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Bequests, Gifts, and Social Security

John Laitner

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DEPARTMENT OF ECONOMICS
University of Michigan
Ann Arbor, Michigan 48109

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Abstract: This paper analyzes the very long run, or "stationary state," impact of an unfunded social security system. We use an overlapping generations model framework. A key feature is that while parents care about their children and can leave non–negative bequests to them, children also care about their parents and can make non–negative "gifts" to them. We show that the possibility of negative "net bequests" may make social security less harmful to private wealth accumulation than would otherwise be the case. A subsidiary finding is that risk–loving behavior may emerge for some households due to the nature of intergenerational transfers within family lines.

John Laitner
The Department of Economics
Lorch Hall
The University of Michigan
Ann Arbor, Michigan 48109

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BEQUESTS, GIFTS, AND SOCIAL SECURITY

The purpose of this paper is to present a theoretical model illustrating some of the possible effects of an unfunded social security system on steady-state national capital accumulation. Such a social security system contemporaneously transfers tax revenues extracted from the labor incomes of young families to elderly households — the first benefits accruing to old families which never paid contributions. Our framework is an overlapping generations model with finite-lived households. All supply labor inelastically.[1] Each cares about other generations within its family line. The latter sympathies may generate intergenerational transfers. The key novel feature of the setup is that transfers can go both from parents to children and from children to parents. We show that the elaborate gift-bequest possibilities can affect household behavior in interesting ways and can have important implications for the influence on the economy as a whole of social security.

As background suppose, for a moment, that net bequests must be non-negative (and always under the complete jurisdiction of parents). Consider two polar cases. In one, described by Barro [1974], all families desire to leave bequests or are on the borderline of doing so, and all anticipate the same circumstances for their successors. For such a model the advent of an unfunded social security system makes little difference: as the system begins, elderly recipients pass their windfalls to their heirs, offsetting the latters' taxes; in time, the heirs receive benefits, which they in turn bequeath — and so on. All original first-order conditions remain valid and aggregative wealth accumulation is unaffected. Only bequests change.

At the other extreme, Laitner [1979a,b] presents a model in which labor incomes vary randomly among households. Sufficiently long repetitions of bad luck within family lines can lead to households which neither desire to leave bequests nor are on the borderline of wanting to do so. In a stationary-state equilibrium, a positive fraction of all time-t family lines have current members falling into this non-bequest category. Alternatively, every family line must anticipate a positive probability, say, ς, of entering such a state within, say, $T < \infty$ generations — see Laitner [1979c].

In the second model the introduction of a social security system can have a large effect on an economy's ability to finance physical capital in the long run. An elderly family receiving a social security windfall and not otherwise desiring to make a bequest will consume the benefit (at least in part). As we follow the economy through time, each family line (with probability 1) hits (many) non-bequest periods. By the time we have (asymptotically) approached a new stationary state, every family line will be separated —via zero bequest generations — from its ancestors who witnessed the social security system's inception.

Consider a family in the new stationary state with social security. As we have just seen, the family is cut off from any inheritance based on the system's start. Social security need not affect its lifetime resources — we can, at this point, think of each household's social security taxes as equaling the present value of its benefits. Thus, bequests need not differ from previous levels (at given factor prices). On the other hand, each family's life-cycle saving in youth presumably falls by the present value of its future social security benefit.

In the second model funded and unfunded social security systems affect steady-state family asset accumulation in identical ways. In the former case, however, accumulation in the social security trust fund exactly counterbalances reductions in private life-cycle saving. In the second

[1] Thus, although some of our results below parallel Feldstein [1974], our assumptions and line of reasoning are entirely different.

instance, in contrast, at given factor prices society's stationary–state stock of wealth will be reduced by the amount of the missing trust fund. This is true regardless of the magnitude of ς.

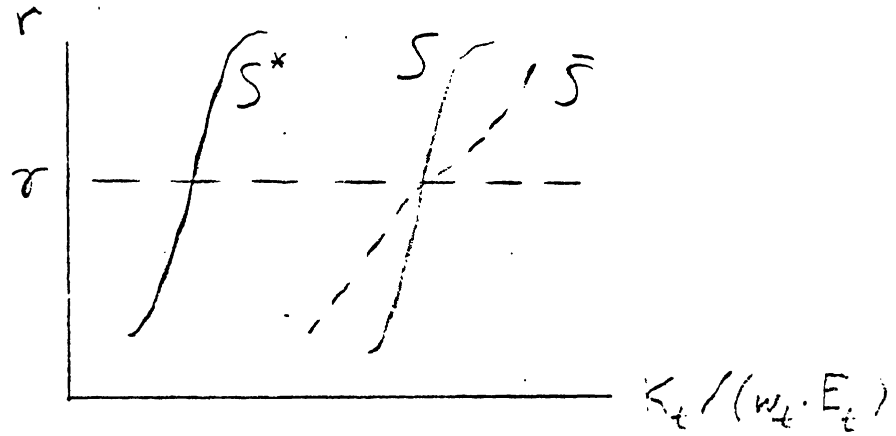

Diagram 1. Steady–state desired per capita wealth holdings normalized by the wage rate

Diagram 1 shows the aggregate supply of financing for physical capital. K_t represents the aggregate of the steady–state wealth holdings of all private individuals and the social security system; w is the steady–state wage rate; r is 1 plus the steady–state interest rate; and, E_t is the "effective" labor force —the natural labor force multiplied by a factor γ^t, with γ equaling 1 plus the rate of labor augmenting technological progress. The supply curve S corresponds to a fully funded social security system. Subtracting (horizontally) the amount of social security trust fund per unit of labor income, S^* corresponds to an economy with an unfunded system and the second bequest framework above. Dropping the assumed equality in present–value terms of each family's social security taxes and benefits, $\overline{S}$ might correspond to the case with no social security at all.[2]

As stated, this paper allows young families to care about their parents and to make nonnegative "gifts" to them. As in Barro and Laitner, parents care about their children and can make nonnegative bequests to them. A parent's "net bequest" (the bequest it gives to its children minus the gift it receives from them), therefore, may turn out to be negative, zero, or positive. Although the possibility of negative net bequests might seem to rule out Laitner's [1979a,c] corner–solution behavior, that turns out not to be true. Thus, in some ways our results resemble the second scenario described above.

On the other hand, the existence of some negative bequests (prior to the advent of social security) can reduce the magnitude of the shift from S to S^* in Diagram 1. First, the possibility of receiving a gift allows households — especially those of average or below–average means — to save less for retirement than an ordinary life–cycle model would predict. In fact, we show that

2 We could endogenize factor prices by adding a second relation between r and $K_t/w \cdot E_t$. If, for example, K_t is held as physical capital and we have a Cobb–Douglas aggregate production function $Q_t = (K_t)^a (E_t)^{1-a}$, the second curve would graph $r = [a/(1-a)]/[K_t/(w \cdot E_t)]$ — see Tobin [1967]. In the case of a Samuelson [1958] consumption loans model, we might fix $r = \gamma$ and think of K_t as money holdings. In a model with international financial flows, r might be fixed in world markets and our K_t would correspond to domestic wealth holdings.

parents may intentionally exaggerate this reduction, because their descendants control the gifts they receive. Families which would otherwise save little for retirement have limited lattitude for cuts under a different social security regime. Second, we show that social security may alter family marginal rates of substitution in such a way as to minimize (or even offset) reductions in life–cycle savings which are feasible.

An interesting sidelight of our gift–bequest transfer setup becomes apparent in the course of the analysis: we find that risk–loving behavior is likely to emerge for households in some wealth classes. This conclusion holds despite the concavity of underlying direct utility functions in our framework. A possible result is that families may intentionally neglect to make full use of life insurance and annuities — even when both are available at actuarially fair prices.

The organization of this paper is as follows. Section 1 presents our model of family utility maximization. Section 2 characterizes desired saving and intergenerational transfers. Section 3 discusses intentional risk taking. Section 4 considers the steady–state distribution of wealth for the economy as a whole and derives our version of the supply curve S for Diagram 1. It establishes the inevitability of some corner solutions. Section 5 introduces an unfunded social security system. Section 6 concludes the paper.

1. Family Utility Maximization

This section describes our model of the behavior of families. As stated, the economy consists of overlapping generations of individual households. Time is discrete. Each household lives two periods. During its first period, a family raises one offspring; in the second, the child forms its own household and passes its first period of life.[3] There is one consumption good in the economy. Its current price is normalized to 1 in every period. Each household supplies one manhour of labor in youth and none in old age. Differences in ability cause the quality of labor supplied to vary across families, however, as explained below. For simplicity we set the rate of labor augmenting technological change, $\gamma - 1$, equal to 0.[4] Savings carried from period t to $t+1$ yield a nonstochastic steady–state return $r - 1 > 0$. Until Section 5 there is no social security system.

Our model of household behavior has three basic elements. One is a direct utility function for each family. Consider a household born at time t. Its replication of the utility function includes as arguments its own two lifetime consumption figures, C_{1t} and C_{2t}; the two figures for its descendant, $C_{1,t+1}$ and $C_{2,t+1}$; the two for its parent family, $C_{1,t-1}$ and $C_{2,t-1}$; the two for its second–generation descendant, $C_{1,t+2}$ and $C_{2,t+2}$; the two for its parents' parents, $C_{1,t-2}$ and $C_{2,t-2}$; and so on. For simplicity, we assume additive separability and homotheticity. Thus, the household utility function is a member of the Bergson class — see Katzner [1970]. If $h \in (0,1)$ is the subjective discount factor, and if

$$u(C) \equiv ln(C) \quad \text{or} \quad C^{\beta}/\beta \quad \text{some } \beta < 1,\ \beta \neq 0,$$

$$v(C) \equiv a \cdot u(C) \quad \text{some} \quad a > 0,$$

[3] We could reinterpret the model to allow two offspring per family and marriage between families with perfectly assortative mating. Laitner [1979b] allows two offspring per family and random mating. The within–generation tax implications of similar models, which has recently been explored in Bernheim and Bagwell [1985], is not a topic here.

[4] Laitner [1979c], in a simplier, but related context, shows how to incorporate technological change. In applications, the magnitude of γ would be a very important determinant of household bequests in our framework.

then family born at time t wants to maximize

$$h \cdot v(C_{2,t-1}) + \sum_{i=0}^{\infty} h^i \cdot [u(C_{1,t+i}) + v(C_{2,t+i})]. \tag{1}$$

Earlier family line consumption figures ($C_{1,t-1}$, $C_{2,t-2}$, etc.) are determined prior to time t and hence superfluous in the time–t maximization process. Formulation (1) assumes subjective discounting based on time proximity. To simplify the analysis in Section 4, we assume

$$a < 1 \tag{2}$$

(a being the parameter in the definition of $v(.)$). This seems sensible since our families have fewer members in their second period of life.

Having each family assign its own lifetime consumption the highest weight in (1) is crucially important in this paper. It embodies an underlying limitation to altruism — families care most about themselves. It leads to conflict between adjacent generations (within the same family line): time–t parents desire more lifetime consumption for themselves than for their children; the children desire more lifetime consumption for themselves than for their parents.

On the other hand, the scope of the disagreement above is restricted: (1) implies that time–t parents will weight the lifetime utility of their descendant of time $t+s$, $s \geq 1$ relative to the lifetime utility for their time–$(t+s+1)$ successor in the same way as their child. For example, the parents will weight the lifetime utility of their child relative to their child's child in the same manner as their child compares himself and his offspring. This degree of agreement enables us to write recursive equation (13) below.

The geometric subjective discounting in (1) is also required for (13). Homotheticity would be essential to our stationary–state analysis if $\gamma \neq 0$ — see note 4. Other than concavity and monotonicity, Section 2 makes heavy use of the property $u'(0) = v'(0) = \infty$; Bergson functions are a convenience for constructing the upper bound in (10); and Section 4 uses the property $u'(C)/u'(C-Z) \to 1$ as $C \to \infty$ any fixed $Z < \infty$. The homotheticity of $u(.)$ and $v(.)$ leads to homogeneity results for saving and net bequest functions which allow us to write the supply of savings as a function of r, as in Diagram 1 — although, in this regard, other ways of presenting our arguments are possible.

A second key element of our model is an exogenous random variable $\tilde{A}$, independent of time, from which every household's ability is drawn.[5] If the steady–state wage is w, $\gamma = 1$, and a family born at time t has ability A_t, its (lifetime) labor earnings are $w \cdot A_t$. A family's ability becomes known to it, and to everyone else, as it first supplies labor. Since there are many families in each cohort, average ability per labor supplier at any time t is nonstochastic. However, the randomness of $\tilde{A}$ is critical — without it, $h < 1$ in (1) would leave no basis for any positive bequests; with it, lucky families may leave estates.

