Center for Research on Economic and Social Theory

CREST Working Paper

# Non-Partitional Information On Dynamic State Spaces and the Possibility of Speculation

*Hyun Song Shin*

**December 1989**
**Number 90–11**

**DEPARTMENT OF ECONOMICS**
University of Michigan
**Ann Arbor, Michigan 48109**

# NON-PARTITIONAL INFORMATION ON DYNAMIC STATE SPACES AND THE POSSIBILITY OF SPECULATION*

Hyun Song Shin,

Department of Economics,
University of Michigan,
Ann Arbor, MI 48109—1220, U. S. A.

and

Magdalen College,
Oxford, OX1 4AU, U.K.

Revised, December 1989

Abstract: We describe an environment in which the development of the world over time is an object of uncertainty for the individual. In this environment, a natural representation of information is in terms of non-partitional structures. However, not all non-partitional information can be justified in this way. We identify a set of conditions which are necessary and jointly sufficient for such a representation —namely, that the individual's information be such that (1) whatever is known is true, (2) whatever is known is <u>known</u> to be known, and (3) that information be nested. Moreover, these three conditions are precisely those identified in a paper by Geanakoplos as precluding speculative trade in games with generalized information. Thus, our discussion provides an alternative perspective on the issue of speculation.

---

* I thank Ken Binmore, John Geanakoplos and Jim Mirrlees for comments and many stimulating discussions on the issues raised here. I record my thanks to the Economics department of the University of Michigan for their hospitality during the academic year 1989-90.

## 1. Introduction

The standard method of representing the information of an individual in an uncertain environment is to suppose that the individual has a partition of the states of the world and that, given a realization of a state of the world, the individual is informed as to which element of his partition contains the realized state. A series of recent papers have explored the possibility of relaxing the assumption that information is partitional (Samet (1987), Shin (1987), Brandenburger *et al.* (1988), Geanakoplos (1988), Rubinstein and Wolinsky (1988)). In these papers, an individual's information is determined by a correspondence $\varphi$ which associates a subset of the state space with each state. The interpretation is that, at the state $\omega$, the individual believes that the true state is in $\varphi(\omega)$.

However, a frequent criticism of this approach is that an individual is not making full use of the informational content of the signals he receives. Consider the following example. There are two states $\omega_1$ and $\omega_2$. When $\omega_2$ is realized, the individual is informed of this fact. However, when $\omega_1$ is realized, the individual is uncertain as to which state is the true state. The problem arises when we try to close the model by assuming that the individual "knows the model". The difficulty is this. Given that he knows the model, the individual at $\omega_1$ ought to infer that $\omega_1$ is the true state since, if $\omega_2$ were the true state he would be informed of this fact. In other words, if the individual understands the model, his own ignorance is an extra signal to be utilized in making inferences about the world. Thus, Geanakoplos (1988) is led to concede that these models describe the information of boundedly rational individuals who ignore the subtle informational content of signals and take signals only at their face value.

In this paper, we describe a dynamic environment in which the possible development of the world over time is an object of uncertainty for the individual. This environment has the distinctive feature that when the world reaches any state during its process of transition from state to state, there is a chance that the transition will halt there. In this environment, a natural representation of the information of an individual is in terms of non-partitional structures. Here, information is non-partitional even though the individual fully understands the model. Thus, the individuals in our framework are not vulnerable to the charge that they are failing to comprehend their environment.

However, not all non-partitional information can be given a representation in terms of our dynamic framework. The main result of this paper is that the following three conditions are each necessary and jointly sufficient for such a representation: (1) that an individual cannot "know" falsehoods, (2) that if an individual knows something, he *knows* that he knows, and (3) that information be nested.

Quite apart from any intrinsic interest in such a characterization, this result gains added significance due to Geanakoplos's(1988) finding that the three conditions above are each necessary and jointly sufficient to preclude speculative trade in games where players have generalized information. Thus, there turns out to be an intimate connection between the possibility of a dynamic representation and the possibility of speculative trade. Indeed, by utilizing the conditions of a dynamic representation, we are able to identify the causes of speculative trade when information is non-partitional. Essentially, the reasons for speculative trade can be traced to the violation of a generalized form of the dominance principle.

We order our discussion as follows. In section 2, we describe the dynamic environment which forms the basis of our analysis. In section 3, we define the notion of dynamic representation and present our main characterization result. The proof of this result is obtained as a consequence of the discussion in sections 4 and 5. In section 4, we set out a systematic procedure for constructing a dynamic representation of any information structure satisfying the three conditions above. This yields a constructive proof of the sufficiency half of our theorem. In section 5, we demonstrate the necessity of each of the three conditions.

Finally, in section 6, we analyse the issue of speculation. By drawing on the results of Geanakoplos (1988) and the criteria for a dynamic representation, we provide intuitive explanations of the causes of speculative trade.

## 2. The Environment

Our environment is based on the following objects. $\Omega$ is an arbitrary set of *states* $\omega$, $M$ is a finite set of *messages* $\mu$, and $\sigma$ is a *signal*, defined to be a function $\sigma : \Omega \rightarrow M$. We shall assume that $M$ is finite and has $m$ elements. The $i$th message is denoted by $\mu_i$.

Time is discrete and is indexed by the non-negative integers. At time 0, no state is realized, and hence, no message is issued by $\sigma$. At time 1, precisely one state is realized, and a message is issued according to the signal $\sigma$. We shall assume that the message $\mu_i$ is issued with probability $u_i$. Let $u = (u_1, u_2, \ldots, u_m)$.

From time 1 onwards, the world is in precisely one of the states at any moment in time, but the world makes transitions from one state to another over time. We shall suppose that these transitions take the following simple form. When the message $\mu_i$ is issued at time $t$, the message $\mu_j$ is issued at time $t+1$ with probability $p_{ij}$. Moreover, we assume that this probability is independent of the history of past messages issued. We denote by $P$ the matrix of transition probabilities $p_{ij}$, where $p_{ij}$ is the $(i,j)$-th element of $P$.

