Center for Research on Economic and Social Theory

and

Department of Economics

Working Paper Series

Multidimensional Mechanism Design for Auctions with Externalities

Philippe Jehiel, Benny Moldovanu and Ennio Stacchetti

May 1994
Number 94-03

DEPARTMENT OF ECONOMICS
University of Michigan
Ann Arbor, MI 48109-1220

Multidimensional Mechanism Design for Auctions with Externalities

Philippe Jehiel
C.E.R.A.S. Paris
Benny Moldovanu
University of Bonn
and
Ennio Stacchetti
The University of Michigan

Version: May 23, 1994

1. Introduction

A fundamental assumption in the vast literature on optimal auction design is that agents' final payoffs are determined solely by whether or not they obtain the auctioned good, and by the payments made as required by the rules of the auction (see, for example, Myerson (1981), and Milgrom and Weber (1982)). The possibility that the auctioned good might play a role in future interaction among the auction's participants is excluded.

There are many situations, however, in which an auction's participants interact after the close of the auction, and where the outcome of the auction affects the nature of their future interaction. Buyers must take this into account when devising bidding strategies; the seller must design the auction accordingly.

In particular, when a buyer does not obtain the object, he is no longer indifferent about the identity of the winner of the auction. Several illustrations are: changes of ownership (such as mergers or privatizations) in oligopolistic markets; the sale of a patent or of important inputs when there is downstream competition between buyers; the award of major projects that lead to the creation of a new technology standard; the location of environmentally hazardous enterprises such as waste management plants or nuclear reactors; the location of a potentially powerful international organization such as the European Central Bank.

The previous discussion suggests to us a model that differs from most of the literature on optimal auction design, and more broadly, mechanism design problems in two important aspects:

(1) To represent buyer i's payoff when buyer j gets the object, $j = 1, \dots, N$, buyer i is characterized by an N-dimensional vector $t^i = (t^i_1, \dots, t^i_N)$. As usual, the coordinate t^i_i can be thought of as his "private value", while each other coordinate t^i_j can be interpreted as his total payoff in the future interaction, or, say, as his losses caused by pollution, when buyer j gets the object. We use the descriptive term "externalities" to refer to these interaction terms. We are then confronted with a multidimensional mechanism design problem.

(2) Since buyers will generally be unable to escape the effect of externalities simply by refusing to participate in the auction, the agents' "reservation utilities" are neither exogenously given, nor type-independent. Consider the event where a buyer refuses to participate in the auction. In contrast to auctions without externalities, where buyers' reservation values can be normalized to zero, when externalities are present, a buyer's reservation value will, in effect, vary with both his type and the seller's action.

This paper studies the model of auctions with externalities, and while doing so develops techniques that we believe will be valuable in the understanding of other mechanism design problems. We turn now to a brief survey of the most relevant literature, before sketching out our line of attack.

Mechanism design problems arise naturally in numerous economic situations, including the regulation of a monopolist, auctions, government procurement, nonlinear pricing, and the provision of public goods. By and large, most of the vast literature has been restricted to the case in which uncertainty is modelled by a single parameter, and the commodity space has dimension two. In many problems, however, a satisfying analysis demands a multidimensional treatment, but the technical difficulties have impeded our progress.

There are many papers concerning multidimensional mechanism design problems. Two strands of this literature are fairly well developed, the first studying nonlinear pricing in multiproduct monopoly, and the second addressing the regulation of multiproduct monopolists. Among the contributions to the former strand are: Champsaur and Rochet (1989), Laffont, Maskin and Rochet (1987), McAfee, McMillan and Whinston (1989), Mirman and Sibley (1980), Palfrey (1983), Spence (1980), and Wilson (1991, 1993). Notable contributions to the second strand include: Laffont and Tirole (1990), Lewis and Sappington (1988), and Rochet (1984). In addition to these papers, there are a few contributions we would like to discuss because they concentrate on the development of general tools and methods. They are: Armstrong (1992, 1993), Matthews and Moore (1987), McAfee and McMillan (1988), and Rochet (1985, 1992).

McAfee and McMillan (1988) extend the notion of single crossing property (SCP) to the case in which the utility function depends on m commodities, and is characterized by n parameters, where $m \geq 2$, $n \geq 1$, and $n \geq m-1$ (the standard SCP is defined for utility functions that depend on two commodities and one parameter; thus their extension involves both the commodity space and the parameter space). They show that for functions that satisfy the SCP, a mechanism is incentive compatible iff an envelope condition is satisfied. They extend the "no-haggling" result of Riley and Zekhauser (1983) for the problem of a monopolist selling *three* indivisible goods.

Matthews and Moore (1987) study a model with a three-dimensional commodity space, and preferences characterized by a single parameter.[1] They define a new SCP, in type space rather than commodity space, and show that it coincides with the standard SCP when the commodity space has dimension two. They also show that when the "contract set" associated with a mechanism is *attribute-ordered* (or monotone), and their SCP is satisfied, the mechanism is incentive compatible iff it satisfies the adjacent (or local) incentive constraints. In the two-dimensional case, if the utility function satisfies the standard SCP, any mechanism that satisfies the adjacent incentive constraints will have an attribute-ordered contract set, but this doesn't extend to higher dimensional commodity spaces. They present a model similar to that of Mussa and Rosen (1978), with a three-dimensional commodity space, and show that the optimal mechanism's contract set is not attribute-ordered, and therefore that nonlocal incentive constraints are binding; both results are at odds with their two-dimensional counterparts.

Rochet (1985) introduces a conjugate duality approach, and presents a very useful characterization result for incentive compatible mechanisms. The analysis is restricted to utility functions that are linear in their characteristic space. Although this is a severe restriction, utility functions do exhibit the required linear property for many economic models, including our auction problem. Rochet (1992) studies three models with linear utility functions. The first is a multiproduct monopolist problem similar to that studied by McAfee and McMillan (1988) and McAfee, McMillan and Whinston (1988); he shows that, contrary to one-dimensional models, for some choices of parameters, all participation constraints are binding at the optimal mechanism, and the seller extracts all the informational rents. The second is an extension of the Mussa and Rosen (1978) model to the case in which the product has n attributes and consumers' tastes are characterized by n parameters (their marginal value for each attribute). Rochet demonstrates that in an optimal mechanism, upward as well as downward incentive constraints are binding.

The recent work of Armstrong (1992, 1993), in addition to using conjugate duality (and the characterization results of Rochet (1985)), employs the divergence theorem to construct a method for the design of optimal mechanisms. His method resembles very closely that of Myerson (1979), Myerson (1981), and Myerson and Satterthwaite (1983) for the one-dimensional characteristic space. The divergence theorem is the multidimensional analog of integration by parts. Armstrong is able to reduce substantially the complexity of the underlying optimization problem, and to obtain solutions in closed form for classes of problems that satisfy certain regularity conditions. In Armstrong (1992) he studies a multiproduct monopolist pricing problem, and in Armstrong (1993) he investigates the optimal regulation of a multiproduct monopolist.

We view our paper as a contribution to the set of new tools being developed to deal with multidimensional mechanism design problems. In particular, we extend the use of conjugate duality and the divergence theorem to multiagent problems with externalities.

The formulation and simplification of the mechanism design problem is carried

[1] They work with a discrete type space. But the dimensionality of the type space is meaningful because they insist on specific functional forms for the utility function.

out in four major steps (roughly corresponding to Sections 2 through 5). We invoke the revelation principle to formulate the design problem in terms of two functions on the space of type profiles: the probability assignment function, determining the random mechanism to be used for assigning the object to a player, and the payment function specifying the transfer from each buyer to the seller. We then use conjugate duality to derive necessary and sufficient conditions for a mechanism to be incentive compatible, similar to the envelope condition for one-dimensional mechanism design problems. This characterization result allows us to state the seller's problem in a relatively compact format. As is usually done in the one-dimensional case, the envelope condition is used to eliminate half the variables (the payment function) in the problem.

The participation constraints involve an additional problem, not present in an auction without externalities. In one dimensional mechanism design problems, the participation constraints bind only for the "lowest" type. But, as the results in Rochet (1992) suggest, this need not be the case in multidimensional problems. Moreover, in our auction problem, the buyers' reservation values are endogeneous. Thus, we must also construct optimal "threats" to provide incentives for the buyers to participate in the auction. The design of optimal threats is intertwined with the identification of the region where the participation constraints bind. We have identified a symmetric setting for which the seller's problem has a symmetric optimal solution. For symmetric auctions, the participation constraint binds only for the "lowest" type of every buyer, and each buyer is induced to participate by the threat that otherwise the object will be assigned randomly to one of his opponents.

The seller's problem has a rather complex objective function, involving for each buyer, both the conditional expected probability assignment function and its antiderivative. The divergence theorem simplifies the complicated optimization problem at hand, by reducing the objective function to an expression that depends on the probability assignment function only. The divergence theorem plays the same role as integration by parts in the one-dimensional case. If a certain boundary condition is satisfied, the seller's objective function reduces to the expected value of the inner product of an "index function" with the probability assignment function. With the exception of two constraints, the seller's problem can be viewed as a linear programming problem. One of these constraints is the analog of the standard monotonicity condition in one-dimensional mechanism design problems. If we relax these conditions, the relaxed problem is extremely easy to solve, and a solution can be obtained by simple pointwise maximization.