We assume the density $p(\cdot)$ for $\tilde{A}$ is 0 except on the interval

$$\mathcal{A} \equiv [A^L, A^U] \subset (-\infty, \infty)$$

[5] An implication is that abilities are not heritable within family lines. Although we could modify this, for the sake of simplicity we will not.

and is continuous and strictly positive on $\mathcal{A}$. The analysis of Section 4 will be streamlined if we also assume

$$A^L < A^U/2. \tag{3}$$

This has the reasonable implication that the best paid families in a generation will earn more than twice as much as the worst.

Our third basic element is a set of rules for bequests and gifts: In its second period of life, a family can make a bequest to its descendant. The transfer must be nonnegative but is otherwise entirely under the (older) family's control. At the same time, the descendant family, in its first period of life, can make a gift to its parents (and next period a bequest to its own offspring). The gift must be nonnegative but is otherwise entirely under the descendant's control. As stated, since parents weight their own lifetime utility relative to their descendant's differently from the descendant, the parties will disagree about both gifts and bequests.

We confine our attention to stationary states. We will take r, w, and $p(.)$ as given and derive an analogue of the S curve in Diagram 1. For convenience, in most of the family behavioral and utility functions below we will include w as an argument but surpress r and $p(.)$.

We use the following notation. The "wealth" of a household started at time t, W_t, is the sum of its labor income, $w \cdot A_t$; its inheritance, B_t; and negative the gift, G_t, it transfers to its parent:

$$W_t = w \cdot A_t + B_t - G_t. \tag{4}$$

All of the variables in (4) are realized at the start of period t. The household divides W_t into life-cycle savings, S_t, and first-period-of-life consumption, C_{1t}:

$$W_t = S_t + C_{1t}. \tag{5}$$

Life-cycle savings S_t can be invested in an actuarially fair lottery $\tilde{L}(S_t)$ — the need for which will become apparent below. Each lottery $\tilde{L}(S_t)$ is specified with a finite payoff-probability vector $(\ell_1, q_1; ...; \ell_m, q_m)$, where $\ell_i \geq 0$ is a payoff, $q_i \in (0, 1]$ is the corresponding probability, $\sum_{i=1}^m q_i = 1$, and $r \cdot S_t = \sum_{i=1}^m q_i \cdot \ell_i$. Let $\mathcal{L}$ be the set of all such lotteries. (Note that one lottery choice, $\ell_1 = r \cdot S$ and $q_1 = 1$, involves no risk.) In the second period of life, our household splits its lottery outcome, say, $L(S_t)$, and its descendant's gift (if any), G_{t+1}, into a bequest, B_{t+1}, and its own retirement consumption, C_{2t}:

$$G_{t+1} + L(S_t) = B_{t+1} + C_{2t}. \tag{6}$$

We assume bankruptcy laws compel $S_t \geq 0$ and $\ell_i \geq 0$ all i any acceptable lottery.

The law of large numbers shows society as a whole incurs no risk per citizen in providing stochastically independent lotteries of the type described above. Questions of why individual families would want to purchase lotteries and of what interpretation such lotteries might have in the real world are topics of Section 3.

Before proceeding, let us digress to show that our household framework can be restated in game theoretic terms. Each family line engages in a separate infinite time horizon game. Consider one line. The "players" are the households of different generations. The "payoffs" have been described. At any time t two players are active: the family line's time-$(t-1)$ household, the "parent," and the time-t household, the "child." At time t, first $w_t \cdot A_t$, the children's earnings, and the outcome of the parent's lottery, $L(S_{t-1})$, are revealed ($w_{t-1} \cdot A_{t-1}$ is already public information). Second, the parent chooses its bequest (to the child), B_t. Simultaneously, the child chooses its gift (to the parent), G_t. The bequest and gift must fall within the sets

$$0 \leq B_t \leq L(S_t) \quad \text{and} \quad 0 \leq G_t \leq w \cdot A_t. \tag{7}$$

The lower limits have been explained. Lines (4) and (6) seem to imply $B_t \leq G_t + L(S_t)$ and $G_t \leq w \cdot A_t + B_t$ would be appropriate upper bounds. However, in describing a game we want a player's time–t stategy space to be independent of rivals' current actions. In fact, in working through our solution below we employ (4)–(6) exclusively, but Proposition 2 deduces the validity of (7). Furthermore, the intuition is straightforward: the conflict in preferences described in preceding paragraphs causes a parent always to desire a smaller "net bequest", $B_t - G_t$, than its child does. Hence, whenever the parent wants $B_t > 0$, the child will certainly set $G_t = 0$. Similarly, whenever the child desires $G_t > 0$, the parent will want $B_t = 0$. Line (7) then follows from (4) and (6).

In the final step of the game at time t, the parent derives $C_{2,t-1}$ from (6) (lagged one period) and the child chooses S_t and C_{1t} subject to $S_t \geq 0$ and (4)–(5). We seek a subgame perfect Nash equilibrium for the infinite time horizon game. What is more, the equilibria on which we focus are also stationary with respect to time.

Turning to our solution procedure, for any $W_t = W$, let $V_t(W; w)$ give a household's utility from its own lifetime consumption plus the consumption of its descendants in all generations (in other words, expression (1) less its first term). If the wealth of the household's child is W_{t+1}, let $V_{t+1}(W_{t+1}; w)$ give the offspring's analogue of $V_t(W_t; w)$. Define

$$\delta(L, N; w; A; V) \equiv h \cdot v(L - N) + V(w \cdot A + N; w), \tag{8}$$

$$\Delta(L, N; w; A; V) \equiv v(L - N) + h \cdot V(w \cdot A + N; w). \tag{9}$$

If L is our household's lottery outcome and $w \cdot A_{t+1}$ the labor income of its descendant, then the bequest it leaves, $B_{t+1} = B$, and the gift it receives, $G_{t+1} = G$, must simultaneously satisfy (see (1))

$$B = \arg \sup_{B' \in [0, L+G]} \Delta(L, B' - G; w; A_{t+1}; V_{t+1}), \tag{10}$$

$$G = \arg \sup_{G' \in [0, w \cdot A_{t+1} + B]} \delta(L, B - G'; w; A_{t+1}; V_{t+1}). \tag{11}$$

If B and G satisfy (10)-(11), the parent household's net bequest, $N = B - G$, depends on w, L, A_{t+1}, and V_{t+1}:

$$N = N(L; w; A_{t+1}; V_{t+1}). \tag{12}$$

The fact that the household and its descendant both care most about themselves creates the difference between $\Delta(.)$ and $\delta(.)$.

Let $E[.]$ be the expected value operator. Below we show $N(.)$ is a unique function, continuous in A, in our model. So, $V_t(.)$ must obey

$$V_t(W; w) = \sup_{S \in [0, W], \tilde{L}(S) \in \mathcal{L}} \{u(W - S) + E[v(\tilde{L}(S) - N(\tilde{L}(S); w; \tilde{A}_{t+1}; V_{t+1})) + h \cdot V_{t+1}(w \cdot \tilde{A}_{t+1} + N(\tilde{L}(S); w; \tilde{A}_{t+1}; V_{t+1}); w)]\}. \tag{13}$$

A household with wealth $W_t = W$ can certainly achieve a utility $V_t(W; w)$ higher than

$$V^L(W; w) \equiv [u(W/2) + v(r \cdot W/2)] + h \cdot [u(w \cdot A^L/2) + v(r \cdot w \cdot A^L/2)] + h^2 \cdot [u(w \cdot A^L/2) + v(r \cdot w \cdot A^L/2)] + ... \tag{14}$$

Selecting consumption figures more favorable than those which are actually feasible, $V_t(W; w)$ should be bounded above by

$$V^U(W; w) \equiv [u(W) + v(r \cdot W + w \cdot A^U)] + h \cdot [u(r \cdot W + w \cdot A^U) + v(r^2 \cdot W + r \cdot w \cdot A^U + w \cdot A^U)] + h^2 \cdot [u(r^2 \cdot W + r \cdot w \cdot A^U + w \cdot A^U) + v(r^3 \cdot W + r^2 \cdot w \cdot A^U + r \cdot w \cdot A^U + w \cdot A^U)] + ... \tag{15}$$

For finiteness, we limit our attention to values of r with

$$r \geq 1 \quad \text{and} \quad r \cdot h < 1. \tag{16}$$

Then

$$V^U(W; w) < [1 + hr + 2(rh)^2 + ...] \cdot u(W + w \cdot A^U) + [r + 2r^2h + 3r^3h^2 + ...] \cdot v(W + w \cdot A^U).$$

Fix an r satisfying (16). With our geometric subjective discounting, unchanging distribution for $\tilde{A}$, and identical direct utility functions through time, we seek a stationary $V(.)$:

$$V_t(W; w) = V(W; w) \quad \text{all} \quad t, \quad \text{all} \quad w > 0, \quad \text{and all} \quad W \geq 0. \tag{17}$$

For $V(.)$ to constitute a "solution" to our model we require

$$V^L(W; w) \leq V(W; w) \leq V^U(W; w) \quad \text{all} \quad W \geq 0. \tag{18}$$

Although the preceding paragraph makes this seem unrestrictive, it rules out nonsense cases such as $V(W; w) = \infty$ all W. Finally, we want (10)-(11) to define at least one function $N(.; V)$, as in (12). For such an $N(.)$, we want (13) to hold (with $V(.)$ in place of $V_t(.)$ and $V_{t+1}(.)$). $V(.)$ is a "solution" to our family model if it satisfies (13) and (17)-(18) in this manner.

A formal proof of the existence of at least one family–model solution function $V(.)$ and corresponding unique first–period of life saving function $S(W; w; V) = S$ and unique (and continuous in A) net–bequest function $N(L; w; A; V) = N$ for each $W \geq 0$, $w > 0$, $A \in \mathcal{A}$, $L \geq 0$, and r satisfying (16) is available from the author — see Laitner [1986].[6] Given Proposition 2 below, this solution defines a subgame–perfect, stationary Nash equilibrium for each family line's infinite time game, as explained. Fortunately our method of solving the game provides detailed characterizations of $V(.)$, $S(.)$, and $N(.)$.

2. Families' Saving and Net Bequest Functions

This section derives several properties of each solution $V(.)$ and, in particular, each corresponding first–period–of–life saving and second–period net bequest function.

[6] The dual maximization in (10)–(11) precludes existence arguments based on conventional dynamic programming theory. Instead, Laitner [1986] resorts to a general fixed–point theorem applied to a mapping defined on a set of candidate solution functions.

We begin with a series of lemmas about $V(.)$ which lay the groundwork for studying $S(.)$ and $N(.)$. Fix any wage $w > 0$ and any interest factor r consistent with (16).

The first lemma shows $V(.;w)$ is monotone increasing. When $V(.)$ is a solution for our family model, it satisfies (13): if

$$\Gamma(L;w;V) \equiv \int_{\mathcal{A}} \Delta(L, N(L;w;A;V);w;A;V) \cdot p(A)\, dA, \tag{19}$$

then

$$V(W;w) = \sup_{S\in[0,W],\tilde{L}(S)\in\mathcal{L}} \{u(W-S) + E[\Gamma(\tilde{L}(S);w;V)]\}. \tag{20}$$

If we raise W in (20), first holding S fixed, then allowing S to vary, each stage raises (or leaves unchanged) the total right–hand side expression. The first step yields

Lemma 1. *If $V(.)$ is a solution, $V(.)$ satisfies $V(W+\xi;w) - V(W;w) > u'(W+\xi)\cdot\xi$ all $W \geq 0$, $\xi > 0$.*

The second lemma uses the "steepness" of $u(.)$, $v(.)$, and $V(.;w)$ to put bounds on the arguments of $v(.)$ and $V(.)$ on the right-hand side of (13).