The most distinctive feature of our environment can now be introduced. When the world reaches a particular state, there is a probability that it will "settle" at that state, and stay there forever. A physical analogy is with a particle moving from one state to another with the feature that, having reached any of the states, there is a probability that it will be "absorbed" by that state. We assume that the settling process takes the following simple form. When the world is at a state which gives rise to the message $\mu_i$, the world settles at this state with probability $q_i$. We assume that each $q_i$ is non-zero, and each $q_i$ is independent of history. Since the world settles on a state if and only if it does not make a transition, the $i$th row of $P$ sums to $1-q_i$. We denote by $q$ the $m$-tuple $(q_1,q_2,\ldots,q_m)$.

We shall assume for the moment that there is a single individual. This individual knows the signal $\sigma$, and observes the messages issued by it. Moreover, the individual knows $u$, $P$ and $q$, and as soon as the world settles on a state, the individual is aware of this fact. In other words, this individual knows everything there is to know about his environment, subject to the imperfect information of his signal $\sigma$.

We shall adopt as a paradigmatic example of our environment the case of an individual facing the uncertainty generated by the spinning of a roulette wheel. A full description of the "state" of the system given by the roulette wheel will be extremely complex. However, the individual is able to observe the position of the ball as it moves between the slots of the roulette wheel over time. The slots correspond to the *messages* received by the individual. The proviso is that our roulette wheel is the rather special one in which the likely position of the ball at any moment in time depends only on its position in the previous period. It also has the feature that the probability of the ball settling on a slot is constant over time.

The comparison with a roulette wheel is intended to emphasize the feature of our environment that the payoffs are determined when, and only when, the world settles on a state. For example, we can imagine a referee taking bets on a spin of the roulette wheel and distributing rewards once the ball has come to rest. Thus, for the individual, it is the long-run settling process which matters. The period to period transitions are important only to the extent that they provide information concerning the long-run settling process.

The long-run behaviour of our environment can be described more tractably by constructing a Markov chain which mimicks the dynamics of our environment. This Markov chain is constructed as follows. Define;

(2.1) $$\theta_i := \left\{\omega \,\middle|\, \sigma(\omega) = \mu_i\right\}.$$

$\theta_i$ is the set of states which give rise to the $i$th message. We denote by $\Theta$ the set of all such $\theta_i$. Clearly, $\Theta$ partitions $\Omega$. The transition matrix $P$ can now be seen as describing the transition between elements of $\Theta$. Thus, $p_{ij}$ is the probability that the true state will be in $\theta_j$ next period given that the true state is currently in $\theta_i$.

For each $\theta_i$, we introduce a *copy* $\overline{\theta}_i$ of $\theta_i$. We denote by $\overline{\Theta}$ the set of all such copies. Let $S$ be the union $\Theta \cup \overline{\Theta}$, and we index the elements $s$ of $S$ so that $\theta_i = s_i$ and $\overline{\theta}_i = s_{m+i}$. Denote by $Q$ the diagonal matrix whose leading diagonal elements are the $m$ absorption probabilities $q_1, q_2, \ldots, q_m$ (in this order). We call this the *absorption matrix*. Consider the square matrix $T$ of order $2m$ given in partitioned form as follows. 0 is the zero matrix of order $m$ and $I$ is the identity matrix of order $m$.

(2.2) $$T = \begin{bmatrix} P & Q \\ 0 & I \end{bmatrix}$$

The $j$th row of $T$, when $j \leq m$, is given by

(2.3) $$[p_{j1}\ p_{j2}\ \cdots\ p_{jm}\ 0\ 0 \ldots 0\ q_j\ 0 \ldots 0]$$

Each element of $T$ is non-negative, and each row of $T$ sums to unity since $\sum_{\ell} p_{j\ell} = 1 - q_j$. Thus $T$ can be seen as the transition matrix associated with a Markov chain on the set $S$. Moreover, this is a process which mimicks our original model. When the world starts out at $\theta_j$, its transition to the first $m$ elements of $S$ (the set $\Theta$) is governed by the matrix $P$. But in addition, there is probability $q_j$ that the world will pass to the copy $\bar{\theta}_j$ of $\theta_j$. By construction, once $\bar{\theta}_j$ has been reached, the world will never leave it. This mimicks the absorption process of our original model. In particular, the long run properties of our environment will be mimicked by the matrix $T^n$ as $n$ becomes large. The following result characterizes the long run behaviour of our environment.

*PROPOSITION 2.1*. The inverse $(I-P)^{-1}$ exists, and $T^n \to T^*$ as $n \to \infty$, where

$$T^* = \begin{bmatrix} 0 & (I-P)^{-1}Q \\ 0 & I \end{bmatrix}$$

Let us remark on the interpretation of $T^*$. Since the top left hand corner of $T^*$ is occupied by the zero matrix, this means that, at whichever element of $\Theta$ the world starts out, it will eventually settle at one of the copies $\bar{\theta} \in \bar{\Theta}$. The matrix $(I-P)^{-1}Q$ governs the probability of this settling process. We obtain this matrix in a similar way to which the inverse of the Leontief activity matrix is obtained as a power series (see, for example, Gale (1960) ch.9).

