(q)$ exists and is a point of E.*

In this paper we address the following question that naturally arises from the preceding results. Let $f \in \mathrm{End}(\mathbb{C}^m, \mathrm{O})$ be tangent to the identity of order $k \geq 2$, let $[v]$ be a non-degenerate characteristic direction, and let the origin be non-dicritical: *If the real part of the directors of $[v]$ are non-negative and at least one of them equals zero, under what conditions does f have an invariant attracting domain tangential to the direction $[v]$?* Another natural question to ask, although we do not address it much in this paper, is: *If $[v]$ is degenerate,* *under what conditions does f have an invariant attracting domain tangential to the direction $[v]$?* Vivas showed similar results to Theorem 1.17 for maps with specific types of degenerate and non-degenerate characteristic directions in dimension 2 [V2, Theorems 1 and 2]. In fact, working independently at around the same time, Vivas and the author showed the existence of invariant attracting domains whose points converge tangentially to a particular direction for an overlapping collection of maps [L, V2]. The assumptions that Vivas makes require additional definitions, so we wait to state these results until §2.6 and Chapter 3.

The two main results in Chapter 2 are Theorem A, a local result, and Theorem B, a global extension of Theorem A. Recall from Remark 1.11 that a generic $f \in \mathrm{End}(\mathbb{C}^2, \mathrm{O})$ that is tangent to the identity of order $k \geq 2$ has $k+1$ characteristic directions at the origin counted with multiplicity. In Theorems A and B, we only consider maps f with a unique characteristic direction at the origin so that these $k+1$ characteristic directions all coincide.

Theorem A. *Let $f \in \mathrm{End}(\mathbb{C}^2, \mathrm{O})$ be tangent to the identity of order $k \geq 2$, and let $[v]$ be a non-degenerate characteristic direction. Assume that $[v]$ is the only characteristic direction of f at the origin. Then there exists an invariant attracting domain $\Omega \subset \mathbb{C}^2$, with the origin in its boundary, in which every point is attracted to the origin tangentially to the direction $[v]$, and such that the restriction of f to Ω is conjugate to translation: $(\tau, \omega) \mapsto (\tau, \omega + 1)$.*

Most of the local results we just discussed can also be extended to global results. Let $f \in \mathrm{End}(\mathbb{C}^2, \mathrm{O})$ be tangent to the identity of order $k \geq 2$, and let $[v]$ be a non-degenerate characteristic direction. In addition, assume that f is a global biholomorphism of $\mathbb{C}^m$. The *attractive basin to* $(\mathrm{O}, [v])$ is the set:

$$\Omega_{(\mathrm{O},[v])} := \left\{ x \in \mathbb{C}^m \setminus \{\mathrm{O}\} \;\middle|\; f^n(x) \to \mathrm{O}, [f^n(x)] \to [v] \right\}, \tag{1.8}$$

so $\Omega_{(\mathrm{O},[v])}$ is the set of points in $\mathbb{C}^m \setminus \{\mathrm{O}\}$ that are attracted to the origin tangentially to the direction $[v]$.

Definition 1.22. A proper subdomain $D \subset \mathbb{C}^m$ is a *Fatou-Bieberbach domain* if it is biholomorphic to $\mathbb{C}^m$.

In the global setting, we are interested in when invariant attracting domains exist and are Fatou-Bieberbach domain. Weickert proved the existence of an automorphism of $\mathbb{C}^2$ that is tangent to the identity with an invariant attracting domain that is biholomorphic to $\mathbb{C}^2$ and on which the automorphism is biholomorphic to conjugation [W1, W2]. Hakim extended this and Theorem 1.17 to the following:

Theorem 1.23 (Hakim, [H1]). *Let $f \in \mathrm{End}(\mathbb{C}^m, \mathrm{O})$ be tangent to the identity, and let $[v]$ be a non-degenerate attracting characteristic direction. If, in addition, f is a biholomorphism* *of* $\mathbb{C}^m$*, then* f *has an invariant attracting Fatou-Bieberbach domain, in particular* $\Omega_{(\mathrm{O},[v])}$*, and on that domain* f *is biholomorphically conjugate to translation:* $(\tau,\omega)\mapsto(\tau,\omega+1)$*.*

If we assume that f is a biholomorphism of $\mathbb{C}^2$, then some of Vivas' results that were mentioned above, and will be discussed further in §2.6, also extend in the same way as in the previous theorem [V3]. On the other hand, Stensønes and Vivas showed that for any $m\geq 3$ there exists a biholomorphism f of $\mathbb{C}^m$ that is tangent to the identity and whose *basin of attraction to the origin*:

$$\Omega := \{x\in\mathbb{C}^m \mid f^n(x)\to \mathrm{O} \text{ as } n\to\infty\}$$

is biholomorphic to $(\mathbb{C}\setminus\{0\})^{m-2}\times\mathbb{C}^2$, so Ω is not a Fatou-Bieberbach domain [StV].

We can extend Theorem A to a more global result, but we must first show that such a biholomorphism f can exist. In order to do so, we use the following result due independently to Weickert [W1, Theorem 2.1.1] and Buzzard-Forstneric [BF, Theorem 1.1]:

Theorem 1.24. *Let* $P=(P_1,\ldots,P_m), m\geq 2$*, be a holomorphic polynomial self-map of* $\mathbb{C}^m$ *with* $P'(0)$ *invertible. Let* $d\geq \max_i\{\deg(P_i)\}$*. Then there exists* $\psi:\mathbb{C}^m\to\mathbb{C}^m$*, a biholomorphism, such that* $|\psi(z)-P(z)| = \mathrm{o}\left(|z|^d\right)$ *near the origin.*

Then, knowing that such an f can exist, we extend Theorem A to the following:

Theorem B. *Let* $f\in\mathrm{End}(\mathbb{C}^2,\mathrm{O})$ *be tangent to the identity of order* $k\geq 2$*, and let* $[v]$ *be a non-degenerate characteristic direction. Assume that* $[v]$ *is the only characteristic direction of* f *at the origin. If, in addition,* f *is a biholomorphism of* $\mathbb{C}^2$*, then* f *has an invariant attracting Fatou-Bieberbach domain* $\Sigma\subset\mathbb{C}^2$*, with the origin in its boundary, and on that domain* f *is conjugate to translation:* $(\tau,\omega)\mapsto(\tau,\omega+1)$*.*

In the following chapters we discuss the existence of invariant attracting domains, submanifolds and parabolic curves for maps $f\in\mathrm{End}(\mathbb{C}^m,\mathrm{O})$ that are tangent to the identity of order $k\geq 2$ and how these existence results depend on different properties of f. The focus of Chapter 2 is proving Theorems A and B. In Chapters 3 and 4 we discuss and provide additional examples of results on the existence of invariant attracting domains, submanifolds, and parabolic curves in $\mathbb{C}^2$ and $\mathbb{C}^m$, respectively, for $m\geq 3$. At the end of Chapters 3 and 4, we summarize these existence results in Tables 3.1 and 4.1, respectively.

Chapter 2

Map in $\mathbb{C}^2$ whose only characteristic direction is non-degenerate

In this chapter, we study maps $f \in \operatorname{End}(\mathbb{C}^2, \mathrm{O})$ that are tangent to the identity of order $k \geq 2$, have exactly one distinct characteristic direction, and its unique characteristic direction is non-degenerate. Later on, in Chapter 3, we will discuss what is known more generally for maps $f \in \operatorname{End}(\mathbb{C}^2, \mathrm{O})$ that are tangent to the identity of order $k \geq 2$.

Recall the statements of Theorems A and B from Chapter 1.

Theorem A. *Let $f \in \operatorname{End}(\mathbb{C}^2, \mathrm{O})$ be tangent to the identity of order $k \geq 2$, and let $[v]$ be a non-degenerate characteristic direction. Assume that $[v]$ is the only characteristic direction of f at the origin. Then there exists an invariant attracting domain $\Omega \subset \mathbb{C}^2$, with the origin in its boundary, in which every point is attracted to the origin tangentially to the direction $[v]$, and such that the restriction of f to Ω is conjugate to translation: $(\tau, \omega) \mapsto (\tau, \omega + 1)$.*

Given the assumptions on f in this theorem, it turns out that the director of f must be zero. Hence, Hakim's Theorem 1.17 does not apply to this type of map. Theorem A partially answers questions raised by Abate about the quadratic map (1_{11}) in [A2], which will be discussed further in the next section.

Given f as in Theorem A, we can express f near the origin as $f = \operatorname{Id} + P_k + P_{k+1} + \ldots$, where $P_k \not\equiv \mathrm{O}$ and P_j is a homogeneous polynomial of degree j. Then, from Theorem 1.24, we know that there exists a biholomorphism that is arbitrarily close to f near the origin [BF, W1]. Hence, a biholomorphism exists that satisfies the assumptions in Theorem A. When we add the assumption that f is a biholomorphism, the local result of Theorem A extends to the following global result.

Theorem B. *Let $f \in \operatorname{End}(\mathbb{C}^2, \mathrm{O})$ be tangent to the identity of order $k \geq 2$, and let $[v]$ be a non-degenerate characteristic direction. Assume that $[v]$ is the only characteristic direction of f at the origin. If, in addition, f is a biholomorphism of $\mathbb{C}^2$, then f has an invariant* *attracting Fatou-Bieberbach domain $\Sigma \subsetneq \mathbb{C}^2$ (i.e. Σ is biholomorphic to $\mathbb{C}^2$), with the origin in its boundary, and on that domain f is conjugate to translation: $(\tau, \omega) \mapsto (\tau, \omega + 1)$.*

Working independently from the author, Vivas in [V3] proved a similar result to Theorems A and B for *irregular* characteristic directions. Given the assumptions on f and $[v]$ in Theorems A and B, it turns out that $[v]$ must be irregular. In §2.6 we discuss this further.

This chapter is devoted to proving Theorems A and B. In §2.1, we conjugate f from Theorem A to a suitable normal form using a linear change of coordinates and show that the director of its unique characteristic direction must be zero. In §2.2, we perform a coordinate change moving the fixed point from the origin to infinity and find an invariant domain. In §2.3, we perform another, more complicated, coordinate change so that f acts on the second coordinate by translation. The technique we employ to find this coordinate change is similar to that used in the degenerate case studied in [V1], but is more complicated here because it requires solving a system of differential equations instead of just one differential equation. In §2.4, we perform a final coordinate change so that f acts as the identity on the first coordinate and translation on the second. Each time we perform a coordinate change, we find an invariant attracting domain for f so that by §2.4 we have finished showing Theorem A. In §2.5, we assume that, in addition, f is a biholomorphism of $\mathbb{C}^2$ and extend our domain from §2.4 to one that is biholomorphic to $\mathbb{C}^2$, concluding with Theorem B. In §2.6, we introduce the necessary terms to understand Vivas' results from [V3] and discuss how these compare to Theorems A and B.

2.1. Preliminaries

Lemma 2.1. *Let $f \in \mathrm{End}(\mathbb{C}^2, \mathrm{O})$ be tangent to the identity of order $k \geq 2$, and let $[v]$ be a non-degenerate characteristic direction. Assume that $[v]$ is the only characteristic direction of f at the origin. Then f is linearly conjugate to:*

$$f_0(x,y) = (x,y)\left(1 + xyR(x,y) + y^{k-1}\right) + \left(P(x,y), x^k + Q(x,y)\right), \tag{2.1}$$

where P, Q are convergent power series vanishing to order at least $k+1$ at the origin and R is a homogeneous polynomial of degree $k-3$ such that $R \equiv 0$ if $k = 2$.

Proof. We can write $f(x,y)$ as a sum of its homogenous polynomials,

$$f(x,y) = (x,y) + \sum_{j=k}^{\infty} P_j(x,y),$$

where $P_j(x,y)$ are homogeneous polynomials of degree j and $P_k \not\equiv 0$. We assume that $[0:1]$ is the characteristic direction of f since, via a linear conjugation of f, we can move the characteristic direction of f to $[0:1]$ without changing the degree of any of the P_j. We can write the k-th degree polynomial as:

$$P_k(x,y) = \left(\sum_{j=0}^{k} a_j x^{k-j} y^j, \sum_{j=0}^{k} b_j x^{k-j} y^j\right),$$

where $a_j, b_j \in \mathbb{C}$. Since $[0:1]$ is the only characteristic direction of f, $P_k[x:y] \neq [x:y]$ for all $x \neq 0$. This restricts the possible values of $\{a_j, b_j\}$:

$$P_k(0,1) = (a_k, b_k) \text{ and } P_k(1,0) = (a_0, b_0), \text{ therefore } a_k = 0, b_k \neq 0 \text{ and } b_0 \neq 0,$$

$$P_k(x,1) = \left(\sum_{j=0}^{k-1} a_j x^{k-j}, \sum_{j=0}^{k} b_j x^{k-j}\right) = \left(x\sum_{j=0}^{k-1} a_j x^{k-1-j}, \sum_{j=0}^{k} b_j x^{k-j}\right) \neq \lambda(x,1)$$

for any $\lambda \in \mathbb{C}$. In addition,

$$P_k[x:1] = [x:1] \Leftrightarrow x\sum_{j=0}^{k-1} a_j x^{k-1-j} = x\sum_{j=0}^{k} b_j x^{k-j} \Leftrightarrow \left(-b_0 x^k + \sum_{j=1}^{k}(a_{j-1} - b_j)x^{k-j}\right)x = 0$$

and, by assumption, the first condition is true only when $x = 0$, then the last condition must only be true when $x = 0$. Since $b_0 \neq 0$, this implies that $a_{j-1} = b_j, \forall 1 \leq j \leq k$. We can now re-write $P_k(x,y)$ as:

$$P_k(x,y) = \left(x\sum_{j=0}^{k-1} a_j x^{k-1-j} y^j, y\sum_{j=0}^{k-1} a_j x^{k-1-j} y^j + b_0 x^k\right) := (xS(x,y), yS(x,y) + b_0 x^k).$$

Now that we have a more explicit form for P_k, we want to simplify it further using linear conjugation. Let l be a linear map that fixes $[0:1]$. Then we can write l as:

$$l(x,y) := (ax, cx + dy) \text{ and } l^{-1}(x,y) = \frac{1}{ad}(dx, -cx + ay), \text{ where } ad \neq 0. \tag{2.2}$$

$$\begin{aligned}
l^{-1} \circ P_k \circ l(x,y) &= l^{-1}(axS(ax, cx+dy), (cx+dy)S(ax, cx+dy) + b_0 a^k x^k) \\
&= \left(xS(ax, cx+dy), yS(ax, cx+dy) + \frac{b_0 a^k}{d} x^k\right) \\
S(ax, cx+dy) &= \sum_{j=0}^{k-1} a_j a^{k-1-j} x^{k-1-j} (cx+dy)^j \\
&= x^{k-1}\left(\sum_{j=0}^{k-1} a_j a^{k-1-j} c^j\right) + xy(\cdots) + y^{k-1}(a_{k-1} d^{k-1})
\end{aligned}$$

We can choose a, c, d so that (1) $\frac{b_0 a^k}{d} = 1$, (2) $a_{k-1} d^{k-1} = 1$, and (3) $\sum_{j=0}^{k-1} a_j a^{k-1-j} c^j = 0$. Therefore $P_k(x, y)$ is linearly conjugate to:

$$\left(x\left(xyR(x,y) + y^{k-1}\right), y\left(xyR(x,y) + y^{k-1}\right) + x^k\right),$$

where $R \equiv 0$ if $k \leq 2$, otherwise $R(x, y) = \frac{1}{xy}\left(S(ax, cx + dy) - y^{k-1}\right)$ is a homogeneous polynomial of degree $k - 3$. Let

$$(P(x,y), Q(x,y)) := l^{-1} \circ \sum_{j=k+1}^{\infty} P_j \circ l(x,y).$$

Since the (P_j) are convergent power series in a neighborhood of the origin, so are P and Q. $\square$

Abate in [A2] studied quadratic maps tangent to the identity up to holomorphic conjugacy. He showed that for quadratic self maps of $\mathbb{C}^2$ tangent to the identity, holomorphic conjugacy was equivalent to linear conjugacy and used this along with the number of characteristic directions of the maps to classify all such maps. In addition, Ueda and Rivi also classified such maps [U3, R1, W2]. If we assume that the map f in Lemma 2.1 is quadratic with no terms of higher degree, then f_0 is the map (1_{11}) in [A2], namely:

$$f_0(x,y) = \left(x(1+y), y(1+y) + x^2\right) = (x + xy, y + y^2 + x^2).$$

We have not made any explicit assumptions on the director of f; however, the following lemma shows that the director of f must be zero, hence Theorem 1.17 does not apply to f.

Lemma 2.2. *The real part of the director of f at $[0:1]$ is zero.*

Proof. Let $U := \left\{[x_0 : x_1] \in \mathbb{CP}^1 \,\middle|\, x_1 \neq 0\right\}$ and define $\pi : U \to \mathbb{C}$ by $\pi([x_0 : x_1]) = \frac{x_0}{x_1}$. Define

$$g(x) := \pi \circ \widehat{P_k} \circ \pi^{-1}(x) = \pi\left[xS(x,1) : S(x,1) + b_0 x^k\right] = \frac{xS(x,1)}{S(x,1) + b_0 x^k},$$

where $x \in \mathbb{C}$ and $\widehat{P_k} : \mathbb{CP}^1 \to \mathbb{CP}^1$ by $[v] \mapsto [P_k(v)]$. So for $x \in \mathbb{C}$, $D(\widehat{P_k})_{[x:1]} - \mathrm{Id} = g'(x) - \mathrm{Id}$. Therefore the director of f at $[0:1]$ is the value of $g'(0) - \mathrm{Id}$ and

$$g'(x) = \frac{S(x,1)^2 + b_0 x^k S(x,1) + b_0 x^{k+1} S'(x,1) - k b_0 x^k S(x,1)}{\left(S(x,1) + b_0 x^k\right)^2},$$

so $g'(0) - \mathrm{Id} = 0$. Therefore the director of f at $[0:1]$ is zero. $\square$

Notation. We will use π_j to denote projection onto the jth coordinate.

2.2. Invariant region

We want to find a domain of attraction for the map f_0, which is equivalent to finding one for f. For $xy \neq 0$, define the new coordinates (u,v) as:

$$(u,v) := \psi_0(x,y) := \left(a\frac{y^k}{x^k}, \frac{b}{y^{k-1}}\right),$$

where $a = -\frac{k-1}{k}, b = -\frac{1}{k-1}$. Define $\Omega^{(u,v)}_{R,\delta,\theta}$ to be:

$$\left\{(u,v) \in \mathbb{C}^2 \;\middle|\; \operatorname{Re}(u) > R, |u|^{\frac{(k-1)(k+1)}{k}} < \delta|v|, |\operatorname{Arg}(u)| < \theta, |\operatorname{Arg}(v)| < \frac{k-1}{k}\theta\right\} \tag{2.3}$$

and

$$\Omega^u_{R,\theta} := \left\{u \in \mathbb{C} \;\middle|\; \operatorname{Re}(u) > R, |\operatorname{Arg}(u)| < \theta\right\} = \pi_1\left(\Omega^{(u,v)}_{R,\delta,\theta}\right), \tag{2.4}$$

for any

$$0 < \theta < \frac{\pi}{4}, \qquad 0 < \delta \ll 1, \qquad \text{and} \qquad R \gg 0.$$

Fix R, δ, θ satisfying the above conditions to define

$$\Omega^{(u,v)} := \Omega^{(u,v)}_{R,\delta,\theta} \qquad \text{and} \qquad \Omega^u := \Omega^u_{R,\theta}. \tag{2.5}$$

Now we can define an inverse to ψ_0 restricted to the domain $\Omega^{(u,v)}$:

$$(x,y) := \psi_0^{-1}(u,v) = \left(\left(\frac{a}{u}\right)^{\frac{1}{k}}\left(\frac{b}{v}\right)^{\frac{1}{k-1}}, \left(\frac{b}{v}\right)^{\frac{1}{k-1}}\right),$$

where we choose the $\frac{1}{k}, \frac{1}{k-1}$ roots that map 1 to 1. Therefore $\psi_0 : \psi_0^{-1}\left(\Omega^{(u,v)}\right) \to \Omega^{(u,v)}$ is a biholomorphism. Note that $0 \in \partial\left(\psi_0^{-1}\left(\Omega^{(u,v)}\right)\right)$.