We finally study the relaxed problem. The index function includes terms that are obtained as the solution of certain partial differential equations. These terms are not

uniquely defined because the boundary conditions are not fully specified. We use this degree of freedom to symmetrize the index function. When the index function is symmetric, the relaxed problem also admits a symmetric optimal solution, and the participation constraints bind only at the origin. Obviously, if the solution satisfies the relaxed constraints, then it is also the seller's optimal solution. Myerson (1981) identifies a "regularity condition" on the hazard rate of the buyers' distributions for the independent and private values auction problem. This condition guarantees that the solution of a relaxed problem, similar to ours, is the optimal solution of the seller's problem. Moreover, for problems that do not satisfy the regularity condition, he has a method to modify the objective function so that the problem satisfies the regularity condition, and the solution of its relaxed problem still is the seller's optimal solution. We have not yet found its multidimensional counterpart.

The paper proceeds as follows: Section 2 presents the model and the seller's optimization problem. In Section 3 we introduce the symmetric setting, and establish results that greatly simplify the participation constraints. In Section 4, we use the divergence theorem to transform the seller's objective function. We symmetrize the index function in Section 5, and present examples in Section 6.

2. The Model

There are N buyers, indexed by $i = 1, \ldots, N$, and a seller, designated as player $i = 0$. We will refer to the "players" when we want to include the seller, and to the "buyers" when we want to exclude her, although the seller is not a player of the auction game. Hence, we let $I_0 := \{0, 1, \ldots, N\}$ be the set of players, and $I := \{1, \ldots, N\}$ be the set of buyers. The seller owns a single unit of an indivisible object.

To represent buyer i's externalities, we assume that his type is an N-dimensional vector[2] $t^i = (t^i_1, \ldots, t^i_N)$, where t^i_j is buyer i's payoff when buyer j gets the object. We focus here on the case of negative externalities, that is, on the situation in which any potential buyer perceives a *positive* payoff when he obtains the object, and a *negative* payoff when anybody else gets it (vis-a-vis the case where the seller keeps the object). The seller has an analogous type represented by the N-dimensional vector t^0. Buyer i's type is drawn from $T_i := (b_0, b_1]^{i-1} \times [a_0, a_1) \times (b_0, b_1]^{N-i}$, where $b_1 \leq 0$ and $a_0 \geq 0$, according with the density f_i, and is independent of all other players' types. Thus, the probability that the buyers' types are given by the N-tuple $(t^1, \ldots, t^N) \in T := T_1 \times \cdots \times T_N$ is

[2] In a richer model, we could assume that player i's type is represented by an $(N+1)$-dimensional vector $(t^i_0, t^i_1, \ldots, t^i_N)$, where t^i_0 represents his payoff when the seller keeps the object. Part of the analysis would apply to this model as well. However, the characterization results of Sections 3 and 4 below for symmetric settings do not extend to the richer model.

$f(t) := f_1(t^1) \times \cdots \times f_N(t^N)$. We allow for the possibility that $b_0 = -\infty$ and/or $a_1 = +\infty$. We assume that $f_i(t^i) > 0$ for all $t^i \in T_i$. The "origin" (or "upper-left" corner) of T_i is denoted by $O^i := (b_1, \ldots, b_1, a_0, b_1, \ldots, b_1)$.

Buyers' types are private information: buyer i knows his own type but he doesn't know anybody else's. The seller's type can be made publicly known. Although the seller's type is revealed (partially) through the choice of the auction mechanism, this information does not affect the buyers' expected payoffs.

Buyer i's utility is additively separable: if he pays x_i to the seller and player j gets the object, his utility is $t^i_j - x_i$, where $t^i_0 := 0$. In general, a buyer's payment need not be zero even if he doesn't get the object The seller's utility is also additively separable: if buyer i pays her x_i, $i \in I$, and she gives the object to player j, her utility is $t^0_j + x_1 + \cdots + x_N$, where $t^0_0 := 0$.

We are concerned with the design of an auction that maximizes revenue for the seller. By the Revelation Principle, there is no loss of generality in restricting attention to direct revelation mechanisms where each buyer reports a type. Moreover, it is enough to consider mechanisms that are incentive compatible and that satisfy the participation constraints, that is, mechanisms for which it is a Bayesian equilibrium for each buyer to report his type truthfully. Since a buyer cannot be forced to "participate" in the auction, "nonparticipation" must be included among his possible reports. Let $\Sigma := \{q \in \mathbf{R}^N_+ \mid \sum q_i \leq 1\}$ be the set of *probability vectors*; the coordinate q_i of a probability vector q represents the probability that buyer i gets the object, while $1 - \sum_{i=1}^N q_i$ represents the probability that the seller keeps the object. The seller specifies the rules of the auction in terms of a *revelation mechanism* (ρ, x, p), where $\rho = (\rho^1, \ldots, \rho^N) \in \Sigma^N$ is a profile of N probability vectors, $x_i : T \to \mathbf{R}$, $i \in I$, and $p : T \to \Sigma$. The seller asks each of the buyers simultaneously to report a type. If all buyers submit a type and the report profile is $(t^1, \ldots, t^N) \in T$, buyer i must pay the seller $x_i(t^1, \ldots, t^N)$, and he gets the object with probability $p_i(t^1, \ldots, t^N)$. If buyer i refuses to participate in the auction while all other buyers submit a report, the object is given to buyer j with probability ρ^i_j, $j \in I$,[3] and no buyer makes a payment to the seller. If two or more buyers refuse to submit a report, then, say,[4] the seller keeps the object with probability 1 and nobody makes any payments.

[3] The probability vector ρ^i is designed to "punish" buyer i when he refuses to report his type. Since $t^i_j \leq 0$ for all $j \neq i$ and $t^i_0 = 0$, the threat is more severe when the seller reduces the probability of keeping the object and/or the probability of giving the object to buyer i. Therefore, without loss of generality, we will require that for each buyer i, $\rho^i_i = 0$ and $1 - \sum_{j=1}^N \rho^i_j = 0$.

[4] We study the Nash equilibria of the game, and disregard the possibility of coalition formation. Hence, profitable multiple deviations are irrelevant. If the domain of the function p were extended to include profiles where some players report "nonparticipation", then the vectors ρ^i could be included as part of the definition of p. We have chosen the domain T mainly to simplify the notation below; in an honest equilibrium, no player reports "nonparticipation".

Suppose player i believes everybody else reports truthfully. Then, to assess the expected value of any of his reports, he only needs to know the conditional expected value, given his own type, of his payment and the probability assignment vector. Define then the functions $y_i : T_i \to \mathbf{R}$ and $q^i : T_i \to \Sigma$ as follows:

$$y_i(t^i) := \int_{T_{-i}} x_i(t^1, \ldots, t^N) f_{-i}(t^{-i}) dt^{-i}$$

$$q^i_j(t^i) := \int_{T_{-i}} p_j(t^1, \ldots, t^N) f_{-i}(t^{-i}) dt^{-i}.$$

We will refer to these functions as buyer i's *conditional expected payment* and *conditional expected probability assignment* in the mechanism (ρ, x, p). If buyer i believes his opponents will report truthfully, and reports type s^i when his type is t^i, his expected utility is $U(s^i, t^i) := q^i(s^i) \cdot t^i - y_i(s^i)$.

The auction mechanism (ρ, x, p) is said to be *incentive compatible* for buyer i if

$$U_i(t^i, t^i) \geq U_i(s^i, t^i) \quad \text{for all } s^i, t^i \in T_i,$$

and to satisfy the *participation constraints* for buyer i if

$$U_i(t^i, t^i) \geq \rho^i \cdot t^i \quad \text{for all } t^i \in T_i.$$

The right hand side of the last inequality is buyer i's expected value when he doesn't make any payments to the seller, and the seller assigns the object randomly according with the probability vector ρ^i. The auction mechanism is *feasible* if it is incentive compatible and satisfies the participation constraints for every buyer.

Clearly, if (ρ, x, p) is a feasible mechanism, so is $(\rho, \overline{x}, p)$, where $\overline{x}_i(t) := y_i(t^i)$ for all $t \in T$. Moreover, with $\overline{x}$ the buyers expect to make the same payment to the seller as with x. Thus, there is no loss of generality in restricting attention to mechanisms for which the payment of each player depends only on his own report. Consequently, we will specify below auction mechanisms directly in terms of (ρ, y, p).

The seller wants to maximize total expected revenue plus the expected value of her externality. Therefore her problem is

$$(P) \qquad \max_{(\rho, y, p)} \sum_{i=1}^N \int_{T_i} y_i(t^i) f_i(t^i) dt^i + \int_T p(t) \cdot t^0 f(t) dt$$
$$s.t. \quad \text{incentive compatibility and participation constraints.}$$

Suppose that there are two types s^i and t^i such that $q^i(s^i) = q^i(t^i)$ and $y_i(s^i) > y_i(t^i)$. If buyer i is of type s^i and believes everybody else reports truthfully, he would strictly prefer to report t^i than to report s^i, and truthtelling would not be incentive compatible. Therefore, for any incentive compatible mechanism, the "contract surface" $C_i := \{ (q^i(t^i), y_i(t^i)) \mid t^i \in T_i \} \subset \mathbf{R}^{N+1}$ for buyer i intersects any vertical line (in the last coordinate) at most once, and hence is also defined *explicitly* by a function $Y_i : \Sigma \to \overline{\mathbf{R}}$, where $\overline{\mathbf{R}} = \mathbf{R} \cup \{ +\infty \}$ is the extended real line. That is, $C_i = \{ (q, Y_i(q)) \mid q \in \Sigma \}$ and $y_i(t^i) = Y_i(q^i(t^i))$.