Lemma 2. *Let $V(.)$ be a solution. Let $N(.;V)$ be consistent with (10)-(11). Then (i) there exists $X_0 \in (0, A^L)$ such that $w\cdot A + N(L;w;A;V) > w\cdot X_0$ any $L \geq 0$ and any $A \in \mathcal{A}$; (ii) if $v(0) = -\infty$, there exists $\Omega(L;w)$ such that $L - N(L;w;A;V) \geq \Omega(L;w) > 0$ all $L \geq 0$ and $A \in \mathcal{A}$. If $v(0) > -\infty$, let $\Omega(L;w) \equiv 0$. $\Omega(.;w)$ is continuous.*

The appendix contains a proof. Part (i) implies that once any solution for our family model has been in force for more than one period, values of $W < w\cdot X_0$ no longer appear in the economy. The lemma together with (17) enable us to bound $\Gamma(.)$:

$$\begin{aligned}\Gamma(L;w;V) \leq& v(L + w\cdot A^U) + \\ & h\cdot V^U(L + w\cdot A^U;w) \equiv \Gamma^U(L;w;V) < \infty \quad \text{all} \quad L \geq 0;\end{aligned} \tag{21}$$

$$\begin{aligned}\Gamma(L;w;V) \geq& v(\Omega(L;w;V)) + \\ & h\cdot V^L(w\cdot X_0;w) \equiv \Gamma^L(L;w;V) > -\infty \quad \text{all} \quad L \geq 0.\end{aligned} \tag{22}$$

We can prove that $V(.;w)$ must be concave as follows. If a set $J \subset R^n$, let conv(J) be its convex hull. We can view conv(J) both (see Rockafellar [1970, p.12]) as (i) the set of all convex combinations of finitely many elements of J, and (ii) the intersection of all convex sets containing J. If $V(.)$ is a solution, and if $\Gamma(.)$ is as in (19), let

$$\psi(w;V) \equiv \text{conv}(\{(L, \Gamma(L;w;V)) \mid L \geq 0\}), \tag{23}$$

$$\Psi(L;w;V) \equiv \sup\{Z \mid (L,Z) \in \psi(w;V)\} \quad \text{each} \quad L \geq 0. \tag{24}$$

Lines (21)-(22) show $\psi(.)$ is well–defined. Since $\Gamma^U(.)$ is concave,

$$\{(L,Z) \mid L \geq 0,\ Z \leq \Gamma^U(L;w;V)\}$$

is a convex set. Line (21) shows it contains $\psi(w;V)$; thus, the second characterization of conv(.) implies

$$\Psi(L;w;V) \leq \Gamma^U(L;w;V) < \infty \quad \text{all} \quad L \geq 0. \tag{25}$$

Lemma 2 and the first characterization yield

$$\Psi(L;w;V) \geq \Gamma^L(L;w;V) > -\infty \quad \text{all} \quad L \geq 0. \tag{26}$$

Thus, $\Psi(.;w;V)$ is concave in L — see Rockafellar [1970, Th.5.3]. The first interpretation of conv(.) and the availability of lotteries yield

$$\sup_{S\in[0,W],\tilde{L}(S)\in\mathcal{L}} \{u(W-S)+E[\Gamma(\tilde{L}(S);w;V)]\} = \sup_{S\in[0,W]} \{u(W-S)+\Psi(r\cdot S;w;V)\}. \tag{27}$$

Equation (27) implies

Lemma 3. *If $V(.)$ is a solution, $V(.;w)$ must be strictly concave.*

The appendix finishes the proof.

The next lemma shows that if $V(.;w)$ is a solution to our family model for $w=1$, we can always extend it to a solution $\overline{V}(.)$ for all w such that $\overline{V}(.)$ manifests the same homotheticity as $u(.)$ and $v(.)$.

Lemma 4. *Let $V(.;w)$ be a solution to our family model for $w=1$. If $u(c)=c^\beta/\beta$, define $\overline{V}(W;w)\equiv w^\beta\cdot V(W/w;1)$ all $W\geq 0$, $w>0$. If $u(c)=ln(c)$, define $\overline{V}(W;w)\equiv V(W/w;1)+(1+a)\cdot ln(w)/(1-h)$ all $W\geq 0$, $w>0$. Then $\overline{V}(.;w)$ is a solution to our family model for all $w>0$.*

The appendix supplies a proof. On the basis of Lemma 4, henceforth we restrict our attention to family–model solutions with the homogeneity property that $\overline{V}(.)$ has.

Lemma 5 is an incarnation of the familiar envelope theorem.

Lemma 5. *If $V(.)$ is a solution, $\partial V(W;w)/\partial W$ exists and equals $u'(W-S(W;w;V))$ all $W\geq 0$, $w>0$, and $A\in\mathcal{A}$.*

The appendix provides a proof.

Lemma 6 collects our final results for $V(.)$. Essentially, we bound $\Gamma(.;V)$ above with a simpler function. Let

$$\phi(L;w;A;V) \equiv \sup_{N\in[-w\cdot A,L]} \Delta(L;N;w;A;V).$$

Lines (21)-(22) yield

$$-\infty < \Gamma^L(L;w;V) \leq \phi(L;w;A;V) \leq \Gamma^U(L;w;V) < \infty \quad \text{all} \quad X \geq 0.$$

We can see that $\phi(.)$ is monotone in A, hence integrable. Define

$$\Phi(L;w;V) \equiv \int_{\mathcal{A}} \phi(L;w;A;V)\cdot p(A)\,dA.$$

A comparison of the definitions of $\Phi(.)$ and $\Gamma(.)$ shows $\Phi(.;w;V) \geq \Gamma(.;w;V)$ — the construction of $\Phi(.)$ gives current-generation parents complete control over N. We can easily see that $\Phi(.;w;V)$ is concave (see the proof below). Therefore, $\Phi(.;w;V)$ bounds $\Psi(.;w;V)$ above.

Intuitively, very wealthy families will not receive gifts. Thus, Lemma 6 will demonstrate that $\Phi(.)$ and $\Gamma(.)$ coincide for large enough values of L. Hence, the same is true for $\Phi(.;w;V)$ and $\Psi(.;w;V)$. As Diagram 2 shows, therefore, we cannot draw a linear section of (the graph of) $\Psi(.)$ induced by a lottery and ending beyond the point at which the graphs of $\Phi(.)$ and $\Psi(.)$ begin to correspond. It follows that wealthy families will not use multi-outcome lotteries.

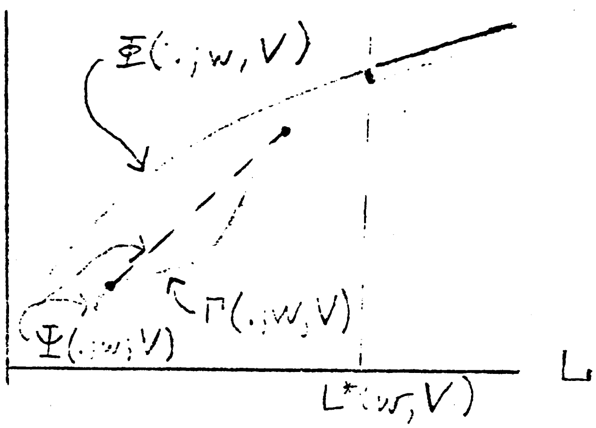

Diagram 2. Graphs of $\Phi(.)$, $\Psi(.)$, and $\Gamma(.)$.

Lemma 6. *Let $V(.)$ be a solution. For all $L \geq 0$, $\Phi(.;w;V)$ is strictly concave and $\Phi(L;w;V) \geq \Psi(L;w;V) \geq \Gamma(L;w;V)$. $L^*(w;V) \equiv \inf\{L \geq 0 \,|\, N(L;w;A;V) > 0 \; \forall A \in \mathcal{A}\} \in [0,\infty)$ exists all $w > 0$. If $L \geq L^*(w;V)$, then $\Psi(L;w;V) = \Phi(L;w;V) = \Gamma(L;w;V)$ and L must originate from a single outcome lottery. In fact, if $w = 1$, there exists some $\epsilon > 0$ such that any $L > L^*(1;V) - \epsilon$ must originate from a single outcome lottery.*

The appendix supplies a proof.

Turning to the first-period-of-life saving function $S(.)$, we find that despite the complexity of our model, conventional properties hold — in particular, $S(.;w;V)$ is monotone nondecreasing in W and the propensity to save on the margin never exceeds 1. Lemma 4 enables us to show that $S(.)$ is linearly homogeneous in its first two arguments.

Proposition 1. *Let $V(.)$ be a solution with the homogeneity property discussed above, and let $\Gamma(.)$ be as in (19). Then if $W' > W \geq 0$, we have $0 \leq S(W';w;V) - S(W;w;V) \leq W' - W$. $S(W;w;V) \to \infty$ as $W \to \infty$. If $\alpha > 0$ is any constant, $S(\alpha \cdot W; \alpha \cdot w; V) = \alpha \cdot S(W;w;V)$.*

The appendix supplies a proof.

The net bequest function is more intricate. If we allowed an elderly family complete control of its net transfer, it would use

$$N_\Delta(L;w;A;V) \equiv \arg \sup_{N \in [-w \cdot A, L]} \Delta(L, N; w; A; V). \tag{28}$$

If we allowed the same family's descendant complete control, the choice would be

$$N_\delta(L;w;A;V) \equiv \arg \sup_{N\in[-w\cdot A,L]} \delta(L,N;w;A;V). \tag{29}$$

(The proof of Lemma 4 in Laitner [1986] shows both $N_\Delta(.)$ and $N_\delta(.)$ are unique and single valued.) The selfishness of both parties implies N_δ will always equal or exceed N_Δ. If N_Δ is positive, the actual net bequest will be positive, as both parties desire. Conversely, if N_δ is negative, the actual N will certainly be negative. If N_Δ is negative and N_δ is positive, the parents would like a gift but cannot compel it, and the offspring would like a bequest but cannot compel it; hence, we should observe $N = 0$. The following proposition, proven in the appendix, makes these points formally. (And, incidentally, justifies line (7).)

Proposition 2. *Let $L \geq 0$, let $A \in \mathcal{A}$, and let $V(.)$ be a solution. Let $N = N(L;w;A;V)$, $N_\Delta = N_\Delta(L;w;A;V)$, and $N_\delta = N_\delta(L;w;A;V)$. Then (i) $N_\delta > N_\Delta$; (ii) if $N_\Delta \geq 0$, $N = N_\Delta$; if $N > 0$, $N = N_\Delta$; (iii) if $N_\delta \leq 0$, $N = N_\delta$; if $N < 0$, $N = N_\delta$; and, (iv) if $N_\delta > 0 > N_\Delta$, $N = 0$. In fact, if $L^L(w;A;V) \equiv \sup\{L \geq -w\cdot A \mid N_\delta(L;w;A;V) < 0\}$ and $L^U(w;A;V) \equiv \inf\{L \geq -w\cdot A \mid N_\Delta(L;w;A;V) > 0\}$, then $-\infty < L^L(w;A;V) < L^U(w;A;V) < \infty$ and $N(L;w;A;V) = 0$ all $L \in [L^L(w;A;V), L^U(w;A;V)]$.*

As in Proposition 1, we can also show $N(.;w;A;V)$ is nondecreasing and not too steep. $N(L;w;.;V)$, on the other hand, is nonincreasing. The homotheticity in line (1) and Lemma 4 imply $N(.)$ is linearly homogeneous in (L,w). The appendix contains the proof.

Proposition 3. *Let $V(.)$ be a solution with the homogeneity property discussed above. Let $L \geq 0$, $A \in \mathcal{A}$, and $\xi > 0$. Let $N \equiv N(L;w;A;V)$. Then (i) $0 \leq N(L+\xi;w;A;V) - N < \xi$ and the same inequalities hold strictly for $N_\Delta(.)$ and $N_\delta(.)$; (ii) if $A+\xi \in \mathcal{A}$, $0 \leq N - N(L;w;A+\xi;V) < w\cdot\xi$ and the same inequalities hold strictly for $N_\Delta(.)$ and $N_\delta(.)$; (iii) if $\alpha > 0$ is a constant, $N(\alpha\cdot L;\alpha\cdot w;A;V) = \alpha\cdot N$; and, (iv) $N \to \infty$ as $L \to \infty$.*

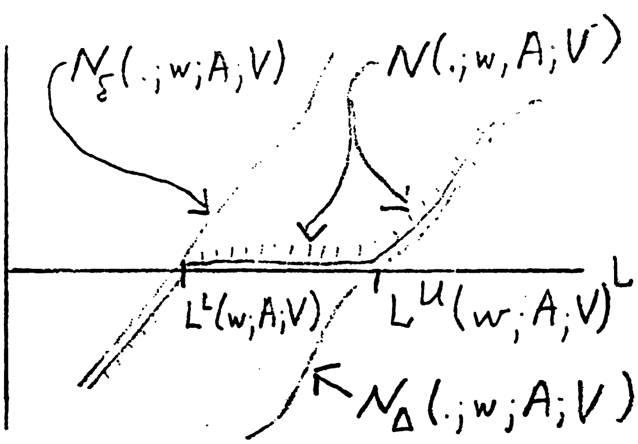

Diagram 3. The graph of the net bequest function

For any $A \in \mathcal{A}$ and any solution $V(.)$ to our family model, Propositions 2 and 3 characterize the shape of the graph of $N(.;w;A;V)$, as illustrated in Diagram 3. For all L, the graph is nondecreasing and its slope is less than 1. For $L \leq L^L(w;A;V)$, both the parent and descendant want a negative net bequest. The descendant chooses the magnitude; the parent would like the absolute amount to be bigger but has no leverage. For $L \in [L^L(w;A;V), L^U(w;A;V)]$, the parent would like a gift and the descendant would like a bequest. The outcome is no net transfer. For $L \geq L^U(w;A;V)$, both parties desire a positive net bequest. The descendant prefers a larger magnitude than the parent, but the parent controls the amount. The shape of the graph in Diagram 3, therefore, directly manifests the conflicting maximization problems of (10)-(11).