Proposition 2.3. *Given $R' < R$ and $\theta < \theta' < \frac{\pi}{4}$, $\exists \kappa > 0$ such that if $u \in \Omega^u_{R,\theta}$, then $B(u, \kappa|u|) \subset \Omega^u_{R',\theta'}$. Furthermore, given $\alpha \neq 0$ and a holomorphic function F on $\Omega^u_{R',\theta'}$ satisfying the bound $F = \mathrm{O}(u^\alpha)$, then $\forall n \in \mathbb{N}$, the nth derivative satisfies the bound $F^{(n)} = \mathrm{O}\left(u^{\alpha-n}\right)$ on $\Omega^u_{R,\theta}$.*

Proof. Given $u \in \Omega^u_{R,\theta}$,

$$\operatorname{Re}(u) - R' > \left(1 - \frac{R'}{R}\right)\operatorname{Re}(u) > \frac{1}{2}\left(1 - \frac{R'}{R}\right)|u|$$

and

$$\sin\left(\theta' - \operatorname{Arg}(u)\right)|u| > \sin(\theta' - \theta)|u|.$$

Hence for any $0 < \kappa \le \min\left\{\frac{1}{2}\left(1-\frac{R'}{R}\right), \sin(\theta'-\theta)\right\}$, the disk $B(u, \kappa|u|) \subset \Omega^u_{R',\theta'}$. On $\Omega^u_{R',\theta'}$, $F(\zeta) = \mathrm{O}(\zeta^\alpha)$ and $\exists C > 0$ such that $|F(\zeta)| < C|\zeta|^\alpha \le \begin{cases} C(1+\kappa)^\alpha |u|^\alpha, & \text{if } \alpha \ge 0 \\ C(1-\kappa)^\alpha |u|^\alpha, & \text{if } \alpha < 0 \end{cases}$.
Therefore, $\forall n \in \mathbb{N}$,

$$\left|F^{(n)}(u)\right| \le \frac{n!}{2\pi}\left|\int_{|\zeta-u|=\kappa|u|} \frac{F(\zeta)}{(\zeta-u)^{n+1}} d\zeta\right| \le \frac{n!}{(\kappa|u|)^n} \sup_{|\zeta-u|=\kappa|u|} |F(\zeta)| = \mathrm{O}\left(u^{\alpha-n}\right),$$

where we used Cauchy estimates to get the first inequality. □

Remark 2.4. On several occasions, we will adjust R, δ, θ to shrink the domain $\Omega^{(u,v)}$ (or Ω^u). In particular, we will choose R', δ', θ' that depend on R, δ, θ so that by making R large enough and δ, θ small enough the domain $\Omega^{(u,v)}_{R',\delta',\theta'}$ (or $\Omega^u_{R',\theta'}$) satisfies all of the properties that had been shown for $\Omega^{(u,v)}$ (respectively, Ω^u) and $\Omega^{(u,v)}_{R',\delta',\theta'} \supsetneq \Omega^{(u,v)}$ (respectively, $\Omega^u_{R',\theta'} \supsetneq \Omega^u$). We will use this and Proposition 2.3 to find domains on which a holomorphic function is defined as well as similar subdomains on which we can bound the derivatives of that holomorphic function.

Let $f_1 := \psi_0 \circ f_0 \circ \psi_0^{-1}$. For any $(u,v) \in \Omega^{(u,v)}$, denote the n-th iterate of the map after the coordinate change by $f_1^n(u,v) := (u_n, v_n)$, where $f_1^0(u,v) = (u,v)$. Later in this section we will prove the following results on invariance of $\Omega^{(u,v)}$ and size of (u_n, v_n).

Lemma 2.5. *$\Omega^{(u,v)}$ is invariant under f_1.*

Lemma 2.6. *For any $(u,v) \in \Omega^{(u,v)}$ and any positive integer n,*

$$\mathrm{Re}(v) + \frac{3n}{2} \ge \mathrm{Re}(v_n) \ge \mathrm{Re}(v) + \frac{n}{2} \tag{2.6}$$

and

$$\mathrm{Re}(u) + 3\log\left(1 + \frac{n}{\mathrm{Re}(v)}\right) \ge \mathrm{Re}(u_n) \ge \mathrm{Re}(u) + \frac{1}{6}\log\left(1 + \frac{n}{\mathrm{Re}(v)}\right). \tag{2.7}$$

It follows from Lemma 2.6 that for any $(u,v) \in \Omega^{(u,v)}$,

$$\begin{gathered} 2|v| + 3n \ge |v_n| \ge \frac{|v| + n}{2}, \text{ and} \\ 2|u| + 6\log\left(1 + \frac{n}{\mathrm{Re}(v)}\right) \ge |u_n| \ge \frac{|u|}{2} + \frac{1}{6}\log\left(1 + \frac{n}{\mathrm{Re}(v)}\right). \end{gathered} \tag{2.8}$$

In order to simplify the coordinate change, we make the following definitions:

$$\widetilde{R}(u) := \left(\frac{v}{b}\right)^{\frac{k-3}{k-1}} R(x,y) = R\left(\left(\frac{a}{u}\right)^{\frac{1}{k}}, 1\right), \tag{2.9}$$
$$\widetilde{P}(u,v) := \left(\frac{v}{b}\right)^{\frac{k+1}{k-1}} P(x,y) = \left(\frac{v}{b}\right)^{\frac{k+1}{k-1}} P\left(\left(\frac{a}{u}\right)^{\frac{1}{k}}\left(\frac{b}{v}\right)^{\frac{1}{k-1}}, \left(\frac{b}{v}\right)^{\frac{1}{k-1}}\right),$$
$$\widetilde{Q}(u,v) := \left(\frac{v}{b}\right)^{\frac{k+1}{k-1}} Q(x,y) = \left(\frac{v}{b}\right)^{\frac{k+1}{k-1}} Q\left(\left(\frac{a}{u}\right)^{\frac{1}{k}}\left(\frac{b}{v}\right)^{\frac{1}{k-1}}, \left(\frac{b}{v}\right)^{\frac{1}{k-1}}\right),$$
$$h(u,v) := kb^{\frac{k}{k-1}}\left(\frac{\widetilde{Q}(u,v)}{u^{\frac{1}{k}}} - \frac{\widetilde{P}(u,v)}{a^{\frac{1}{k}}}\right) := \sum_{j=0}^{\infty} \frac{h_j(u)}{v^{\frac{j}{k-1}}}.$$

Then $\widetilde{P}, \widetilde{Q}, h$ are convergent power series in $u^{-\frac{1}{k}}, v^{-\frac{1}{k-1}}$, h_j is a convergent power series in $u^{-\frac{1}{k}}$, and $h_j, \widetilde{R}$ are holomorphic on Ω^u. Now we find an expression for u_1:

$$\begin{aligned}
u_1 &= u\left(1 + \frac{\frac{x^k}{y} + \frac{Q(x,y)}{y} - \frac{P(x,y)}{x}}{1 + xyR(x,y) + y^{k-1} + \frac{P(x,y)}{x}}\right)^k \\
&= u\left(1 + \frac{1}{v}\frac{\frac{ab}{u} + b\left(\frac{b}{v}\right)^{\frac{1}{k-1}}\widetilde{Q}(u,v) - b\left(\frac{b}{v}\right)^{\frac{1}{k-1}}\left(\frac{u}{a}\right)^{\frac{1}{k}}\widetilde{P}(u,v)}{1 + \frac{b}{v}\left(\left(\frac{a}{u}\right)^{\frac{1}{k}}\widetilde{R}(u) + 1 + \left(\frac{b}{v}\right)^{\frac{1}{k-1}}\left(\frac{u}{a}\right)^{\frac{1}{k}}\widetilde{P}(u,v)\right)}\right)^k \\
&= u\left(1 + \frac{1}{v}\left(\frac{ab}{u} + \frac{u^{\frac{1}{k}}}{v^{\frac{1}{k-1}}}\frac{h(u,v)}{k}\right)\left[1 + \mathrm{O}\left(\frac{1}{v}\right)\right]\right)^k \\
&= u\left(1 + \frac{k}{v}\left(\frac{ab}{u} + \frac{u^{\frac{1}{k}}}{v^{\frac{1}{k-1}}}\frac{h(u,v)}{k}\right) + \mathrm{O}\left(\frac{1}{uv^2}, \frac{u^{\frac{1}{k}}}{v^{2+\frac{1}{k-1}}}\right)\right) \\
&= u + \frac{1}{v} + \frac{u^{\frac{k+1}{k}}}{v^{\frac{k}{k-1}}}h(u,v) + \mathrm{O}\left(\frac{1}{v^2}\right) \\
&= u + \frac{1}{v} + \frac{u^{\frac{k+1}{k}}}{v^{\frac{k}{k-1}}}\sum_{j=0}^{k-2}\frac{h_j(u)}{v^{\frac{j}{k-1}}} + \mathrm{O}\left(\frac{1}{v^2}\right),
\end{aligned}$$

Similarly:

$$
\begin{aligned}
v_1 &= v\left(1+xyR(x,y)+y^{k-1}+\frac{x^k}{y}+\frac{Q(x,y)}{y}\right)^{-(k-1)}\\
&= v\left(1+\frac{b}{v}\left[\left(\frac{a}{u}\right)^{\frac{1}{k}}\widetilde{R}(u)+1+\frac{a}{u}+\left(\frac{b}{v}\right)^{\frac{1}{k-1}}\widetilde{Q}(u,v)\right]\right)^{-(k-1)}\\
&= v\Bigg(1-(k-1)\frac{b}{v}\left[1+\left(\frac{a}{u}\right)^{\frac{1}{k}}\widetilde{R}(u)+\frac{a}{u}+\left(\frac{b}{v}\right)^{\frac{1}{k-1}}\widetilde{Q}(u,v)\right]\\
&\qquad\quad +k(k-1)\frac{b^2}{v^2}\left[1+\left(\frac{a}{u}\right)^{\frac{1}{k}}\widetilde{R}(u)+\frac{a}{u}+\left(\frac{b}{v}\right)^{\frac{1}{k-1}}\widetilde{Q}(u,v)\right]^2+\mathrm{O}\left(\frac{1}{v^3}\right)\Bigg)\\
&= v+1+\left[\frac{a}{u}\right]^{\frac{1}{k}}\widetilde{R}(u)+\frac{a}{u}+\left[\frac{b}{v}\right]^{\frac{1}{k-1}}\widetilde{Q}(u,v)+\frac{k}{k-1}\frac{1}{v}\left[1+\left(\frac{a}{u}\right)^{\frac{1}{k}}\widetilde{R}(u)+\frac{a}{u}\right]^2+\mathrm{O}\left(\frac{1}{v^{\frac{k}{k-1}}}\right)\\
&= v+1+\sum_{j=0}^{k-1}\frac{g_j(u)}{v^{\frac{j}{k-1}}}+\mathrm{O}\left(\frac{1}{v^{\frac{k}{k-1}}}\right),
\end{aligned}
$$

where $a=-\frac{k-1}{k}, b=-\frac{1}{k-1}$ and g_0 is a polynomial in $u^{-\frac{1}{k}}$ with no constant term and g_j are power series in $u^{-\frac{1}{k}}$. We will frequently use these properties of the $\{g_j\}$ in what follows.

To summarize, for $(u,v)\in\Omega^{(u,v)}$ we have derived the following equations for u_1, v_1:

$$
u_1 = u+\frac{1}{v}+\frac{u^{\frac{k+1}{k}}}{v^{\frac{k}{k-1}}}h(u,v)+\mathrm{O}\left(\frac{1}{v^2}\right)=u+\frac{1}{v}+\frac{u^{\frac{k+1}{k}}}{v^{\frac{k}{k-1}}}\sum_{j=0}^{k-2}\frac{h_j(u)}{v^{\frac{j}{k-1}}}+\mathrm{O}\left(\frac{1}{v^2}\right) \tag{2.10}
$$

$$
v_1 = v+1+\sum_{j=0}^{k-1}\frac{g_j(u)}{v^{\frac{j}{k-1}}}+\mathrm{O}\left(\frac{1}{v^{\frac{k}{k-1}}}\right)=v+1+\mathrm{O}\left(\frac{1}{u^{\frac{1}{k}}},\frac{1}{v^{\frac{1}{k-1}}}\right). \tag{2.11}
$$

Proof of Lemma 2.5. Fix any $(u,v)\in\Omega^{(u,v)}$. First we show that $\mathrm{Re}(u_1)>R$.

$$
\begin{aligned}
\mathrm{Re}(u_1) &= \mathrm{Re}(u)+\mathrm{Re}\left(\frac{1}{v}\left[1+\frac{u^{\frac{k+1}{k}}}{v^{\frac{1}{k-1}}}h(u,v)+\mathrm{O}\left(\frac{1}{v}\right)\right]\right)\\
&> \mathrm{Re}(u)+\frac{1}{2|v|}-\frac{2\delta^{\frac{1}{k-1}}C}{|v|}>\mathrm{Re}(u)>R
\end{aligned}
$$

where C is some constant such that $|h(u,v)|<C$, R is sufficiently large and δ is chosen to be suitably small. Next we check that the arguments of u_1, v_1 remain small enough.

$$
|\operatorname{Arg}(v_1)|\le\max\left\{|\operatorname{Arg}(v)|,\left|\operatorname{Arg}\left(1+\sum_{j=0}^{k-1}\frac{g_j(u)}{v^{\frac{j}{k-1}}}+\mathrm{O}\left(\frac{1}{v^{\frac{k}{k-1}}}\right)\right)\right|\right\}\le\frac{k-1}{k}\theta,
$$

for R large enough.

$$|\operatorname{Arg}(u_1)| \leq \max\left\{|\operatorname{Arg}(u)|, |\operatorname{Arg}(v)| + \left|\operatorname{Arg}\left(1 + \frac{u^{\frac{k+1}{k}}}{v^{\frac{1}{k-1}}} h(u,v) + \mathrm{O}\left(\frac{1}{v}\right)\right)\right|\right\} < \theta,$$

for δ suitably small. Finally we verify that $|u_1|$ remains small enough relative to $|v_1|$.

$$\begin{aligned}
|u_1|^{\frac{k^2-1}{k}} &< |u|^{\frac{k^2-1}{k}} \left|1 + \frac{1}{uv}\left(1 + \frac{u^{\frac{k+1}{k}}}{v^{\frac{1}{k-1}}} h(u,v) + \mathrm{O}\left(\frac{1}{v}\right)\right)\right|^{\frac{k^2-1}{k}} \\
&< \delta|v| \left|1 + \frac{k^2-1}{k}\frac{1}{uv}\left(1 + \frac{u^{\frac{k+1}{k}}}{v^{\frac{1}{k-1}}} h(u,v) + \mathrm{O}\left(\frac{1}{v}\right)\right)\right| \\
&< \delta|v|\left|1 + \frac{1}{2v}\right| < \delta|v_1|,
\end{aligned}$$

for large enough R. Therefore, $f_1\left(\Omega^{(u,v)}\right) \subset \Omega^{(u,v)}$. ☐

Proof of Lemma 2.6.

$$\operatorname{Re}(v_n) = \operatorname{Re}(v) + n + \sum_{j=0}^{n-1} \mathrm{O}\left(u_j^{-\frac{1}{k}}, v_j^{-\frac{1}{k-1}}\right)$$

$$u_n = u + \sum_{j=0}^{n-1} \frac{1}{v_j}\left[1 + \frac{u_j^{\frac{k+1}{k}}}{v_j^{\frac{1}{k-1}}} h(u_j, v_j) + \mathrm{O}\left(\frac{1}{v_j}\right)\right]$$

In the expression for $\operatorname{Re}(v_n)$, the O-terms are bounded independently of j and so we can choose R sufficiently large that, for each j, the term $\mathrm{O}\left(u_j^{\frac{-1}{k}}, v_j^{\frac{-1}{k-1}}\right)$ is bounded above by $\frac{1}{2}$. Hence, $|\operatorname{Re}(v_n - v) - n| \leq \frac{n}{2}$. Now we find bounds on $\frac{|u_n|}{|v_n|^\alpha}$ and $\operatorname{Re}(u_n)$:

$$\operatorname{Re}(u_n - u) \leq \sum_{j=0}^{n-1} \frac{1 + 2\delta^{\frac{1}{k-1}} C}{|v_j|} \leq \sum_{j=0}^{n-1} \frac{\frac{3}{2}}{\operatorname{Re}(v) + \frac{1}{2}j} \leq \int_{-1}^{n-1} \frac{3dx}{x + 2\operatorname{Re}(v)} < 3\log\left(1 + \frac{n}{\operatorname{Re}(v)}\right)$$

$$\operatorname{Re}(u_n - u) \geq \sum_{j=0}^{n-1} \frac{1 - 2\delta^{\frac{1}{k-1}} C}{|v_j|} \geq \sum_{j=0}^{n-1} \frac{\frac{1}{4}}{\operatorname{Re}(v) + \frac{3}{2}j} \geq \frac{1}{6}\int_0^n \frac{dx}{x + \frac{2}{3}\operatorname{Re}(v)} \geq \frac{1}{6}\log\left(1 + \frac{n}{\operatorname{Re}(v)}\right)$$

Therefore we have the desired bounds on $\operatorname{Re}(u_n), \operatorname{Re}(v_n)$. ☐

2.3. Fatou coordinate

In this section we will perform a coordinate change to simplify the expression for (2.11). For any $(u,v) \in \Omega^{(u,v)}$ we define:

$$(u,w) := \widetilde{\psi}_1(u,v) := \left(u, v\left[1 + \sum_{j=0}^{k-1} \frac{\phi_j(u)}{v^{\frac{j}{k-1}}}\right]\right),$$

where $\{\phi_j\}$ are holomorphic functions in Ω^u that we will define later as solutions of certain differential equations. Let $w_n := \pi_2 \circ \widetilde{\psi}_1 \circ f_1^n(u,v) = \pi_2 \circ \widetilde{\psi}_1(u_n, v_n)$. The goals of this section are to show that the sequence $(w_n - n)_{n=1}^{\infty}$ converges uniformly to a Fatou coordinate ω, that is $\omega \circ f = \omega + 1$, and that the map $(u,v) \mapsto (u, \omega(u,v))$ defines a coordinate change, that is a biholomorphism onto its image. Before introducing the Fatou coordinate, we need to simplify the expression for w_1 in terms of u, w and define the holomorphic functions $\{\phi_j\}$. Using Taylor series expansion (or integration by parts):

$$\phi_j(u_1) = \phi_j(u) + (u_1 - u)\phi_j'(u) + \int_u^{u_1} \phi_j''(\zeta)(u_1 - \zeta)d\zeta, \tag{2.12}$$

where we are integrating along the line $\gamma_u(t) = (1-t)u + tu_1$ for $0 \le t \le 1$, which is contained in Ω^u. Let $\hat{t} \in [0,1]$ be such that $\left|\phi_j''\left(\gamma_u\left(\hat{t}\right)\right)\right| = \max_{0 \le t \le 1} |\phi_j''(\gamma_u(t))|$ and $\hat{u} := \gamma_u(\hat{t})$. We can bound the integral from (2.12) as follows:

$$\int_u^{u_1} \phi_j''(\zeta)(u_1 - \zeta)d\zeta = \mathrm{O}\left(\phi_j''(\hat{u})(u_1 - u)^2\right)$$