Buyer i's *surplus function* $S_i : T_i \to \mathbf{R}$ is the conjugate of the function Y_i:

$$S_i(t^i) := \sup \{ q \cdot t^i - Y_i(q) \mid q \in \Sigma \}.$$

This optimization problem determines buyer i's optimal report decision when he is presented with the "menu" C_i and his type is t^i. Various properties of S_i are readily available and have been previously recorded by Armstrong (1992) and Rochet (1985) (for a general reference on conjugate duality and properties of convex functions, the reader should consult the classical books by Rockafellar (1972, 1974)). S_i is convex, continuous, and monotonically increasing. Let $\partial S_i(t^i)$ denote the subdifferential of S_i at t^i. Then $q \in \partial S_i(t^i)$ iff q attains the sup in the definition of $S_i(t^i)$, that is, iff $q \in \Sigma$ and $S_i(t^i) = q \cdot t^i - Y_i(q)$. Thus, (ρ, y, p) is incentive compatible for buyer i iff $q^i(t^i) \in \partial S_i(t^i)$ for all $t^i \in T_i$ and

$$y_i(t^i) = Y_i(q^i(t^i)) = q^i(t^i) \cdot t^i - S_i(t^i).$$

Moreover, since S_i is convex, S_i is differentiable almost everywhere, and if S_i is differentiable at t^i, $\partial S_i(t^i) = \{\nabla S_i(t^i)\}$. Thus $q^i(t^i) = \nabla S_i(t^i)$ almost everywhere in T_i.

The property that $q^i(t^i) \in \partial S_i(t^i)$ for every $t^i \in T_i$ is the familiar "envelope condition". Let (ρ, y, p) be an incentive compatible auction mechanism. If S_i is differentiable at t^i, we have

$$\frac{dS_i}{dt^i_j}(t^i) = \frac{dU_i}{dt^i_j}(t^i, t^i) = \frac{\partial U_i}{\partial t^i_j}(s^i, t^i)\Big|_{s^i = t^i} = q^i_j(t^i).$$

Since S_i is differentiable and $\nabla S_i(t^i) = q^i(t^i)$ almost everywhere,

$$S_i(t^i) = S_i(s^i) + \int_{s^i}^{t^i} q^i(\tau) \cdot d\tau \quad \text{for all } s^i, t^i \in T_i.$$

The integral in the right hand side is a line integral, which does not depend on the specific path from s^i to t^i. That is, the vector field q^i must be *conservative*.[5] The following

[5] A vector field (or function) $\varphi : \mathbf{R}^N \to \mathbf{R}^N$ is *conservative* if it is the gradient of a function $\Phi : \mathbf{R}^N \to \mathbf{R}$. If φ is differentiable, φ is conservative iff $\partial\varphi_i/\partial\tau_j \equiv \partial\varphi_j/\partial\tau_i$ for all $i \neq j$.

proposition summarizes these results. They are stated in terms of the *potential function* $Q_i(t^i) := S_i(t^i) - S_i(O^i)$, where $S_i(O^i) = q^i(O^i) \cdot O^i - y_i(O^i)$. The term $q^i(O^i) \cdot O^i$ appears numerous times, and will be denoted by $z_i(O^i)$ (obviously, it depends on p).

PROPOSITION 1: *Consider the auction mechanism (ρ, y, p), and let q^i, $i \in I$, be the corresponding conditional probability assignment functions. Then,*

(1) The mechanism is incentive compatible for buyer i iff the vector field $q^i : T_i \to \mathbf{R}^N$ is conservative, the potential function $Q_i : T_i \to \mathbf{R}$, defined by

$$Q_i(t^i) := \int_{O^i}^{t^i} q^i(s^i) \cdot ds^i \qquad \text{for each } t^i \in T_i$$

is convex, and

$$y_i(t^i) = q^i(t^i) \cdot t^i - Q_i(t^i) + y_i(O^i) - z_i(O^i) \qquad \text{for each } t^i \in T_i.$$

(2) The mechanism satisfies the participation constraints for buyer i iff

$$Q_i(t^i) - y_i(O^i) + z_i(O^i) \geq \rho^i \cdot t^i \qquad \text{for all } t^i \in T_i.$$

Note that the potential function Q_i is well defined when q^i is conservative, and is nondecreasing in each of its arguments because $q^i(t^i)$, being a probability vector, is always nonnegative. The condition that Q_i be convex is equivalent to the condition that q^i be monotone. The function q^i is *monotone* if for every $s^i, t^i \in T_i$, $(s^i - t^i) \cdot (q^i(s^i) - q^i(t^i)) \geq 0$. This is the standard extension to vector-valued functions of the familiar notion of monotonicity for real-valued functions.

Part (1) of Proposition 1 states that for incentive compatible mechanisms, the expected payment function y_i is uniquely determined by $y_i(O^i)$ and the probability assignment function p. Therefore it can be eliminated from the seller's design problem, as it is usually done in one-dimensional mechanism design problems.

By Proposition 1, if the auction is incentive compatible, the seller expects an average payment from buyer i equal to

$$\int_{T_i} y_i(t^i) f_i(t^i) dt^i = \int_{T_i} \Big(q^i(t^i) \cdot t^i - Q_i(t) + y_i(O^i) - z_i(O^i)\Big) f_i(t) dt^i.$$

Let $y(O) := (y_1(O^1), \ldots, y_N(O^N))$, $z(O) := (z_1(O^1), \ldots, z_N(O^N))$, and for later use as well, let $e := (1, \ldots, 1) \in \mathbf{R}^N$. The seller's problem is then

$$(P) \max_{(\rho, y(O), p)} \sum_{i=1}^N \int_{T_i} \left(q^i(t^i) \cdot t^i - Q_i(t^i)\right) f_i(t^i) dt^i + \int_T p(t) \cdot t^0 f(t) dt + e \cdot (y(O) - z(O))$$

$$\text{s.t.} \quad Q_i(t^i) - y_i(O^i) + z_i(O^i) \geq \rho^i \cdot t^i \quad t^i \in T_i, \quad i \in I$$

$$q^i \text{ is a monotone and conservative field}, \quad i \in I.$$

Obviously, the seller would like to maximize $y_i(O^i)$ for each buyer i. However, the participation constraints set an upper bound on $y_i(O^i)$. Given a probability assignment function p and a probability vector ρ^i, buyer i's participation constraints are satisfied iff $y_i(O^i) \le z_i(O^i) - Q_i^*(\rho^i)$, where

$$Q_i^*(\rho^i) := \sup_{t^i \in T_i} \left[\rho^i \cdot t^i - Q_i(t^i)\right].$$

(Q_i^* denotes the conjugate function of Q_i in the convex sense.) Optimally, the seller chooses $y_i(O^i) = z_i(O^i) - Q_i^*(\rho^i)$, and the seller's objective function becomes

$$\int_T p(t) \cdot \left(t^0 + \cdots + t^N\right) f(t)dt - \sum_{i=1}^N \int_{T_i} Q_i(t^i) f_i(t^i) dt^i - \sum_{i=1}^N Q_i^*(\rho^i).$$

In the next section we study a symmetric case, for which we can find ρ and $y(O)$ explicitly.

3. Symmetry and the Participation Constraints

The seller's problem has a rather large and complex set of constraints. Similarly to the techniques used to solve one-dimensional mechanism design problems, we will relax the last two constraints. Concerning the participation constraints, we will prove below that for a class of symmetric problems, it is enough to check these constraints at the origin. Note that since $Q_i(O^i) = 0$ by definition, buyer i's participation constraint at the origin is equivalent to $y_i(O^i) \le z_i(O^i) - b_1$. For general problems (symmetric or otherwise), it is sufficient to verify the participation constraints at the "upper" boundary of every type set. The *upper boundary* of T_i is the set of points t^i in T_i for which $t^i + \alpha e \notin T_i$ for any $\alpha > 0$.

LEMMA 1: *Suppose the auction mechanism (ρ, y, p) is incentive compatible. Then, for each i, the potential function Q_i satisfies the inequality $Q_i(t^i + \alpha e) \le Q_i(t^i) + \alpha$ for each $t^i \in T_i$ and $\alpha \in \mathbf{R}_+$ such that $t^i + \alpha e \in T_i$.*

PROOF: Since $\nabla Q_i(t^i) = q_i(t^i) \in \Sigma$ almost everywhere, we have

$$\sum_{j=1}^N \frac{\partial Q_i}{\partial t_j^i} \le 1$$

almost everywhere. This implies that $\frac{d}{d\alpha} Q_i(t^i + \alpha e) \le 1$ almost everywhere. ∎

PROPOSITION 2: *Suppose the auction mechanism (ρ, y, p) is incentive compatible. If (ρ, y, p) satisfies the participation constraints at the upper boundary of T_i, then (ρ, y, p) satisfies the participation constraints everywhere in T_i.*

PROOF: Suppose that the mechanism (ρ, y, p) satisfies the participation constraints at the upper boundary, and consider a type t^i not on the upper boundary of T_i. There exists a type s^i at the upper boundary of T_i and a scalar $\alpha > 0$ such that $t^i = s^i - \alpha e$. Therefore, since $\rho^i \cdot e = 1$,

$$Q_i(t^i) - y_i(O^i) + z_i(O^i) \ge Q_i(s^i) - y_i(O^i) + z_i(O^i) - \alpha \ge \rho^i \cdot s^i - \alpha = \rho^i \cdot t^i. \quad ∎$$

A *permutation* of I is any bijection $\pi : I \to I$. A permutation π *fixes* i if $\pi(i) = i$. The set of all permutations is denoted by Φ, and Φ_i denotes the set of all permutations that fix i. For each permutation π of I, we define the function $\Pi : \mathbf{R}^N \to \mathbf{R}^N$ as follows: $\Pi(\tau) := (\tau_{\pi^{-1}(1)}, \ldots, \tau_{\pi^{-1}(N)})$ for each $\tau \in \mathbf{R}^N$.