*PROOF*.

$$T^n = \begin{bmatrix} P^n & (I+P+\cdots+P^{n-1})Q \\ 0 & I \end{bmatrix}$$

We show first that $P^n \to 0$ as $n \to \infty$. Since $\sum_{\ell} p_{j\ell} = 1 - q_j$ and $q_j > 0$ for all $j$, each row sum of $P$ is strictly less than 1. Thus, there exists $0 \leq r < 1$ such that $\sum_{\ell} p_{j\ell} \leq r < 1$, for all $j$. Then,

$$\sum_k p_{jk}^{(2)} = \sum_k \sum_{\ell} p_{j\ell}\, p_{\ell k} = \sum_{\ell} \left( \sum_k p_{\ell k} \right) p_{j\ell} \leq r \sum_{\ell} p_{j\ell} \leq r^2. \tag{2.4}$$

That is, each row sum of $P^2$ is bounded by $r^2$. In general, each row sum of $P^n$ is bounded by $r^n$. Since $r^n \to 0$ as $n \to \infty$, we have $P^n \to 0$. Next, let $B^n = (I + P + \cdots + P^{n-1})$. Then,

$$(I - P)B^n = I - P^n \tag{2.5}$$

Taking determinants, $|I - P|\,|B^n| = |I - P^n|$. But $|I - P^n| \to |I| = 1$ as $n \to \infty$, so that $|I - P| \neq 0$ and $(I - P)^{-1}$ exists. Then, pre-multiplying (2.5) by $(I - P)^{-1}$,

$$B^n = (I - P)^{-1} - (I - P)^{-1}P^n \tag{2.6}$$

But $B^n \to (I - P)^{-1}$ as $n \to \infty$, since $P^n \to 0$. Thus, $B^n Q \to (I - P)^{-1}Q$. □

Consider the matrix $(I - P)^{-1}Q$. This matrix gives the long-run settling probabilities over the copy set $\bar{\Theta}$, and these probabilities mirror the long-run settling process in our original model over the set $\Theta$. Thus, the $(i, j)$-th element of this matrix gives the probability that, when the world starts out at a state in $\theta_i$, it will eventually settle on a state in $\theta_j$. For an individual whose payoffs are determined when the world settles on a state, the matrix $(I - P)^{-1}Q$ provides the appropriate characterization of uncertainty. More precisely, the $i$th row of $(I - P)^{-1}Q$ represents the individual's *interim* uncertainty when the true state of the world is in $\theta_i$.

In addition, there is a clear sense in which our dynamic environment has a "prior" over $\Theta$ — namely, the distribution of settling probabilities viewed from time 0. This prior, denoted by $\pi$, is obtained as the weighted average of the rows of

$(I-P)^{-1}Q$, where the weights are given by the initial realization probabilities $(u_1, \ldots, u_m)$.

$$\pi = u(I-P)^{-1}Q \tag{2.7}$$

We therefore have a concise characterization of the uncertainty facing the individual at all points in time and at all states. At time 0, when no state is realized, the appropriate characterization of uncertainty is in terms of the prior $\pi$. At time 1, uncertainty at $\theta_i$ is characterized by the $i$th row of $(I-P)^{-1}Q$. At all subsequent periods, when the world has not yet settled on a state, the $i$th row of $(I-P)^{-1}Q$ remains the appropriate characterization of uncertainty at $\theta_i$.

Having prepared this background, we proceed to the main discussion of this paper. We shall characterize the class of non-partitional information structures on $\Omega$ which can be given a reconstruction in terms of the dynamic scenario described in this section.

## 3. Dynamic Representation

Let a non-partitional information structure $(\Omega, \varphi, \pi)$ be given, where $\Omega$ is an arbitrary state space, $\varphi$ is a correspondence $\varphi : \Omega \to 2^{\Omega}$, and $\pi$ is a probability measure on $\Omega$ such that $\varphi(\omega)$ is measurable, for all $\omega \in \Omega$. We shall, however, assume that the range of $\varphi$ is finite, and denote by $\Phi_1$, $\Phi_2$, ..., $\Phi_m$ the subsets of $\Omega$ in the range of $\varphi$.

The first step in attempting to give the triple $(\Omega, \varphi, \pi)$ an interpretation in terms of our dynamic framework is to introduce a finite message space and a signal. For this, we define;

$$\theta_i := \left\{ \omega \,\middle|\, \varphi(\omega) = \Phi_i \right\}. \tag{3.1}$$

$\theta_i$ consists of those states at which the individual's information is $\Phi_i$. The set of all such $\theta_i$ partitions $\Omega$, and we denote this partition by $\Theta$. As the notation suggests, the intended interpretation of $\Theta$ is as the partition generated by some signal $\sigma:\Omega \rightarrow M$, where $M$ is a finite message space. $\theta_i$ has the interpretation of the set of states which gives rise to the $i$th message. (Compare with (2.1)).

We denote by $\pi_i$ the probability $\pi(\theta_i)$. When no confusion is likely, we shall also use the symbol $\pi$ to denote the $m$-tuple $(\pi_1, \pi_2, \ldots, \pi_m)$. The context should make clear whether we intend $\pi$ to denote the vector or the measure. Also, for the rest of this paper, we shall confine our attention to $\pi$ such that $\pi(\theta_i) > 0$ for all $i$.

It was shown in the last section how the uncertainty facing the individual at each state could be characterized by the appropriate row of the matrix $(I-P)^{-1}Q$. There is a matrix which plays an analogous role when we are given the non-partitional structure $(\Omega, \varphi, \pi)$. At a state $\omega \in \theta_i$, the individual forms beliefs by conditioning on $\varphi(\omega) = \Phi_i$. Thus, the uncertainty at $\theta_i$ is characterized by the $i$th row of the following matrix.

$$(3.2) \qquad \begin{bmatrix} \pi(\theta_1|\Phi_1) & \pi(\theta_2|\Phi_1) & \cdots & \pi(\theta_m|\Phi_1) \\ \pi(\theta_1|\Phi_2) & \pi(\theta_2|\Phi_2) & \cdots & \pi(\theta_m|\Phi_2) \\ \vdots & \vdots & \ddots & \vdots \\ \pi(\theta_1|\Phi_m) & \pi(\theta_2|\Phi_m) & \cdots & \pi(\theta_m|\Phi_m) \end{bmatrix}$$

We denote this matrix by $R$. Since the $i$th row of $R$ characterizes the uncertainty of the individual at $\theta_i$, the matrix $R$ plays the same role as the matrix $(I-P)^{-1}Q$ of the last section. The interesting question is whether the matrix $R$ could have arisen from the dynamic scenario described in the previous section. That is, whether there are appropriate matrices $P$, $Q$ such that $R=(I-P)^{-1}Q$. Also, we would need to interpret the vector $\pi = (\pi_1, \pi_2, \ldots, \pi_m)$ as having arisen from some initial probability distribution $u=(u_1,u_2,\ldots,u_m)$ via the dynamic process determined by $P$ and $Q$. That is, whether $\pi=u(I-P)^{-1}Q$. Finally, in order to complete the scenario, we would have to capture the fact that the signal which generates the partition $\Theta$ is the *only* signal that the individual has access to. For this, the correspondence $\varphi:\Omega\rightarrow 2^{\Omega}$ must be at least as coarse as the partition $\Theta$. That is, each $\varphi(\omega)$ must be a union of elements of $\Theta$. Drawing together these considerations, we state the following definition.