Then we can write w_1 as:

$$\begin{aligned}
w_1 &= v_1 + \sum_{j=0}^{k-1} \phi_j(u_1) v_1^{1-\frac{j}{k-1}} \\
&= v_1 + \sum_{j=0}^{k-1} \left[\phi_j(u) + \frac{\phi_j'(u)}{v}\left(1 + \frac{u^{\frac{k+1}{k}}}{v^{\frac{1}{k-1}}} \sum_{l=0}^{k-2} \frac{h_l(u)}{v^{\frac{l}{k-1}}} + \mathrm{O}\left(\frac{1}{v}\right)\right) + \mathrm{O}\left(\frac{\phi_j''(\hat{u})}{v^2}\right)\right] \cdot \\
&\qquad\qquad v^{1-\frac{j}{k-1}} \left[1 + \frac{1}{v}\left(1 + \sum_{m=0}^{k-1} \frac{g_m(u)}{v^{\frac{m}{k-1}}} + \mathrm{O}\left(\frac{1}{v^{\frac{k}{k-1}}}\right)\right)\right]^{1-\frac{j}{k-1}} \\
&= v_1 + \sum_{j=0}^{k-1} v^{1-\frac{j}{k-1}} \left[\phi_j(u) + \frac{\phi_j'(u)}{v}\left(1 + \frac{u^{\frac{k+1}{k}}}{v^{\frac{1}{k-1}}} \sum_{l=0}^{k-2} \frac{h_l(u)}{v^{\frac{l}{k-1}}}\right) + \mathrm{O}\left(\frac{\phi_j'(u)}{v^2}, \frac{\phi_j''(\hat{u})}{v^2}\right)\right] \cdot \\
&\qquad\qquad \left[1 + \left(1 - \frac{j}{k-1}\right)\frac{1}{v}\left(1 + \sum_{m=0}^{k-1} \frac{g_m(u)}{v^{\frac{m}{k-1}}}\right) + \frac{(k-1-j)}{v^2} \mathrm{O}\left(j, \frac{1}{v^{\frac{1}{k-1}}}\right)\right] \\
&= w + 1 + \sum_{j=0}^{k-1} \frac{1}{v^{\frac{j}{k-1}}}\left(g_j(u) + \frac{k-1-j}{k-1}(1 + g_0(u))\phi_j(u) + \phi_j'(u)\right) \\
&\quad + \sum_{j=0}^{k-1}\sum_{m=1}^{k-1} \frac{1}{v^{\frac{j+m}{k-1}}}\left(\frac{k-1-j}{k-1}\phi_j(u) g_m(u) + \phi_j'(u) u^{\frac{k+1}{k}} h_{m-1}(u)\right) \\
&\quad + \sum_{j=0}^{k-1} \mathrm{O}\left(\frac{1}{v^{\frac{k}{k-1}}}, \frac{\phi_0(u)}{v^{\frac{k}{k-1}}}, j(k-1-j)\frac{\phi_j(u)}{v^{\frac{k+j-1}{k-1}}}, \frac{\phi_j'(u)}{v^{\frac{k+j-1}{k-1}}}, \frac{\phi_j''(\hat{u})}{v^{\frac{k+j-1}{k-1}}}\right)
\end{aligned}$$

In order to simplify the equation for w_1, define F_j, G_j as:

$$F_j(u) := \frac{k-1-j}{k-1}\left(1+g_0(u)\right), \tag{2.13}$$
$$G_j(u) := -\left[g_j(u) + \sum_{l=0}^{j-1}\left(\frac{k-1-l}{k-1}\phi_l(u)g_{j-l}(u) + \phi_l'(u)u^{\frac{k+1}{k}}h_{j-l-1}(u)\right)\right].$$

for all integers $0 \leq j \leq k-1$. Therefore,

$$w_1 = w + 1 + \sum_{j=0}^{k-1}\frac{\phi_j'(u) + F_j(u)\phi_j(u) - G_j(u)}{v^{\frac{j}{k-1}}} \tag{2.14}$$
$$+ \mathrm{O}\left(\frac{1}{v^{\frac{k}{k-1}}}, \frac{\phi_0'(u)}{v}\right) + \sum_{j=0}^{k-1}\mathrm{O}\left(\frac{(k-1-j)\phi_j(u)}{v^{\frac{k}{k-1}}}, \frac{ju^{\frac{k+1}{k}}\phi_j'(u)}{v^{\frac{k}{k-1}}}, \frac{\phi_j''(\hat{u})}{v^{\frac{k+j-1}{k-1}}}\right)$$

for $0 \leq j \leq k-1$. Now we want to show that we can find $\{\phi_j\}$ such that $\phi_j' + F_j\phi_j - G_j \equiv 0$ on $\Omega^{(u,v)}$ and all of the O-terms involving $\{\phi_j, \phi_j', \phi_j''\}$ are small enough.

Proposition 2.7. *If R is large enough and δ, θ are small enough, then there exist holomorphic functions $\{\phi_j\}_{0\leq j\leq k-1}$ on Ω^u such that*

$$\phi_j' + F_j\phi_j - G_j \equiv 0 \tag{2.15}$$

on Ω^u, where F_j, G_j are defined using (2.13). *Furthermore, we have the bounds*

$$\begin{aligned}
&G_j(u) = \mathrm{O}\left(u^{\frac{j-1}{k}}\right), && G_j'(u) = \mathrm{O}\left(u^{\frac{j-1-k}{k}}\right), && \\
&\phi_m(u) = \mathrm{O}\left(u^{\frac{m-1}{k}}\right), && \phi_m'(u) = \mathrm{O}\left(u^{\frac{m-1-k}{k}}\right), && \phi_m''(u) = \mathrm{O}\left(u^{\frac{m-1-2k}{k}}\right), \\
&\phi_{k-1}(u) = \mathrm{O}\left(u^{\frac{2k-2}{k}}\right), && \phi_{k-1}'(u) = \mathrm{O}\left(u^{\frac{k-2}{k}}\right), && \phi_{k-1}''(u) = \mathrm{O}\left(u^{-\frac{2}{k}}\right),
\end{aligned} \tag{2.16}$$

on Ω^u, where $0 \leq m < k-1$.

Proof. First we verify that for each j, given the maps F_j and G_j then (2.15) has a solution. Given $\{\phi_l\}_{0\leq l<j}$, we can define G_j as in (2.13) and use this to define ϕ_j. Since G_0 is already defined, we can start this process. We follow the techniques used in Remark 2.4 to choose $0 \ll R_2 < R_1 < R$ and $0 < \theta < \theta_1 < \theta_2 < \frac{\pi}{4}$ so that F_j, G_j are holomorphic on $\Omega^u_{R_2,\theta_2}$, ϕ_j is holomorphic on $\Omega^u_{R_1,\theta_1}$, and the derivatives of G_j, ϕ_j are bounded on the subset $\Omega^u \subset \Omega^u_{R_1,\theta_1} \subset \Omega^u_{R_2,\theta_2}$, where these domains were defined in (2.4). Fix u_0 such that $R_2 < u_0 < R_1$. A solution to (2.15) is:

$$\phi_j(u) = e^{-\int_{u_0}^u F_j(\nu)d\nu}\int_{u_0}^u G_j(\nu)e^{\int_{u_0}^{\nu} F_j(\zeta)d\zeta}d\nu, \tag{2.17}$$

where $u \in \Omega^u_{R_1,\theta_1}$ and the integral is taken along any simple, smooth curve between u_0 and u contained in $\Omega^u_{R_2,\theta_1} \subset \Omega^u_{R_2,\theta_2}$. Note that when $j = k-1$, $F_{k-1} \equiv 0$ so (2.17) simplifies to $\phi_{k-1}(u) = \int_{u_0}^u G_{k-1}(\nu)d\nu$. There are only finitely many j, so we can repeat this process to get $\{\phi_j\}_{0\le j\le k-1}$ holomorphic on $\Omega^u_{R_1,\theta_1}$ that satisfy the differential equation (2.15) and so that $\{G_j, G_j', \phi_j, \phi_j', \phi_j''\}_{0\le j\le k-1}$ are all defined and bounded on Ω^u as in (2.16).

Now we want to verify the orders in (2.16). When $j = 0$ and $u \in \Omega^u$ we know:

$$G_0(u) = -g_0(u) = \mathrm{O}\left(u^{-\frac{1}{k}}\right) \qquad \text{and} \qquad G_0'(u) = -g_0'(u) = \mathrm{O}\left(u^{-\frac{k+1}{k}}\right).$$

Suppose that the orders on $\{G_l, G_l', \phi_l, \phi_l', \phi_l''\}_{0\le l<j}$ given in (2.16) hold for some $1 \le j \le k-1$. Then for $u \in \Omega^u$,

$$G_j(u) = \mathrm{O}\left(g_j(u), \left\{\phi_l(u), \phi_l'(u)u^{\frac{k+1}{k}}\right\}_{0\le l<j}\right) = \mathrm{O}\left(u^{\frac{j-1}{k}}\right) \qquad \text{and} \qquad G_j'(u) = \mathrm{O}\left(u^{\frac{j-1-k}{k}}\right)$$

where G_j' can be bounded on Ω^u using Cauchy estimates as describe in Remark 2.4. Note that if $j = k-1$, then for $u \in \Omega^u$ and $n \in \mathbb{N}$,

$$\phi_{k-1}^{(n)}(u) = \frac{d^n}{du^n}\int_{u_0}^u G_{k-1}(\nu)d\nu = \mathrm{O}\left(u^{\frac{k-2-(n-1)k}{k}}\right).$$

It remains to show that if, for $0 \le j < k-1$, the orders in (2.16) are satisfied by $\{G_l, G_l', \phi_l, \phi_l', \phi_l''\}_{0\le l<j}$ and hence by $\{G_j, G_j'\}$, then they must also be satisfied by $\phi_j, \phi_j', \phi_j''$. Recall that $\{G_l\}_{0\le l\le j}$ are holomorphic on $\Omega^u_{R_2,\theta_2}$ and we want ϕ_j to be holomorphic on $\Omega^u_{R_1,\theta_1}$. Given any $u \in \Omega^u_{R_1,\theta_1}$, define c_u so that $\mathrm{Arg}(c_u) = \mathrm{Arg}(u)$ and $\mathrm{Re}(c_u) = u_0$. Then $c_u \in \Omega^u_{R_2,\theta_1}$. Parametrize the line segment between c_u and u by $\gamma(t) := tu$, where $\frac{c_u}{u} \le t \le 1$. By using integration by parts once in (2.17) and this parametrization, we can express ϕ_j as:

$$\begin{aligned}
\phi_j(u) &= e^{-\int_{u_0}^u F_j(\nu)d\nu}\left(\frac{G_j(\nu)}{F_j(\nu)}e^{\int_{u_0}^{\nu} F_j(\zeta)d\zeta}\Bigg|_{u_0}^{u} - \int_{u_0}^u \frac{G_j'(\nu)F_j(\nu) - G_j(\nu)F_j'(\nu)}{F_j(\nu)^2}e^{\int_{u_0}^{\nu} F_j(\zeta)d\zeta}d\nu\right) \\
&= \frac{G_j(u)}{F_j(u)} + \mathrm{O}\left(e^{-\int_{u_0}^u F_j(\nu)d\nu}\right) - e^{-\int_{u_0}^u F_j(\nu)d\nu}\int_{c_u}^u \widetilde{G}_j(\nu)e^{\int_{u_0}^{\nu} F_j(\zeta)d\zeta}d\nu \\
&= \frac{G_j(u)}{F_j(u)} + \mathrm{O}\left(e^{-u}\right) - u\int_{\frac{c_u}{u}}^1 \widetilde{G}_j(tu)e^{-u\int_t^1 F_j(\tau u)d\tau}dt
\end{aligned}$$

where $u \in \Omega^u_{R_1,\theta_1}$, $\widetilde{G}_j(u) := \frac{G_j'(u)F_j(u) - G_j(u)F_j'(u)}{F_j(u)^2} = \mathrm{O}\left(u^{\frac{j-1-k}{k}}\right)$, $F_j(u) = \mathrm{O}(1)$, and we are integrating along $\gamma(t)$. Assume $\widetilde{G}_j \not\equiv 0$, otherwise there is nothing left to prove. Then:

$$\begin{aligned}
\left|\int_{\frac{c_u}{u}}^1 \widetilde{G}_j(tu)e^{-u\int_t^1 F_j(\tau u)d\tau}dt\right| &\le \left(1 - \frac{c_u}{u}\right)\max_{\frac{c_u}{u}\le t\le 1}\left(\left|\widetilde{G}_j(tu)\right|e^{-\frac{k-1-j}{k-1}\mathrm{Re}\left(u\int_t^1(1+g_0(\tau u))d\tau\right)}\right) \\
&\le C\max_{\frac{c_u}{u}\le t\le 1}(|u|t)^{\frac{j-1-k}{k}}e^{-\frac{k-1-j}{k-1}\frac{1-t}{2}|u|}
\end{aligned}$$

for some constant $C > 0$. When $t \neq 1$, the exponential term's exponent is a negative multiple of $|u|$ so the exponential term can be bounded above by a constant times an arbitrarily small power of $|u|$ whereas the other term remains bounded above by a constant. When $t = 1$, the integral is bounded above by $C|u|^{\frac{j-1-k}{k}}$. Therefore,

$$\phi_j(u) = \frac{G_j(u)}{F_j(u)} + \mathrm{O}\left(e^{-u}\right) + u\,\mathrm{O}\left(u^{\frac{j-1-k}{k}}\right) = \mathrm{O}\left(u^{\frac{j-1}{k}}\right)$$

on $\Omega^u_{R_1,\theta_1}$. By shrinking the domain of ϕ_j', ϕ_j'' to Ω^u, as discussed in Remark 2.4, we can use Cauchy's estimates and the order of ϕ_j to obtain the desired orders on the derivatives of ϕ_j. In particular, for any $u \in \Omega^u$ and $n \in \mathbb{N}$:

$$\phi_j^{(n)}(u) = \mathrm{O}\left(\frac{u^{\frac{j-1}{k}}}{u^n}\right) = \mathrm{O}\left(u^{\frac{j-1-kn}{k}}\right).$$

If we take $n = 1, 2$ we get the desired results. $\square$

Let $(u, v) \in \Omega^{(u,v)}$. In the expression for w_1 given in (2.14), we want to bound the $\phi_j''(\hat{u})$ terms using u instead of $\hat{u}$. Since $\hat{u}$ is on the line segment between u and u_1, $\exists c > 0$ that is independent of u such that $|\hat{u}| > c|u|$. Combining this with the previous result we get:

$$\phi_j''(\hat{u}) = \mathrm{O}\left(u^{\frac{j-1-2k}{k}}\right).$$

Now that we have bounds on ϕ_j and its derivatives, we can re-write w_1 from (2.14) as:

$$w_1 = w + 1 + \epsilon_1(u, v) + \epsilon_2(u, v),$$

where ϵ_1, ϵ_2 are functions of u, v with the following orders:

$$\epsilon_1(u, v) = \mathrm{O}\left(\frac{\phi_0'(u)}{v}, \frac{\phi_0''(\hat{u})}{v}\right) = \mathrm{O}\left(\frac{1}{u^{\frac{k+1}{k}} v}\right)$$
$$\epsilon_2(u, v) = \mathrm{O}\left(\frac{1}{v^{\frac{k}{k-1}}}\right) + \sum_{j=0}^{k-1} \mathrm{O}\left(\frac{(k-1-j)\phi_j(u)}{v^{\frac{k}{k-1}}}, \frac{j u^{\frac{k+1}{k}} \phi_j'(u)}{v^{\frac{k}{k-1}}}, \frac{j\phi_j''(\hat{u})}{v^{\frac{k+j-1}{k-1}}}\right) = \mathrm{O}\left(\frac{u^{\frac{2k-1}{k}}}{v^{\frac{k}{k-1}}}\right).$$

Proposition 2.8. *For any $(u, v) \in \Omega^{(u,v)}$ and $w = w(u, v)$, we have*

$$\lim_{n\to\infty} \frac{u_n}{\log n} = 1 \qquad \textit{and} \qquad \lim_{n\to\infty} \frac{w_n}{n} = 1.$$

Proof. The sequence $\left(\frac{w_n}{n}\right)$ converges to 1 because, for some constants $C, l > 0$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \left(\epsilon_1(u_j, v_j) + \epsilon_2(u_j, v_j)\right) \leq \lim_{n\to\infty} \frac{1}{n} \sum_{j=l}^{n-1} \left(\frac{C}{(\log j)^{\frac{k+1}{k}} j} + \frac{C(\log j)^{\frac{2k-1}{k}}}{j^{\frac{k+1}{k}}}\right) = 0,$$

where we are using the bounds on u_n, v_n from Proposition 2.6, and

$$\lim_{n\to\infty}\frac{w_n}{n} = \lim_{n\to\infty}\frac{1}{n}\left[w+n+\sum_{j=0}^{n-1}\left(\epsilon_1(u_j,v_j)+\epsilon_2(u_j,v_j)\right)\right] = 1.$$

In order to show $\lim_{n\to\infty}\frac{u_n}{\log n}=1$, we first need to replace the v_j terms in u_n with w_j terms:

$$\frac{1}{v}=\frac{1}{w}\left(1+\sum_{j=0}^{k-1}\frac{\phi_j(u)}{v^{\frac{j}{k-1}}}\right)=\frac{1}{w}\left(1+\phi_0(u)+\mathrm{O}\left(\frac{1}{v^{\frac{1}{k-1}}}\right)\right).$$

We use the bounds on u_n, v_n from Lemma 2.6 to show that the sequence $\left(\frac{u_n}{\log n}\right)$ converges to 1.

$$\begin{aligned}
\lim_{n\to\infty}\frac{u_n}{\log n} &= \lim_{n\to\infty}\frac{1}{\log n}\left[u+\sum_{j=0}^{n-1}\left(\frac{1}{v_j}+\frac{u_j^{\frac{k+1}{k}}}{v_j^{\frac{k}{k-1}}}h(u_j,v_j)+\mathrm{O}\left(\frac{1}{v_j^2}\right)\right)\right]\\
&= \lim_{n\to\infty}\frac{1}{\log n}\sum_{j=0}^{n-1}\frac{1}{v_j} = \lim_{n\to\infty}\frac{1}{\log n}\sum_{j=0}^{n-1}\frac{1}{w_j}\left(1+\phi_0(u_j)\right)\\
&= \lim_{n\to\infty}\frac{1}{\log n}\sum_{j=m}^{n-1}\frac{1}{j}+\lim_{n\to\infty}\frac{1}{\log n}\sum_{j=m}^{n-1}\frac{\phi_0(u_j)}{j}=1,
\end{aligned}$$

where we recall that $h(u,v)=\mathrm{O}(1)$ and we choose $m<n-1$ large enough so that we can replace w_j by j for $j\ge m$. We arrive at the final equality using:

$$\log n-\log m=\int_{m-1}^{n}\frac{dx}{x}\le\sum_{j=m}^{n-1}\frac{1}{j}\le\int_{m-1}^{n-1}\frac{dx}{x}=\log(n-1)-\log(m-1),$$

that $\phi_0(u)=\mathrm{O}\left(u^{-\frac{1}{k}}\right)$, and

$$\begin{aligned}
\frac{1}{\log n}\sum_{j=m}^{n-1}\frac{1}{j(\log j)^{\frac{1}{k}}} &\le \frac{1}{(\log n)^{\frac{1}{2k}}}\sum_{j=m}^{n-1}\frac{1}{j(\log j)^{\frac{2k+1}{2k}}}\\
&\le \frac{1}{(\log n)^{\frac{1}{2k}}}\int_{m-1}^{n-1}\frac{dx}{x(\log x)^{\frac{2k+1}{2k}}}=\mathrm{O}\left(\frac{1}{(\log n)^{\frac{1}{2k}}}\right).
\end{aligned}$$

□

Proposition 2.9. *The sequence $(w_n-n)_{n=0}^{\infty}$ converges uniformly on compact subsets of $\Omega^{(u,v)}$. Let ω be the holomorphic limit function. Then $\omega(u_1,v_1)=\omega(u,v)+1$.*

The function ω is usually referred to as a Fatou coordinate.