Let π_{ij} be the *simple permutation* that switches the indices i and j: $\pi_{ij}(i) = j$, $\pi_{ij}(j) = i$, and $\pi_{ij}(k) = k$ for all $k \notin \{i, j\}$, and let $\Pi_{ij} : \mathbf{R}^N \to \mathbf{R}^N$ be the corresponding map switching coordinates i and j. Clearly, $\pi_{ij}^{-1} = \pi_{ij}$; this observation is used repeatedly below. Any permutation of I can be decomposed into simple permutations. For $t \in (\mathbf{R}^N)^N$, $\Pi_{ij}^*(t)$ will denote the profile of vectors $(s^1, \ldots, s^N)$, where

$$s^k = \Pi_{ij}(t^k) \quad \text{for all } k \notin \{i, j\},$$
$$s^i = \Pi_{ij}(t^j), \quad \text{and} \quad s^j = \Pi_{ij}(t^i).$$

Note that in this profile $\Pi_{ij}(t^i)$ is in the j-th position, while $\Pi_{ij}(t^j)$ is in the i-th. Let π be any permutation of I, and suppose it can be obtained as the composition $\pi_{ij} \circ \pi_{kl} \circ \cdots \circ \pi_{rs}$ of simple permutations. Then, $\Pi^* : (\mathbf{R}^N)^N \to (\mathbf{R}^N)^N$ denotes the function obtained as the composition $\Pi_{ij}^* \circ \Pi_{kl}^* \circ \cdots \circ \Pi_{rs}^*$.

For the rest of the paper, we confine attention to *symmetric settings* that satisfy the following properties:

(S_1) $t_i^0 = t_j^0$ for all $i, j \in I$.
(S_2) for each $i \in I$, type t^i, and permutation π that fixes i, $f_i(\Pi(t^i)) = f_i(t^i)$.
(S_3) for any $i \ne j$ and type t^i, $f_i(t^i) = f_j(\Pi_{ij}(t^i))$.

Condition (S_1) says that no matter which buyer gets the object, the seller suffers the same externality. (S_2) states that the probabilities that buyer i is of type t^i and that buyer i is of another type equal to t^i but with coordinates j and k (where $j \ne i$ and $k \ne i$) switched, are the same. Finally, (S_3) says that types for each player are drawn from the "same" distribution.

The mechanism (ρ, y, p) is *symmetric* for buyer i if for each permutation $\pi \in \Phi_i$,

(1) $\rho^i_i = 0$ and $\rho^i_j = \frac{1}{N-1}$ for all $j \neq i$.

(2) $y_i(\Pi(t^i)) = y_i(t^i)$, for all $t^i \in T_i$.

(3) $q^i(\Pi(t^i)) = \Pi(q^i(t^i))$, for all $t^i \in T_i$.

We will say that the function q^i is *symmetric* if it satisfies condition (3) above, and that p is *symmetric* if each q^i is symmetric.

Condition (3) says that if buyer i's report were changed from t^i to s^i, where s^i is the vector t^i with coordinates $j \neq i$ and $k \neq i$ switched, then player i's conditional probabilities of assigning the object to players j and k would be switched while all other assignment probabilities remain the same. Part (1) of Proposition 1 implies that if (ρ, y, p) is incentive compatible and q^i is symmetric, then y_i must satisfy condition (2) above.

Lemma 2 below shows that the following condition implies that p is symmetric. For each profile $t = (t^1, \ldots, t^N)$, when buyers j and k trade places, each player's probability of getting the object is unaffected. That is, $p_i(\Pi_{jk}(t)) = p_i(t)$ for each i different from j and k, and $p_j(\Pi_{jk}(t)) = p_k(t)$ and $p_k(\Pi_{jk}(t)) = p_j(t)$. Recall that for a player i different from j and k, the i-th vector in the profile $\Pi^{\pi}_{jk}(t)$ has entries j and k switched. Roughly then, since players j and k have the "same" probability distribution over types, player i's expected probability assignment vector is the same when his type is $\Pi_{jk}(t^i)$ as when his type is t^i.

LEMMA 2: *Suppose $p : T \to \Sigma$ has the property that for every simple permutation π_{ij} with $i < j$, and every $t \in T$, $p(\Pi^{\pi}_{ij}(t)) = \Pi_{ij}(p(t))$. Then p is symmetric.*

PROOF: Since a permutation π that fixes i can be decomposed into simple permutations that never move i, to prove that p is symmetric, it is enough to check that for any three different indices i, j, and k, $q^i(\Pi_{jk}(t^i)) = \Pi_{jk}(q^i(t^i))$ for all $t^i \in T_i$.

Let $t \in T$ and i, j, and k be three different indices. We have

$$\begin{aligned}
q^i(\Pi_{jk}(t^i)) &= \int_{T_{-i}} p(\Pi_{jk}(t^i), t^{-i})\, f_{-i}(t^{-i})\, dt^{-i} \\
&= \int_{T_{-i}} p(\Pi^{\pi}_{jk}(t))\, f_{-i}(\Pi^{\pi}_{jk}(t)^{-i})\, dt^{-i} \\
&= \int_{T_{-i}} \Pi_{jk}(p(t))\, f_{-i}(\Pi^{\pi}_{jk}(t)^{-i})\, dt^{-i} && \text{(by assumption)} \\
&= \int_{T_{-i}} \Pi_{jk}(p(t))\, f_{-i}(t^{-i})\, dt^{-i} && \text{(by } (S_2) \text{ and } (S_3)) \\
&= \Pi_{jk}(q^i(t^i))
\end{aligned}$$

∎

It is easy to see that the set of feasible triples (ρ, y, p) for problem (P) is convex.

That is, for any two (ρ, y, p) and $(\hat{\rho}, \hat{y}, \hat{p})$ satisfying the constraints of problem (P), and any $\lambda \in (0,1)$, the triple $\lambda(\rho, y, p) + (1-\lambda)(\hat{\rho}, \hat{y}, \hat{p})$ also satisfies the constraints of (P) (including the constraint that for each i, $\lambda q^i + (1-\lambda)\hat{q}^i$ is a monotone and conservative field). Moreover, the objective function of (P) is linear in p and $y(O)$, and thus its value at $\lambda(\rho, y, p) + (1-\lambda)(\hat{\rho}, \hat{y}, \hat{p})$ is equal to the convex combination of its values at (ρ, y, p) and $(\hat{\rho}, \hat{y}, \hat{p})$. This observations are used in the following proposition to construct a symmetric optimal mechanism.

PROPOSITION 3: *Suppose the setting is symmetric. Then, for any feasible auction mechanism (ρ, y, p) there exists a symmetric auction mechanism $(\overline{\rho}, \overline{y}, \overline{p})$ which is feasible and gives the seller the same expected revenue.*

PROOF: Suppose (ρ, y, p) is a feasible auction mechanism. We now construct a symmetric mechanism $(\overline{\rho}, \overline{y}, \overline{p})$ as follows: $\overline{\rho}^i_i = 0$ and $\overline{\rho}^i_j = \frac{1}{N-1}$ for all $j \neq i$,

$$\overline{p}_k(t) := \frac{1}{N!} \sum_{j=1}^{N} \sum_{\pi \in \Phi_j} p_j(\Pi^{\pi} \circ \Pi^{\pi}_{kj}(t)) \quad k = 1, \ldots, N, \quad t \in T,$$

and $\overline{y}_i(t^i) := \overline{q}^i(t^i) \cdot t^i - \overline{Q}_i(t^i) + y_0$ for each $i = 1, \ldots, N$ and $t^i \in T_i$, where $y_0 := \frac{1}{N}\sum_{j=1}^{N}(y_j(O^j) - z_j(O^j))$. Clearly, $\overline{p}(t) \geq 0$, for all $t \in T$, and for any j,

$$\{\, \Pi^{\pi} \mid \pi \in \Phi \,\} = \{\, \Pi^{\pi} \circ \Pi^{\pi}_{kj} \mid \pi \in \Phi_j, \text{ and } k = 1, \ldots, N \,\}.$$

Therefore, for each $t \in T$,

$$\sum_{k=1}^{N} \overline{p}_k(t) = \frac{1}{N!} \sum_{\pi \in \Phi} \sum_{j=1}^{N} p_j(\Pi^{\pi}(t)) = 1.$$

Tedious but simple computations show that for each $i \neq j$, $\overline{p}(\Pi^{\pi}_{ij}(t)) = \Pi_{ij}(\overline{p}(t))$ for each $t \in T$. Hence, by Lemma 2, $\overline{p}$ is symmetric.