*DEFINITION.* $(\Omega,\varphi,\pi)$ has a *dynamic representation* if;

($D1$) Each $\varphi(\omega)$ is a union of elements of $\Theta$.

($D2$) There exist transition matrix $P$ and absorption matrix $Q$ such that;

$$R=(I-P)^{-1}Q.$$

($D3$) There exists probability distribution $u$ such that;

$$\pi=uR.$$

We present two examples of non-partitional information structures for which we can construct a dynamic representation. In section 6, we shall encounter examples where no dynamic representation exists.

*EXAMPLE 3.1.* $\Omega=\{\omega_1,\omega_2\}$, $\varphi(\omega_1)=\Omega$, $\varphi(\omega_2)=\{\omega_2\}$, and $\pi(\{\omega_1\})=\pi(\{\omega_2\})=\frac{1}{2}$. Let $\theta_1=\{\omega_1\}$ and $\theta_2=\{\omega_2\}$. Since each $\theta_i$ is a singleton set, ($D1$) is satisfied. For ($D2$),

$R=\begin{bmatrix}\frac{1}{2} & \frac{1}{2}\\ 0 & 1\end{bmatrix}$. Then for $P=\begin{bmatrix}\frac{8}{10} & \frac{1}{10}\\ 0 & \frac{9}{10}\end{bmatrix}$ and $Q=\begin{bmatrix}\frac{1}{10} & 0\\ 0 & \frac{1}{10}\end{bmatrix}$,

$$P=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}-\begin{bmatrix}\frac{2}{10} & -\frac{1}{10}\\ 0 & \frac{1}{10}\end{bmatrix}=I-R^{-1}Q \tag{3.3}$$

so that $R=(I-P)^{-1}Q$. Finally, $u=[1,0]$ yields $\pi=uR$, and so ($D3$) is satisfied. Thus, $(\Omega,\varphi,\pi)$ has a dynamic representation.

*EXAMPLE 3.2.* $\Omega=\left\{\omega_1,\omega_2,\ldots,\omega_7\right\}$, $\pi(\{\omega_i\})=\frac{1}{7}$ for all $i$. $\varphi$ is as given in figure 3.1.

[Figure 3.1 here]

For each $\omega$, $\varphi(\omega)$ is represented as the smallest balloon containing $\omega$. Since $\varphi(\omega)=\varphi(\omega') \Rightarrow \omega=\omega'$, each element of $\Theta$ is a singleton, and $(D1)$ is satisfied. Define $\theta_i=\left\{\omega_i\right\}$, for all $i$. For $(D2)$, we note that;

$$R=\begin{bmatrix} \frac{1}{7} & \frac{1}{7} & \frac{1}{7} & \frac{1}{7} & \frac{1}{7} & \frac{1}{7} & \frac{1}{7} \\ & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 \\ & & 1 & 0 & 0 & 0 & 0 \\ & & & 1 & 0 & 0 & 0 \\ & & & & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ & \mathbf{0} & & & & 1 & 0 \\ & & & & & & 1 \end{bmatrix} \qquad R^{-1}=\begin{bmatrix} 7 & -3 & 0 & 0 & -3 & 0 & 0 \\ & 3 & -1 & -1 & 0 & 0 & 0 \\ & & 1 & 0 & 0 & 0 & 0 \\ & & & 1 & 0 & 0 & 0 \\ & & & & 3 & -1 & -1 \\ & \mathbf{0} & & & & 1 & 0 \\ & & & & & & 1 \end{bmatrix}$$

By setting all the absorption probabilities equal to $\frac{1}{10}$, we have $QR^{-1}=\frac{1}{10}R^{-1}$. Thus,

$$P=I-QR^{-1}=\frac{1}{10}\begin{bmatrix} 3 & 3 & 0 & 0 & 3 & 0 & 0 \\ & 7 & 1 & 1 & 0 & 0 & 0 \\ & & 9 & 0 & 0 & 0 & 0 \\ & & & 9 & 0 & 0 & 0 \\ & & & & 7 & 1 & 1 \\ & \mathbf{0} & & & & 9 & 0 \\ & & & & & & 9 \end{bmatrix}$$

which is a matrix of probabilities. Thus, $(D2)$ is satisfied. Lastly, for $(D3)$, we let $u=(1,0,\ldots,0)$. Then, $uR=(\frac{1}{7},\frac{1}{7},\ldots,\frac{1}{7})=\pi$, so that $(D3)$ is satisfied. Thus, the above structure has a dynamic representation.

From the examples above, we can see a pattern emerging. Notice that the matrices $P$, $R$ and $R^{-1}$ are all upper triangular, and that each of them is obtained by "nesting" the matrices associated with each substructure in a recursive way. The formal treatment which follows is a generalization of these themes.

What properties of the correspondence $\varphi$ allow us to carry out this construction? As we show below, this construction turns on three properties of the correspondence $\varphi$. They are;

(C1) $$\omega \in \varphi(\omega), \ \forall \omega$$

(C2) $$\omega' \in \varphi(\omega) \ \Rightarrow \ \varphi(\omega') \subseteq \varphi(\omega), \ \forall \omega, \omega'$$

(C3) $$\varphi(\omega) \cap \varphi(\omega') \neq \varnothing \ \Rightarrow \ [\varphi(\omega) \subseteq \varphi(\omega') \text{ or } \varphi(\omega') \subseteq \varphi(\omega)], \ \forall \omega, \omega'$$

The first two conditions have been examined in some detail by Samet (1987) and Shin (1987). (C1) corresponds to the principle that whatever is known is true, while (C2) corresponds to the principle that whatever is known is *known* to be known. (C3) is the condition known as *nestedness* introduced by Geanakoplos (1988). The correspondences $\varphi$ which satisfy these three conditions encompass the standard case where information is partitional. In particular, Geanakoplos (1988) has shown that these three conditions are necessary and jointly sufficient to preclude speculation in the context of games with generalized information structures.