3. Lotteries

The potential importance of lotteries is evident in Section 2: if $\Gamma(.;w;V)$ — see (19) — is not concave, young families will sometimes be able to enhance their total utility by investing their life–cycle savings in lotteries. In this section, we first show that convexities are likely to arise for some families of low wealth. This occurs despite the concavity of direct utility functions (see (1)); it is due to the fact that when parents receive a gift, the gift is under the control of their offspring — and the offspring have different preferences from the parents. Second, we suggest a possible empirical counterpart for the lotteries in $\mathcal{L}$.

For a solution $V(.)$, $w > 0$, and $A \in \mathcal{A}$, consider

$$\Upsilon(L;w;A;V) \equiv \Delta(L;N(L;w;A;V);w;A;V). \tag{30}$$

The expected value of (30) —taking expectations over A — gives a family's second–period–of–life total utility —see (19)-(20). Let $L^L(w;A;V)$ be as in Proposition 2.

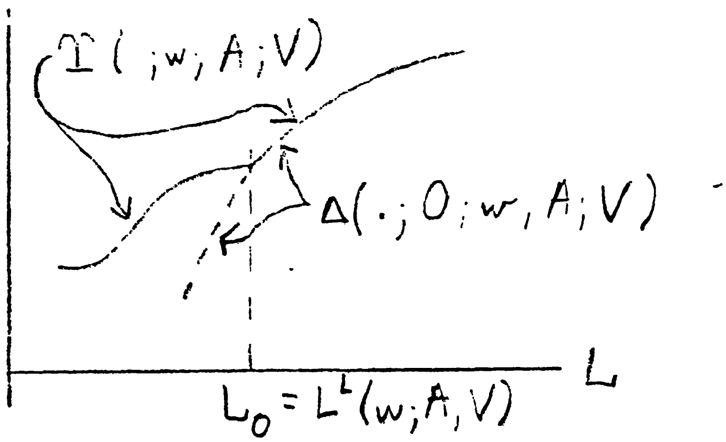

Diagram 4. The graph of (30) with respect to L

Lotteries seem likely to play a role for some families because the graph of (30) —see Diagram 4 — will tend to have a convexity at $L_0 \equiv L^L(w;A;V)$. Let $\partial^+\Upsilon(.)/\partial L$ be the derivative from the right for $\Upsilon(.)$. Moving to the right from L_0, the envelope theorem — see the graph of $N(.;w;A;V)$ in Diagram 3 — shows

$$\partial^+\Upsilon(L_0;w;A;V)/\partial L = v'(L_0 - N(L_0;w;A;V)). \tag{31}$$

This is a conventional outcome: for $L \geq L_0$, a parent family's net bequest is under its own, albeit constrained, control; thus, an extra unit of resources yields an increment to total utility equal to second–period–of–life marginal utility from consumption.

Moving left from L_0 the story is quite different. For $L < L_0$, N is under the offspring's control and is the wrong magnitude from the parents' viewpoint. Thus, in evaluating changes in L, parents will care not only about $v'(.)$ but also about how L is manipulating $N(L; w; A; V)$. In particular, if N is constant, a smaller L will lead to a decrease in $\Delta(.)$ equal, as before, to $v'(.)$. But, Diagram 3 shows N will drop as $L < L_0$ does. The latter change will yield a partially offsetting increase in utility to the parents because, in their evaluation, N is too high. Mathematically, if $\partial^- \Upsilon(.)/\partial L$ exists,[7]

$$\partial^- \Upsilon(L_0; w; A; V)/\partial L = v'(L_0 - N(L_0; w; A; V)) - [v'(L_0 - N(L_0; w; A; V)) - h \cdot \partial V(w \cdot A + N(L_0; w; A; V); w)/\partial W] \cdot [\partial^- N(L_0; w; A; V)/\partial L]. \tag{32}$$

From Diagram 3, we expect

$$\partial^- N(L_0; w; A; V)/\partial L = \partial N_\delta(L_0; w; A; V)/\partial L > 0.$$

Because $N = N_\delta > N_\Delta$,

$$-[v'(L_0 - N(L_0; w; A; V)) - h \cdot \partial V(w \cdot A + N(L_0; w; A; V); w)/\partial W] < 0.$$

If $\partial^- \Upsilon(.)/\partial L < \partial^+ \Upsilon(.)/\partial L$, we must, however, have a convexity at L_0.

Provided the integration in (19) does not iron out the convexity at $L^L(w; A; V)$, agents considering savings levels near or below $L^L(w; A; V)/r$ will desire to use fair lotteries. Lemma 6 shows this will not apply to wealthy families. Notice in Diagram 4 that lotteries such as $(\ell_1, q_1; \ell_2, q_2)$ some q_1 and q_2 which leave parents either well enough off to be independent of their offspring's (restricted) generosity or precommitted to low enough resources to extract a sizable gift may be quite popular.

In practice not many families of any wealth group seem to spend large fractions of their incomes on lottery tickets. However, a "real world" analogue for lottery purchases might be families not taking full advantage of insurance and annuities. Life insurance may be the best example. Our life–cycle model is severely simplified. Suppose we temporarily deviate slightly from our framework and think of a second–period–of–life labor income component $\alpha \cdot w \cdot A$ for families. Presumably a family's sampling A is known in the first period of life. Uncertain time of death will induce uncorrelated randomness in α. With full use of life insurance, the insured value of $\tilde{\alpha} \cdot w \cdot A$ can simply be incorporated into second–period–of–life family resources. A household desiring a lottery, on the other hand, might leave a portion of $\tilde{\alpha} \cdot w \cdot A$, say, $\tilde{\alpha}^* \cdot w \cdot A$, uninsured. We can then interpret the family's $\tilde{L}(S)$ as manifesting $r \cdot S + E[\tilde{\alpha} \cdot w \cdot A] + \tilde{\alpha}^* \cdot w \cdot A - E[\tilde{\alpha}^* \cdot w \cdot A]$.

The same type of argument applies to annuities. Uncertainties about life spans presumably make $v(.)$ a random function. A perfect annuities market would provide a full set of contingent securities — see, for example, Yaari [1965]. In our problem, however, families might well choose not to make full use of such instruments.

4. The Economywide Distribution of Wealth

For any given r satisfying (16) and $w > 0$, we can find a solution $V(.)$ to our family line maximization conditions — see Section 2. $V(.)$, in turn, yields unique life–cycle saving and net bequest functions $S(.)$ and $N(.)$. The present section uses these functions to study the evolution

[7] Kolmogorov and Fomin [1970, p. 321] show this derivative exists almost everywhere.

through time of the national distribution of "wealth" values. The randomness of $\tilde{A}$ prevents the distribution from collapsing to a single value. In fact, we prove convergence to a stationary distribution. We also show that when the economy has reached a stationary distribution, provided there are any positive bequests, a positive fraction of families in each cohort are at corner solutions of their maximization problems — a result having important implications for the study of social security in Section 5.

For some $w > 0$ and r obeying (16) let $V(.)$ solve our family model as in Section 2. If a given family line has wealth W_t at time t and W_{t+1} at time $t+1$, the connection between W_t and W_{t+1} involves a composition of $S(.;V)$ and $N(.;V)$: W_t leads to youthful savings of $S(W_t;w;V)$, which, together with a realization A_{t+1} from $\tilde{A}$ and a lottery $\tilde{L}(S(W_t;w;V))$, yield W_{t+1} —

$$W_{t+1} = w \cdot A_{t+1} + N(L(S(W_t;w;V));w;A_{t+1};V). \tag{33}$$

For each W_t think of a corresponding normalized variable $X_t = W_t/w$. Then dividing through (33) by w and using the linear homogeneity properties established in Propositions 1 and 3,

$$X_{t+1} = A_{t+1} + N(L(S(X_t;1;V));1;A_{t+1};V), \tag{34}$$

with $\tilde{L}(S(X_t;1;V)) = \tilde{L}(S(W_t;w;V))/w$. Our analysis focuses on (34).

The preferred lottery may not be unique. If not, we simply assume the family uses the lottery $(\ell_1, q_1; \ell_2, q_2)$ with ℓ_1 the minimum and ℓ_2 the maximum payoff among preferred mixed strategies.[8]

Line (34) then induces a Markov transition function for X as follows. Given a solution $V(.)$, with each X_t we can associate a unique life–cycle savings figure $S(X_t;1;V)$. The latter now implies a unique lottery $(\ell_1, q_1; \ell_2, q_2)$. We employ the notation $\ell_1 \equiv \ell^L(X_t;V) < \ell_2 \equiv \ell^U(X_t;V)$, $q_1 = q(X_t;V)$, and $q_2 = 1 - q_1$. If the descendant from a young household with normalized wealth X_t receives normalized labor income A_{t+1}, the normalized wealth of the descendant, X_{t+1}, will equal

$$\begin{aligned} &A_{t+1} + N(\ell^L(X_t;V);1;A_{t+1};V) \quad \text{with probability} \quad q(X_t;V), \\ &A_{t+1} + N(\ell^U(X_t;V);1;A_{t+1};V) \quad \text{with probability} \quad 1 - q(X_t;V). \end{aligned}$$

Proposition 3 shows $A + N(L;1;A;V)$ is continuous in A. Thus, if $\mathcal{X}$ is a Borel (or "Lebesgue measurable") set in R^1, for any $L \geq 0$

$$\Lambda(L;\mathcal{X};V) \equiv \{A \in \mathcal{A} \mid A + N(L;1;A;V) \in \mathcal{X}\} \tag{35}$$

is Lebesgue measurable. So, we have a well–defined Markov transition rule $\lambda(.)$ for X_t:

$$\begin{aligned} \lambda(X_t;\mathcal{X}_{t+1};V) \equiv & q(X_t;V) \cdot \int_{\Lambda(\ell^L(X_t;V);\mathcal{X}_{t+1};V)} p(A)\, dA + \\ & [1 - q(X_t;V)] \cdot \int_{\Lambda(\ell^U(X_t;V);\mathcal{X}_{t+1};V)} p(A)\, dA. \end{aligned} \tag{36}$$

Lemma 2 (part (i)) shows values of $X < X_0$ are transitory — thus they play no role in a steady state. Lemma 6 and condition (16) enable us to establish an upper bound. The idea is that there is some normalized wealth figure so large that a parent family possessing it would set aside for its own consumption a portion larger, in–second–period–of–life terms, than the fixed amount A^U. In such a case, even the most favorable (normalized) labor income for the offspring would not enable the descendant to achieve as high a (normalized) wealth figure as its parent.

[8] Alternatives, such as assuming all possible pairs of equally desirable lottery outcomes are chosen with equal probability, would not affect our outcomes substantively.

Lemma 7. *There exists* $X^U(V) < \infty$ *such that* $X \geq X^U(V)$ *implies* $S(X;1;V)$ *will not be invested in a multi-outcome lottery and* $A + N(r \cdot S(X;1;V);1;A;V) < X$ *all* $A \in \mathcal{A}$. *Thus, all ergodic sets for* $\lambda(.)$ *must lie in the interval* $\mathcal{I}(V) \equiv [X_0, X^U(V)] \subset (0,\infty)$.

The appendix provides a proof. For future reference, let the class of Borel subsets of $\mathcal{I}(V)$ be $\mathcal{B}(V)$.

We now turn to the well-known analysis of Markov models in Doob [1953, ch.v]. Let $\mu(.)$ be the Lebesgue measure. Doob shows that $\lambda(.;V)$ will have a finite number of ergodic sets in $\mathcal{B}(V)$ if: (i) $\lambda(X;.;V)$ each $X \in \mathcal{I}(V)$ is a probability measure on $\mathcal{B}(V)$; (ii) $\lambda(.;\mathcal{X};V)$ is a measurable function on $\mathcal{I}(V)$ each $\mathcal{X} \in \mathcal{B}(V)$; and, (iii) (hypothesis of Doeblin) there exists $\epsilon > 0$ such that if $\mathcal{X} \in \mathcal{B}(V)$ and $\mu(\mathcal{X}) \leq \epsilon$, then $\lambda(X;\mathcal{X};V) \leq 1-\epsilon$ all $X \in \mathcal{I}(V)$. The first two properties are strictly technical. The third requires, roughly speaking, that $\lambda(X_t;.;V)$ not assign probability 1 to a transition into any set $\mathcal{X}_{t+1} \in \mathcal{B}(V)$ of very small radius. Looking at (36) and recalling our assumptions about the density function $p(A)$, in economic terms the third condition will hold if a descendant's (normalized) first-period-of-life wealth figure is dependent on his/her realization of $\tilde{A}$. Independence could only occur if the descendant's net inheritance offset changes in its labor income realization dollar-for-dollar — which is inconsistent with the selfishness inherent in our direct utility function.