Proof. For some $C > 0$ and any $n, m \in \mathbb{N}$ with $n > m$,

$$\begin{aligned}\big|(w_n - n) - (w_m - m)\big| &= \left|\sum_{l=m}^{n-1} [\epsilon_1(u_l, v_l) + \epsilon_2(u_l, v_l)]\right| \le \sum_{l=m}^{n-1}\left(\frac{C}{|u_l|^{\frac{k+1}{k}}|v_l|} + \frac{C|u_l|^{\frac{2k-1}{k}}}{|v_l|^{\frac{k}{k-1}}}\right) \\ &\le \sum_{l=m}^{n-1}\left[\frac{\frac{2C}{\mathrm{Re}(v)}}{\left(\mathrm{Re}(u) + \frac{1}{6}\log\left(1 + \frac{l}{\mathrm{Re}(v)}\right)\right)^{\frac{k+1}{k}}\left(1 + \frac{l}{\mathrm{Re}(v)}\right)} + \frac{C\left(\mathrm{Re}(u) + 3\log\left(1 + \frac{l}{\mathrm{Re}(v)}\right)\right)^{\frac{2k-1}{k}}}{\left(\mathrm{Re}(v) + \frac{l}{2}\right)^{\frac{k}{k-1}}}\right] \\ &\le -12Ck\left(\mathrm{Re}(u) + \frac{1}{6}\log\left(1 + \frac{x}{\mathrm{Re}(v)}\right)\right)^{-\frac{1}{k}}\Bigg|_{x=m-1}^{n-1} + C\frac{\mathrm{Re}(u)^{\frac{2k-1}{k}}}{\mathrm{Re}(v)^{\frac{k}{k-1}}}\int_{m-1}^{n-1}\frac{dx}{\left(1 + \frac{x}{2\,\mathrm{Re}(v)}\right)^{\frac{k+1}{k}}} \\ &\le 12Ck\left(\mathrm{Re}(u) + \frac{1}{6}\log\left(1 + \frac{m-1}{\mathrm{Re}(v)}\right)\right)^{-\frac{1}{k}} + 2Ck\frac{\mathrm{Re}(u)^{\frac{2k-1}{k}}}{\mathrm{Re}(v)^{\frac{1}{k-1}}}\left(1 + \frac{m-1}{2\,\mathrm{Re}(v)}\right)^{-\frac{1}{k}}\end{aligned}$$

where we used bounds from Lemma 2.6. For any compact $K \subset \Omega^{(u,v)}$, $\exists S$ such that $|u|, |v| < S, \forall (u, v) \in K$. So

$$|(w_n - n) - (w_m - m)| < \frac{12Ck}{\left(R + \frac{1}{6}\log\left(1 + \frac{m-1}{S}\right)\right)^{\frac{1}{k}}} + \frac{2CkS^{\frac{2k-1}{k}}R^{-\frac{1}{k-1}}}{\left(1 + \frac{m-1}{2S}\right)^{\frac{1}{k}}}.$$

Therefore $\forall \epsilon > 0, \exists M \in \mathbb{N}$ such that $\forall n, m > M$ and $\forall (u, v) \in K$, $|(w_n - n) - (w_m - m)| < \epsilon$. Hence, the sequence of holomorphic functions $(w_n - n)$ converges uniformly on compact subsets of $\Omega^{(u,v)}$ to a holomorphic limit function:

$$\omega(u, v) = \lim_{n\to\infty}(w_n - n) = \lim_{n\to\infty}\sum_{j=0}^{n-1}(w_{j+1} - w_j - 1) + w.$$

Finally we show that $\omega \circ f_1 = \omega + 1$ on $\Omega^{(u,v)}$:

$$\begin{aligned}\omega(u_1, v_1) &= \sum_{j=0}^{\infty}(w_{j+2} - w_{j+1} - 1) + w_1 \\ &= \sum_{j=0}^{\infty}(w_{j+1} - w_j - 1) + w + 1 \\ &= \omega(u, v) + 1\end{aligned}$$

□

Define the function η as follows for any $(u, v) \in \Omega^{(u,v)}$:

$$\eta(u, v) := \omega(u, v) - w = \lim_{n\to\infty}((w_n - n) - (w - 0)) \tag{2.18}$$

From the proof of Proposition 2.9, we can bound $\eta(u,v)$ for any $(u,v) \in \Omega^{(u,v)}$:

$$|\eta(u,v)| \leq 12Ck\,(\mathrm{Re}(u)-1)^{-\frac{1}{k}} + 4Ck\,\mathrm{Re}(u)^{\frac{2k-1}{k}}\,\mathrm{Re}(v)^{-\frac{1}{k-1}}. \tag{2.19}$$

Now we will show that the following map is a coordinate change from coordinates (u,v) to (u,ω):

$$\psi_1(u,v) := (u, \omega(u,v)).$$

First we need to define several domains that contain $\Omega^{(u,v)}$. Recall the definition of $\Omega^{(u,v)}_{R',\theta',\delta'}$ from (2.3). We choose appropriate constants $0 \ll R_2 < R_1 < R, 0 < 2\delta < \delta_2 \ll 1,$ and $0 < 3\theta < \frac{\pi}{4}$ so that

$$\Omega^{(u,v)} \subsetneq \Omega_1 := \Omega^{(u,v)}_{R_1,2\theta,2\delta} \subsetneq \Omega_2 := \Omega^{(u,v)}_{R_2,3\theta,\delta_2}$$

and all of these domains have been shown to satisfy the properties depicted thus far for $\Omega^{(u,v)}$.

Proposition 2.10. *Let*

$$\Omega_2^{(u,\omega)} := \left\{ (u,\omega) \;\middle|\; \mathrm{Re}(u) > R, |u|^{\frac{(k-1)(k+1)}{k}} < \delta|\omega|, |\operatorname{Arg}(u)| < \theta, |\operatorname{Arg}(\omega)| < \frac{k-1}{k}\theta \right\}$$
$$\Omega_2^{(u,v)} := \Omega_1 \cap \psi_1^{-1}\left(\Omega_2^{(u,\omega)}\right) = \Omega^{(u,v)}_{R_1,2\theta,2\delta} \cap \psi_1^{-1}\left(\Omega_2^{(u,\omega)}\right)$$

Then $\psi_1 : \Omega_2^{(u,v)} \to \Omega_2^{(u,\omega)}$ is a biholomorphism.

Proof. First of all, ψ_1 is holomorphic on Ω_2 since it is holomorphic in each component. Now we want to find a domain on which ψ_1 is injective. For any $(u,v), (\hat{u},\hat{v}) \in \Omega^{(u,v)}$,

$$\psi_1(u,v) = \psi_1(\hat{u},\hat{v}) \Leftrightarrow u = \hat{u} \text{ and } \omega(u,v) = \omega(\hat{u},\hat{v}).$$

Fix $u_0 \in \Omega^u$ and let $\omega_{u_0}(v) := \omega(u_0, v)$. Define

$$\begin{aligned} \Omega_{2,u_0} &:= \{v \in \mathbb{C} \mid (u_0,v) \in \Omega_2\} \\ \Omega_{1,u_0} &:= \{v \in \mathbb{C} \mid (u_0,v) \in \Omega_1\} \\ \Omega_{u_0} &:= \left\{v \in \mathbb{C} \mid (u_0,v) \in \Omega^{(u,v)}\right\} \end{aligned}$$

Then ω_{u_0} is holomorphic on Ω_{2,u_0}. Fix any $y \in \Omega_{u_0}$. For any $v \in \Omega_{2,u_0}$, let

$$g(v) := \omega_{u_0}(v) - y \qquad \text{and} \qquad h(v) := v - y.$$

Then g, h are holomorphic on Ω_{2,u_0}. Let γ be the curve that is the boundary of the region:

$$\Omega_{1,u_0} \cap \{v \in \mathbb{C} \mid |v| < 2|y|\} = \left\{ v \in \mathbb{C} \;\middle|\; (2\delta)^{-1}|u_0|^{\frac{k^2-1}{k}} < |v| < 2|y| \text{ and } |\operatorname{Arg}(v)| < 2\theta \right\}.$$

The point y lies inside the region bounded by the curve γ. Note that:

$$|g(v)-h(v)| = \left|\sum_{j=0}^{k-1} \frac{\phi_j(u_0)}{v^{\frac{j}{k-1}}} + \frac{\eta(u_0,v)}{v}\right| |v| < \frac{c|v|}{|u_0|^{\frac{1}{k}}}$$

for some constant $c > 0$. We want to show that on γ:

$$|g(v)-h(v)| < |h(v)|, \text{ which we can prove by showing that } \frac{c}{|u_0|^{\frac{1}{k}}} < \frac{|h(v)|}{|v|} = \frac{|v-y|}{|v|}.$$

We bound $\frac{|h(v)|}{|v|}$ from below on the segments of γ:

$$(1)\ |v| = (2\delta)^{-1}|u_0|^{\frac{k^2-1}{k}}, \qquad (2)\ |v| = 2|y|, \qquad (3)\ |\operatorname{Arg}(v)| = 2\theta.$$

On the first segment of γ:

$$|h(v)| = |v-y| \geq |y| - |v| > \left(\delta^{-1} - (2\delta)^{-1}\right)|u_0|^{\frac{k^2-1}{k}} = (2\delta)^{-1}|u_0|^{\frac{k^2-1}{k}} = |v|$$

On the second segment of γ:

$$|h(v)| = |v-y| \geq |v| - |y| = |y| = \frac{1}{2}|v|.$$

On the third segment of γ, $|\operatorname{Arg}(v)| = 2\theta$. Fix any v on this segment and without loss of generality assume $\operatorname{Arg}(v) = 2\theta$. The distance $|v-y|$ is greater than the shortest distance from v to the line of angle θ from the origin, therefore

$$|h(v)| = |v-y| \geq |v|\sin(\theta).$$

Hence on γ we have:

$$\frac{|h(v)|}{|v|} = \frac{|v-y|}{|v|} \geq \min\left\{1, \frac{1}{2}, \sin\theta\right\} = \sin\theta.$$

By requiring that $R > \left(\frac{c}{\sin\theta}\right)^k$, we get

$$|g(v)-h(v)| < |h(v)| = |v-y| \text{ on } \gamma.$$

From this inequality, we know that neither $g(v)$ nor $h(v)$ has a zero on γ. The region Ω_{2,u_o} contains the closed curve γ and g, h are holomorphic on Ω_{2,u_o} with no zeros on γ, so we can extend the closed, connected set bounded by γ to a region slightly larger that contains no extra zeros of g or h. By Rouché's theorem, g and h have the same number of zeros on this region. Since $h(v) = v - y$ has exactly one zero in this region, $g(v)$ must as well. Note that if $v \in \Omega_{1,u_o}$ and $|v| \geq 2|y|$, then it is not possible for $\omega(u_o, v) = y$ (we can see this from the calculation for y in terms of v). Therefore g is injective on Ω_{1,u_0} for any $u_0 \in \Omega^u$ and so ψ_1 is injective on Ω_1. Furthermore, $\forall (u_o, y) \in \Omega_2^{(u,\omega)}$ we know that $(u_o, y) \in \Omega^{(u,v)}$ and so there exists a unique element $(u_o, v) \in \Omega_1$ such that $\psi_1(u_o, v) = (u_o, y)$. Therefore, $\psi_1 : \Omega_2^{(u,v)} \to \Omega_2^{(u,\omega)}$ is a biholomorphism. $\square$

2.4. Conjugacy to translation

In this section we make a coordinate change so that composition with f acts as the identity map on the first component. We can re-write (2.10) and (2.18) for any $(u,v)\in\Omega^{(u,v)}$ as:

$$\omega = w+\eta(u,v) = v\left(1+\sum_{l=0}^{k-1}\frac{\phi_l(u)}{v^{\frac{l}{k-1}}}+\frac{\eta(u,v)}{v}\right) \tag{2.20}$$

$$\frac{1}{v}=\frac{1}{\omega}\left(1+\sum_{l=0}^{k-1}\frac{\phi_l(u)}{v^{\frac{l}{k-1}}}+\frac{\eta(u,v)}{v}\right)=\mathrm{O}\left(\frac{1}{\omega}\right)$$

$$u_1 = u+\frac{1}{v}+\frac{u^{\frac{k+1}{k}}}{v^{\frac{k}{k-1}}}h(u,v)+\mathrm{O}\left(\frac{1}{v^2}\right)$$

where $h=\mathrm{O}(1)$. Let $f_2 := \psi_1\circ f_1\circ\psi_1^{-1}$. Then for any $(u,\omega)\in\Omega_2^{(u,\omega)}$, we can express $f_2(u,\omega)=(u_1,\omega_1)$ as:

$$u_1 = u+\frac{1}{\omega}+\frac{\phi_0(u)}{\omega}+\sum_{l=1}^{k-1}\mathrm{O}\left(\frac{\phi_l(u)}{\omega^{\frac{l+k-1}{k-1}}}\right)+\mathrm{O}\left(\frac{u^{\frac{k+1}{k}}}{\omega^{\frac{k}{k-1}}},\frac{1}{\omega^2}\right) \tag{2.21}$$

$$= u+\frac{1}{\omega}+\frac{\phi_0(u)}{\omega}+\mathrm{O}\left(\frac{u^{\frac{k+1}{k}}}{\omega^{\frac{k}{k-1}}}\right)$$

$$\omega_1 = \omega+1, \tag{2.22}$$

where we use (2.16) and (2.19) to bound $\{\phi_l\}$ and η. By employing the same techniques as in the proof of Lemma 2.5, we can show that if $(u,\omega)\in\Omega_2^{(u,\omega)}$, then $(u_1,\omega_1)\in\Omega_2^{(u,\omega)}$. Since ψ is surjective, if $(u,\omega)\in\Omega_2^{(u,\omega)}$, then $\exists(u,v)\in\Omega_2^{(u,v)}$ such that $\psi_1(u,v)=(u,\omega)$ and $\psi_1(u_1,v_1)=(u_1,\omega_1)$. We can use (2.20) to roughly bound $\operatorname{Re} v$ by $\operatorname{Re}\omega$: $\frac{\operatorname{Re}\omega}{2}<\operatorname{Re} v<2\operatorname{Re}\omega$. Combining this with (2.7), we derive the inequality:

$$3\log\left(1+\frac{2n}{\operatorname{Re}(\omega)}\right)\geq \operatorname{Re}(u_n)-\operatorname{Re}(u)\geq\frac{1}{6}\log\left(1+\frac{n}{2\operatorname{Re}(\omega)}\right). \tag{2.23}$$

Proposition 2.11. *Fix some $u_0\in\pi_1\left(\Omega_2^{(u,\omega)}\right)$ and let $1\leq l\leq k$. Define:*

$$\alpha_l(u) := \int_{u_0}^{u}\phi_0^l(\zeta)d\zeta,\quad \alpha(u) := \sum_{l=1}^{k}(-1)^l\alpha_l(u),\quad \text{and}\quad t_n := u_n-\log\omega_n+\alpha(u_n) \tag{2.24}$$

for any $(u,\omega)\in\Omega_2^{(u,\omega)}$ and $n\in\mathbb{N}$. Then the sequence $(t_n)_{n=0}^{\infty}$ converges uniformly on compact subsets of $\Omega_2^{(u,\omega)}$. Let τ be the holomorphic limit function of the sequence. Then $\tau(u_1,\omega_1)=\tau(u,\omega)$.

Proof. For any $(u,\omega) \in \Omega_2^{(u,\omega)}$ and $j \in \mathbb{N}$, we use Taylor series expansion as in (2.12) to get:

$$\alpha_l(u_{j+1}) = \alpha_l(u_j) + \alpha_l'(u_j)(u_{j+1} - u_j) + \int_{u_j}^{u_{j+1}} \alpha_l''(\zeta)(u_{j+1} - \zeta) d\zeta, \tag{2.25}$$

where we are integrating along the line $\gamma_{u_j}(t) = (1-t)u_j + tu_{j+1}$ for $0 \le t \le 1$, which is contained in $\pi_1\left(\Omega_2^{(u,\omega)}\right) = \Omega^u$ by (2.5). After substituting the definition of α_l into (2.25), we get:

$$\int_{u_j}^{u_{j+1}} \phi_0^l(\zeta) d\zeta = \phi_0^l(u_j)(u_{j+1} - u_j) + \int_{u_j}^{u_{j+1}} l\phi_0^{l-1}(\zeta)\phi_0'(\zeta)(u_{j+1} - \zeta) d\zeta. \tag{2.26}$$

Let $\hat{t} \in [0,1]$ be such that:

$$\left|\phi_0^{l-1}\left(\gamma_{u_j}\left(\hat{t}\right)\right)\phi_0'\left(\gamma_{u_j}\left(\hat{t}\right)\right)\right| = \max_{0 \le t \le 1}\left|\phi_0^{l-1}\left(\gamma_{u_j}(t)\right)\phi_0'\left(\gamma_{u_j}(t)\right)\right|,$$

and let $\hat{u} := \gamma_{u_j}(\hat{t})$. Since $\hat{u}$ is on the line between u_j and u_{j+1} we can bound $\hat{u}$ so that $C|u_j| > |\hat{u}| > c|u_j|$ for some constants $C > c > 0$ that are independent of u and j. Then we can bound the integral from (2.26) as follows:

$$\int_{u_j}^{u_{j+1}} l\phi_0^{l-1}(\zeta)\phi_0'(\zeta)(u_{j+1} - \zeta) d\zeta = \mathrm{O}\left((u_{j+1} - u_j)^2 \phi_0^{l-1}(\hat{u})\phi_0'(\hat{u})\right) = \mathrm{O}\left(\omega_j^{-2} u_j^{-\frac{k+l}{k}}\right)$$

Combining this bound with (2.26) we get:

$$\int_{u_j}^{u_{j+1}} \phi_0^l(\zeta) d\zeta = \phi_0^l(u_j)(u_{j+1} - u_j) + \mathrm{O}\left(u_j^{-\frac{k+l}{k}} \omega_j^{-2}\right).$$

For any $n, m \in \mathbb{N}$ with $n > m$,

$$\begin{aligned}
u_n - u_m &= \sum_{j=m}^{n-1}\left(\frac{1}{\omega_j} + \frac{\phi_0(u_j)}{\omega_j} + \mathrm{O}\left(\frac{u_j^{\frac{k+1}{k}}}{\omega_j^{\frac{k}{k-1}}}\right)\right) \\
\alpha(u_n) - \alpha(u_m) &= \sum_{j=m}^{n-1}\left(\alpha(u_{j+1}) - \alpha(u_j)\right) = \sum_{j=m}^{n-1}\sum_{l=1}^{k}(-1)^l \int_{u_j}^{u_{j+1}} \phi_0^l(\zeta) d\zeta \\
&= \sum_{j=m}^{n-1}\sum_{l=1}^{k}(-1)^l \left(\phi_0^l(u_j)\left(\frac{1}{\omega_j} + \frac{\phi_0(u_j)}{\omega_j}\right) + \mathrm{O}\left(\frac{u_j^{\frac{k+1}{k}}}{\omega_j^{\frac{k}{k-1}}}\right)\right)
\end{aligned}$$