Fix i; we have that

$$\begin{aligned}
\overline{q}^i_k(t^i) &= \frac{1}{N!} \sum_{j=1}^{N} \sum_{\pi \in \Phi_j} \int_{T_{-i}} p_j(\Pi^{\pi} \circ \Pi^{\pi}_{kj}(t)) f_{-i}(t^{-i}) dt^{-i} \\
&= \frac{1}{N!} \sum_{j=1}^{N} \sum_{\pi \in \Phi_j} q_j^{\pi \circ \pi_{kj}(i)}(\Pi \circ \Pi_{kj}(t^i)).
\end{aligned}$$

But, for any k (in particular for i), the collection of sets

$$\{\, \pi \circ \pi_{kj} \mid \pi \in \Phi_j \,\} \quad j = 1, \ldots, N$$

is a partition of Φ; the set for a given j contains all the permutations that send k to j. Therefore, for each j and k, and each $\pi \in \Phi_j$, there exists l and $\hat{\pi} \in \Phi_l$ such that $\hat{\pi} \circ \pi_{il} \equiv \pi \circ \pi_{kj}$. Moreover, $\hat{\pi} \circ \pi_{il}(i) = l$, and also $j = \pi \circ \pi_{kj}(k) = \hat{\pi} \circ \pi_{il}(k)$. Hence

$$\overline{q}^i_k(t^i) = \frac{1}{N!} \sum_{l=1}^{N} \sum_{\hat{\pi} \in \Phi_l} q^l_{\hat{\pi} \circ \pi_{il}(k)}(\hat{\Pi} \circ \Pi_{il}(t^i)),$$

and

$$\overline{q}^i(t^i) = \frac{1}{N!} \sum_{l=1}^{N} \sum_{\hat{\pi} \in \Phi_l} \Pi_{il} \circ \hat{\Pi}^{-1} \circ q^l(\hat{\Pi} \circ \Pi_{il}(t^i)).$$

This implies that

$$\overline{Q}_i(t^i) = \frac{1}{N!} \sum_{j=1}^{N} \sum_{\pi \in \Phi_j} Q_j(\Pi \circ \Pi_{ij}(t^i)).$$

Since the mechanism satisfies the participation constraints for each player, we have that for each j and $\pi \in \Phi_j$,

$$Q_j(\Pi \circ \Pi_{ij}(t^i)) - y_j(O^j) + z_j(O^j) \geq \rho^j \cdot \Pi \circ \Pi_{ij}(t^i) = \Pi_{ij} \circ \Pi^{-1}(\rho^j) \cdot t^i.$$

It is easy to see that

$$\frac{1}{(N-1)!} \sum_{\pi \in \Phi_j} \Pi^{-1}(\rho^j) = \overline{\rho}^j \quad \text{and} \quad \Pi_{ij}(\overline{\rho}^j) = \overline{\rho}^i.$$

Hence, $(\overline{\rho}, \overline{y}, \overline{p})$ satisfies the participation constraints:

$$\overline{Q}_i(t^i) - y_0 \geq \frac{1}{N} \sum_{j=1}^{N} \overline{\rho}^i \cdot t^i = \overline{\rho}^i \cdot t^i,$$

If $g, h : T_i \to \mathbf{R}^N$ are two monotone and conservative vector fields, then $\lambda g + \mu h$ is also a monotone and conservative vector field for all $\lambda, \mu > 0$. Therefore, part (1) of Proposition 1 implies that $(\overline{\rho}, \overline{y}, \overline{p})$ satisfies the incentive compatibility constraints. Finally, the objective function of the seller evaluated at $\overline{p}$ and $y(O)$ is just the average over all permutations $\pi \in \Phi$ of the objective function evaluated at each $\Pi(p)$ and $\Pi(y(O))$. But, the value of the objective function at $\Pi(p)$ and $\Pi(y(O))$ is the same as the value at p and $y(O)$. Therefore, the symmetric mechanism $(\overline{\rho}, \overline{y}, \overline{p})$ gives the seller the same expected revenue as (ρ, y, p). ∎

Proposition 3 implies that there is an optimal solution $(\overline{\rho}, \overline{y}, \overline{p})$ of problem (P) which is symmetric. Therefore, the seller can restrict attention to symmetric auction mechanisms without any loss in expected revenue.

Proposition 4: *Let (ρ, y, p) be any incentive compatible auction mechanism such that for a given i, ρ^i satisfies the requirements of footnote 3, q^i is symmetric, and buyer i's participation constraint at the origin is satisfied: $y_i(O^i) \leq z_i(O^i) - b_1$. Then (ρ, y, p) satisfies all the participation constraints for player i.*

Proof: We will show that if the mechanism violates the participation constraint for a buyer i at a point other than the origin, then it also violates the participation constraint for buyer i at the origin. We only study the case in which $i = 1$. Suppose that t^1 is such that $Q_1(t^1) - y_1(O^i) + z_1(O^1) < \rho^1 \cdot t^1$. Consider the sequence of $N-1$ vectors $t^{1,1}, t^{1,2}, \ldots, t^{1,N-1}$ obtained from t^1 by sequentially rotating its coordinates $2, \ldots, N$, while coordinate 1 remains fixed: $t^{1,1} := t^1$, $t^{1,2} := (t^1_1, t^1_3, \ldots, t^1_N, t^1_2), \ldots, t^{1,N-1} := (t^1_1, t^1_N, t^1_2, \ldots, t^1_{N-1})$. Since q^1 is symmetric, $Q_1(t^{1,k}) = Q_1(t^1)$ for all k, and since $\rho^1 \cdot t^{1,k} = \rho^1 \cdot t^1$, the participation constraint is violated for each $t^{1,k}$. Define

$$s^1 := \frac{1}{N-1} \sum_k t^{1,k} = (t^1_1, \alpha, \ldots, \alpha), \quad \text{where} \quad \alpha := \frac{1}{N-1} \sum_{j \neq 1} t^1_j < b_1.$$

Since Q_1 is convex,

$$\begin{aligned} Q_1(s^1) - y_1(O^1) + z_1(O^1) &\leq \frac{1}{N-1} \sum_k \left[Q_1(t^{1,k}) - y_1(O^1) + z_1(O^1) \right] \\ &< \frac{1}{N-1} \sum_k \rho^1 \cdot t^{1,k} = \rho^1 \cdot s^1. \end{aligned}$$

Hence, the participation constraint is violated at s^1. Let $r^1 := s^1 - (\alpha - b_1)e = (t^1_1 - \alpha + b_1, b_1, \ldots, b_1)$. By Lemma 1, r^1 also violates the participation constraint: $Q_1(r^1) - y_1(O^1) + z_1(O^1) \leq Q_1(s^1) - (\alpha - b_1) - y_1(O^1) + z_1(O^1) < \rho^1 \cdot s^1 - (\alpha - b_1) = \rho^1 \cdot r^1 = b_1$ (the last two equalities attain because $\sum_{j=1}^{N} \rho^1_j = 1$ and $\rho^1_1 = 0$). Finally, since Q_1 is monotone and $b_1 - \alpha > 0$, $y_1(O^1) - z_1(O^1) > Q_1(r^1) - b_1 \geq Q_1(O^1) - b_1 = -b_1$; this is a contradiction. ∎

Proposition 4 shows that for any incentive compatible auction mechanism (ρ, y, p) for which q^i is symmetric, the participation constraints of buyer i bind only at the origin. But, the reservation value of a buyer i with type O^i is b_1, regardless of how ρ^i is chosen, as long as ρ^i satisfies the assumptions of footnote 3. So, for *any* ρ^i satisfying these assumptions, if q^i is symmetric and $y_i(O^i) - z_i(O^i) \leq -b_1$, (ρ, y, p) satisfies *all* the participation constraints for player i (that is, we don't really need condition (1) in the definition of a symmetric mechanism).

The seller wants to choose $y_i(O^i)$ as large as possible, but the participation constraint for buyer i at the origin requires that $y_i(O^i) \leq z_i(O^i) - b_1$. Hence, the seller will optimally set $y(O^i) = z_i(O^i) - b_1$. Buyer i's payment to the seller can be decomposed into two terms: an *entry fee*, represented by $y(O^i)$, and a transfer $q^i(t^i)\cdot t^i - Q_i(t^i) - z_i(O^i)$ required to increase buyer i's chances of getting the object and/or to decrease the chances that harmful opponents get the object. In general, the latter could be negative, in which case the principal returns money to the buyer. However, each buyer's net payment in an optimal symmetric auction is always positive.

PROPOSITION 5: *In a symmetric optimal auction, no buyer is ever subsidized by the seller.*

PROOF: . Consider the straight path from O^i to t^i defined by $h(x) := O^i + x(t^i - O^i)$, $x \in [0,1]$. For each $t^i \in T_i$, we have

$$q^i(t^i)\cdot t^i - Q_i(t^i) = q^i(t^i)\cdot O^i + \int_0^1 \left(q^i(t^i) - q^i(h(x))\right)\cdot (t^i - O^i)\, dx,$$

and since q^i is monotone, $(q^i(t^i)-q^i(h(x)))\cdot(t^i-O^i) = \frac{1}{1-x}(q^i(t^i)-q^i(h(x)))\cdot(t^i-h(x)) \geq 0$ for each $x \in (0,1)$. Therefore,

$$\begin{aligned} y_i(t^i) &= y_i(O^i) - z_i(O^i) + q^i(t^i)\cdot t^i - Q_i(t^i) \\ &\geq -b_1 + q^i(t^i)\cdot O^i = q^i_i(t^i)(a_0 - b_1) \geq 0. \end{aligned}$$

∎

In a symmetric setting, the seller's problem reduces to

$$(P) \qquad \begin{aligned} \max_p \quad & \int_T p(t)\cdot\left(t^0 + \cdots + t^N\right) f(t)dt - \sum_{i=1}^N \int_{T_i} Q_i(t^i) f_i(t^i) dt^i - Nb_1 \\ \text{s.t.} \quad & p : T \to \Sigma \\ & q^i \text{ is a monotone and conservative field, } i \in I. \end{aligned}$$

A standard technique to construct solutions of one-dimensional mechanism design problems is to relax the monotonicity constraint. Let (R) denote the relaxed problem obtained from (P) when all constraints, except that requiring that $p(T) \subset \Sigma$, are dropped. A similar argument to that of Proposition 3 shows that (R) has a symmetric optimal solution. Moreover, if p is any solution of (R) such that q^i is a monotone and conservative field for each i, then a symmetric solution $\overline{p}$ of (R) can be constructed so that $\overline{q}^i$ is monotone and conservative for each i. Suppose $\overline{p}$ is a symmetric solution of (R) that satisfies these constraints, and let $(\overline{p}, \overline{y}, \overline{p})$ be the auction mechanism where for each $i \in I$, $\overline{\rho}^i_i := 0$, $\overline{\rho}^i_j := \frac{1}{N-1}$ for all $j \neq i$, and $\overline{y}_i(t^i) := \overline{q}^i(t^i)\cdot t^i - \overline{Q}_i(t^i) - b_1$ for each $t^i \in T_i$. Then, $(\overline{p}, \overline{y}, \overline{\rho})$ is an optimal auction mechanism.