Proposition 4. *Let* $V(.)$ *be a solution to our family model. Doob's properties (i)-(iii) hold for* $\lambda(.;V)$. *Thus,* $\lambda(.;V)$ *has a finite number of ergodic sets on* $\mathcal{I}(V)$. *Each has a single cyclic subset.*

The appendix supplies a proof.

Inequalities (2)–(3), assumed in Section 1, immediately streamline our discussion to one ergodic set.[9]

Proposition 5. *Let* $V(.)$ *be a solution to our family model. Then (2)-(3) imply* $\lambda(.;V)$ *has a single ergodic set.*

The appendix supplies a proof.

Returning to Doob [1950, ch.v],

Proposition 6. *Let* $V(.)$ *be a solution to our family model. Let* $\mathcal{E}$ *be the unique ergodic set for* $\lambda(.;V)$. *Then a unique stationary distribution* $D(.)$ *corresponds to* $\mathcal{E}$. *Let* $E \in \mathcal{B}(V)$. *Define* $\lambda^0(X;\mathcal{X}) \equiv \lambda(X;\mathcal{X};V)$ *and* $\lambda^i(X;\mathcal{X}) \equiv \int_{\mathcal{E}} \lambda^{i-1}(Z;\mathcal{X}) \cdot \lambda^0(X;dZ)$ *all* $\mathcal{X} \in \mathcal{B}(V)$ *and* $i = 1,2,\ldots$. *Then* $\lim_{i\to\infty} \lambda^i(X;E) = \int_E D(dX)$ *any* $X \in \mathcal{E}$. *Also,* $\int_{\mathcal{E}} D(dX) = 1$.

Proposition 4 and Doob [1953, ch.v] establish this result. Proposition 6 shows that over time the cross sectional distribution of normalized wealth must converge to $D(.)$. After convergence, with probability 1 realizations of X outside of $\mathcal{E}$ do not occur.

We can proceed to our version of the supply curve in Diagram 1. An r satisfying (16) leads to a family-model solution $V(.)$. Proposition 6 then gives the steady-state cross-sectional distribution of normalized wealth, $D(.)$. Let $\tilde{X}$ have the latter distribution. It is independent of the wage rate. For any w, however, the distribution of wealth, $\tilde{W}$, is generated from $w \cdot \tilde{X}$. The function $S(.;1;V)$

[9] Laitner [1979a] discusses the implications of multiple ergodic sets. Without (2) and (3) a multiplicity is possible in the present context. It would complicate our discussion but leave its substance unchanged.

is uniquely defined and continuous. Thus, the mean normalized savings figure for the economy is well–defined:

$$E[S(w \cdot \tilde{X}; w; V)]/w = E[S(\tilde{S}; 1; V)]. \tag{37}$$

The distribution for $\tilde{X}$ is bounded (see Lemma 7); hence, if there are many families in each generation, we can appeal to the law of large numbers to derive an excellent proxy for $K_t/(w \cdot E_t)$:

$$K_t/(w \cdot E_t) = E[S(\tilde{X}; 1; V)] \quad \text{all} \quad t. \tag{38}$$

(Recall that E_t corresponds to the number of current young families in the economy.) Considering different values of r, we can then plot out our S curve.

As the introduction shows, the effect of social security may depend heavily on whether or not family lines inevitably experience over time corner–solution net bequests. Proposition 2 sets the stage for an affirmative answer. We must prove, however, that the stationary distribution $D(.)$ puts positive probability weight on family–line (normalized) wealth figures in the flat section of $N(.)$ in Diagram 3. Lemma 6 provides the key.

Proposition 7. *Let $V(.)$ be a solution to our family model. Suppose some families leave positive bequests. Then there exist $\varsigma > 0$ and $T < \infty$ such that in the steady state any family must assign a probability $\geq \varsigma$ to the event that one of its descendants within T generations will have second–period–of–life normalized wealth L and an offspring with ability $A \in \mathcal{A}$ such that $L \in (L^L(1; A; V), L^U(1; A; V))$ — see the notation of Proposition 2.*

The appendix provides a proof. Proposition 7 implies a crucial distinction exists in the long–run between funded and unfunded social security systems.

5. Social Security

This section considers the introduction of an unfunded social security system at time $t = 0$. We want to compare the stationary state of the economy before $t = 0$ with the new one emerging after time 0 — a stationary state consisting of fixed factor prices r and w; a solution $V(.)$ to our family model; and, a stationary distribution of normalized wealth, $\tilde{X}$. Specifically, we want to compare the curves giving the steady–state supply of financing for physical capital — see Diagram 1 — before and after the advent of social security.

To simplify the discussion, think of social security as follows. When the wage rate is w, young families face a lump sum social security tax of $w \cdot \tau$. To be sure everyone can pay it, assume

$$\tau < A^L.$$

Each family's benefit in old age is $\gamma \cdot w \cdot \tau$ — with an unfunded system stationary–state benefits grow at the sum of the rates of labor–augmenting technological progress and natural labor force increase, $\gamma - 1$. Thus, the present value of each household's tax less benefit is $\sigma \cdot w$ with

$$\sigma \equiv [(r - \gamma)/r] \cdot \tau.$$

We assume bankruptcy laws prevent young families from using their future social security benefits as collateral for loans.[10]

[10] In the U.S., future social security benefits are not legal collateral for a loan — see Diamond [1977].

In Diagram 1, the $\overline{S}$ curve gives the supply of financing for physical capital in the absence of a social security system. As we introduce social security and complete the transition to a new stationary state, S^* becomes the supply curve. We can distinguish several components of the total shift.

At the first level of analysis, social security changes a given household's net lifetime labor earnings from $w \cdot A$ to $w \cdot [A - \sigma]$. Replace each A_{t+1} in (34) with $A_{t+1} - \sigma$. Assume (16) and (36) hold. We can then compute a stationary distribution of normalized wealth $\tilde{X}(\sigma)$. $\tilde{X}(0)$ corresponds to the case without social security.

In the absence of social security, average normalized stationary–state financial holdings per worker carried between periods of life equal

$$E[S(\tilde{X}(0); 1; V)]. \tag{39}$$

With an unfunded social security system we might imagine they are

$$E[S(\tilde{X}(\sigma); 1; V)] - (\tau \cdot \gamma / r). \tag{40}$$

The last term reflects the fact that social security's tax–in–youth and benefit–in–old–age nature reduces (by $\tau \cdot \gamma \cdot w/r$) each household's need for life–cycle saving. Subtracting (39) from (40),

$$\{E[S(\tilde{X}(\sigma); 1; V)] - E[S(\tilde{X}(0); 1; V)]\} - (\tau \cdot \gamma / r). \tag{41}$$

The $\{.\}$ term might be thought of as causing the twisting of S relative to $\overline{S}$ in Diagram 1. The definition of σ shows

$$\sigma \geq (<) 0 \quad \text{as} \quad r \geq (<) \gamma.$$

The $-(\tau \cdot \gamma / r)$ term in (41) causes the displacement from S to S^* in Diagram 1.

At the next two levels, the story depends on Proposition 7. We assume some families leave positive bequests. Proposition 7 then shows that when the economy reaches a new stationary state after time 0, all families will be completely cut off from the windfalls distributed to elderly households (at time 0) as the social security system started up.

Because bankruptcy laws prevent social security participants from borrowing against their future benefits, and because of our constraints $S \geq 0$ and $\ell_i \geq 0$ all i in the definition of $\mathcal{L}$, in general our household–model solution $V(.)$ and resulting saving and net bequest functions and stationary distribution of normalized wealth will change after time $t = 0$. Specifically, line (20) becomes

$$V(W; w) = \sup_{S \in [0, W], \tilde{L}(S) \in \mathcal{L}} \{u(W - S) + E[\Gamma(\gamma \cdot w \cdot \tau + \tilde{L}(S); w; V)]\}. \tag{42}$$

The consequences may be very complicated; the following discussion is only designed to suggest intuitively what may happen.[11]

Line (40) presupposes that households which would have set $S = S_0 < (\tau \cdot \gamma / r)$ in the absence of social security borrow the present value of their system benefits, registering life–cycle savings of $S_0 - (\tau \cdot \gamma / r)$. As stated in the preceeding paragraph, however, the constraint $S \in [0, W]$ and the

[11] As suggested in Section 3, in the real world families may have some labor earnings comparatively late in life. Our story only depends on having social security benefits being significant in magnitude relative to such earnings.

definition of $\mathcal{L}$ prevent such behavior in (42). Thus, the savings of such families cannot be reduced (as in (40)) by the full amount of social security taxes, and the horizontal shift from S to S^* in Diagram 1 will be cut back.

Although this effect could emerge in a life–cycle saving model with a nonnegativity constraint on asset holdings but no gifts or bequests, households in such a model would be almost sure to set aside for retirement in the absence of social security a larger sum than the benefits of a modest system. In our framework, in contrast, young households have the option of consuming their labor earnings and relying on gifts to finance their retirement. In the absence of social security, households of low and moderate means — especially in an economy with a positive rate of technological progress — may well find this course of action attractive. Further, the derivative calculations of (31)–(32) show that parents anticipating partial reliance on a gift may intentionally choose a very low S in order to extract a larger gift than their self–directed descendant would otherwise be willing to make — see Section 3. (Or, parents may gamble with a lottery having some chance of a very low payoff.) Social security will limit such families' precommitment possibilities instead of displacing saving they would otherwise want.

A third factor is even more subtle. Consider Diagram 5. In the pre–social security world, a low income family might have selected a life–cycle savings level yielding the outcome at L_0. Social security benefits might equal L_1. Roughly speaking, if the curves are as shown, social security will prevent the family from extracting a gift from its descendant — it cannot reach outcomes with $L + \gamma \cdot r \cdot w < L_1$.[12] What is more, left to its own devices, the household's marginal utility for further saving at L_1 is higher than at L_0. Thus, the household may save a positive amount, moving to L_2, for example.[13] Total private saving for the household in the world with social security, $(L_2 - L_1)/r$, may exceed, equal, or fall short of a corresponding household's saving L_0/r in the society before $t = 0$. Thus, we have a second reason for the shift between S and S^* in Diagram 1

[12] We must be careful here. The analysis at this point is only meant to suggest intuitively various possible outcomes. In particular, comparing (20) and (42) we see that in general $V(.)$, and hence the graph in Diagram 5, will change after $t = 0$ (even for the same r and w).

[13] As I wrote the final draft of this paper, I became aware of a related manuscript by Hansson and Stuart [1986]. Their framework differs from ours in not allowing parents to leave bequests and in not allowing families to have different labor incomes. Although gifts similar to ours are possible, Hansson and Stuart do not allow lotteries — see Section 3 above. Their model may sometimes permit a desirable stationary state with no gifts and an undesirable one with (universally) positive gifts. A proper social security system can force the economy from the latter to the former type of outcome. Such a switch is somewhat analogous to what is happening above in Diagram 5.

to be less than $-(\tau \cdot \gamma / r)$ — conceivably S^* even lies to the right of S in some ranges.

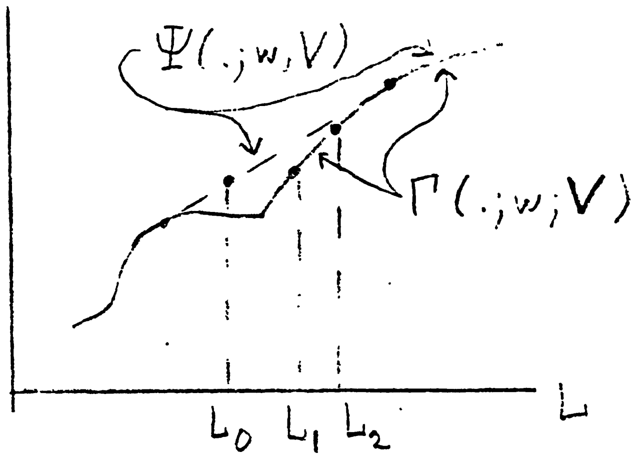

Diagram 5. Desired lottery realizations

An interesting sidelight is that if social security benefits are large enough, they may force households into the concave region of $\Gamma(.;w;V)$ — recall Lemma 6. This happens in Diagram 5 if benefits are L_1. (But, see note 40.) Returning to the discussion of lotteries in Section 3, this might imply a difference in households' behavior toward risk after the advent of social security: private–sector demand for life insurance and annuities might well rise.