For our part, the main result of our paper is that the three conditions above are necessary and jointly sufficient for the existence of a dynamic representation. This result takes on additional significance in the light of Geanakoplos's result on speculation, since we have the corollary that the existence of a dynamic representation is necessary and sufficient for the absence of speculation.

*THEOREM 1*. $(\Omega, \varphi, \pi)$ has a dynamic representation if and only if $\varphi$ satisfies (C1), (C2) and (C3).

The proof of this result will be obtained as a consequence of the discussion in the next two sections. Let us first consider necessity. The necessity of (C1) and (C2) are easy to show. (C1) follows from the property of our dynamic environment that, when the world starts out from a state, there is a non-zero probability that it will settle there. (C2) is a consequence of the transitivity of the accessibility relation. That is, if a transition is possible from $\theta_i$ to $\theta_j$ and also from $\theta_j$ to $\theta_k$, then the transition from $\theta_i$ to $\theta_k$ is always possible.

The necessity of (C3) is harder to show. We rely on an argument which shows that, if (C3) does *not* hold, then either $u$ or $P$ has a negative entry. Since the entries in $u$ and $P$ must be probabilities, we conclude from this that no dynamic representation exists when (C3) fails.

For the sufficiency part of the proof, we exhibit an effective procedure for constructing $u$, $P$ and $Q$ which satisfy the appropriate properties. This boils down to checking that $u$ and $P$ have entries which are probabilities. This is the subject of the following section.

## 4. Sufficiency

In this section, we shall set out a systematic procedure for constructing a dynamic representation of a structure $(\Omega, \varphi, \pi)$ where $\varphi$ satisfies (C1), (C2) and (C3). As a by-product of this discussion we will have proved the sufficiency half of theorem 1. Let us start with some preliminary results.

*LEMMA 4.1.* If $\varphi$ satisfies (C1) and (C2), each $\varphi(\omega)$ is a union of elements of $\Theta$.

*PROOF.* Suppose not. Then, there exist $\theta_j$ and $\Phi_i$ such that $\theta_j \cap \Phi_i \neq \varnothing$ but $\theta_j \not\subseteq \Phi_i$. Take any $\omega \in \theta_j \cap \Phi_i$. Then, $\varphi(\omega) = \Phi_j$. By (C1), $\theta_j \subseteq \Phi_j$. Thus, $\Phi_j \not\subseteq \Phi_i$. But since $\omega \in \Phi_i$, we have, by (C2), $\varphi(\omega) = \Phi_j \subseteq \Phi_i$, which is a contradiction. □

Let us denote by $\theta(\omega)$ the element of $\Theta$ which contains $\omega$. When each $\varphi(\omega)$ is a union of elements of $\Theta$, $\omega' \in \varphi(\omega) \Leftrightarrow \theta(\omega') \subseteq \varphi(\omega)$. We shall define the binary relation $\preceq$ on $\Theta$ so that $\theta_i \preceq \theta_j$ is taken to mean $\theta_j \subseteq \Phi_i$. Together with the assumption that each $\theta_i$ is non-null, we have;

$$(4.1) \qquad \theta_i \preceq \theta_j \;\Leftrightarrow\; \theta_j \subseteq \Phi_i \;\Leftrightarrow\; \pi(\theta_j \mid \Phi_i) > 0$$

*LEMMA 4.2.*

(i) If $\varphi$ satisfies (C1) and (C2), $\preceq$ is a partial ordering.

(ii) Suppose $\varphi$ satisfies (C1) and (C2). Then $\varphi$ satisfies (C3) if and only if the structure $(\Theta, \preceq)$ is a union of trees.

*PROOF*. (i) From (C1), $\theta_i \subseteq \Phi_i$ for all $i$. Thus, $\preceq$ is reflexive. Next, (C2) implies that when $\theta_j \subseteq \Phi_i$, we have $\Phi_j \subseteq \Phi_i$. This implies the transitivity and anti-symmetry of $\preceq$. For transitivity, note that when $\theta_j \subseteq \Phi_i$ and $\theta_k \subseteq \Phi_j$, we have $\theta_k \subseteq \Phi_j \subseteq \Phi_i$, so that $\preceq$ is transitive. Also, when $\theta_i \preceq \theta_j$ and $\theta_j \preceq \theta_i$, we have $\Phi_i = \Phi_j$, so that $\theta_i = \theta_j$. Thus, $\preceq$ is anti-symmetric.

(ii) $\varphi$ satisfies (C3) whenever, for all $\omega$, $\omega'$, $\omega''$,

$$(4.2) \qquad \omega \in \varphi(\omega') \cap \varphi(\omega'') \;\Rightarrow\; \left[\varphi(\omega') \subseteq \varphi(\omega'') \text{ or } \varphi(\omega'') \subseteq \varphi(\omega')\right].$$

By (C1) and (C2), $\omega \in \varphi(\omega') \;\Leftrightarrow\; \varphi(\omega) \subseteq \varphi(\omega')$. Thus, (4.2) is equivalent to;

$$(4.3) \qquad \omega \in \varphi(\omega') \cap \varphi(\omega'') \;\Rightarrow\; \left[\omega' \in \varphi(\omega'') \text{ or } \omega'' \in \varphi(\omega')\right].$$

We know that $\omega \in \varphi(\omega') \Leftrightarrow \theta(\omega) \subseteq \varphi(\omega') \Leftrightarrow \theta(\omega') \preceq \theta(\omega)$. Thus, (4.3) is equivalent to;

$$(4.4) \qquad \left[\theta(\omega') \preceq \theta(\omega) \text{ and } \theta(\omega'') \preceq \theta(\omega)\right] \;\Rightarrow\; \left[\theta(\omega') \preceq \theta(\omega'') \text{ or } \theta(\omega'') \preceq \theta(\omega')\right]$$

That is, for any $\theta \in \Theta$, all its predecessors according to $\preceq$ are totally ordered by $\preceq$. Thus, each connected component of $(\Theta, \preceq)$ is a tree. □

By this lemma, when $\varphi$ satisfies (C1), (C2) and (C3), the ordered structure $(\Theta, \preceq)$ is a collection of trees. When $(\Theta, \preceq)$ consists of more than one tree, each tree can be regarded as a separate information structure and be treated in isolation. Thus, we shall confine our attention to the case where $(\Theta, \preceq)$ is a *single* tree.