4. The Divergence Theorem

We now simplify the objective function of the seller's problem. We use the divergence theorem and adapt the techniques of Armstrong (1993) to deal with the seller's problem.

Let $\Psi : \mathbf{R}^k \to \mathbf{R}^k$ be a continuously differentiable vector field. The *divergence* of Ψ is the function $\operatorname{div}\Psi : \mathbf{R}^k \to \mathbf{R}$ defined by

$$\operatorname{div}\Psi(\tau) := \sum_{i=1}^k \frac{\partial \Psi_i}{\partial \tau_i}(\tau).$$

If U is a bounded region in $\mathbf{R}^k$ having a piecewise smooth boundary ∂U, the divergence theorem asserts that

$$\int_U \operatorname{div}\Psi dV = \int_{\partial U} \Psi \cdot dS,$$

where dS denotes the *surface element*, and is equal to $n\, d\sigma$ (n denotes the unitary exterior normal vector of the boundary ∂U, and $d\sigma$ is the surface area differential). The divergence theorem also holds in some cases for which U is unbounded. For example, if $U = [0, +\infty)^k$, the divergence theorem is valid, provided that

$$\lim_{R\to\infty} \int_{S(R)} \Psi \cdot dS = 0,$$

where $S(R)$ is the intersection of the sphere of radius R and center at the origin with the positive orthant.

Let $F^i : \mathbf{R}^N \to \mathbf{R}^N$ be a solution of the differential equation $\operatorname{div} F^i = f_i$ in T_i. Then, $\operatorname{div}(Q_i F^i) = \nabla Q_i \cdot F^i + Q_i f_i$, and the divergence theorem implies that

$$\int_{T_i} Q_i f_i dt^i = -\int_{T_i} \nabla Q_i \cdot F^i dt^i + \int_{\partial T_i} Q_i F^i \cdot dS.$$

Therefore,

$$\int_{T_i} \left(q^i(t^i)\cdot t^i - Q_i(t^i)\right) f_i(t^i) dt^i = \int_{T_i} q^i(t^i)\cdot\left(t^i + \frac{F^i(t^i)}{f_i(t^i)}\right) f_i(t^i) dt^i - \int_{\partial T_i} Q_i F^i \cdot dS.$$

If over a portion of the boundary ∂T_i, the vector field F^i is orthogonal to ∂T_i, while over its complement, Q_i is 0, the surface integral in the right hand side would be 0. The partial differential equation $\operatorname{div} F^i = f_i$ does not determine a unique solution; to specify a unique solution one can, for example, impose boundary conditions of the form $F^i \cdot n = g$ on ∂T_i, where g is a real-valued function defined on the boundary ∂T_i. Trying to make the surface integral equal to 0, one would like to choose $g \equiv 0$. However, this is not possible: by the divergence theorem, the function g can be chosen arbitrarily, provided it satisfies the constraint

$$\int_{\partial T_i} g d\sigma = \int_{T_i} f_i dV = 1.$$

In some cases it is possible to construct a solution of $\operatorname{div} F^i = f_i$ in T_i such that $F^i \cdot n = 0$ almost everywhere on ∂T_i. But such a solution will necessarily have singularities on ∂T_i.

For each buyer i pick a solution F^i of $\operatorname{div} F^i = f_i$ and define the *index function* $\xi : T \to \mathbf{R}^N$ by

$$\xi(t) := \left[t^0 + \sum_{i=1}^N \left(t^i + \frac{F^i(t^i)}{f_i(t^i)} \right) \right] f(t) \qquad t \in T.$$

The seller's objective function then becomes

$$\int_T \xi(t) \cdot p(t) dt - \sum_{i=1}^N \int_{\partial T_i} Q_i F^i \cdot dS - N b_1.$$

A strategy for solving problem (R) is to solve the modified (and relaxed) problem

$$(M) \qquad \max_{p: T \to \Sigma} \int_T \xi(t) \cdot p(t) dt.$$

This problem is extremely simple and a solution can be constructed directly by taking $p(t) \in \arg\max \{ \xi(t) \cdot q \mid q \in \Sigma \}$ for each $t \in T$. If

$$\int_{\partial T_i} Q_i F^i \cdot dS = 0 \quad \text{for each } i \in I,$$

then p is also a solution of problem (R).

5. Symmetrization of the Index Function

In this Section we show that for symmetric settings, the index function ξ defined in Section 4 can be chosen to be symmetric. When the index function is symmetric, problem (M) has an optimal solution that is symmetric. The index function is constructed in terms of F^i, where F^i solves the *linear* differential equation $\operatorname{div} F^i = f_i$, $i = 1, \ldots, N$. Fix i and suppose $\operatorname{div} F^i = f_i$. If $f_i(\Pi_{jk}(t^i)) = f_i(t^i)$ for all $t^i \in T_i$, $j \neq i$ and $k \neq i$, then $\operatorname{div}(\Pi_{jk} \circ F^i \circ \Pi_{jk}) = f_i \circ \Pi_{jk} = f_i$ for all $j \neq i$ and $k \neq i$. Thus, any convex combination of $\{\Pi_{jk} \circ F^i \circ \Pi_{jk} \mid j \neq i, k \neq i\}$ solves the same partial differential equation as F^i. Also, for symmetric settings, any solution of the partial differential equation for $i = 1$ composed with the permutation Π_{1j} gives a solution of the equation for $j \neq 1$. Lemmas 3 and 4 below use these facts to construct solutions to these equations that satisfy certain symmetry properties.

Lemma 3: *Let $i \in I$, $g : \mathbf{R}^N \to \mathbf{R}$ satisfy $g(\Pi(\tau)) = g(\tau)$ for all τ and permutation $\pi \in \Phi_i$, and $G : \mathbf{R}^N \to \mathbf{R}^N$ be such that $\operatorname{div} G = g$. Define*

$$G^*(\tau) := \frac{1}{(N-1)!} \sum_{\pi \in \Phi_i} \Pi(G(\Pi(\tau))) \qquad \text{for each } \tau.$$

Then, $\operatorname{div} G^* = g$, and $G^*(\Pi(\tau)) = \Pi(G^*(\tau))$ for all τ and permutation $\pi \in \Phi_i$.

Proof: To avoid complex notation, let us give the proof for the case $N = 3$ and $i = 1$. We have that

$$G_1^*(\tau) = (G_1(\tau) + G_1(\tau_1, \tau_3, \tau_2))/2$$
$$G_2^*(\tau) = (G_2(\tau) + G_3(\tau_1, \tau_3, \tau_2))/2$$
$$G_3^*(\tau) = (G_3(\tau) + G_2(\tau_1, \tau_3, \tau_2))/2.$$

Therefore,

$$\frac{\partial G_1^*}{\partial \tau_1}(\tau) = \frac{1}{2} \left[\frac{\partial G_1}{\partial \tau_1}(\tau) + \frac{\partial G_1}{\partial \tau_1}(\tau_1, \tau_3, \tau_2) \right],$$
$$\frac{\partial G_2^*}{\partial \tau_2}(\tau) = \frac{1}{2} \left[\frac{\partial G_2}{\partial \tau_2}(\tau) + \frac{\partial G_3}{\partial \tau_3}(\tau_1, \tau_3, \tau_2) \right], \text{ and}$$
$$\frac{\partial G_3^*}{\partial \tau_3}(\tau) = \frac{1}{2} \left[\frac{\partial G_3}{\partial \tau_3}(\tau) + \frac{\partial G_2}{\partial \tau_2}(\tau_1, \tau_3, \tau_2) \right],$$

and thus $\operatorname{div} G^*(\tau) = [\operatorname{div} G(\tau) + \operatorname{div} G(\tau_1, \tau_3, \tau_2)]/2 = [g(\tau) + g(\tau_1, \tau_3, \tau_2)]/2 = g(\tau)$. It is also easy to see that $G^*(\Pi_{23}(\tau)) \equiv \Pi_{23}(G^*(\tau))$. ∎

Under assumption (S_2), Lemma 3 asserts that if $\operatorname{div} F^i = f_i$ admits a solution, then there is a solution F^i that has the additional property that $F^i(\Pi(t^i)) = \Pi(F^i(t^i))$ for all t^i and permutation π that fixes i.