Finally, suppose conditions (2)–(3) fail to hold. Then there may be multiple ergodic sets — $\mathcal{E}_1, \ldots, \mathcal{E}_m$. The proof of Proposition 4 shows they can indexed so that $\mathcal{E}_i > \mathcal{E}_{i+1}$ all i, where the inequality means $X_i \in \mathcal{E}_i$ and $X_{i+1} \in \mathcal{E}_{i+1}$ imply $X_i > X_{i+1}$. Assuming there are some positive bequests, they must come from family lines in $\mathcal{E}_1$ — the proof of Proposition 5 will yield inf $\mathcal{E}_1 \leq A^L$. The latter result means families in set $\mathcal{E}_i$ any $i \geq 2$ will give and receive positive gifts. Thus, the key condition of no corner solutions which Barro's [1974] analysis assumes holds for such households, and, as explained in the introduction, a modest enough social security system will not affect their saving. Again the magnitude of the shift in Diagram 1 from S to S^* is reduced. (Proposition 7 holds for all families in $\mathcal{E}_1$.)

6. Conclusion

We have presented a model of intergenerational wealth transfers without conventional nonnegativity constraints on net bequests. Instead, we allow nonnegative bequests controlled by parents and nonnegative gifts controlled by offspring, the parents' "net bequest" being the difference between the bequest they choose and the gift they receive. The characteristic net bequest function which we derive is negative but rising at low wealth levels (for the parent family), zero over an intermediate range of wealth values, and then positive and increasing.

We show that if there are any positive bequests in the economy, a positive fraction of families will be in the flat section of their net bequest function. Over time the economy will then be cut off from the windfall transfer payments to old families accompanying the start of an unfunded social security system. This leads us to expect the same deleterious effects on national wealth accumulation from social security as in much simpler life–cycle models.

On the other hand, the possibility of gifts, and their specific nature, suggests that our intergenerational transfers will motivate some low and middle income families to precommit themselves to very low retirement resources. Social security will force them to a different program — there being little or no savings to be offset in the process. Life–cycle saving may be enhanced in some wealth catagories as marginal rates of substitution change.

We also showed that risk–loving behavior may well emerge (as a result of the way gifts are chosen), but that social security may constrain households in such a manner that risk–averse behavior is restored.

A projected next step in this research is to expand our life–cycle model to many–period lives, to incorporate technological change, and to generate simulation examples which attempt to measure the quantitative importance of the theoretical issues identified here. An additional interpretation of portions of the empirical literature on social security and national wealth accumulation (see, for example, Feldstein [1983], Leimer and Lesnoy [1982], and Darby [1979]) may then be possible.

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APPENDIX

This appendix uses the following notational conventions: for the function $F(X_1, X_2, ...)$, $F'(X_1, X_2, ...) \equiv \partial F(X_1, X_2, ...)/\partial X_1$; for the one-sided derivatives with respect to X_1,

$$F^-(X_1, X_2, ...) \equiv \partial^- F(X_1, X_2, ...)/\partial X_1, \quad F^+(X_1, X_2, ...) \equiv \partial^+ F(X_1, X_2, ...)/\partial X_1.$$

Proof of Lemma 2:

Part i. Define $N^*(.)$ from $h \cdot v'(-w \cdot N^*(A)) = u'(w \cdot A + w \cdot N^*(A))$. Define $f(A) \equiv A + N^*(A)$. Then $f(.) \in C^0$. $A \geq A^L$ and $u'(0) = -\infty$ imply $f(A) > 0$. Hence, $0 < X_1 \equiv \min_{A \in \mathcal{A}}\{f(A)\}$. Define $X_0 \equiv \min\{X_1/2, A^L/2\}$.

If $N(L; w; A; V) \geq 0$, we are done because then $w \cdot A + N(L; w; A; V) \geq w \cdot A > w \cdot X_0$. If not, $N = B - G$ with $G > B \geq 0$. Suppose the latter. Define $F(A) \equiv w \cdot f(A)$. Let $0 < \epsilon < F(A)$. Note that $-N^*(A) > 0$ because $v'(0) = \infty$. Integrating over $[L - w \cdot N^*(A), L - w \cdot N^*(A) + \epsilon]$, and using Lemma 1,

$$\begin{aligned} &V(F(A); w) - V(F(A) - \epsilon; w) > u'(F(A)) \cdot \epsilon = h \cdot v'(-w \cdot N^*(A)) \cdot \epsilon = \\ &h \cdot \int v'(-w \cdot N^*(A))\, dZ \geq h \cdot \int v'(L - w \cdot N^*(A))\, dZ > h \cdot \int v'(Z)\, dZ = \\ &h \cdot [v(L - w \cdot N^*(A) + \epsilon) - v(L - w \cdot N^*(A))]. \end{aligned}$$

Hence, $\delta(L; w \cdot N^*(A); w; A; V) > \delta(L; w \cdot N^*(A) - \epsilon; w; A; V)$. So, the arbitrariness of ϵ implies $N(L; w; A; V) \geq N^*(A)$. Thus, $w \cdot A + N(L; w; A; V) \geq w \cdot X_1 > w \cdot X_0$.

Part ii. Suppose $v(0) = -\infty$. Define $f(w, L) \equiv V^U(w \cdot A^U + L) - V^L((w \cdot A^L + L)/2)$. There exists $\Omega \equiv \Omega(L; w) < (w \cdot A^L + L)/2$ such that $\Omega > 0$ and $h \cdot [v((w \cdot A^L + L)/2) - v(\Omega)] = f(w, L)$. Hildenbrand [1974, p.30] shows $\Omega(.; w)$ is continuous. Let $A \in \mathcal{A}$, $N \equiv N(L; w; A; V)$, and $Z \equiv (w \cdot A + L)/2$. Then $N \leq L$. When $L - N < \Omega$, $V(w \cdot A + N) - V(Z) \leq V(w \cdot A + L) - V(Z) < f(w, L) \leq h \cdot [v(Z) - v(\Omega)] < h \cdot [v(Z) - v(L - N)]$ — hence, $\delta(L, N; w; A; V) < \delta(L, L - Z; w; A; V)$. The same inequality certainly holds for $\Delta(.)$. Thus, $L - N \geq \Omega$. ∎

Proof of Lemma 3:

Let $W' > 0$, $W'' > 0$, $0 < \theta < 1$, $W \equiv \theta \cdot W' + (1-\theta) \cdot W''$, $S' = S(W'; w; V)$, $S'' = S(W''; w; V)$, and $S \equiv \theta \cdot S' + (1 - \theta) \cdot S''$. Then the strict concavity of $u(.)$ and the concavity of $\Psi(.)$ imply (see (20) and (27))

$$\begin{aligned} V(W; w) \geq u(W - S) + \Psi(r \cdot S; w; V) > \theta \cdot u(W' - S') + (1 - \theta) \cdot u(W'' - S'') + \\ \theta \cdot \Psi(r \cdot S'; w; V) + (1 - \theta) \cdot \Psi(r \cdot S''; w; V) = \theta \cdot V(W'; w) + (1 - \theta) \cdot V(W''; w). \end{aligned}$$

∎

Proof of Lemma 4

Define $\overline{N}(L; w; A) \equiv w \cdot N(L/w; 1; A; V)$ and $\overline{S}(W; w) \equiv w \cdot S(W/w; 1; V)$. Then (10), (11), and (13) show $\overline{S}(.)$ and $\overline{N}(.)$ give maximizing values of S and N when $\overline{V}(.)$ replaces $V_{t+1}(.)$. (13) also shows that $V_t(.) = \overline{V}(.)$ in the latter case. $\overline{V}(.)$ satisfies (17). ∎

Proof of Lemma 5.

Let $W' > W > 0$, $S \equiv S(W;w;V)$, and $S' \equiv S(W';w;V)$. Rockafellar [1970, Th. 24.1] shows the one–sided derivatives of $\Psi(L;.)$ with respect to L are defined, nonincreasing, and obey $\Psi^-(L;w;V) \geq \Psi^+(L;w;V)$. Define $\xi \equiv W' - W$.

Step 1. We show $0 \leq S' - S \leq W' - W$.

The definition of S shows $-u'(W'-S)+r\cdot\Psi^-(r\cdot S;w;V) > -u'(W-S)+r\cdot\Psi^-(r\cdot S;w;V) \geq 0$ or $S = 0$. Thus, $S' \geq S$. Then $-u'(W' - S - \xi) + r \cdot \Psi^+(r \cdot S + r \cdot \xi;w;V) = -u'(W - S) + r \cdot \Psi^+(r \cdot S + r \cdot \xi;w;V) \leq -u'(W - S) + r \cdot \Psi^+(r \cdot S;w;V) \leq 0.$[1] Thus, $S + \xi \geq S'$.

Step 2. We show $V^+(W;w) = u'(W - S)$.

Note that $u(W'-S')-u'(W-S) = [u(W'-S)-u(W-S)]+[u(W'-S')-u(W'-S)]$. Using (27), Step 1, and the mean value theorem, $[V(W+\xi;w) - V(W;w)]/\xi = u'(W - S+\eta) + T1\cdot[S' - S]/\xi$ some $\eta \in [0,\xi]$ where $T1 \equiv [-u(W+\xi-S)+u(W+\xi-S')+\Psi(r\cdot S';w;V)-\Psi(r\cdot S;w;V)]/(S'-S)$. Let $\xi \downarrow 0$. Suppose $S(W + \xi';w;V) = S$ some $\xi' > 0$. Then Step 1 shows $S(W + \xi'';w;V) = S$ all $\xi'' \in [0,\xi']$. Canceling $S' - S$ terms, we have $V^+(W;w) = u'(W - S)$. If not, as $\xi \downarrow 0$, $T1 \to -u'(W-S)+r\cdot\Psi^+(r\cdot S;w;V)$. The definition of S shows the last sum ≤ 0. Strict inequality implies $S(W + \xi';w;V) = S$ some $\xi' > 0$. Equality and Step 1 yield $V^+(W;w) = u'(W - S)$.

Step 3. We show $V^-(W;w) = u'(W - S)$. This plus Step 2 complete the proof.

Let $0 < W - \xi < W$. Let $S' \equiv S(W - \xi;w;V)$. Then $V(W - \xi) - V(W) = [u(W - \xi - S') - u(W - S')] + [u(W - S') - u(W - S)] + [\Psi(r\cdot S';w;V) - \Psi(r\cdot S;w;V)]$. Forming difference quotients as before, $V^-(W;w) = u'(W - S)$. ∎

Proof of Lemma 6:

The text shows $\Phi(L;w;V) \geq \Gamma(L;w;V)$. $\phi(.;w;A;V)$ is strictly concave — an argument analogous to the proof of Lemma 3 holds. If $L = \theta\cdot L' + (1-\theta)\cdot L''$,

$$
\begin{aligned}
\Phi(L;w;V) &= \int_{\mathcal{A}} \phi(L;w;A;V)\cdot p(A)\,dA > \\
&\int_{\mathcal{A}} [\theta\cdot\phi(L';w;A;V) + (1-\theta)\cdot\phi(L'';w;A;V)]\cdot p(A)\,dA \\
&= \theta\cdot\Phi(L';w;V) + (1-\theta)\cdot\Phi(L'';w;V).
\end{aligned}
$$

Thus, $\Phi(.;w;V)$ is strictly concave. Hence, $\Phi(.;w;V) \geq \Psi(.;w;V)$.

The definition of $\Phi(.)$ shows $\Phi(L;w;V) = \Gamma(L;w;V)$ for $L \geq L^*(w;V)$. Hence, for such L we have $\Phi(L;w;V) = \Psi(L;w;V) = \Gamma(L;w;V)$. Since $\Phi(.;w;V)$ is concave, Diagram 2 shows such L values must originate from single–outcome lotteries.