Thus, suppose $(\Theta, \preceq)$ is a tree with root $\overset{o}{\theta}$. We shall associate a matrix with each node of the tree by means of the function $f$ defined as follows.

(i) For each terminal node $\theta$, $f(\theta) = [1]$. That is, we associate with each terminal node the matrix consisting of the single entry 1.

(ii) Suppose $\theta$ is a non-terminal node, and suppose $\underline{\theta}_1, \underline{\theta}_2, \ldots, \underline{\theta}_k$ are the immediate successors of $\theta$. Suppose also that $f(\underline{\theta}_1) = S_1$, $f(\underline{\theta}_2) = S_2$, ..., $f(\underline{\theta}_k) = S_k$. Then $f(\theta)$ is defined as the matrix;

$$\left[\begin{array}{c|cccc} 1 & \cdots & & & 1 \\ \hline 0 & S_1 & & & \\ \vdots & & S_2 & & \\ \vdots & & & \ddots & \\ 0 & & & & S_k \end{array}\right] \tag{4.5}$$

This is the matrix obtained from the block diagonal matrix which has $(S_1, S_2, \ldots, S_k)$ along the leading diagonal by adding a column of zeros on the left, and then adding a row of 1's on the top. Define the matrix $D$ to be $f(\overset{o}{\theta})$. That is, $D$ is the matrix associated with the root of the tree. We then index the set $\Theta$ as follows. If the first row of $f(\theta)$ appears in the $i$th row of $D$, we let $\theta = \theta_i$.

[Figure 4.1 here]

We shall say that a matrix $X$ is *nested inside* a matrix $Y$ if there is a partitioning of $Y$ such that $X$ appears as a component in this partitioned matrix. By construction, $\theta_i \preceq \theta_j$ if and only if $f(\theta_j)$ is nested inside $f(\theta_i)$. Since the top row of $f(\theta_i)$ consists of 1's, $\theta_i \preceq \theta_j$ implies that $d_{ij} = 1$. Conversely, the $i$th row of $D$ consists of zeros except for those entries which form the first row of $f(\theta_i)$. Thus, $d_{ij} = 1$ implies that $\theta_i \preceq \theta_j$. Together, (4.6) $$d_{ij} = \begin{cases} 1 & \text{if } \theta_i \preceq \theta_j \\ 0 & \text{otherwise} \end{cases}$$

We note that each matrix in the range of $f$ is an upper triangular matrix with 1's along the leading diagonal. This ensures the non-singularity of any matrix associated with a node of our tree. We then note the following feature of the matrices $[f(\theta)]^{-1}$.

*LEMMA 4.3* For any $\theta \in \Theta$, the first column of $[f(\theta)]^{-1}$ has precisely one non-zero entry— namely, 1. For $i \geq 2$, the $i$-th column of $[f(\theta)]^{-1}$ has precisely two non-zero entries— namely, 1 and $-1$.

*PROOF.* The proof is by induction. We show that the conditions of the lemma hold for each terminal node, and then show that when the conditions of the lemma hold for all immediate successors of a node $\theta$, they hold for $\theta$ also.

For a terminal node, $f(\theta) = [f(\theta)]^{-1} = [1]$, and so the conditions of the lemma hold trivially. Next, let $\theta$ be a non-terminal node. Suppose $\underline{\theta}_1$, $\underline{\theta}_2$, ..., $\underline{\theta}_k$ are the immediate successors of $\theta$, and that $f(\underline{\theta}_1) = S_1$, $f(\underline{\theta}_2) = S_2$, ..., $f(\underline{\theta}_k) = S_k$. Then $f(\theta)$ is given by (4.5). Consider the following matrix.

(4.7) $$\left[\begin{array}{c|cccc} \multicolumn{5}{c}{\text{——— } x \text{ ———}} \\ \hline 0 & S_1^{-1} & & & \\ | & & S_2^{-1} & & \\ | & & & \ddots & \\ 0 & & & & S_k^{-1} \end{array}\right]$$

This is the matrix obtained from the block diagonal matrix which has $(S_1^{-1}, S_2^{-1}, \ldots, S_k^{-1})$ along the leading diagonal by first adding a column of zeros on the left, and then adding a top row given by $x = (x_1, x_2, \ldots, x_\ell)$, where $x_1 = 1$, and for $i \geq 2$,

$$x_i = \begin{cases} -1 & \text{if there is precisely one non-zero element below } x_i \\ 0 & \text{otherwise} \end{cases}$$

By the induction hypothesis, each $S_j^{-1}$ satisfies the conditions of the lemma. By construction, $x_i = -1$ if and only if the first column of one of the matrices $S_j^{-1}$ appears in the $i$-th column of (4.7). Thus, in each column of (4.7) except the first, there are precisely two non-zero elements (1 and $-1$). Since $x_1 = 1$, the first column of (4.7) has this element as the only non-zero element. Thus, (4.7) satisfies the conditions of the lemma. Therefore, our proof will be complete when we have shown that (4.7) is the inverse of (4.5).