Lemma 4: *Assume (S_2) and (S_3) are satisfied and let F^1 be such that $\operatorname{div} F^1 = f_1$ and $F^1(\Pi(t^1)) = \Pi(F^1(t^1))$ for all t^1 and permutation π that fixes 1. For each $i > 1$, define $F^i(t^i) := F^1(\Pi_{1i}(t^i))$, $t^i \in T_i$. Then, for each $i > 1$, $\operatorname{div} F^i = f_i$ and $F^i(\Pi(t^i)) = \Pi(F^i(t^i))$ for all t^i and permutation π that fixes i.*

Lemmas 3 and 4 can be used to construct a *symmetric* index function, and for a symmetric index function, problem (M) has a symmetric optimal solution.

Lemma 5: *Assume (S_1)–(S_3) are satisfied and let F^1 be such that $\operatorname{div} F^1 = f_1$ and $F^1(\Pi(t^1)) = \Pi(F^1(t^1))$ for all t^1 and permutation $\pi \in \Phi_1$. For each $i > 1$, let $F^i(t^i) := F^1(\Pi_{1i}(t^i))$, $t^i \in T_i$. Then, the index function*

$$\xi(t) = \left[t^0 + \sum_{i=1}^N \left(t^i + \frac{F^i(t^i)}{f_i(t^i)} \right) \right] f(t) \qquad t \in T$$

satisfies: $\xi(\Pi^*_{ij}(t)) = \Pi_{ij}(\xi(t))$ *for all* $t \in T$ *and* $i \neq j$.

PROOF: Conditions (S_2) and (S_3) imply that $f(\Pi^*_{ij}(t)) = f(t)$ for all $t \in T$, and $\sum_k \Pi^*_{ij}(t)^k = \Pi_{ij}\left(\sum_k t^k\right)$. For all $k \notin \{i,j\}$, $F^k(\Pi^*_{ij}(t)^k) = \Pi_{ij}(F^k(t^k))$. Moreover $F^i(\Pi_{ij}(t)^i) = F^i(\Pi_{ij}(t^j)) = F^1(\Pi_{1i}(\Pi_{ij}(t^j))) = \Pi_{ij}(F^1(\Pi_{1j}(t^j))) = \Pi_{ij}(F^j(t^j))$, and similarly, $F^j(\Pi_{ij}(t)^j) = \Pi_{ij}(F^i(t^i))$. Therefore,

$$\sum_k \frac{F^k(\Pi^*_{ij}(t)^k)}{f_k(\Pi^*_{ij}(t)^k)} = \sum_k \frac{\Pi_{ij}(F^k(t^k))}{f_k(t^k)}. \qquad \blacksquare$$

Lemmas 2 and 5 imply that when (S_1)–(S_3) are satisfied, there is an optimal solution p of (M) which is *symmetric*. If $\xi(\Pi^*_{ij}(t)) = \Pi_{ij}(\xi(t))$ for all $t \in T$ and $i \neq j$, and $p(t) \in \operatorname{argmax}_{q\in\Sigma} \xi(t)\cdot q$, then clearly $\Pi_{ij}(p(t)) \in \operatorname{argmax}_{q\in\Sigma} \xi(\Pi^*_{ij}(t))\cdot q$. Therefore, problem (M) admits a symmetric solution $p : T \to \Sigma$. The next proposition summarizes the conditions for the relaxation procedure proposed in Section 4.

PROPOSITION 6: *Suppose* (S_1)–(S_3) *hold, and let* F^1 *be such that* $\operatorname{div} F^1 = f_1$ *and* $F^1(\Pi(t^1)) = \Pi(F^1(t^1))$ *for all* $t^1 \in T_1$ *and* $\pi \in \Phi_1$. *Define* $F^i(t^i) := F^1(\Pi_{1i}(t^i))$, $t^i \in T_i$, *and*

$$\xi(t) := \left[t^0 + \sum_{i=1}^N \left(t^i + \frac{F^i(t^i)}{f_i(t^i)}\right)\right] f(t) \qquad t \in T.$$

Let p *be symmetric and such that* $p(t) \in \arg\max\{\,\xi(t)\cdot q \mid q \in \Sigma\}$ *for each* $t \in T$. *Consider the auction mechanism* (ρ, y, p), *where* $\rho^i_i = 0$, $\rho^i_j = (N-1)^{-1}$ *for all* $j \neq i$, *and* $y_i(t^i) := q^i(t^i)\cdot t^i - Q_i(t^i) - b_1$, $t^i \in T_i$. *Suppose that*

(1) $\int_{\partial T_1} Q_1 F^1 \cdot dS = 0$, *and*

(2) q^i *is conservative and monotone for each* i.

Then (ρ, y, p) *is an optimal auction mechanism.*

6. EXAMPLES

EXAMPLE 1: As mentioned above, the techniques developed in this paper can be adapted to deal with the case in which the buyers'types have an additional coordinate to represent the externality they suffer when the seller keeps the object. We consider here an example with just one buyer whose type $t = (t_0, t_1)$ is distributed in $T_1 := (-\infty, 0] \times [0, \infty)$ with density

$$f_1(t) := \lambda^2 e^{-\lambda(t_1 - t_0)}.$$

Since there is only one buyer, we omit the superindex in his type. Assume the seller has type $t^0 = (t^0_0, t^0_1) = 0$. The vector field

$$F^1(t) := h(t_1 - t_0) f_1(t) \begin{pmatrix} t_0 \\ t_1 \end{pmatrix} \quad \text{where} \quad h(z) := -\frac{\lambda z + 1}{(\lambda z)^2},\ z \in \mathbf{R},$$

solves the equation $\operatorname{div} F^1 = f_1$ in T_1. Therefore, the seller problem is

$$(P) \qquad \begin{aligned} \max \quad & \int_{T_1} p(t)\cdot\left(t + \frac{F^1(t)}{f_1(t)}\right) f_1(t)dt - \int_{\partial T_1} Q_1 F^1 \cdot dS \\ \text{s.t.} \quad & p : T_1 \to \Sigma \\ & p \text{ is a monotone and conservative field.} \end{aligned}$$

Since player 1 is the only buyer, his conditional expected probability assignment function coincides with p (that is, $q^1 \equiv p$), and since $q^1(t) \in \Sigma$ for all $t \in T_1$,

$$|Q_1(t)| \leq \int_0^1 |p(xt)\cdot t|\, dx \leq t_1 - t_0, \quad \text{for all } t \in T_1.$$

The vector field F^1 has a singularity at the origin, and T_1 is unbounded. However, we can show that the the second term in the seller's objective function (involving the integral over the boundary ∂T_1) is equal to 0. Let $T_1(k)$ be the subset of T_1 contained between the lines $L_{1/k}$ and L_k, where $L_z := \{t \in T_1 \,|\, t_1 - t_0 = z\}$, $z \geq 0$. We have

$$\int_{\partial T_1} Q_1 F^1 \cdot dS = \lim_{k\to\infty} \int_{\partial T_1(k)} Q_1 F^1 \cdot dS.$$

The boundary $\partial T_1(k)$ is made of four line segments: $L_{1/k}$, L_k, the "vertical" line $\{(0, t_1) \,|\, \frac{1}{k} \leq t_1 \leq k\}$, and the "horizontal" line $\{(t_0, 0) \,|\, -k \leq t_0 \leq -\frac{1}{k}\}$. The unitary exterior normal vector for each segment is, respectively,

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 0 \\ -1 \end{pmatrix}.$$

The vector field F^1 is perpendicular to the exterior normal on the horizontal and vertical line segments, and since the length of L_z is $z\sqrt{2}$,

$$\left| \int_{L_z} Q_1 F^1 \cdot dS \right| \leq -h(z) z^3 \lambda^2 e^{-\lambda z},$$

and

$$\left| \lim_{k\to\infty} \int_{\partial T_1(k)} Q_1 F^1 \cdot dS \right| \leq \lim_{k\to\infty} \left[\frac{1}{k}\left(1 + \frac{\lambda}{k}\right) e^{-\lambda/k} + k(1 + \lambda k) e^{-\lambda k} \right] = 0.$$

This example is representative: in our model of Section 2 with N buyers, when $a_1 = +\infty$ and $b_0 = -\infty$, it is always possible to find a solution of $\operatorname{div} F^1 = f_1$ such that $F^1 \cdot n = 0$ at the boundary of T_1. As we pointed out in Section 4, such a solution will have singularities on ∂T_1. As it is done in this example, it is enough to place just one singularity at the origin. If the sigularity is such that

$$\lim_{z \to 0} z^2 \max \{\|F^1(O^1 + t^1)\| \mid t^1_1 - (t^1_2 + \cdots + t^1_N) = z\} = 0,$$

then for any $p : T \to \Sigma$, $\int_{\partial T_1} Q_1 F^1 \cdot dS = 0$.

Define $\gamma(z) := 1 + h(z) = \frac{(\lambda z)^2 - \lambda z - 1}{(\lambda z)^2}$, and $\xi(t) := \gamma(t_1 - t_0) f_1(t) t$. Then, the seller's objective function is

$$\int_{T_1} \xi(t) \cdot p(t) dt$$

The function $\gamma : \mathbf{R}_+ \to \mathbf{R}$ is increasing, concave, and has a root at $z^* := \frac{1+\sqrt{5}}{2\lambda}$. Therefore, the optimal solution of problem (R) (problem (P) with the last two constraints relaxed) is

$$p(t) = \begin{cases} (1,0) & \text{if } t_1 - t_0 < z^* \\ (0,1) & \text{if } t_1 - t_0 \geq z^*. \end{cases}$$

It is easy to see that this is a conservative and monotone vector field, and hence it is also the solution of problem (P). One can also verify that $y(t) = z^*$ for $t_1 - t_0 \geq z^*$, and $y(t) = 0$ otherwise. The solution represents a "take-it-or-leave-it" offer by the seller at price z^*. The buyer accepts the offer iff his type t is such that $t_1 - t_0 \geq z^*$; otherwise the seller keeps the object.