We show $L^*(w;V) < \infty$ exists. Define L^0 from $v'(L^0) = (h/2)\cdot u'(w\cdot A^U)$. Then $L^0 \in (0,\infty)$. Let $L \geq L^0$ and $A \in \mathcal{A}$. Suppose $N \equiv N(L;w;A;V) \leq 0$. Then $(B,G) = (0,-N)$ solves (10)-(11). Using Lemmas 1 and 5,

$$
\begin{aligned}
0 &\geq -v'(L - N) + h\cdot V'(w\cdot A + N;w) \geq -v'(L) + h\cdot V'(w\cdot A;w) \geq \\
&- v'(L^0) + h\cdot u'(w\cdot A^U) = (h/2)\cdot u'(w\cdot A^U) > 0,
\end{aligned}
$$

[1] Note: $S = W > 0$ makes $-u'(W - S) + r\cdot\Psi^+(r\cdot S;w;V) = -\infty$ and, hence, is never a chosen savings level.

a contradiction. Hence, $L \geq L^0$ implies $N > 0$. So, $L^*(w;V) \leq L^0$.

Define

$$\phi^*(L;A;V) \equiv \sup_{N \in [0,L]} \Delta(L;N;1;A;V).$$

$\phi^*(.)$ is bounded and monotone in A just as $\phi(.)$ is. Define

$$\Phi^*(L;V) \equiv \int_A \phi^*(L;A;V) \cdot p(A)\, dA.$$

The arguments above show $\Phi^*(.;V)$ is strictly concave. Propositions 2-3 below (established independently) show that for some $\epsilon' > 0$, $\Phi^*(L;V) = \Gamma(L;1;V)$ all $L \geq L^*(1;V) - \epsilon'$. As in Lemma 5, the envelope theorem shows $\Phi^{*\prime}(L;1;V)$ exists all $L > L^*(1;V) - \epsilon'$. So, $\Gamma'(L;1;V) = \Phi^{*\prime}(L;1;V)$. Let $\epsilon \equiv \epsilon'/2$. Since any line segment connecting two lottery outcomes and ending at $L^*(1;V) - \epsilon$ would then have to be tangent to $\Phi^*(.;1;V)$ at $L^*(1;V) - \epsilon$, a graph such as Diagram 2 plus the strict concavity of $\Phi^*(.;w;V)$ establish the last part of Lemma 6. ∎

Proof of Proposition 1:

Step 1 in the proof of Lemma 5 establishes the inequalities; the construction of $\overline{V}(.)$ in the proof of Lemma 4 establishes homogeneity.

It remains to show $S(W;w;V) \to \infty$ as $W \to \infty$. Step 1 in the proof of Lemma 5 shows $S(.;w;V)$ is nondecreasing. Let lim stand for $\lim_{W\to\infty}$. Then we can write $S^0 \equiv \lim S(W;w;V)$. If $S^0 = \infty$, we are done. Suppose $S^0 < \infty$. We have seen $\Psi^-(.;w;V)$ exists and is nonincreasing. Lemma 1 and the first part of Lemma 6 imply $\Gamma(L;w;V)$ must be strictly increasing for large L. Hence, $\Psi^-(L;w;V) > 0$ all L. So, using Rockafellar [1970, th.24.1], $-\lim u'(W - S(W;w;V)) + \lim r \cdot \Psi^+(r \cdot S(W;w;V);w;V)) = 0 + r \cdot \Psi^-(r \cdot S^0;w;V) > 0$ — contradicting the first-order condition $-u'(W - S(W;w;V)) + r \cdot \Psi^+(r \cdot S(W;w;V);w;V) \leq 0$. ∎

Proof of Proposition 2:

Part (i). Recall Lemma 5. Using Lemma 1, we can see $-w \cdot A < N_\Delta < L$. So, $v'(L - N_\Delta) = h \cdot V'(w \cdot A + N_\Delta;w)$. So, $h \cdot v'(L - N_\Delta) < V'(w \cdot A + N_\Delta;w)$. Then since $h \cdot v'(L - N_\delta) = V'(w \cdot A + N_\delta;w)$, $N_\delta > N_\Delta$.

Parts (ii)-(iii). The concavity of $V(.)$ and $v(.)$, (10)-(11), and (28) show $N_\Delta \geq 0$ implies $N = N_\Delta$. Further, if $N > 0$, $(N,0)$ solves (10)-(11) and the same properties show $N = N_\Delta$. The same reasoning works for part (iii).

Part (iv). Suppose $N_\delta > 0 > N_\Delta$. If $N > 0$, $N = N_\Delta$ — see the preceding paragraph. This contradicts $0 > N_\Delta$. If $N < 0$, $N = N_\delta > 0$. Thus, $N = 0$.

If $\xi > 0$, an analog to the first step in the proof of Lemma 5 shows $0 \leq N_x(L + \xi;w;A;V) - N_x(L;w;A;V) \leq \xi$ for $x = \Delta$ or δ. Thus, part (i) shows $L^L(w;A;V) < L^U(w;A;V)$. Lemma 6 shows the latter is bounded above. The former is bounded below (by definition) by $-w \cdot A$. The preceding paragraph completes the last step in the proof. ∎

Proof of Proposition 3:

Let $\xi > 0$. Define $N_x^* \equiv N_x(L+\xi;w;A;V)$, $N_x^{**} \equiv N_x(L;w;A+\xi;V)$, and $N_x \equiv N_x(L;w;A;V)$ for $x = \delta$ or Δ. Define $N^* \equiv N(L+\xi;w;A;V)$. The definition of $\overline{V}(.)$ in the proof of Lemma 4 implies part (iii).

Parts (i)-(ii). $v'(L-N_\Delta) = h \cdot V'(w \cdot A + N_\Delta; w)$ and $v'(L+\xi-N_\Delta^*) = h \cdot V'(w \cdot A + N_\Delta^*; w)$. Both $v(.)$ and $V(.;w)$ are strictly concave. Hence, $N_\Delta^* > N_\Delta$ and $L - N_\Delta < L + \xi - N_\Delta^*$. So, $N_\Delta^* - N_\Delta < \xi$. The same reasoning implies $0 < N_\delta^* - N_\delta < \xi$. Recall Proposition 2. If $N > 0$, $N = N_\Delta \leq N_\Delta^* = N^*$. Suppose $N = 0$. If $N^* < 0, 0 > N^* = N_\delta^* \geq N_\delta = N = 0$ — a contradiction. If $N^* > 0$, $0 < N^* - N = N_\Delta^* - N \leq N_\Delta^* - N_\Delta < \xi$. Suppose $N < 0$. If $N^* < 0$, $N^* - N = N_\delta^* - N_\delta \in (0, \xi)$. If $N^* \geq 0$, $0 < N^* - N \leq N_\delta^* - N = N_\delta^* - N_\delta < \xi$. Part (ii) is strictly analogous.

Part (iv). Recall Lemma 6. Let $L > L^*(w;V)$. Then $v'(L - N(L;w;A;V)) = h \cdot V'(w \cdot A + N(L;w;A;V); w)$. Given Lemma 1 and the monotonicity of $N(.;w;A;V)$, as $L \to \infty$, this first-order condition requires $N(L;w;A;V) \to \infty$. ▮

Proof of Lemma 7:

Proposition 1, Lemma 6, and Proposition 3 imply $\exists X^0(V) < \infty$ such that $X \geq X^0(V)$ implies $S(X;1;V)$ will not be invested in a multi–outcome lottery and $r \cdot S(X;1;V) > L^*(w;V)$ (see Lemma 6).

Step 1. Let lim stand for $\lim_{X\to\infty}$. For $0 < \xi < \infty$ and we show $\lim V'(X;1)/V'(X-\xi;1) = 1$.

Define $C(X;V) \equiv X - S(X;1;V)$. Lemma 5 shows $V'(X;1) = u'(C(X;V))$. Since $V(.;1)$ is increasing and concave, $\lim V'(X;1)$ exists. $V(X;1) \leq V^U(X;1)$ and $V^U(X;1)$ is bounded above — see the inequality after (11) — by a function with slope $\to 0$ as $X \to \infty$. Thus, $\lim V'(X;1) = 0$. So, $\lim u'(C(X;V)) = 0$. Hence, $\lim C(X;V) = \infty$. Proposition 1 shows $\xi \geq C(X+\xi;V) - C(X;V) \geq 0$. So,

$$\begin{aligned}1 &\geq \lim V'(X;1)/V'(X-\xi;1) = \lim u'(C(X;V))/u'(C(X-\xi;V)) \geq \\ &\lim u'(C(X;V))/u'(C(X;V)-\xi) = \lim u'(C(X;V)/[C(X;V)-\xi]) = u'(1) = 1.\end{aligned}$$

Step 2. If $X \geq X^0(V)$, the envelope theorem (see Section 3) and Apostol [1974, th.7.40] imply $\Gamma'(L;w;V) = \int_{\mathcal{A}} v'(L - N(L;1;A;V)) \cdot p(A)\, dA$. Define $\xi \equiv A^U - A^L + 1$. Step 1 and (11) show $\exists X^U(V) \geq X^0(V)$ such that $X \geq X^U(V)$ implies $r \cdot h \cdot V'(X-\xi;1) \leq V'(X;1)$. Let $X \geq X^U(V)$, $S \equiv S(X;1;V)$, and $L \equiv r \cdot S$. Then

$$\begin{aligned}&r \cdot h \cdot V'(X-\xi;1) \leq V'(X;1) = u'(X-S) = r \cdot \Gamma'(r \cdot S;1;V) = \\ &r \cdot \int_{\mathcal{A}} v'(L - N(L;1;A;V)) \cdot p(A)\, dA = r \cdot h \cdot \int_{\mathcal{A}} V'(A + N(L;1;A;V);1) \cdot p(A)\, dA \leq \\ &r \cdot h \cdot V'(A^L + N(L;1;A^L;V);1).\end{aligned}$$

So, $X - \xi \geq A^L + N(L;1;A^L;V)$. Hence, $X > A^U + N(L;1;A^L;V) \geq A + N(L;1;A;V)$ all $A \in \mathcal{A}$.
▮

Proof of Proposition 4:

Step 1. Property (i) follows from the definition of $\lambda(.;V)$ and Kolmogorov and Fomin [1970, p. 298].

Step 2. For property (ii), let $c > 0$. $S(X;1;V)$ and, by construction, $\ell^L(X;V)$ and $\ell^U(X;V)$ are nondecreasing in X. If $q(X;V) \in (0,1)$, $q(.;V)$ is decreasing. Proposition 3 shows $A + N(X;1;A;V)$ is strictly increasing in A. Hence, if $I = [X_0, c]$ or $[X_0, c)$, $\lambda(X;I;V)$ is nonincreasing in X — thus, measurable in X. $\lambda(X,[c_1,c_2];V) = \lambda(X,[X_0,c_2];V) - \lambda(X,[X_0,c_1);V)$ is then also measurable. Clearly the same is true for $\lambda(X,I;V)$ with $I = [c_1,c_2)$, (c_1,c_2), or $(c_1,c_2]$.

Let $\mathcal{X}_n$ be a countable union of disjoint intervals. Then the paragraph above and Halmos [1974, p.84] show $\lambda(.,\mathcal{X}_n;V)$ is measurable. Kolmogorov and Fomin [1970, p.261] shows we can construct a sequence of such sets with $\mathcal{X}_1 \supset \mathcal{X}_2 \supset ...$ and $\cap \mathcal{X}_n = \mathcal{X}$ any $\mathcal{X} \in \mathcal{B}(V)$. Kolmogorov and Fomin [1970, p.266] shows $\lim_{n\to\infty} \lambda(X,\mathcal{X}_n;V) = \lambda(X,\mathcal{X};V)$. So, Kolmogorov and Fomin [1970, p.285] shows $\lambda(.,\mathcal{X};V)$ is a measurable function.