Denote by $A$ the matrix obtained by post-multiplying (4.5) by (4.7). We verify that $A$ is the identity matrix. It is clear from inspection that all rows of $A$ from the second row to the last coincide with the corresponding row of the identity matrix. Thus, it remains to check that the top row of $A$ is given by $(1, 0, \ldots, 0)$. Let $(a_{11}, a_{12}, \ldots, a_{1\ell})$ be the top row of $A$. For all $i$, $a_{1i}$ is the sum of the elements of the $i$-th column of (4.7) since the top row of (4.5) consists of 1's. Thus, $a_{11} = 1$ and $a_{1i} = 0$ for $i \geq 2$. This shows that (4.7) is the inverse of (4.5) and completes the proof of the lemma. $\square$

Now, since each $f(\theta)$ is an upper triangular matrix with 1's along the leading diagonal, so is its inverse $[f(\theta)]^{-1}$. But the above lemma implies that each column of $[f(\theta)]^{-1}$ has precisely one positive element — namely, 1. Thus, the diagonal entries are the *only* positive entries in $[f(\theta)]^{-1}$. This gives us the following corollary.

*COROLLARY.* $D^{-1}$ has 1's along the leading diagonal and has no positive entries off the leading diagonal.

With this corollary, we can construct the matrices $P$, $Q$ and vector $u$ which give us the dynamic representation. Consider the matrix $R$. From (4.1), we know that the $(i,j)$-th entry of $R$ is positive if and only if $\theta_i \preceq \theta_j$. Denote by $r_{ij}$ the $(i,j)$-th entry of $R$. From (4.6), we can express $R$ in terms of $D$ as;

(4.8) $$r_{ij} = d_{ij} \frac{\pi_j}{\pi(\Phi_i)}$$

Then, $R^{-1}$ can be expressed in terms of $D^{-1}$ as follows.

(4.9) $$r_{ij}^{(-1)} = d_{ij}^{(-1)} \frac{\pi(\Phi_j)}{\pi_i}$$

To see this, we verify that $\sum_k r_{ik} r_{kj}^{(-1)}$ is given by;

$$\sum_k d_{ik} \frac{\pi_k}{\pi(\Phi_i)} d_{kj}^{(-1)} \frac{\pi(\Phi_j)}{\pi_k} = \sum_k d_{ik} d_{kj}^{(-1)} \frac{\pi(\Phi_j)}{\pi(\Phi_i)} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$$

Then, by the above corollary, $R^{-1}$ has positive leading diagonal entries and has no positive entries off the leading diagonal. Let $\epsilon > 0$ be a number small enough so that $\epsilon R^{-1}$ has entries whose absolute value is strictly less than 1. Let $Q$ be the diagonal matrix $\epsilon I$. $Q$ then qualifies as an absorption matrix. Define $P = I - QR^{-1}$. The matrix $QR^{-1}$ has the following features. The entries along the leading diagonal are positive, and there are no positive entries off the leading diagonal. Moreover, all entries have absolute value strictly less than 1. Thus, $P$ is a matrix of probabilities. So, there are matrix $P$, $Q$ satisfying the appropriate conditions such that $R = (I-P)^{-1}Q$. This satisfies (D2).

For (D3), let $u = (1,0,0,\dots,0)$. Since the top row of $D$ consists of 1's, the top row of $R$ is the vector $\pi = (\pi_1, \pi_2, \dots, \pi_m)$. Thus, $uR = \pi$, and (D3) is satisfied. Lastly, lemma 4.1 shows that (D1) is satisfied. Thus, when $\varphi$ satisfies (C1), (C2) and (C3), the structure $(\Omega, \varphi, \pi)$ has a dynamic representation.

## 5. Necessity

In this section, we shall show that each of the conditions (C1), (C2) and (C3) is necessary for the existence of a dynamic representation. We proceed by considering the necessity of (C1), (C2) and (C3) in turn.

Necessity of (C1). Suppose $(\Omega,\varphi,\pi)$ has a dynamic representation. Then $R=(I-P)^{-1}Q$ for some $P$ and $Q$. We know that $(I-P)^{-1}Q$ is the limit of the sequence of sums $Q+PQ+P^2Q+\cdots+P^nQ$, where each term in this sum is a matrix of non-negative entries. Since $Q$ is a diagonal matrix of positive entries, $(I-P)^{-1}Q$ has positive leading diagonal entries. That is, $\pi(\theta_i|\Phi_i)>0$ for all $i$. But since $(\Omega,\varphi,\pi)$ has a dynamic representation, (D1) is satisfied, and each $\Phi_i$ is a union of elements of $\Theta$. Thus, $\theta_i\subseteq\Phi_i$ for all $i$. So, for any $\omega\in\theta_i$, $\omega\in\theta_i\subseteq\Phi_i=\varphi(\omega)$. Thus, (C1) holds.

Necessity of (C2). Suppose $(\Omega,\varphi,\pi)$ has a dynamic representation. Denote by $r_{ij}$ the $(i,j)$-th element of $R$. That is, $r_{ij}=\pi(\theta_j|\Phi_i)$. Since $R=(I-P)^{-1}Q$ and $Q$ is a diagonal matrix of positive entries, $r_{ij}$ has the same sign as the corresponding element of $(I-P)^{-1}$. Thus, $r_{ij}>0 \Leftrightarrow \sum_{n=0}^{\infty}p_{ij}^{(n)}>0 \Leftrightarrow p_{ij}^{(n)}>0$ for some $n\geq 0$. Now, suppose $r_{ij}>0$ and $r_{jk}>0$. $r_{ij}>0 \Leftrightarrow p_{ij}^{(n)}>0$ for some $n\geq 0$, and $r_{jk}>0 \Leftrightarrow p_{jk}^{(m)}>0$ for some $m\geq 0$. Then, $p_{ik}^{(n+m)}=\sum_{\ell}p_{i\ell}^{(n)}p_{\ell k}^{(m)}\geq p_{ij}^{(n)}p_{jk}^{(m)}>0$ so that $r_{ik}>0$. Therefore, when $r_{ij}>0$ and $r_{jk}>0$, we have $r_{ik}>0$. By (D1), $r_{ij}>0 \Leftrightarrow \theta_j\subseteq\Phi_i$. Thus,

$$\text{(5.1)}\qquad \left[\theta_j\subseteq\Phi_i \;\&\; \theta_k\subseteq\Phi_j\right] \Rightarrow \theta_k\subseteq\Phi_i$$

Take any three states $\omega$, $\omega'$, $\omega''$. Let $\omega\in\theta_i$, $\omega'\in\theta_j$ and $\omega''\in\theta_k$. Then $\varphi(\omega)=\Phi_i$ and $\varphi(\omega')=\Phi_j$, and (4.1) implies;

$$\text{(5.2)}\qquad \left[\omega'\in\varphi(\omega) \;\&\; \omega''\in\varphi(\omega')\right] \Rightarrow \omega''\in\varphi(\omega),$$

which is the statement of (C2).