Now we bound $|t_n - t_m|$ using the previous equations:

$$
\begin{aligned}
\big|t_n - t_m\big| &= \Big|(u_n - u_m) - (\log\omega_n - \log\omega_m) + (\alpha(u_n) - \alpha(u_m))\Big| \\
&\le \left|\sum_{j=m}^{n-1}\left(\sum_{l=0}^{k}(-1)^l\phi_0^l(u_j)\left(\frac{1}{\omega_j} + \frac{\phi_0(u_j)}{\omega_j}\right) - \log\left(\frac{\omega_{j+1}}{\omega_j}\right)\right)\right| + c\sum_{j=m}^{n-1}\frac{|u_j|^{\frac{k+1}{k}}}{|\omega_j|^{\frac{k}{k-1}}} \\
&\le \left|\sum_{j=m}^{n-1}\left(\frac{1}{\omega_j} - \log\left(1 + \frac{1}{\omega_j}\right)\right)\right| + c\sum_{j=m}^{n-1}\left(\frac{|u_j|^{\frac{k+1}{k}}}{|\omega_j|^{\frac{k}{k-1}}} + \frac{1}{|u_j|^{\frac{k+1}{k}}|\omega_j|}\right) \\
&\le c\sum_{j=m}^{n-1}\left(\frac{|u_j|^{\frac{k+1}{k}}}{|\omega_j|^{\frac{k}{k-1}}} + \frac{2}{|u_j|^{\frac{k+1}{k}}|\omega_j|}\right) \\
&\le \frac{2c\operatorname{Re}(u)^{\frac{k+1}{k}}}{\operatorname{Re}(\omega)^{\frac{k}{k-1}}}\sum_{j=m}^{n-1}\frac{\left(1 + \frac{3}{\operatorname{Re}(u)}\log\left(1 + \frac{2j}{\operatorname{Re}(\omega)}\right)\right)^{\frac{k+1}{k}}}{\left(1 + \frac{j}{\operatorname{Re}(\omega)}\right)^{\frac{k}{k-1}}} \\
&\quad + 2c\sum_{j=m}^{n-1}\left[\left(\operatorname{Re}(u) + \frac{1}{6}\log\left(1 + \frac{j}{2\operatorname{Re}(\omega)}\right)\right)^{\frac{k+1}{k}}\left(1 + \frac{j}{2\operatorname{Re}(\omega)}\right)\operatorname{Re}(\omega)\right]^{-1} \\
&\le \frac{2c\operatorname{Re}(u)^{\frac{k+1}{k}}}{\operatorname{Re}(\omega)^{\frac{k}{k-1}}}\int_{m-1}^{n-1}\left[1 + \frac{x}{\operatorname{Re}(\omega)}\right]^{-\frac{k+1}{k}}dx \\
&\quad - 24kc\left[\operatorname{Re}(u) + \frac{1}{6}\log\left(1 + \frac{x}{2\operatorname{Re}(\omega)}\right)\right]^{-\frac{1}{k}}\Bigg|_{x=m-1}^{n-1} \\
&\le 2kc\frac{\operatorname{Re}(u)^{\frac{k+1}{k}}}{\operatorname{Re}(\omega)^{\frac{1}{k-1}}}\left(1 + \frac{m-1}{\operatorname{Re}(\omega)}\right)^{-\frac{1}{k}} + 24kc\left(\operatorname{Re}(u) + \frac{1}{6}\log\left(1 + \frac{m-1}{2\operatorname{Re}(\omega)}\right)\right)^{-\frac{1}{k}}
\end{aligned}
$$

for some constant $c > 1$ independent of (u, ω). For any compact $K \subset \Omega_2^{(u,\omega)}$, $\exists S$ such that $|u|, |v| < S, \forall (u, v) \in K$. Then

$$
\big|t_n - t_m\big| < \frac{2kcS^{\frac{k+1}{k}}}{R^{\frac{1}{k-1}}}\left(1 + \frac{m-1}{S}\right)^{-\frac{1}{k}} + 24kc\left(R + \frac{1}{6}\log\left(1 + \frac{m-1}{2S}\right)\right)^{-\frac{1}{k}}.
$$

Therefore $\forall\epsilon > 0, \exists M \in \mathbb{N}$ such that $\forall n, m > M$ and $\forall (u, v) \in K$, $|t_n - t_m| < \epsilon$. Hence, the sequence of holomorphic functions (t_n) converges uniformly on compact subsets of $\Omega_2^{(u,\omega)}$ to the limit function τ, which also must be holomorphic on $\Omega_2^{(u,\omega)}$. So

$$
\tau(u, \omega) = \lim_{n\to\infty} t_n = \sum_{j=0}^{\infty}(t_{j+1} - t_j) + t.
$$

Finally we show that $\tau \circ f_2 = \tau$ on $\Omega_2^{(u,\omega)}$:

$$
\tau(u_1, \omega_1) = \sum_{j=0}^{\infty}(t_{j+2} - t_{j+1}) + t_1 - (t - t) = \sum_{j=0}^{\infty}(t_{j+1} - t_j) + t = \tau(u, \omega)
$$

$\square$

Now we show that the following map is a coordinate change:

$$\psi_2(u,\omega) := (\tau(u,\omega),\omega).$$

Define the function μ as follows for any $(u,\omega)\in\Omega_2^{(u,\omega)}$:

$$\mu(u,\omega) := \tau(u,\omega) - u + \log\omega - \alpha(u) = \lim_{n\to\infty}(t_n - t)\,. \tag{2.27}$$

We can bound $\mu(u,\omega)$ for any $(u,\omega)\in\Omega_2^{(u,\omega)}$ using the proof of Proposition 2.11 and $m=0$:

$$|\mu(u,\omega)| \le 4kc\frac{\mathrm{Re}(u)^{\frac{k+1}{k}}}{\mathrm{Re}(\omega)^{\frac{1}{k-1}}} + \frac{24kc}{(\mathrm{Re}(u)-1)^{\frac{1}{k}}}. \tag{2.28}$$

Let $\Omega_{R',\delta',\theta'}^{(u,\omega)}$ be defined as:

$$\left\{(u,\omega)\;\middle|\; \mathrm{Re}(u) > R', |u|^{\frac{(k-1)(k+1)}{k}} < \delta'|\omega|, |\operatorname{Arg}(u)| < \theta', |\operatorname{Arg}(\omega)| < \frac{k-1}{k}\theta'\right\} \tag{2.29}$$

for any R',δ',θ'. To simplify notation, replace (R,δ,θ) by (R_2,δ_2,θ_2) in the preceding work so that $\Omega_2^{(u,\omega)} = \Omega_{R_2,\delta_2,\theta_2}^{(u,\omega)}$. Given (R_2,δ_2,θ_2), we choose appropriate constants $R_2 < R_1 < R$, $0<\delta<\delta_1<\delta_2$, and $0<\theta<\theta_1<\theta_2$ so that

$$\Omega_0 := \Omega_{R,\delta,\theta}^{(u,\omega)} \subsetneq \Omega_1 := \Omega_{R_1,\delta_1,\theta_1}^{(u,\omega)} \subsetneq \Omega_2^{(u,\omega)}$$

and all of these domains have satisfied the properties shown thus far for $\Omega_2^{(u,\omega)}$.

Proposition 2.12. *Let*

$$\Omega_0^{(\tau,\omega)} := \left\{(\tau-\log(\omega),\omega)\;\middle|\; \mathrm{Re}(\tau) > R, |\tau| < \delta|\omega|^{\frac{k}{(k-1)(k+1)}}, |\operatorname{Arg}(\tau)| < \theta, |\operatorname{Arg}(\omega)| < \frac{k-1}{k}\theta\right\}$$
$$\Omega_0^{(u,\omega)} := \Omega_1 \cap \psi_2^{-1}\left(\Omega_0^{(\tau,\omega)}\right)$$

Then $\psi_2 : \Omega_0^{(u,\omega)} \to \Omega_0^{(\tau,\omega)}$ is a biholomorphism onto its image.

Proof. We use a similar strategy as in the proof of Proposition 2.10. ψ_2 is holomorphic on $\Omega_2^{(u,\omega)}$ since it is holomorphic in each component. For any $(u,w),(\hat{u},\hat{w})\in\Omega_2^{(u,\omega)}$,

$$\psi_2(u,w) = \psi_2(\hat{u},\hat{w}) \Leftrightarrow \tau(u,w) = \tau(\hat{u},\hat{w}) \text{ and } w = \hat{w}.$$

Fix $w_0\in\pi_2(\Omega_0)$ and let $\tau_{w_0}(u) := \tau(u,w_0)$. Define

$$\Omega_{2,w_0} := \left\{u\in\mathbb{C} \mid (u,w_0)\in\Omega_2^{(u,\omega)}\right\}$$
$$\Omega_{1,w_0} := \{u\in\mathbb{C} \mid (u,w_0)\in\Omega_1\}$$
$$\Omega_{0,w_0} := \{u\in\mathbb{C} \mid (u,w_0)\in\Omega_0\}$$

Then τ_{w_0} is holomorphic on Ω_{2,w_0}. Fix any $y \in \Omega_{0,w_0}$. Let

$$g(u) := \tau_{w_0}(u) + \log(w_0) - y \qquad \text{and} \qquad h(u) := u - y$$

Then g, h are holomorphic on Ω_{2,w_0}. Let γ be the curve that is the boundary of the region:

$$\Omega_{1,w_0} = \left\{ u \in \mathbb{C} \;\middle|\; \mathrm{Re}(u) > R_1, |u| < (\delta_1 |w_0|)^{\frac{k}{(k-1)(k+1)}}, |\operatorname{Arg}(u)| < \theta_1 \right\}.$$

The point y lies inside the curve γ. Using (2.16), we bound α:

$$\alpha(u) = \sum_{l=1}^{k} (-1)^l \int_{u_0}^{u} \phi_0^l(\zeta) d\zeta = \mathrm{O}\left(\int_{u_0}^{u} \phi_0(\zeta) d\zeta \right) = \mathrm{O}\left(u^{\frac{k-1}{k}} \right).$$

Then for $u \in \Omega_{2,w_0}$, using this bound and the one on μ given in (2.28), we get:

$$|g(u) - h(u)| = |\mu(u, w_0) + \alpha(u)| < c|u|^{1-\frac{1}{k}} < |u|^{1-\frac{1}{k^2-1}},$$

for some $c > 0$ and R_2 large enough. We want to show that on γ:

$$|g(u) - h(u)| < |h(u)|.$$

We bound $|h(u)|$ from below on the segments of γ:

$$(1)\ \mathrm{Re}(u) = R_1, \qquad (2)\ |u| = (\delta_1 |w_0|)^{\frac{k}{(k-1)(k+1)}}, \qquad (3)\ |\operatorname{Arg}(u)| = \theta_1.$$

On the first segment of γ:

$$|h(u)| \geq \left(\frac{|y|}{|u|} - 1 \right) |u| > \left(\frac{R_0}{\sqrt{2} R_1} - 1 \right) |u| \geq |u|^{1-\frac{1}{k^2-1}},$$

where the last inequality follows if we assume that $R_0 \geq \sqrt{2} R_1 \left(1 + R_1^{-\frac{1}{k^2-1}} \right)$.
On the second segment of γ:

$$|h(u)| \geq |u| - |y| > |u| \left(1 - \left(\frac{\delta_0}{\delta_1} \right)^{\frac{k}{k^2-1}} \right) > |u|^{1-\frac{1}{k^2-1}},$$

where the last inequality follows if we assume that $R_1 \geq \left(1 - \left(\frac{\delta_0}{\delta_1} \right)^{\frac{k}{k^2-1}} \right)^{-(k^2-1)}$.
On the third segment of γ:

$$|h(u)| = |u - y| > |u| \sin(\theta_1 - \theta_0) \geq |u|^{1-\frac{1}{k^2-1}},$$

where the last inequality follows if we assume that $R_1 \geq (\sin(\theta_1 - \theta_0))^{-(k^2-1)}$.
Given $\{\theta_j, \delta_j\}_{0 \leq j \leq 2}$, we can choose $\{R_j\}_{0 \leq j \leq 2}$ large enough. Hence

$$|h(u)| > |g(u) - h(u)| \text{ on } \gamma.$$

From this inequality, we know that neither g nor h has a zero on γ. Since the region Ω_{2,w_0} contains the closed curve γ and both g and h are holomorphic on Ω_{2,w_0} with no zeros on γ, we can extend the closed, connected set bounded by γ to a region slightly larger that contains no extra zeros of g or h. By Rouché's theorem, g and h have the same number of zeros on this region; h has exactly one zero in this region, hence so does g. So $\forall (y, w_0) \in \Omega_0$, there is a unique element $u \in \Omega_{1,w_0}$ such that $\tau(u, w_0) = y - \log(w_0)$. Consequently, for any $(y - \log(w_0), w_0) \in \Omega_0^{(\tau,\omega)}$ there is a unique $u \in \Omega_{1,w_0}$ such that $\psi_2(u, w_0) = (\tau(u, w_0), w_0) = (y - \log(w_0), w_0)$. Therefore $\psi_2 : \Omega_0^{(u,\omega)} \to \Omega_0^{(\tau,\omega)}$ is a biholomorphism. $\square$

Let

$$\begin{aligned} \Omega_0^{(u,v)} &:= \psi_1^{-1}\left(\Omega_0^{(u,\omega)}\right), \qquad \Omega_0^{(x,y)} := \psi_0^{-1}\left(\Omega_0^{(u,v)}\right), \qquad \Omega := l\left(\Omega_0^{(x,y)}\right) \\ \Psi &:= \psi_2 \circ \psi_1 \circ \psi_0, \qquad \text{and} \qquad f_3 := \Psi \circ f_0 \circ \Psi^{-1}. \end{aligned} \tag{2.30}$$

Lemma 2.13. *$f_3\left(\Omega_0^{(\tau,\omega)}\right) \subseteq \Omega_0^{(\tau,\omega)}$ and $f_0\left(\Omega_0^{(x,y)}\right) \subseteq \Omega_0^{(x,y)}$.*

Proof. For any $(\hat{\tau}, \omega) \in \Omega_0^{(\tau,\omega)}$, let $\tau := \hat{\tau} + \log(\omega)$. Then

$$f_3\left(\hat{\tau}, \omega\right) = (\hat{\tau}, \omega + 1) = (\tau - \log(\omega), \omega + 1).$$

Let

$$t := \hat{\tau} + \log(\omega + 1) = \tau + \log(1 + \omega^{-1}).$$

To show that $(\hat{\tau}, \omega + 1) = (t - \log(\omega + 1), \omega + 1) \in \Omega_0^{(\tau,\omega)}$, we need:

$$(1)\ \operatorname{Re}(t) > R, \qquad (2)\ |t| < (\delta|\omega + 1|)^{\frac{k}{(k-1)(k+1)}}, \qquad (3)\ |\operatorname{Arg}(t)| < \theta.$$

First of all,

$$\operatorname{Re}(t) = \operatorname{Re}(\tau) + \log\left|1 + \omega^{-1}\right| > R.$$

Secondly,

$$|t| = |\tau|\left|1 + \tau^{-1}\log\left(1 + \omega^{-1}\right)\right| < (\delta|\omega|)^{\frac{k}{(k-1)(k+1)}}\left|1 + \tau^{-1}\log\left(1 + \omega^{-1}\right)\right| < (\delta|\omega + 1|)^{\frac{k}{(k-1)(k+1)}},$$

where the last inequality follows from the Taylor series expansions:

$$\begin{aligned} 1 + \tau^{-1}\log\left(1 + \omega^{-1}\right) &= 1 + \tau^{-1}\omega^{-1} + \mathrm{O}\left(\tau^{-1}\omega^{-2}\right) \\ \left(1 + \omega^{-1}\right)^{\frac{k}{(k-1)(k+1)}} &= 1 + \frac{k}{k^2 - 1}\omega^{-1} + \mathrm{O}\left(\omega^{-2}\right) \end{aligned}$$

Finally,

$$|\operatorname{Arg}(t)| \leq \max\left\{|\operatorname{Arg}(\tau)|, \left|\operatorname{Arg}\log\left(1 + \omega^{-1}\right)\right|\right\} < \theta,$$

because

$$\left|\operatorname{Arg}\log\left(1+\omega^{-1}\right)\right| = \left|\operatorname{Arg}(\omega^{-1}) + \operatorname{Arg}\left(1-\omega^{-1}\left(\frac{1}{2}+\mathrm{O}\left(\omega^{-1}\right)\right)\right)\right|$$
$$< \left|\operatorname{Arg}(\omega)\right| < \theta.$$

Note that the last inequality follows because $\operatorname{Arg}(\omega^{-1})$ and $\operatorname{Arg}(1-\omega^{-1})$ have opposite signs. Therefore,

$$f_3\left(\Omega_0^{(\tau,\omega)}\right) \subset \Omega_0^{(\tau,\omega)} \qquad \Rightarrow \qquad f_0\left(\Omega_0^{(x,y)}\right) \subset \Omega_0^{(x,y)}.$$

□

Also the origin is in the boundary of $\Omega = l\left(\Omega_0^{(x,y)}\right)$, where l is the bilinear map (2.2). Given any $(x,y) \in \Omega_0^{(x,y)}$, we showed that $(x_n,y_n) \in \Omega_0^{(x,y)}, \forall n \in \mathbb{N}$. Since $(x_n,y_n) = \psi_0^{-1}(u_n,v_n)$ and $|u_n|,|v_n| \to \infty$, the sequence $\{(x_n,y_n)\}$ converges to the origin and so $(0,0) \in \partial\Omega_0^{(x,y)} \cap \partial\Omega$.

To summarize, we have shown that there is a domain Ω with the origin in its boundary that is biholomorphic to $\Omega_0^{(\tau,\omega)} \subset \mathbb{C}^2$ and on the latter, f acts as the identity on the first coordinate and translation on the second. Figure 2.1 illustrates all of the coordinate changes we performed to get that result.