The change of variables $\tau = t_1 - t_0$ would allow us to solve this example using the existing techniques for one-dimensional mechanism design problems. Our result then replicates an example of Myerson (1981), showing that the optimal auction in this case is a modified Vickrey auction, in which the seller submits a bid $\tau_0 = z^*$. Of course, this is possible only in problems with just one buyer.

EXAMPLE 2: Now consider the standard problem of this paper with two buyers, where $T_1 = [0, \infty) \times (-\infty, 0]$, $T_2 = (-\infty, 0] \times [0, \infty)$, $f_1(t^1) = \lambda^2 \exp\{-\lambda(t^1_1 - t^1_2)\}$, and $f_2(t^2) = \lambda^2 \exp\{-\lambda(t^2_2 - t^2_1)\}$. Also, suppose $t^0 = 0$. This is a symmetric setting.

From Example 1 we have that

$$F^1(t^1) := h(t^1_1 - t^1_2) f_1(t^1) \begin{pmatrix} t^1_1 \\ t^1_2 \end{pmatrix}, \quad t^1 \in T_1,$$

and $F^2(t^2) := F^1(\Pi_{12}(t^2))$, $t^2 \in T_2$, solve $\operatorname{div} F^i = f_i$, $i = 1, 2$. The same argument above shows that for any $q^i : T_i \to \Sigma$,

$$\int_{\partial T_i} Q_i F^i \cdot dS = 0, \quad i = 1, 2.$$

Therefore, the seller's objective function is

$$\int_T \xi(t) \cdot p(t) dt,$$

where $\xi(t) := \gamma(t^1_1 - t^1_2) t^1 + \gamma(t^2_2 - t^2_1) t^2$.

We focus again on the relaxed problem. The function $p : T \to \Sigma$ that maximizes $\xi(t) \cdot p(t)$ for each $t \in T$ is given by

$$p(t) = \begin{cases} (0,0) & \text{if } \xi_1(t) < 0 \text{ and } \xi_2(t) < 0 \\ (1,0) & \text{if } \xi_1(t) \geq 0 \text{ and } \xi_1(t) > \xi_2(t) \\ (0,1) & \text{if } \xi_2(t) \geq 0 \text{ and } \xi_1(t) < \xi_2(t). \end{cases}$$

Clearly, if $s^1 = t^1 + \alpha e$ for some $\alpha \in \mathbf{R}$, then $\gamma(s^1_1 - s^1_2) = \gamma(t^1_1 - t^1_2)$. Therefore, if (t^1, t^2) solves $\xi_1(t) = \xi_2(t)$ (that is, if $t^1_1\gamma(t^1_1 - t^1_2) + t^2_1\gamma(t^2_2 - t^2_1) = t^1_2\gamma(t^1_1 - t^1_2) + t^2_2\gamma(t^2_2 - t^2_1)$), so does $(t^1 + \alpha e, t^2)$ and $(t^1, t^2 + \alpha e)$. Moreover, if $t^2 = \Pi_{12}(t^1) = (t^1_2, t^1_1)$, then (t^1, t^2) also solves $\xi_1(t) = \xi_2(t)$. Therefore, the set of $t \in T$ that solve $\xi_1(t) = \xi_2(t)$ is the intersection of T with the hyperplane H generated by the vectors

$$\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \text{ and } \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

Since $(1, -1, 1, -1)$ is perpendicular to H, H is also described by the equation

$$t^1_1 - t^1_2 + t^2_1 - t^2_2 = 0 \quad \text{or} \quad t^1_1 - t^1_2 = t^2_2 - t^2_1.$$

Thus, the solution of the relaxed problem is given by

$$p(t) = \begin{cases} (0,0) & \text{if } t^1_1\gamma(t^1_1 - t^1_2) + t^2_1\gamma(t^2_2 - t^2_1) < 0 \text{ and } t^1_2\gamma(t^1_1 - t^1_2) + t^2_2\gamma(t^2_2 - t^2_1) < 0 \\ (1,0) & \text{if } t^1_1\gamma(t^1_1 - t^1_2) + t^2_1\gamma(t^2_2 - t^2_1) \geq 0 \text{ and } t^1_1 - t^1_2 > t^2_2 - t^2_1 \\ (0,1) & \text{if } t^1_2\gamma(t^1_1 - t^1_2) + t^2_2\gamma(t^2_2 - t^2_1) \geq 0 \text{ and } t^1_1 - t^1_2 < t^2_2 - t^2_1. \end{cases}$$

To verify whether the corresponding functions q^1 and q^2 are conservative and monotone is a formidable task. We don't attempt this here. In a subsequent paper we expect to give a more general characterization of the functions $p : T \to \Sigma$ that produce a monotone and conservative vector field q^i for each $i = 1, \ldots, N$. However, this example suggests that in contrast with the private and independent values standard auction, the solution cannot be viewed as a modified Vickrey aution with a reservation price. Instead of inducing the buyers to raise their bids by the threat of keeping the object if their bids are too low, here the seller produces similar incentives by pitching the buyers against each other. Note that $(t^1, t^2) = ((\alpha, -\alpha), (-\alpha, \alpha))$ satisfies $\xi_1(t) = 0$ and $\xi_2(t) = 0$ for all $\alpha > 0$. Hence, no matter how small they are, when the reports of the buyers are on this ray, the seller never keeps the object.

References

Armstrong, M. (1992): "Optimal Nonlinear Pricing by a Multiproduct Firm," mimeo, Cambridge.

Armstrong, M. (1993): "Regulating a Multiproduct Firm with Unknown Costs," mimeo, Cambridge.

Champsaur, P. and J. C. Rochet (1989): "Multiproduct Duopolists," *Econometrica*, **57**, 533–558.

Laffont, J. J., E. Maskin and J. C. Rochet (1987): "Optimal Nonlinear Pricing with Two-Dimensional Characteristics," in *Information, Incentives and Economic Mechanisms*, ed. Groves et al. : University of Minnesota Press, 256–266.

Laffont, J. J. and J. Tirole (1990): "The Regulation of Multiproduct Firms, Part I: Theory," *Journal of Public Economics*, **43**, 1–36.

Lewis, T. and D. Sappington (1988): "Regulating a Monopolist with Unknown Demand and Cost Functions," *Rand Journal of Economics*, **19**, 438–457.

Matthews, S. and J. Moore (1987): "Monopoly Provision of Quality and Warranties: An Exploration in the Theory of Multidimensional Screening," *Econometrica*, **55**, 441–468.

McAfee, R. P. and J. McMillan (1988): "Multidimensional Incentive Compatibility and Mechanism Design," *Journal of Economic Theory*, **46**, 335–354.

McAfee, R. P., J. McMillan, and M. D. Whinston (1989): "Multiproduct Monopoly, Commodity Bundling, and Correlation of Values," *The Quarterly Journal of Economics*, **93**, 371–383.

Milgrom, P. and R. Weber (1982): "A Theory of Auctions and Competitive Bidding," *Econometrica*, **50**, 1089–1122.

Mirman, L. J. and D. Sibley (1980): "Optimal Nonlinear Prices for Multiproduct Monopolies," *Bell Journal of Economics*, **11**, 659–670.

Mussa, M. and S. Rosen (1978): "Monopoly and Product Quality," *Journal of Economic Theory*, **18**, 301–317.

Myerson, R. (1979): "Incentive Compatibility and the Bargaining Problem," *Econometrica*, **47**, 61–73.

Myerson, R. (1981): "Optimal Auction Design," *Mathematics of Operations Research*, **6**, 58–63.

Myerson, R. and M. Satterthwaite (1983): "Efficient Mechanisms for Bilateral Trading," *Journal of Economic Theory*, **29**, 265–281.

Palfrey, T. R. (1983): "Bundling Decisions by a Multiproduct Monopolist with Incomplete Information," *Econometrica*, **51**, 463–483.

Riley, J. and R. Zeckhauser (1983): "Optimal Selling Strategies: When to Haggle, When to Hold Firm," *Quarterly Journal of Economics*, **98**, 267–289.

Rochet, J. C. (1984): "Monopoly Regulation with Two-Dimensional Uncertainty," CEREMADE Discussion Paper.

Rochet, J. C. (1985): "The Taxation Principle and Multitime Hamilton-Jacobi Equations," *Journal of Mathematical Economics*, **14**, 113–128.

Rochet, J. C. (1992): "Optimal Screening of Agents with Multiple Characteristics," mimeo, Université de Toulouse.

Rockafellar, R. T. (1972): *Convex Analysis.* : Princeton University Press.

Rockafellar, R. T. (1974): *Conjugate Duality and Optimization.* : SIAM, Philadelphia.

Spence, A. M. (1980): "Multi-Product Quantity-Dependent Prices and Profitability Constraints," *Review of Economic Studies*, **47**, 821–841.

Wilson, R. (1991): "Multiproduct Tariffs," *Journal of Regulatory Economics*, **3**, 5–26.

Wilson, R. (1993): *Nonlinear Pricing.* : Oxford University Press.

Key Words: Meeting Competition Clauses, Facilitating Practices