Step 3. For property (iii), define $L^U(V) \equiv \max\{L^*(1;V), r \cdot S(X^U(V);1;V)\}$ — recall Lemma 6. Then for $X \in I(V)$ we need only consider lottery outcomes in $[0, L^U(V)]$. Define $f(L,A) \equiv A + N(L;1;A;V)$ and $J(L) \equiv [f(L,A^L), f(L,A^U)]$. Proposition 3 shows $f(.)$ is strictly increasing in A and continuous in L. Hence, $\mu(J(L)) = f(L,A^U) - f(L,A^L) > 0$ is continuous in L. So, $J^*(V) \equiv \inf\{\mu(J(L)) \mid L \in [0, L^U(V)]\}$ satisfies $J^*(V) > 0$. By assumption the density $p(.)$ is positive on $\mathcal{A}$ and continuous; therefore, $\exists\, p^L > 0$ with $p(A) \geq p^L$ all $A \in \mathcal{A}$. The definition of $\lambda(.)$ then shows that if $\mathcal{X} \in \mathcal{B}(V)$, $\epsilon \equiv \min\{J^*(V)/2, J^*(V) \cdot p^L/2\}$, $\mu(\mathcal{X}) \leq \epsilon$, and $X \in J(V)$, we have

$$\lambda(X,\mathcal{X};V) \leq \sup\{\int_{A(L;\mathcal{X};V)} p(A)\, dA \mid L \in [0, L^U(V)]\} \leq$$

$$1 - p^L \cdot (J^*(V) - \mu(\mathcal{X})) \leq 1 - p^L \cdot J^*(V)/2 \leq 1 - \epsilon.$$

Step 4. Let ergodic set $\mathcal{E}$ have cyclic subsets C_i, $i = 1, ..., n$. Then $\mathcal{E} = \cup_{i=1}^n C_i$ and, rearranging indices if necessary, $X_t \in C_i$ implies $\lambda(X_t;C_{i+1};V) = 1$ (or, if $X_t \in C_n$, $\lambda(X_t;C_1;V) = 1$). Suppose $X \equiv \inf \mathcal{E} = \inf C_i$. Define $I(c) \equiv [X,c]$. Step 2 shows that for any $c > X$, $\lambda(X,I(c);V) \geq \lambda(X',I(c);V)$ all $X' \in \mathcal{E}$. So, $\inf C_{i+1} = \inf \mathcal{E} = X$. Thus, (i) we can find $Z_i \in C_i$ and $Z_{i+1} \in C_{i+1}$ arbitrarily close to each other and to X. But, (ii) Step 3 shows there are intervals I_i and I_{i+1} each of length $\geq J^*(V) > 0$ such that $Z_i \in I_i \subset C_i$ and $Z_{i+1} \in I_{i+1} \subset C_{i+1}$. Statements (i) and (ii) are consistent only if $n = 1$. ∎

Proof of Proposition 5:

Let $\mathcal{E}_i > \mathcal{E}_j$ mean $X_i > X_j$ all $X_i \in \mathcal{E}_i$, $X_j \in \mathcal{E}_j$. Let $\mathcal{E}_i$, $i = 1, ..., m$ be the ergodic sets of $\lambda(.;V)$.

Step 1. We show that rearranging indices if necessary, we can write $\mathcal{E}_1 > \mathcal{E}_2 > ... > \mathcal{E}_m$. The monotonicity established in Propositions 1 and 3 shows $X_i \in \mathcal{E}_i$ and $X_j \in \mathcal{E}_j$ and $X_i \leq X_j$ imply $\sup \mathcal{E}_i \leq \sup \mathcal{E}_j$. Then X_i, $X_i^* \in \mathcal{E}_i$ and $X_j \in \mathcal{E}_j$ and $X_i < X_j < X_i^*$ imply $\sup \mathcal{E}_i = \sup \mathcal{E}_j$. So, we can find $Z_i \in \mathcal{E}_i$ and $Z_j \in \mathcal{E}_j$ with Z_i and Z_j arbitrarily close together. Hence, the preceding proof shows $\mu(\mathcal{E}_i \cap \mathcal{E}_j) \geq J^*(V)/2 > 0$. This contradicts the fact that for distinct ergodic sets we must have $\mu(\mathcal{E}_i \cap \mathcal{E}_j) = 0$.

Step 2. We show $\inf \mathcal{E}_1 \leq A^L$. Suppose $X^* \equiv \inf \mathcal{E}_1 > A^L$. Let $X_t = X^*$. To preserve $X_{t+1} \geq X^*$, if $A_{t+1} = A^L$ the family line in question will need a positive net bequest at time $t+1$; hence, since $v'(0) = \infty$, we must have $S(X^*; 1; V) \equiv S^* > 0$. Thus, recalling Lemma 5, $V'(X^*; 1) = u'(X^* - S^*) \leq r \cdot \Psi^-(r \cdot S^*; 1; V)$. Let $\Upsilon(.)$ be as in (25).

Let L^* be the minimal lottery outcome associated with S^*. We have seen $N(L^*; 1; A^L; V) > 0$. Hence, $L^* > 0$. In fact, $v'(0) = \infty$ implies $L^* - N(L^*; 1; A; V) > 0$ all $A \in \mathcal{A}$. Choose any $\epsilon > 0$ with $\epsilon < L^* - N(L^*; 1; A^L; V) \leq L^* - N(L^*; 1; A; V)$ all $A \in \mathcal{A}$ — see Proposition 3. Since L^* is a minimal lottery outcome, $\Gamma(L^*; 1; V) = \Psi(L^*; 1; V)$. Rockafellar [1970, th.24,1] implies $\Psi^-(.; 1; V)$ is nonincreasing. Then using a simple graph,

$$\Psi^-(r \cdot S^*; 1; V) \leq \Psi^-(L^*; 1; V) \leq \frac{\Gamma(L^*; 1; V) - \Gamma(L^* - \epsilon; 1; V)}{\epsilon} =$$
$$\int_{\mathcal{A}} [\frac{\Upsilon(L^*; 1; A; V) - \Upsilon(L^* - \epsilon; 1; A; V)}{\epsilon}] \cdot p(A)\, dA \leq$$
$$\int_{\mathcal{A}} [\frac{\Upsilon(L^*; 1; A; V) - v(L^* - \epsilon - N(L^*; 1; A; V)) - h \cdot V(A + N(L^*; 1; A; V); 1)}{\epsilon}] \cdot p(A)\, dA =$$
$$\int_{\mathcal{A}} [\frac{v(L^* - N(L^*; 1; A; V)) - v(L^* - \epsilon - N(L^*; 1; A; V))}{\epsilon}] \cdot p(A)\, dA.$$

Proposition 3 shows $N(L^*; 1; A^L; V) \geq N(L^*; 1; A; V)$ all $A \in \mathcal{A}$. Letting $\epsilon \downarrow 0$,

$$\Psi^-(r \cdot S^*; 1; V) \leq \int_{\mathcal{A}} v'(L^* - N(L^*; 1; A; V)) \cdot p(A)\, dA \leq v'(L^* - N(L^*; 1; A^L; V)).$$

So, because $N(L^*; 1; A^L; V) > 0$,

$$V'(X^*; 1) \leq r \cdot \Psi^-(r \cdot S^*; 1; V) \leq r \cdot v'(L^* - N(L^*; 1; A^L; V)) = r \cdot h \cdot V'(A^L + N(L^*; 1; A^L; V)).$$

Hence, $V(.; 1)$ concave and (11) imply $X^* > A^L + N(L^*; 1; A^L; V) \geq X^*$— a contradiction. Thus, $\inf \mathcal{E}_1 \leq A^L$.

Step 3. We show $m = 1$. Define $X^* \equiv \sup \mathcal{E}_i$ any $i \geq 2$. Suppose (2)–(3) hold. Within a family line let $X_{t-1} = X_t = X^*$. Then Proposition 3 implies $A_{t-1} = A_t = A^U$. Define $S^* \equiv S(X^*; 1; V)$. Let L^* be the maximal lottery outcome associated with S^*. Define $N^* \equiv N(L^*; 1; A^U; V)$. Because $A^U > X^*$, $N^* < 0$. Hence, using Lemma 5,

$$h \cdot v'(L^* - N^*) = V'(A^U + N^*; 1) = u'(A^U + N^* - S^*).$$

Also,

$$h \cdot v'(L^* - N^*) \leq h \cdot v'(r \cdot S^* - N^*) = h \cdot a \cdot u'(r \cdot S^* - N^*).$$

Combining lines, $A^U + N^* - S^* > r \cdot S^* - N^*$. Hence, $A^U > -2 \cdot N^*$. So, $A^U/2 > -N^*$. Thus, $X^* = A^U + N^* = (A^U/2) + [(A^U/2) + N^*] > A^U/2 > A^L$ — a contradiction of $X^* \leq A^L$ from Step 2. ▮

Proof of Proposition 7:

First, some notation. We say a random variable $\tilde{Z}$ has property $P(\bar{Z})$ if $Pr\{\tilde{Z} \in [c, \bar{Z}]\} > 0$ all $c < \bar{Z}$. Let $\tilde{X}$ have the distribution $D(.)$ from Proposition 6. Let $\bar{X} \equiv \sup \mathcal{E}$. We can assume $\tilde{X}$ has $P(\bar{X})$ — if not, $Pr\{\tilde{X} \in [c, \bar{X}]\} = 0$ some $c < \bar{X}$, and lopping $[c, \bar{X}]$ off $\mathcal{E}$ does not affect its ergodicity. Our model induces a Markov process for second–period–of–life lottery outcomes. Let $\tilde{L}$ be the corresponding stationary random variable. It has a single ergodic set, $\mathcal{F}$, because $\tilde{X}$ does. With $\bar{L} \equiv \sup \mathcal{F}$, the argument above shows $\tilde{L}$ has $P(\bar{L})$. Let $F(L, A) \equiv N(L; 1; A; V) + A$. Proposition 3 shows $F(\bar{L}; A^U) = \bar{X}$.

Let $L^* \equiv L^*(1; V)$ — see Lemma 6. Let $\ell^U(.)$ be as in Section 4, and using the notation of Proposition 2, define $I(A) \equiv (L^L(1; A; V), L^U(1; A; V))$. We say result "R" holds if there exists $\rho > 0$ such that

$$Pr\{(\tilde{X}, \tilde{A}) \mid A_{t+1} = \tilde{A},\ X_t = \tilde{X},\ \ell^U(X_t; V) \in I(A_{t+1})\} \geq \rho.$$

Step 1. We show "R" holds if $\bar{L} \leq L^*$. Suppose the latter. Since there are some positive bequests, $N(\bar{L}; 1; A^L; V) > 0$. The definition of L^* yields $N(\bar{L}; 1; A^U; V) \leq 0$. Let $\bar{A} \equiv \sup\{A \in \mathcal{A} \mid N(\bar{L}; 1; A; V) > 0\}$. If $\bar{A} < A^U$, then $P(\bar{L})$, the nature of $\tilde{A}$, and Proposition 2 show "R" holds. Suppose $\bar{A} = A^U$. Then $\bar{L} = L^*$. The last part of Lemma 6, $P(\bar{L})$, and the nature of $\tilde{A}$ imply there exist $L_0 < L_1 < \bar{L}$ with $Pr\{\tilde{L} \in [L_0, L_1]\} > 0$. Proposition 2 shows $N(\bar{L}; 1; \bar{A}; V) = N_\Delta(\bar{L}; 1; \bar{A}; V)$. Propositions 2–3 show we can pick L_0 and L_1 so that $N_\Delta(L; 1; \bar{A}; V) < 0 < N_\delta(L; 1; \bar{A}; V)$ all $L \in [L_0, L_1]$. Propositions 2–3 then establish "R".

Step 2. We show "R" holds if $\bar{L} > L^*$. Suppose the latter. Define $f(X) \equiv r \cdot S(X; 1; V)$, $X^* \equiv F(L^*; A^U)$, $X^1 \equiv F(\bar{L}; A^U)$, and $X^2 \equiv F(\bar{L}; A^L)$. The nature of $\tilde{A}$ and $P(\bar{L})$ show $P(X)$ holds for $\tilde{X}$ each $X \in [X^2, X^1]$. The proof of Proposition 4 shows $X^1 - X^2 \geq J^*(V) > 0$. Set $\bar{L} = \max\{[f(X^1) + f(X^2)]/2, L^*\}$. Then $P(X)$ above, Lemma 6, and the continuity of $f(.)$ show $P(\bar{L})$ holds. If $\bar{L} = L^*$, proceed as in Step 1. Otherwise redefine X^1 and X^2 and repeat the present step. Since $J^*(V) > 0$, we are done in a finite number of iterations.

Step 3. We finish the proof. Steps 1–2 and the independence of $\tilde{X}_t$ and $\tilde{A}_{t+1}$ imply there exist $E \in \mathcal{B}(V)$ and Lebesgue measurable $F \subset \mathcal{A}$ such that $\int_E D(dX) \equiv \rho_1$, $\int_F p(A)\, dA \equiv \rho_2$, $\rho_1 \cdot \rho_2 = \rho > 0$, and $(X, A) \in E \times F$ implies $\ell^U(X; V) \in I(A)$. Doob [1953, ch.v] shows the convergence of Proposition 6 is uniform with respect to X. Thus, there exists $T < \infty$ such that $\forall X \in \mathcal{E}$ we have $|\lambda^T(X; E) - \int_E D(dX)| < \rho_1/2$. Set $\varsigma = \rho_2 \cdot \rho_1/2$. Then for any time–0 family normalized wealth X_0, if $\tilde{X}_t$ is the family line's wealth at time t, conditional on information at time 0,

$$Pr\{(\tilde{X}_T, \tilde{A}_{T+1}) \mid \ell^U(\tilde{X}_T; V) \in I(\tilde{A}_{t+1})\} \geq \varsigma > 0.$$

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