2.5. Fatou-Bieberbach domain

Now that we have finished demonstrating Theorem A, we turn to Theorem B. In order to apply our previous results, we want to assume that f as in Theorem A is an automorphism. From Theorem 1.24, due independently to Weickert [W1] and Buzzard-Forstneric [BF], we know there exist automorphisms of $\mathbb{C}^2$ that approximate f very closely near the origin. In the power series expansions of f and f_0 near the origin, we have only explicitly used terms up to degree at most $2k-1$. By Theorem 1.24, there is an automorphism whose Taylor series expansion near the origin agrees with that of f up through its degree $2k$ terms. We now assume that f is an automorphism of $\mathbb{C}^2$ since it is possible to have an automorphism f as in Theorem A. Then $f_0 = l^{-1}\circ f\circ l$ is an automorphism with the same general form as before in (2.1). Let

$$\Sigma_0^{(x,y)} := \bigcup_{n\geq 0} f_0^{-n}\left(\Omega_0^{(x,y)}\right) \qquad \text{and} \qquad \Sigma := l\left(\Sigma_0^{(x,y)}\right) = \bigcup_{n\geq 0} f^{-n}\left(l\left(\Omega_0^{(x,y)}\right)\right).$$

We can extend the domains of definition of ω and τ to $\Sigma_0^{(x,y)}$:

$$\tau\circ\psi_1\circ\psi_0(x,y) := \tau\circ\psi_1\circ\psi_0(x_n,y_n),$$
$$\omega\circ\psi_0(x.y) := \omega\circ\psi_0(x_n,y_n) - n,$$

$$
\begin{tikzcd}
(z,w) \in \arrow[d, mapsto, "l^{-1}"'] & \Omega \arrow[r, hook, "f"] \arrow[d, "l^{-1}"', "\text{bilinear}"] & \Omega \arrow[d, "l^{-1}"'] & \ni (z_1,w_1) \arrow[d, mapsto] & \\
(x,y) \in \arrow[ddd, mapsto, "\Psi"'] & \Omega_0^{(x,y)} \arrow[r, hook, "f_0"] \arrow[d, "\psi_0"', "\text{biholo.}"] & \Omega_0^{(x,y)} \arrow[r, hook, "\ni (x_1,y_1)", "i"'] \arrow[d, "\psi_0"'] & \psi_0^{-1}\left(\Omega_2^{(u,v)}\right) \arrow[r, hook, "i"] \arrow[d, "\psi_0"'] & \psi_0^{-1}\left(\Omega^{(u,v)}\right) \arrow[d, "\psi_0"'] \\
 & \Omega_0^{(u,v)} \arrow[r, hook, "f_1"] \arrow[d, "\psi_1"', "\text{biholo.}"] & \Omega_0^{(u,v)} \arrow[r, hook, "i"] \arrow[d, "\psi_1"'] & \Omega_2^{(u,v)} \arrow[r, hook, "i"] \arrow[d, "\psi_1"'] & \Omega^{(u,v)} \\
 & \Omega_0^{(u,\omega)} \arrow[r, hook, "f_2"] \arrow[d, "\psi_2"', "\text{biholo.}"] & \Omega_0^{(u,\omega)} \arrow[r, hook, "i"] \arrow[d, "\psi_2"'] & \Omega_2^{(u,\omega)} & \\
(\tau,\omega) \in & \Omega_0^{(\tau,\omega)} \arrow[r, hook, "f_3"] & \Omega_0^{(\tau,\omega)} \ni (\tau,\omega+1) & &
\end{tikzcd}
$$

Figure 2.1: Coordinate changes performed for Theorem A.

where $f_0^n(x,y) = (x_n, y_n) \in \Omega_0^{(x,y)}$ for some $n \in \mathbb{N}$. These are well-defined so we can use them to extend the domain of definition of Ψ to $\Sigma_0^{(x,y)}$:

$$\Psi(x,y) := \psi_2 \circ \psi_1 \circ \psi_0(x,y) = \Psi(x_n,y_n) - (0,n),$$

where $f_0^n(x,y) = (x_n,y_n) \in \Omega_0^{(x,y)}$ for some $n \in \mathbb{N}$. Since Ψ is holomorphic on $\Omega_0^{(x,y)}$ and f_0 is biholomorphic on $\mathbb{C}^2$, this extension of Ψ is holomorphic on $\Sigma_0^{(x,y)}$. In the following, we want to show that $\Psi\left(\Sigma_0^{(x,y)}\right) = \mathbb{C}^2$.

$$
\begin{aligned}
\pi_2 \circ \Psi\left(\Sigma_0^{(x,y)}\right) &= \omega \circ \psi_0\left(\Sigma_0^{(x,y)}\right) \\
&= \bigcup_{n \geq 0} \left\{ \omega \circ \psi_0(x_n,y_n) - n \mid (x_n,y_n) \in \Omega_0^{(x,y)} \right\} \\
&= \bigcup_{n \geq 0} \left\{ \omega\left(\Omega_0^{(u,v)}\right) - n \right\} \\
&= \bigcup_{n \geq 0} \left\{ \pi_2\left(\Omega_0^{(\tau,\omega)}\right) - n \right\} \\
&= \mathbb{C},
\end{aligned}
$$

where the third equality follows because f_0 is an automorphism of $\mathbb{C}^2$ and the final equality follows from the definition of $\Omega_0^{(\tau,\omega)}$ given in Proposition 2.12. For any $w \in \mathbb{C}$ and $n \in \mathbb{N}$, let

$$W_n^w := (\omega \circ \psi_0)^{-1}(w) \cap f_0^{-n}\left(\Omega_0^{(x,y)}\right).$$

Then $(\omega \circ \psi_0)^{-1}(w) \cap \Sigma_0^{(x,y)} = \bigcup_{n\geq 0} W_n^w$.

Theorem 2.14. *Fix any* $w \in \pi_2\left(\Omega_0^{(\tau,\omega)}\right)$. *Then*

$$\tau \circ \psi_1 \circ \psi_0 = \pi_1 \circ \Psi : (\omega \circ \psi_0)^{-1}(w) \cap \Sigma_0^{(x,y)} \to \mathbb{C}$$

is a biholomorphism.

Proof. First we show that the extension of $\tau \circ \psi_1 \circ \psi_0$ to $\Sigma_0^{(x,y)}$ is holomorphic. For any $(x,y) \in \Sigma_0^{(x,y)}$, $\exists n \in \mathbb{N}$ such that $(x_n, y_n) \in \Omega_0^{(x,y)}$. Let U be a connected open neighborhood of (x,y) small enough that $f_0^n(U) \subset \Omega_0^{(x,y)}$, which is then a connected open neighborhood of (x_n, y_n). Therefore $\tau \circ \psi_1 \circ \psi_0$ is holomorphic on $f_0^n(U)$. Since f_0 is holomorphic and $\tau \circ \psi_1 \circ \psi_0 = \tau \circ \psi_1 \circ \psi_0 \circ f_0^n$, it follows that $\tau \circ \psi_1 \circ \psi_0$ is holomorphic on U. Hence $\tau \circ \psi_1 \circ \psi_0$ is holomorphic on $\Sigma_0^{(x,y)}$.
Now we show injectivity. Suppose $\tau \circ \psi_1 \circ \psi_0(x,y) = \tau \circ \psi_1 \circ \psi_0(\widetilde{x}, \widetilde{y})$ for some $(x,y), (\widetilde{x}, \widetilde{y}) \in (\omega \circ \psi_0)^{-1}(w) \cap \Sigma_0^{(x,y)}$. For $n \in \mathbb{N}$ large enough we have $(x_n, y_n), (\widetilde{x}_n, \widetilde{y}_n) \in \Omega_0^{(x,y)}$ and

$$\tau \circ \psi_1 \circ \psi_0(x_n, y_n) = \tau \circ \psi_1 \circ \psi_0(x,y) = \tau \circ \psi_1 \circ \psi_0(\widetilde{x}, \widetilde{y}) = \tau \circ \psi_1 \circ \psi_0(\widetilde{x}_n, \widetilde{y}_n).$$

Then

$$\pi_2 \circ \Psi(x,y) = w = \pi_2 \circ \Psi(\widetilde{x}, \widetilde{y}) \qquad \Rightarrow \qquad \pi_2 \circ \Psi(x_n, y_n) = w + n = \pi_2 \circ \Psi(\widetilde{x}_n, \widetilde{y}_n).$$

Ψ is injective on $\Omega_0^{(x,y)}$ and f_0 is injective on $\mathbb{C}^2$, therefore

$$\Psi(x_n, y_n) = \Psi(\widetilde{x}_n, \widetilde{y}_n) \quad \Rightarrow \quad (x_n, y_n) = (\widetilde{x}_n, \widetilde{y}_n) \quad \Rightarrow \quad (x,y) = (\widetilde{x}, \widetilde{y}).$$

Finally we show surjectivity.
Claim: $\tau \circ \psi_1 \circ \psi_0 (W_n^w) = \tau \circ \psi_1 \circ \psi_0\left((\omega \circ \psi_0)^{-1}(w+n) \cap \Omega_0^{(x,y)}\right)$.
For any $(x,y) \in W_n^w$, it follows that $(x_n, y_n) \in \Omega_0^{(x,y)}$ and $\omega \circ \psi_0(x_n, y_n) = w + n$. Hence $(x_n, y_n) \in (\omega \circ \psi_0)^{-1}(w+n) \cap \Omega_0^{(x,y)}$ and $\tau \circ \psi_1 \circ \psi_0(x,y) = \tau \circ \psi_1 \circ \psi_0(x_n, y_n)$.
Conversely, for any $(x,y) \in (\omega \circ \psi_0)^{-1}(w+n) \cap \Omega_0^{(x,y)}$, $\exists(\widetilde{x}, \widetilde{y}) \in \mathbb{C}^2$ such that $(\widetilde{x}_n, \widetilde{y}_n) = (x,y)$ since f_0 is an automorphism of $\mathbb{C}^2$. Then $\omega \circ \psi_0(\widetilde{x}, \widetilde{y}) = w$ so $(\widetilde{x}, \widetilde{y}) \in W_n^w$ and $\tau \circ \psi_1 \circ \psi_0(x,y) = \tau \circ \psi_1 \circ \psi_0(\widetilde{x}, \widetilde{y})$.

Therefore, $\tau \circ \psi_1 \circ \psi_0(W_n^w) = \tau \circ \psi_1 \circ \psi_0 \left((\omega \circ \psi_0)^{-1}(w+n) \cap \Omega_0^{(x,y)} \right)$.

Fix any $n \in \mathbb{N}$.

$$\begin{aligned}
\tau \circ \psi_1 \circ \psi_0(W_n^w) &= \tau \circ \psi_1 \circ \psi_0 \left((\omega \circ \psi_0)^{-1}(w+n) \cap \Omega_0^{(x,y)} \right) \\
&= \tau \circ \psi_1 \left((\pi_2 \circ \psi_1)^{-1}(w+n) \cap \Omega_0^{(u,v)} \right) \\
&= \left\{ \tau(u, w+n) \;\middle|\; (u, w+n) \in \Omega_0^{(u,\omega)} \right\} \\
&= \left\{ \tau - \log(w+n) \;\middle|\; \mathrm{Re}(\tau) > R, |\tau| < (\delta|w+n|)^{\frac{k}{(k-1)(k+1)}}, |\operatorname{Arg}(\tau)| < \theta \right\},
\end{aligned}$$

where the last equality follows because $\psi_2 : \Omega_0^{(u,\omega)} \to \Omega_0^{(\tau,\omega)}$ is a biholomorphism. For fixed w, $|w+n|^{\frac{k}{(k-1)(k+1)}}$ grows much faster than $|\log(w+n)|$ as $n \to \infty$. Therefore,

$$\pi_1 \circ \Psi \left((\omega \circ \psi_0)^{-1}(w) \cap \Sigma_0^{(x,y)} \right) = \bigcup_{n \geq 0} \tau \circ \psi_1 \circ \psi_0(W_n^w) = \mathbb{C}.$$

□

For $n \in \mathbb{N}$, let

$$\Omega_n := \bigcup_{w \in \pi_2\left(\Omega_0^{(\tau,\omega)}\right)} (\omega \circ \psi_0)^{-1}(w-n) \cap \Sigma_0^{(x,y)}$$

$$\Psi_n := (\pi_1 \circ \Psi \circ f_0^n, \pi_2 \circ \Psi) = \Psi \circ f_0^n - (0,n).$$

Then

$$\Psi_n : \Omega_n \to \mathbb{C} \times \left\{ \pi_2 \left(\Omega_0^{(\tau,\omega)} \right) - n \right\}$$

is a biholomorphism and $\bigcup_{n\in\mathbb{N}} \Omega_n = \Sigma_0^{(x,y)}$. So $\{\Omega_n\}_{n\in\mathbb{N}}$ is an open cover of $\Sigma_0^{(x,y)}$ with coordinate functions $\{\Psi_n\}_{n\in\mathbb{N}}$ which agree with each other on overlaps. This defines on $\Sigma_0^{(x,y)}$ a structure of a locally trivial fiber bundle with base $\mathbb{C}$ and fiber $\mathbb{C}$. Hence,

$$\Psi : \Sigma_0^{(x,y)} \to \mathbb{C}^2$$

is a biholomorphism and

$$\Psi(x,y) = \Psi \circ f_0^n(x,y) - (0,n)$$

for any $(x,y) \in \Sigma_0^{(x,y)}$ and $n \in \mathbb{N}$. In addition, f_0 acts as translation:

$$\Psi \circ f_0(x,y) = \Psi(x,y) + (0,1)$$

for any $(x, y) \in \Sigma_0^{(x,y)}$. Recall that $\Sigma = l\left(\Sigma_0^{(x,y)}\right)$ and let $\Phi := \Psi \circ l^{-1}$. The following commutative diagram illustrates Theorem B:

$$
\begin{tikzcd}
\Sigma \arrow[r, "f"] \arrow[d, "\Phi"'] & \Sigma \arrow[d, "\Phi"] \\
\mathbb{C}^2 \arrow[r, "\mathrm{Id} + (0,1)"'] & \mathbb{C}^2
\end{tikzcd}
$$

where all maps are biholomorphisms.

We have now completed the proofs of Theorems A and B, extending Hakim's Theorems 1.17 and 1.23 to a type of map whose director is zero.

2.6. Comparing results

We now compare the two main theorems in this chapter to a theorem proven by Vivas [V2, Theorem 1]. In order to understand the statement of Vivas' Theorem, we must first introduce a few terms that distinguish between characteristic directions in $\mathbb{C}^2$; these distinctions were made by Abate and Tovena [AT]. Before introducing these terms, let's recall some notation from Chapter 1.

Let $f \in \mathrm{End}(\mathbb{C}^2, \mathrm{O})$ be tangent to the identity of order $k \geq 2$ with characteristic direction $[v]$. Near the origin,

$$f(z) = z + P_k(z) + P_{k+1}(z) + \dots,$$

where $z = (x, y) \in \mathbb{C} \times \mathbb{C}$, $P_k \not\equiv 0$, and $P_l := (p_l, q_l) : \mathbb{C}^2 \to \mathbb{C} \times \mathbb{C}$ are homogenous polynomials of degree l. Assume, without loss of generality, that $[v] = [1 : u_o]$. Let:

$$r(u) := q_k(1, u) - p_k(1, u)u.$$

Let $m \geq 0$ and $n \geq 0$ be the order of vanishing of $p_k(1, u)$ and $r(u)$ at $u = u_o$, respectively.

Definition 2.15. The origin is *dicritical* when $n = \infty$. When the origin is non-dicritical, the characteristic direction $[v]$ is:

1. *Fuschian* if $1 + m = n$,
2. *irregular* if $1 + m < n$, or
3. *apparent* if either $1 + m > n > 0$ or $m = \infty$.

These definitions are invariant under a holomorphic change of coordinates.

Remark 2.16. The direction $[v]$ is a characteristic direction of f if and only if $n > 0$. Furthermore, $[v]$ is a non-degenerate characteristic direction of f if and only if $m = 0 < n$. The origin is dicritical if and only if $r \equiv 0$.

Now we can state Vivas' aforementioned result from [V3].

Theorem 2.17 (Vivas, [V3])**.** *Let $f \in \mathrm{End}(\mathbb{C}^2, \mathrm{O})$ be tangent to the identity. Assume that $[v]$ is an irregular characteristic direction. Then there exists an invariant attracting domain $\Omega \subsetneq \mathbb{C}^2$ in which every point is attracted to the origin tangentially to the direction $[v]$ and, such that the restriction of f to V is conjugate to translation. In addition, if f is a biholomorphism of $\mathbb{C}^2$, then such an Ω exists that is also a Fatou-Bieberbach domain.*

In Theorems A and B, we assume that $f \in \mathrm{End}(\mathbb{C}^2, \mathrm{O})$ is tangent to the identity of order $k \geq 2$, that $[v]$ is a non-degenerate characteristic direction, and that $[v]$ is the only characteristic direction of f at the origin. Given these assumptions, in Lemma 2.1 we showed that f is linearly conjugate to:

$$f_0(x, y) = (x, y)\left(1 + xyR(x, y) + y^{k-1}\right) + \left(P(x, y), x^k + Q(x, y)\right).$$

It is then straightforward to show that the unique characteristic direction of f_0 must be irregular, hence $[v]$ is irregular. Theorems A and B can then follow by using Theorem 2.17 and that $[v]$ must be irregular. As was mentioned in the introduction, the author and Vivas arrived at these results while working independently from each other [L, V2].

Chapter 3

Invariant attracting domains and curves in $\mathbb{C}^2$

In this chapter, we survey what is known in dimension two about the existence of invariant attracting domains and curves for maps that are tangent to the identity. For the entirety of this chapter, let $f \in \mathrm{End}(\mathbb{C}^2, \mathrm{O})$ be tangent to the identity of order $k \geq 2$ with characteristic direction $[v]$ and assume, without loss of generality, that $[v] = [1:0]$.

In order to better understand the dynamics of f, we again look at the lift of f to the blowup of $\mathbb{C}^2$ at the origin. As we saw in (1.4), the lift of f can be expressed in the local coordinates (x, u) as follows:

$$\begin{aligned} x_1 &= x\left(1 + p_k(1,u)x^{k-1} + \mathrm{O}(x^k, ||u||x^k)\right) \\ u_1 &= u\left(1 + x^{k-1}(\alpha + u\tilde{r}(u) + \mathrm{O}(x))\right) + \mathrm{O}(x^k), \end{aligned} \tag{3.1}$$

where

$$r(u) := q_k(1,u) - up_k(1,u),$$

$\alpha = r'(0)$, and $\tilde{r}$ is a polynomial in u defined so that:

$$r(u) = \alpha u + u^2\tilde{r}(u).$$

For both degenerate and non-degenerate characteristic directions we can define the following:

Definition 3.1. *Abate's index of f at $[v] = [1 : u_o]$, $Ind(\tilde{f}, \mathbb{P}^1, [v])$, equals* $\mathrm{Res}_{u=u_o} \frac{p_k(1,u)}{r(u)}$.

If $[v]$ is non-degenerate, we can further simplify the expression for (x_1, u_1) by performing the same coordinate change as we did to get (1.5). Then:

$$\begin{aligned} x_1 &= x\left(1 - \frac{1}{k-1}x^{k-1} + \mathrm{O}\left(||u||x^{k-1}, x^{2k-2}\right)\right) \\ u_1 &= u\left(1 - x^{k-1}(A + u\tilde{R}(u) + \mathrm{O}(x))\right) + \mathrm{O}(x^k), \end{aligned} \tag{3.2}$$

where

$$A := \frac{\alpha}{(k-1)p_k(1,0)} \qquad \text{and} \qquad \tilde{R}(u) := \frac{1}{(k-1)p_k(1,0)}\tilde{r}(u).$$

Recall that A is a holomorphic invariant associated to the direction $[v] = [1:0]$ called the *director of f at $[v]$*.

When $[v]$ is non-degenerate, Abate's index of f at $[v]$ and the director of f at $[v]$ are clearly closely related. We use Abate's index when $[v]$ is degenerate and directors when $[v]$ is non-degenerate.

In §2.6 we defined two constants associated to the characteristic direction $[1:0]$; in particular, the constants m and n are the orders of vanishing of $p_k(1,u)$ and $r(u)$ at $u=0$, respectively. As we saw in Definition 2.15, the origin is dicritical precisely when $n=\infty$ and, when the origin was non-dicritical, characteristic directions were divided into three different types:

$$\text{Fuchsian } (1+m=n), \text{ irregular } (1+m<n), \text{ and apparent } (1+m>n \text{ or } m=\infty).$$

We use this categorization to discuss how the existence of invariant attracting domains whose points converge to the origin tangentially to $[v]$ depends on the properties of that direction.

Remark 3.2. Apparent characteristic directions must be degenerate because a characteristic direction is degenerate if and only if $m \neq 0$. However, Fuchsian and irregular characteristic directions can be either non-degenerate or degenerate.

Before discussing the existence of invariant attracting domains in $\mathbb{C}^2$, we first discuss the existence of complex curves in $\mathbb{C}^2$. More specifically, the existence of parabolic curves (see Definition 1.13).

3.1. Existence of parabolic curves in $\mathbb{C}^2$

Ecalle and Hakim showed that there exist (at least) $k-1$ parabolic curves for f tangent to $[v]$ (see Theorem 1.14 or [E, H2]). By replacing the assumption that $[v]$ is non-degenerate with some other assumptions on f, Abate and Tovena showed that parabolic curves for f at the origin still exist. In particular, if the origin is an isolated fixed point, Abate showed that there exist (at least) $k-1$ parabolic curves for f at the origin tangent to some singular direction (see Theorem 1.15 or [A1, AT]). In addition, Abate proved that when the origin is dicritical, f admits parabolic curves. In particular,

Theorem 3.3 (Abate, [A1])**.** *Let $f \in \mathrm{End}(\mathbb{C}^2, \mathrm{O})$ be tangent to the identity. If the origin is dicritical, then f admits infinitely many parabolic curves.*

Therefore a map $f \in \mathrm{End}(\mathbb{C}^2, \mathrm{O})$ tangent to the identity admits at least $k-1$ parabolic curves if: f has a non-degenerate characteristic direction, the origin is an isolated fixed point, or the origin is dicritical.

Now we discuss the existence of invariant attracting domains whose points converge to the origin tangentially to the direction $[v]$ and how this depends on properties of $[v]$.

3.2. Fuschian characteristic directions

We divide the discussion of the existence of invariant attracting domains for a Fuchsian characteristic direction $[v]$ into five different cases, depending on the values of m, n and the director $[v]$. Recall that when $[v]$ is Fuchsian, $1 + m = n < \infty$.

Case 1. $m = 0$ **(so** $[v]$ **is non-degenerate) and** $\mathrm{Re}(A(v)) > 0$**.**

These are precisely the assumptions for Hakim's Theorems 1.17 and 1.18 in dimension 2 [AR, H2]. Hence, there exist $k-1$ invariant attracting domains, each with the origin in its boundary, in which every point is attracted to the origin tangentially to the direction $[v]$ [AR, H1, H2]. Furthermore, Hakim showed that there exists one such invariant domain Ω on which $f|_\Omega$ is holomorphically conjugate to translation (see Theorem 1.17 or [H2]). In addition, if f is a biholomorphism of $\mathbb{C}^2$, then f has such a domain Ω that is also a Fatou-Bieberbach domain and the restriction of f to Ω is conjugate to translation (see Theorem 1.23 or [H2]).

Case 2. $m = 0$ **(so** $[v]$ **is non-degenerate) and** $\mathrm{Re}(A(v)) = 0$**.**

These assumptions imply that $n = 1$ and $A(v) = \mathrm{Im}(A(v)) \neq 0$. As we see in the following example, there does not necessarily exist an invariant attracting domain for f whose points converge tangentially to $[v]$.

Example 3.4.
Consider the following map:

$$f(x, y) = (\ x(1-x),\ y(1-(1+i\alpha)x)\ ),$$

where $\alpha \in \mathbb{R}\setminus\{0\}$*. Note that this map is also discussed in [R1]. The direction* $[1:0]$ *is a non-degenerate, Fuchsian characteristic direction of* f *with director* $i\alpha$*. The first coordinate is independent of the second coordinate, so we can use the one dimensional results to understand the dynamics of the first coordinate.*

Let $\mathcal{C} \subset \mathbb{C}$ be the standard cauliflower set, which is the parabolic basin for the map $x \mapsto x(1-x)$. It is well-known that for this map, $\{x_n\}$ diverges as $n \to \infty$ if and only if $x \notin \overline{\mathcal{C}}$ and $\{x_n\}$ converges to 0 if and only if $x \in \mathcal{C}$. Furthermore, if $x \in \mathcal{C}$, then $\lim_{n\to\infty} nx_n = 1$, so $\mathrm{Re}(x_n) \sim \frac{1}{n}$ and $|\mathrm{Im}(x_n)| \ll \frac{1}{n}$. We bound $|\mathrm{Im}(x_n)|$ more precisely, for $x \in \mathcal{C}$ and $\mathrm{Re}(x) \neq 0$, by doing the following:

$$\begin{aligned}
\mathrm{Re}(x_1) &= \mathrm{Re}(x)\,(1-\mathrm{Re}(x)) + (\mathrm{Im}(x))^2 \geq \mathrm{Re}(x)\,(1-\mathrm{Re}(x)) \\
|\mathrm{Im}(x_1)| &= |\mathrm{Im}(x)\,(1-2\,\mathrm{Re}(x))| \leq |\mathrm{Im}(x)|\,(1-\mathrm{Re}(x))^2 \\
|\mathrm{Im}(x_n)| &\leq |\mathrm{Im}(x)| \prod_{j=0}^{n-1} (1-\mathrm{Re}(x_j))^2 \leq \frac{|\mathrm{Im}(x)|}{\mathrm{Re}(x)^2}\,\mathrm{Re}(x_n)^2.
\end{aligned}$$

Note that if $\mathrm{Re}(x) = 0$, all of the inequalities except for the last one hold and we can replace $\mathrm{Re}(x)$ by $\mathrm{Re}(x_1) \neq 0$ in the last inequality. Hence, for $x \in \mathcal{C}$ and n large, $\mathrm{Im}(x_n) = \mathrm{O}\left(\frac{1}{n^2}\right)$. Let $(x_n, y_n) := f^n(x,y)$ and $w = \frac{y}{x}$. For $x_n \neq 0$, if

$$(x_n, y_n) \to (0,0) \textit{ along } [1:0], \textit{ then } \frac{1}{x_n}(x_n, y_n) = (1, w_n) \to (1, 0).$$

Notice that:

$$y_n = y \prod_{l=0}^{n-1} (1-(1+i\alpha)x_l) \qquad \textit{and} \qquad w_n = w \prod_{l=0}^{n-1} (1 - i\alpha x_l(1+\mathrm{O}(x_l))).$$

Assume that $x \in \mathcal{C}$ and $y \neq 0$. Then

$$\mathrm{Re}(\log y_n) \sim \mathrm{Re}\left(\sum_{l=1}^{n-1} -(1+i\alpha)x_l\right) \sim \sum_{l=1}^{n-1} -\frac{1}{l} \to -\infty,$$

so $y_n \to 0$ as $n \to \infty$ and

$$\mathrm{Re}(\log w_n) \sim \mathrm{Re}\left(\sum_{l=1}^{n-1} -i\alpha x_l(1+\mathrm{O}(x_l))\right) \sim \sum_{l=1}^{n-1} \left(\alpha\,\mathrm{Im}(x_l) + \mathrm{O}(x_l^2)\right) \nrightarrow -\infty,$$

so $w_n \nrightarrow 0$ as $n \to \infty$. Hence, $\mathcal{C} \times \mathbb{C}$ is attracted to the origin, but its points (excluding $\mathcal{C} \times \{0\}$) do not converge along $[1:0]$. Thus, since $\{x_n\}$ converges to zero if and only if $x \in \mathcal{C}$, f has no invariant attracting domain whose points converge to the origin tangentially to $[1:0]$.

Question. Given the assumptions on f, under what conditions (if any) does there exist an invariant attracting domain whose points converge to the origin tangentially to $[v]$?

Case 3. $m = 0$ **(so** $[v]$ **is non-degenerate) and** $\mathrm{Re}(A(v)) < 0$**.**

There does not exist an invariant attracting domain whose points converge to the origin tangentially to $[v]$ (see Theorem 1.19).

Case 4. $m > 0$ **(so** $[v]$ **is degenerate) and** $\mathrm{Re}\left(Ind(\tilde{f}, \mathbb{P}^1, [v])\right) \in R$**, where**

$$R = \left\{ z \in \mathbb{C}, \mathrm{Re}(z) > -\frac{m}{k-1}, \left| z - \frac{m+1-\frac{m}{k-1}}{2} \right| > \frac{m+1+\frac{m}{k-1}}{2} \right\} \subset \mathbb{C}.$$

Vivas showed that there exists an invariant attracting domain whose points converge to the origin tangentially to $[v]$ [V3].

Question. Given the assumptions on f, if f is an automorphism, can such an invariant attracting domain be a Fatou-Bieberbach domain?

Case 5. $m > 0$ **(so** $[v]$ **is degenerate) and** $\mathrm{Re}\left(Ind(\tilde{f}, \mathbb{P}^1, [v])\right) \notin R$**.**

As we see in the following example, there are maps that do not have an invariant attracting domain whose points converge to the origin along $[v]$.

Example 3.5.
Consider the following map:

$$f(x, y) = \left(x + ay^2 + P_{>2}(x, y), y + Q_{>2}(x, y)\right),$$

where $a \neq 0$*,* $P_{>2}$ *and* $Q_{>2}$ *are convergent power series each with degree at least 3, and* $[1:0]$ *is a degenerate Fuchsian characteristic direction. Then* $m = 2$ *and* $\mathrm{Re}(Ind(\tilde{f}, \mathbb{P}^1, [v]) = -1 \notin R$*. When* $P_{>2} = Q_{>2} \equiv 0$*, it is clear that* f *has no invariant attracting domain tangential to* $[1:0]$*.*

Question. Given the assumptions on f, under what conditions (if any) does there exist an invariant attracting domain whose points converge to the origin tangentially to $[v]$?

3.3. Irregular characteristic directions

In the main results from Chapter 2 (Theorems A and B) we assume that $f \in \mathrm{End}(\mathbb{C}^2, \mathrm{O})$ has a non-degenerate characteristic direction $[v]$ and that $[v]$ is the only characteristic direction of f at the origin. Given these assumptions, it turns out that $[v]$ must irregular. Furthermore, the assumption that $[v]$ is irregular is precisely the assumption made in Vivas' Theorem 2.17. Hence, there exists an invariant attracting domain $\Omega \subset \mathbb{C}^2$, with the origin in its boundary, in which every point is attracted to the origin tangentially to the direction $[v]$, and such that the restriction of f to Ω is conjugate to translation. If, in addition, f is an automorphism of $\mathbb{C}^2$, then there exists such an Ω that is also a Fatou-Bieberbach domain and the restriction of f to Ω is conjugate to translation [L, V3] .

3.4. Apparent characteristic direction

Recall from Definition 2.15 that $[v]$ is apparent when $1+m>n$. Vivas showed in [V3] that when $[v]$ is an apparent characteristic direction of a map f, there are sometimes, but not always, invariant attracting domains tangential to $[v]$.

In particular, suppose f is of the form $f(z)=z+P_k(z)$ with an apparent characteristic direction $[v]$. Then there exist, for $j=1,2$, $f_j(z)=f(z)+Q_j(z)$, where $Q_j(z)=\mathrm{O}(|z|^{k+1})$, such that:

f_1 has an invariant attracting domain tangential to $[v]$; and

f_2 does not have any orbit converging towards the origin along $[v]$.

Question. Given the assumptions on f, under what conditions does there exist an invariant attracting domain whose points converge to the origin tangentially to $[v]$?

3.5. The origin is dicritical

When the origin is dicritical ($n=\infty$), all directions must be characteristic directions. Brochero Martínez showed in [Bro2] that there is an open set U whose points converge to the origin under f and the orbit of each point $p\in U$ is attracted to the origin along some direction, however that direction may not be the same for all points in U. For more details, refer back to Theorem 1.21.

The following example demonstrates how a map f that is dicritical at the origin can have an invariant attracting domain whose points converge to the origin tangentially to some direction, but in which not all points converge along the same direction.

Example 3.6.
Suppose $f=\mathrm{Id}+P_k$ is dicritical at the origin and $P_k\not\equiv 0$ is a homogeneous polynomial of degree k. Then $P_k=\lambda\,\mathrm{Id}$, where $\lambda:\mathbb{C}^2\to\mathbb{C}$ is a non-trivial ($\lambda\not\equiv 0$) homogeneous polynomial of degree $k-1$ (see Remark 1.11). Rewriting f we get:

$$(x_1,y_1):=f(x,y)=(x,y)\left(1+\lambda(x,y)\right).$$

Suppose the orbit of a point $(x,y)\in(\mathbb{C}\setminus\{0\})^2$ converges to the origin. Then the orbit of (x,y) must converge to the origin tangentially to the direction $[x_n:y_n]=[x:y]$ since $\frac{x_n}{x}=\frac{y_n}{y}$. Hence, any invariant attracting domain whose points converge to the origin under iteration by f cannot have all of its points converge along the same direction, however each of its points converges along a particular direction.

3.6. Summary of results in $\mathbb{C}^2$

Let $f \in \operatorname{End}(\mathbb{C}^2, \mathrm{O})$ be tangent to the identity of order $k \geq 2$ with characteristic direction $[v]$. Below we summarize the results from this chapter on the existence of parabolic curves and invariant attracting domains.

There exist (at least) $k-1$ parabolic curves for f if:

- $[v]$ is non-degenerate, in which case the curves are tangent to $[v]$ (see [E, H2]),
- the origin is an isolated fixed point, in which case the curves are tangent to some singular direction (see [A1, AT]), or
- the origin is dicritical, in which case infinitely many parabolic curves exist (see [A1]).

In the following table, we summarize the results on the existence of invariant attracting domains whose points converge to O tangentially to $[v]$. In order to simplify the table, let:

$\hat{A}$ be Abate's Index of f at $[v]$.

A be the director of f corresponding to $[v]$ if $[v]$ is non-degenerate.

U be an open set whose points converge to O, each point along some direction.

$\widehat{\Omega}$ be an invariant attracting domain whose points converge tangentially to $[v]$.

Ω be $\widehat{\Omega}$ such that f is conjugate to translation on $\widehat{\Omega}$.

Σ be Ω and, if f is an automorphism, it is a Fatou-Bieberbach domain.

$[v]$ **is**	**Non-Degenerate** $(m=0)$	**Degenerate** $(m>0)$
Fuchsian $(1+m=n)$	• $\operatorname{Re} A > 0 \Rightarrow \begin{cases} \exists \Sigma & \text{[H1]} \\ \exists (k-1)\ \widehat{\Omega} & \text{[AR]} \end{cases}$ • $\operatorname{Re} A = 0 \neq A$ $\Rightarrow$ sometimes $\nexists \widehat{\Omega}$ Ex. 3.4 • $\operatorname{Re} A < 0 \Rightarrow \nexists \widehat{\Omega}$ [H1]	• $\operatorname{Re} \hat{A} \in R \Rightarrow \exists \widehat{\Omega}$ [V3] • $\operatorname{Re} \hat{A} \notin R \Rightarrow$ sometimes $\nexists \widehat{\Omega}$ Ex. 3.5
Irregular $(1+m<n)$	$(A=0)$ always $\Rightarrow \exists \Sigma$ [L, V3]	always $\Rightarrow \exists \Sigma$ [V3]
Apparent $(1+m>n)$	Does not apply	sometimes $\Rightarrow \begin{cases} \exists \widehat{\Omega} & \text{[V3]} \\ \nexists \widehat{\Omega} & \text{[V3]} \end{cases}$
O is **Dicritical** $(m=\infty)$	$\left. \begin{array}{l} (r \equiv \mathrm{O}) \\ (A=0) \end{array} \right\} \Rightarrow \exists U$ [Bro2]	$(r \equiv \mathrm{O}) \Rightarrow \exists U$ [Bro2]

Table 3.1: Summary of the existence of invariant attracting domains in $\mathbb{C}^2$.

Chapter 4

The higher-dimensional case

In this chapter, we survey what is known about the existence of invariant attracting domains, submanifolds, and curves for maps that are tangent to the identity in dimension $m \geq 3$. For the entirety of this chapter, let $f \in \mathrm{End}(\mathbb{C}^m, O)$ be tangent to the identity of order $k \geq 2$ with characteristic direction $[v]$ and assume, without loss of generality, that $[v] = [1 : O]$. Unless otherwise stated, assume that $[v]$ is *non-degenerate*.

In order to better understand the dynamics of f, we again look at the lift of f to the blowup of $\mathbb{C}^m$ at the origin. As we saw in (1.4), the lift of f can be expressed in the local coordinates $(x, u) \in \mathbb{C} \times \mathbb{C}^{m-1}$ as follows:

$$\begin{aligned} x_1 &= x\left(1 + x^{k-1} p_k(1, u) + O(x^k, ||u|| x^k)\right) \\ u_1 &= u + x^{k-1} r(u) + O(x^k, ||u|| x^k), \end{aligned}$$

where $r(u) := q_k(1, u) - p_k(1, u)u$, so r is a polynomial in u whose terms have degree between 1 and $k + 1$. The previous coordinate change did not require $[v]$ to be non-degenerate, but the next one does require $[v]$ to be non-degenerate. We can further simplify the expression for (x_1, u_1) by performing the same coordinate change as we did to get (1.5) as well as a linear change of the u coordinates so that A is in Jordan canonical form. Then:

$$\begin{aligned} x_1 &= x\left(1 - \frac{1}{k-1} x^{k-1} + O(x^k, ||u|| x^k)\right) \\ u_1 &= (1 - x^{k-1} A)u - x^{k-1} \tilde{r}(u) + O(x^k, ||u|| x^k), \end{aligned} \tag{4.1}$$

where $r(u)$ corresponds to $-Au - \tilde{r}(u)$ after the coordinate change, $\tilde{r}$ is a polynomial in u whose terms have degrees between 2 and $k + 1$, and A is in Jordan canonical form. Recall that the eigenvalues of A are the directors corresponding to the characteristic direction $[v]$. Let $\{\alpha_1, \ldots, \alpha_{m-1}\} \subset \mathbb{C}$ be the directors corresponding to $[v]$.

As we saw in §1.2 and Chapter 3, directors play a significant role in the existence of invariant attracting domains or submanifolds whose points converge tangentially to a direction. In this chapter, we discuss how directors and other factors affect the existence and characteristics of invariant attracting domains or submanifolds. In addition, we discuss the existence of parabolic curves.

4.1. Existence of parabolic curves in $\mathbb{C}^m, m \geq 3$

Most of the results we discussed in Section 3.1 on the existence of parabolic curves in $\mathbb{C}^2$ also hold in higher dimensions. In particular, a map $f \in \mathrm{End}(\mathbb{C}^m, \mathrm{O})$ that is tangent to the identity admits (at least) $k-1$ parabolic curves along some direction if: f has a non-degenerate characteristic direction (Ecalle [E], Hakim, [H2]) or the origin is an isolated fixed point (Abate [A1], Abate-Tovena [AT]). However, we do not have an extension of Abate's result in $\mathbb{C}^2$ for maps where the origin is dicritical.

4.2. All directors have strictly positive real part

When all directors have strictly positive real part (i.e., $\mathrm{Re}(\alpha_j) > 0$ for all j), Hakim showed that there exist $k-1$ invariant attracting domains, each with the origin in its boundary, in which every point is attracted to the origin tangentially to the direction $[v]$ [H1, H2, AR]. Furthermore, Hakim showed that there exists one such invariant attracting domain D on which $f|_D$ is holomorphically conjugate to translation (see Theorem 1.17 or [H2]). In addition, if f is a biholomorphism of $\mathbb{C}^m$, then f has such a domain D that is also a Fatou-Bieberbach domain and $f|_D$ is conjugate to translation (see Theorem 1.23 or [H2]).

4.3. At least one director has strictly negative real part

When at least one director has strictly negative real part, Hakim showed that there does not exist an invariant attracting domain in which every point is attracted to the origin tangentially to the direction $[v]$ (Theorem 1.19 or [H1]). However, if some of the directors have strictly positive real part, an invariant submanifold whose points are attracted to the origin tangentially to $[v]$ can exist (see Theorem 1.20 or [H1]). We discuss this further in the next section.

4.4. Some, but not all, directors have strictly positive real part

Suppose $[v]$ has exactly d directors with strictly positive real part. Then Hakim showed that there exists M^{d+1}, an invariant piece of analytic manifold of dimension $d+1$, with the origin in its boundary, that is tangent to $\mathbb{C}V + E$ at the origin, and such that every point of M^{d+1}

is attracted to the origin tangentially to the direction $[v]$ (see Theorem 1.20 or [H1]). In the following, we see that such a manifold of dimension greater than $d+1$ sometimes exists.

All directors have non-negative real part

When all of the directors corresponding to $[v]$ have non-negative real part, an invariant attracting domain whose points converge to the origin tangentially to the direction $[v]$ sometimes exists.

The following is an example of a map that has an invariant attracting domain whose points converge to the origin, but the map cannot have such a domain in which its points all converge tangentially to the given direction.

Example 4.1. *Let*

$$f(x,y,z) = (x(1-x), y(1-x), z(1-2x))\,.$$

Then $[1:0:0]$ is a non-degenerate characteristic direction with corresponding directors 0 and 1. In addition, f has infinitely many characteristic directions, but the origin is non-dicritical. As we mentioned in Example 3.4, the map $x \mapsto x(1-x)$ has a parabolic basin $\mathcal{C} \subset \mathbb{C}$ such that $\{x_n\}$ diverges as $n \to \infty$ if and only if $x \notin \overline{\mathcal{C}}$ and $\{x_n\}$ converges to 0 if and only if $x \in \mathcal{C}$. In addition, if $x \in \mathcal{C}$, then $x_n \sim \frac{1}{n}$ for large n. For $x \in \mathcal{C}$,

$$\frac{y_n}{y} = \frac{x_n}{x} \sim \frac{1}{n} \text{ for large } n$$

and, as $n \to \infty$,

$$\frac{z_n}{x_n} = \frac{z}{x}\prod_{j=0}^{n-1}\left(1 - \frac{x_j}{1-x_j}\right) \to 0 \text{ since } \operatorname{Re}\left(\sum_{j=0}^{n-1}\frac{x_j}{1-x_j}\right) \to \infty.$$

Hence, if f has an invariant attracting domain whose points converge to the origin, then it must be contained in $\mathcal{C} \times \mathbb{C}^2$. For any point $(x,y,z) \in \mathcal{C} \times \mathbb{C}^2$, as $n \to \infty$,

$$(x_n, y_n, z_n) \to (0,0,0),$$

but

$$[x_n : y_n : z_n] = \left[1 : \frac{y}{x} : \frac{z}{x}\prod_{j=0}^{n-1}\left(1 - \frac{x_j}{1-x_j}\right)\right] \to \left[1 : \frac{y}{x} : 0\right].$$

Therefore f has an invariant attracting domain whose points converge to the origin, but not one whose points converge tangentially to the direction $[1:0:0]$.

4.5. All directors are zero

We divide this section into two parts, each of which depends on properties of the matrix A from (4.1). In the first part, we assume that A is the zero matrix. In the second part, we assume that all eigenvalues of A are zero, but A is not the zero matrix. In each part, we discuss results relating to when there exists an invariant attracting domain whose points converge tangentially to the direction $[v]$.

Part 1. A is the zero matrix

We split this into two cases: (1) the origin is dicritical and (2) the origin is non-dicritical.

Case 1. *The origin is dicritical*

When the origin is dicritical, all directions must be characteristic directions. The same result that we discussed in Section 3.5 also holds in higher dimensions. In particular, Brochero Martínez showed in [Bro2] that there is an open set U whose points converge to the origin under f and the orbit of each point $p \in U$ is attracted to the origin along a particular direction, however that direction may not be the same for all points in U. For more details, see Theorem 1.21 or [Bro2].

Case 2. *The origin is non-dicritical.*

For some maps, an invariant attracting domain whose points converge to the origin tangentially to the direction $[v]$ exists, while for other maps such a domain does not exist.

Let's consider a particular class of maps for which A is the zero matrix, but the origin is non-diciritical. Suppose that $m = 3$, $k = 2$, and f has no higher-order terms. Then $f = \mathrm{Id} + P_2 \in \mathrm{End}(\mathbb{C}^3, \mathrm{O})$ and $(x_1, y_1, z_1) := f(x, y, z)$ equals:

$$\big(\, x\,[1 - x - l_1(y,z)] - p(y,z),\ y[1 - x - l_2(y,z)] - a_2 z^2,\ z[1 - x - l_3(y,z)] - a_3 y^2) \,\big)\,,$$

where l_j and p are homogeneous polynomials of degree 1 and 2, respectively, and $a_j \in \mathbb{C}$. Notice that the expression for f is missing a few terms: (1) there is no x^2 term in y_1, z_1, (2) the coefficients of x^2, xy, and xz in x_1, y_1, and z_1, respectively, is -1, and (3) the coefficients of xz in y_1 and xy in z_1 are both zero. We can express f in this way for the following reasons:

(i) there is no x^2 term in y_1, z_1 because $[v] = [1 : 0 : 0]$ is a characteristic direction of f,

(ii) the coefficient of x^2 in x_1 is non-zero because $[v]$ is non-degenerate and, after a linear change of coordinates, we can assume the coefficient is -1,

(iii) $r'(\mathrm{O}) \equiv \mathrm{O}$ because A is the zero matrix, so the coefficients of: (a) xy in y_1 and xz in z_1 must be the same as x^2 in x_1 and (b) xz in y_1 and xy in z_1 must both be zero.

Given these assumptions, such a map f may or may not have an invariant attracting domain whose points converge to the origin tangentially to $[v] = [1 : 0 : 0]$. One problem that arises when determining whether such an invariant attracting domains exists for f is that it is not clear how to control the relative sizes of y and z upon repeated iteration of f. We can avoid this problem by making more assumptions on f, as we do in the following two examples. In the first example we show that f has an invariant attracting domain whose points converge to the origin tangentially to $[v]$, while in the second example we show that no such domain exists.

Example 4.2.
Let f be the map:

$$f(x, y, z) = (\ x(1 - x - ay - bz) - cyz,\ \ y(1 - x - dy - bz),\ \ z(1 - x - ay - ez)\ )\,, \quad (4.2)$$

for any constants $a, b, c, d, e \in \mathbb{C}$. Assume that $a \neq d$ and $b \neq e$, which is equivalent to assuming $l_1(y, z) \not\equiv l_2(y, z)$ and $l_1(y, z) \not\equiv l_3(y, z)$. When we lift f to the blowup of $\mathbb{C}^3$ at the origin, we can successfully avoid terms of the form $\frac{y}{z}$ and $\frac{z}{y}$ so we do not need to know how y and z behave relative to one another. Using many of the same techniques as in the proofs of Theorems A and B, we can show that f has an invariant attracting domain whose points converge to the origin tangentially to the direction $[1 : 0 : 0]$. First we perform the coordinate change $(y, z) \mapsto \left(\frac{y}{d-a}, \frac{z}{e-b}\right)$, renaming our constants $a, b, c,$ our new coordinates (x, y, z), and our new map f so that:

$$f(x, y, z) = (x(1 - x - ay - bz) - cyz, y(1 - x - (a + 1)y - bz), z(1 - x - ay - (b + 1)z))\,.$$

Then we define the following coordinate change $\phi : (\mathbb{C} \setminus \{0\})^3 \to (\mathbb{C} \setminus \{0\})^3$ by:

$$\phi(x, y, z) = \left(\frac{1}{x}, \frac{x}{y}, \frac{x}{z}\right) := (u, v, w) \quad \Rightarrow \quad \phi^{-1}(u, v, w) = \left(\frac{1}{u}, \frac{1}{uv}, \frac{1}{uw}\right) = (x, y, z).$$

Let $\tilde{f} := \phi \circ f \circ \phi^{-1}$. When $|u|, |v|, |w| \gg 1$, $\tilde{f}$ acts on the new coordinates (u, v, w) as follows:

$$\begin{aligned}
u_1 &= u + 1 + \left(\frac{a}{v} + \frac{b}{w} + \frac{c}{vw}\right) + \frac{1}{u}\left(1 + \frac{a}{v} + \frac{b}{w} + \frac{c}{vw}\right)^2 + \mathrm{O}\left(\frac{1}{u^2}\right) \\
v_1 &= v + \frac{1}{u}\left(1 - \frac{c}{w}\right) + \mathrm{O}\left(\frac{1}{u^2}\right) \\
w_1 &= w + \frac{1}{u}\left(1 - \frac{c}{v}\right) + \mathrm{O}\left(\frac{1}{u^2}\right).
\end{aligned}$$

Define Ω *to be:*

$$\left\{(u,v,w)\in\mathbb{C}^3 \;\middle|\; |\operatorname{Arg}(v)|,|\operatorname{Arg}(w)|<\theta,\ |\operatorname{Arg}(u)|<\theta_0,\ \text{and } R<|v|,|w|<|u|^{\frac{1}{N}}\right\},$$

for $R \gg N > 2$ *and* $0 < 2\theta_0 < \theta < \frac{\pi}{8}$. *It is straightforward to show that* $\tilde{f}(\Omega) \subset \Omega$, *so* $f(\phi^{-1}(\Omega)) \subset \phi^{-1}(\Omega)$. *In addition, we can see that* $|u_n|,|v_n|,|w_n| \to \infty$ *as* $n \to \infty$ *so for any point* $(x,y,z) \in \phi^{-1}(\Omega)$, $(x_n,y_n,z_n) \to (0,0,0)$ *tangentially to* $[1:0:0]$ *as* $n \to \infty$. *It remains to be seen if* f *is conjugate to translation* $(x,y,z) \mapsto (x+1,y,z)$.

Instead of choosing f as in the previous example, we can choose f so that it cannot have an invariant attracting domain whose points converge to the origin tangentially to the direction $[1:0:0]$.

Example 4.3.
Let f *be the map:*

$$f(x,y,z) = (\ x(1-x),\ y(1-x-y),\ z(1-x+y)\ ).$$

Then $[1:0:0]$ *is a non-degenerate characteristic direction whose corresponding matrix* A *is zero. Although* f *has infinitely many characteristic directions, the origin is non-dicritical since not every direction is a characteristic direction. For any* $n \in \mathbb{N}$, *let* $t_n := \frac{y_n}{x_n}$ *and* $w_n := \frac{z_n}{x_n}$ *so that* $(1,t_n,w_n) = \frac{1}{x_n}(x_n,y_n,z_n)$. *Suppose* $(x_n,y_n,z_n) \to \mathrm{O}$, *but* $x_l,y_l,z_l \neq 0$ *for all* $l \in \mathbb{N}$. *We want to show that* $[x_n:y_n:z_n] \not\to [1:0:0]$ *or, equivalently, that* $(1,t_n,w_n)$ *cannot converge to* $(1,0,0)$. *Consider the following expressions for* t_n, w_n *as* $n \to \infty$*:*

$$t_n = \frac{y\prod_{l=0}^{n-1}(1-x_l-y_l)}{x\prod_{l=0}^{n-1}(1-x_l)} = t_0\prod_{l=0}^{n-1}\left(1-\frac{y_l}{1-x_l}\right) \to 0 \quad \Leftrightarrow \quad \operatorname{Re}\left(\sum_{l=0}^{n-1}\frac{-y_l}{1-x_l}\right) \to -\infty$$
$$w_n = \frac{z\prod_{l=0}^{n-1}(1-x_l+y_l)}{x\prod_{l=0}^{n-1}(1-x_l)} = w_0\prod_{l=0}^{n-1}\left(1+\frac{y_l}{1-x_l}\right) \to 0 \quad \Leftrightarrow \quad \operatorname{Re}\left(\sum_{l=0}^{n-1}\frac{y_l}{1-x_l}\right) \to -\infty.$$

We cannot have both sums diverge to $-\infty$, *so* $(1,t_n,w_n)$ *cannot converge to* $(1,0,0)$. *Hence,* (x_n,y_n,z_n) *cannot converge to the origin tangentially to the direction* $[1:0:0]$. *Thus,* f *cannot have an invariant attracting domain whose points converge to the origin tangentially to the direction* $[1:0:0]$.

Given the assumptions that $A = A(v)$ is the zero matrix and the origin is non-dicritical, we have seen an example in which there exists an invariant attracting domain whose points converge to the origin tangentially to $[v]$ and another example in which such a domain does not exist.

Question. Given these assumptions on f, under what conditions does there exist an invariant attracting domain whose points converge to the origin tangentially to the direction $[v]$?

Part 2. A is not the zero matrix

As we saw in (4.1), f is of the form:

$$\begin{aligned} x_1 &= x\left(1 - \frac{1}{k-1}x^{k-1} + \mathrm{O}(x^k, ||u||x^k)\right) \\ u_1 &= (1 - x^{k-1}A)u - x^{k-1}\tilde{r}(u) + \mathrm{O}(x^k, ||u||x^k), \end{aligned}$$

where $r(u)$ corresponds to $-Au - \tilde{r}(u)$ after the coordinate change and $\tilde{r}$ is a polynomial in u whose terms have degree at least 2 and at most $k+1$. In addition, A is in Jordan canonical form and its eigenvalues are all zero, so it must be of the form:

$$A = \begin{pmatrix} 0 & \epsilon_1 & 0 & \dots & 0 \\ 0 & 0 & \epsilon_2 & \dots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \epsilon_{m-2} \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}$$

where $\epsilon_j \in \mathbb{C}$. Express $u \in \mathbb{C}^{m-1}$ as $u = (u^{(1)}, \dots, u^{(m-1)})$ and $\tilde{r} : \mathbb{C}^{m-1} \to \mathbb{C}^{m-1}$ as $\tilde{r} = (\tilde{r}^{(1)}, \dots, \tilde{r}^{(m-1)})$. Then we can rewrite f as:

$$\begin{aligned} x_1 &= x\left(1 - \frac{1}{k-1}x^{k-1} + \mathrm{O}(x^k, ||u||x^k)\right) \\ u_1^{(j)} &= u^{(j)} - x^{k-1}\left(\epsilon_j u^{(j+1)} + \tilde{r}^{(j)}(u) + \mathrm{O}(x, ||u||x)\right), \\ u_1^{(m-1)} &= u^{(m-1)} - x^{k-1}\left(\tilde{r}^{(m-1)}(u) + \mathrm{O}(x, ||u||x)\right), \end{aligned} \tag{4.3}$$

where $1 \le j < m-1$. A major problem in finding an invariant attracting domain for f that converges to the origin tangentially to $[1 : \mathrm{O}]$ arises from the behavior we see in the $u_1^{(j)}$ coordinates for $1 \le j < m-1$. In particular, for $1 \le j < m-1$, $u_1^{(j)}$ depends significantly on $u^{(j+1)}$ when $\epsilon_j \ne 0$. As we see in the following example, an invariant attracting domain that converges to the origin tangentially to $[v] = [1 : \mathrm{O}]$ might not exist for f.

Example 4.4.
For $\epsilon \ne 0$, let:

$$f(x, y, z) := (x(1-x), y(1-x) - \epsilon xz, z(1-x-z))$$

Then $[1:0:0]$ *is a non-degenerate characteristic direction of* f *whose corresponding matrix* A *equals* $\begin{pmatrix} 0 & \epsilon \\ 0 & 0 \end{pmatrix}$. *Define* ψ *to be the following map:*

$$\psi : (x, y, z) \mapsto (x, u, v) := \left(x, \frac{y}{x}, \frac{z}{x}\right), \text{ where } \psi : \mathbb{C} \setminus \{0\} \times \mathbb{C}^2 \to \mathbb{C} \setminus \{0\} \times \mathbb{C}^2.$$

When $0 < |x| < 1$, $\psi \circ f \circ \psi^{-1}$ *acts on our new coordinates* u, v *as follows:*

$$\begin{aligned} u_1 &= u - \epsilon \frac{xv}{1-x} = u - \epsilon xv + \mathrm{O}\left(x^2 v\right) \\ v_1 &= v - \frac{xv^2}{1-x} = v\left(1 - xv + \mathrm{O}\left(x^2 v\right)\right). \end{aligned} \tag{4.4}$$

Suppose for some $x, y, z \neq 0$, *the orbit* $(x_n, y_n, z_n) \to (0,0,0)$. *We want to show that* (x_n, y_n, z_n) *cannot converge tangentially to the direction* $[1:0:0]$*; in particular, we want to show that* $[x_n : y_n : z_n] = [1 : u_n : v_n] \not\to [1:0:0]$. *We perform another coordinate change, moving the origin to* ∞, *by inverting each coordinate. In particular, let* $\phi : (\mathbb{C} \setminus \{0\})^3 \to (\mathbb{C} \setminus \{0\})^3$ *be given by:*

$$\phi : (x, u, v) \to \left(\frac{1}{x}, \frac{1}{u}, \frac{1}{v}\right) := (r, s, t).$$

For $|x|, |v| \ll 1$, *hence* $|r|, |t| \gg 1$, f *acts on the coordinates* r, t *as follows:*

$$\begin{aligned} r_1 &= r + 1 + \mathrm{O}\left(\frac{1}{r}\right) & &\Rightarrow & r_n &= r + n + \sum_{j=0}^{n-1} \mathrm{O}\left(\frac{1}{r_j}\right) \\ t_1 &= \frac{t}{1 - \frac{1}{rt} + \mathrm{O}\left(\frac{1}{r^2 t}\right)} = t + \frac{1}{r} + \mathrm{O}\left(\frac{1}{r^2}\right) & &\Rightarrow & t_n &= t + \sum_{j=0}^{n-1} \frac{1}{r_j}\left(1 + \mathrm{O}\left(\frac{1}{r_j}\right)\right) \end{aligned}$$

Therefore $r_n \sim n$ *and* $t_n \sim \log n$ *for large* n*; hence* $x_n \sim \frac{1}{n}$ *and* $v_n \sim \frac{1}{\log n}$ *for large* n. *Then we see that* $u_n \not\to 0$*:*

$$u_n = u - \epsilon \sum_{j=0}^{n-1} x_j v_j \left(1 + \mathrm{O}(x_j)\right) \sim -\epsilon \log(\log(n)),$$

for large n. *Therefore* f *cannot have a domain of attraction to the origin whose points converge along the direction* $[1:0:0]$.

Question. Given the assumptions on f, under what conditions (if any) does there exist an invariant attracting domain whose points converge to the origin tangentially to the direction $[v]$?

4.6. Summary of results in $\mathbb{C}^m$, for $m \geq 3$

Let $f \in \mathrm{End}(\mathbb{C}^m, \mathrm{O})$ be tangent to the identity of order $k \geq 2$ with non-degenerate characteristic direction $[v]$. Let $\alpha_1, \ldots, \alpha_{m-1} \in \mathbb{C}$ be the directors corresponding to $[v]$, and let $A = A(v)$ be in Jordan canonical form. In the following table, we summarize the results from this chapter on the existence of invariant attracting domains and submanifolds. In order to simplify the table, let:

U be an open set whose points converge to O, each point along some direction.

M^{d+1} be a submanifold of dimension $d+1$ whose points converge to O tangentially to $[v]$.

$\widehat{\Omega}$ be an invariant attracting domain whose points converge to O tangentially to $[v]$.

Ω be $\widehat{\Omega}$ such that f is conjugate to translation on $\widehat{\Omega}$.

Σ be Ω and, if f is an automorphism, a Fatou-Bieberbach domain.

Assumption	Existence Result	
$\mathrm{Re}(\alpha_j) > 0$ for all j	$\exists\Sigma$	[H1]
	$\exists(k-1)\ \widehat{\Omega}$	[AR, H1]
$\mathrm{Re}(\alpha_j) \geq 0$ for all j, $\mathrm{Re}(\alpha_j) = 0$ for only some j	sometimes $\nexists\widehat{\Omega}$	Ex. 4.1
$\mathrm{Re}(\alpha_j) = 0$ for all j		
• $A \equiv 0$		
• O is dicritical (extends to $[v]$ degenerate)	$\exists U$	[Bro2]
• O non-dicritical	sometimes $\exists\widehat{\Omega}$ & $\nexists\widehat{\Omega}$	Ex. 4.2 & 4.3
• $A \not\equiv 0$	sometimes $\nexists\widehat{\Omega}$	Ex. 4.4
$\mathrm{Re}(\alpha_j) > 0$ for d of the j's	$\exists M^{d+1}$	[H1]
$\mathrm{Re}(\alpha_j) < 0$ for some j	$\nexists\ \widehat{\Omega}$	[H1]
$[v]$ is non-degenerate *or*	$\exists$ at least $k-1$ parabolic	[E, H2]
O is an isolated fixed point	curves along a direction	[A1, AT]

Table 4.1: Summary of the existence of invariant attracting domains, invariant attracting submanifolds, and parabolic curves in $\mathbb{C}^m$ for $m \geq 3$.

There are two main differences in the study of the existence of an invariant attracting domain whose points converge to the origin tangentially to a given direction when we consider maps in $\mathbb{C}^2$ versus maps in $\mathbb{C}^m$, for $m \geq 3$. Table 4.1 and Table 3.1, its $\mathbb{C}^2$ counterpart, clearly illustrate these differences. First of all, there is a classification of characteristic directions in $\mathbb{C}^2$ that has not been extended to $\mathbb{C}^m$; in particular, Fuchsian, irregular, and apparent characteristic directions. Secondly, for a map $f \in \mathrm{End}(\mathbb{C}^m, \mathrm{O})$ that is tangent to the identity, a given non-degenerate characteristic direction of f has only one director when $m = 2$ where as it has multiple directors when $m \geq 3$; directors play a significant role in the existence of invariant attracting domains, so this allows for many more possible situations in higher dimensions. However, in both cases, as we can see from Tables 3.1 and 4.1, there are still many open questions on the existence of invariant attracting domains whose points converge to the origin along a particular direction.

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