Necessity of (C3). Suppose that $(\Omega,\varphi,\pi)$ has a dynamic representation but that $\varphi$

does *not* satisfy (C3). From this, we derive a contradiction. By the necessity of (C1) and (C2) and part (ii) of lemma 4.2, $(\Theta, \preceq)$ contains at least one connected component which is not a tree. Thus, there is an element $\underline{\theta} \in \Theta$ and two immediate predecessors of $\underline{\theta}$ (denoted by $\overline{\theta}_1$ and $\overline{\theta}_2$) such that $\overline{\theta}_1$ and $\overline{\theta}_2$ are unrelated by $\preceq$. We then have the following lemma concerning the indexing of $\Theta$.

*LEMMA 5.1*. There is an indexing of the set $\Theta$ so that;

(5.3) (i) $\theta_i \preceq \theta_j \Rightarrow i \leq j$, for all $i, j \in \{1, \dots, m\}$

(ii) $\overline{\theta}_1 = \theta_i$, $\overline{\theta}_2 = \theta_{i+1}$, $\underline{\theta} = \theta_{i+2}$, for some $i \in \{1, \dots, m\}$.

*PROOF*. Partition the set $\Theta$ into $A$ and $\Theta \backslash A$, where

$$A = \left\{\theta \middle| \overline{\theta}_1 \preceq \theta \text{ or } \overline{\theta}_2 \preceq \theta \text{ or } \underline{\theta} \preceq \theta\right\} \tag{5.4}$$

Suppose $\Theta \backslash A$ has $(q-1)$ elements. Index this set with $\left\{1, 2, \dots q-1\right\}$ as follows. Pick any $\preceq$-minimal element and assign it the index 1. In general, assign the smallest available index to a state whose predecessors have all been assigned an index. This indexing of $\Theta \backslash A$ satisfies 5.3(i) for $i, j \in \{1, \dots q-1\}$. Next, let $\overline{\theta}_1 = \theta_q$, $\overline{\theta}_2 = \theta_{q+1}$, $\underline{\theta} = \theta_{q+2}$, so that 5.3(ii) is satisfied. Then, index the rest of the states in $A$ with the set $\left\{q+3, \dots, m\right\}$ in the same manner as for $\Theta \backslash A$. That is, we assign the smallest available index to a state whose predecessors have all be assigned an index. As a result, the indexing of $A$ satisfies 5.3(i) for $i, j \in \left\{q, q+1, \dots m\right\}$.

Now, take any $\theta_j \in \Theta$. We show that $\theta_i \preceq \theta_j$ implies $i \leq j$. There are two cases. First, suppose $\theta_j \in \Theta \backslash A$. Then, all predecessors of $\theta_j$ are in $\Theta \backslash A$. (Since, if $\theta_j$ has a predecessor in $A$, then $\theta_j$ must also be in $A$ by the transitivity of $\preceq$). In this case, $\theta_i \preceq \theta_j$ implies $i \leq j$ since the indexing of $\Theta \backslash A$ satisfies 5.3(i). Next, suppose $\theta_j \in A$ and $\theta_i \preceq \theta_j$. If $\theta_i \in A$, then $i \leq j$ since the indexing of $A$ satisfies 5.3(i). If $\theta_i \in \Theta \backslash A$, then $i \leq q-1$ and $j \geq q$ so that $i < j$. Thus, we conclude that the indexing of $\Theta$ as a whole satisfies 5.3(i). □

Consider the matrix $R$ with the indexing of states 5.3 (i) and (ii). $R$ has the following features. Firstly, $R$ is an upper triangular matrix. To see this, recall that $r_{ij} > 0 \Leftrightarrow \theta_i \preceq \theta_j$. Hence, by 5.3(i), $r_{ij} > 0 \Rightarrow i \leq j$. This is the definition of $R$ being upper triangular. Secondly, all the elements along the leading diagonal of $R$ are positive as we showed in the proof of necessity of (C1). Together with the fact that $R$ is upper triangular, this ensures the non-singularity of $R$.

Let $D$ be the matrix whose $(i,j)$-th element is defined as follows.

$$d_{ij} = \begin{cases} 1 & \text{if } r_{ij} > 0 \\ 0 & \text{otherwise} \end{cases} \tag{5.5}$$

$D$ inherits the following features from $R$. $D$ is upper triangular. $D$ has 1's along the leading diagonal. Clearly, $D$ is non-singular, and we denote by $d_{ij}^{(-1)}$ the $(i,j)$-th element of $D^{-1}$. The inverse of a triangular matrix is itself triangular, so that;

$$\sum_k d_{ik} d_{kj}^{(-1)} = \sum_{\{k \mid i \leq k \leq j\}} d_{ik}\, d_{kj}^{(-1)} \tag{5.6}$$

Thus, $D^{-1}$ has 1's along the leading diagonal. Consider two elements of $DD^{-1}$ in particular — the $(i,i+2)$-th element and the $(i+1,i+2)$-th element. From (5.6), we have,

$$\begin{bmatrix} d_{ii} & d_{i,i+1} & d_{i,i+2} \\ 0 & d_{i+1,i+1} & d_{i+1,i+2} \end{bmatrix} \begin{bmatrix} d_{i,i+2}^{(-1)} \\ d_{i+1,i+2}^{(-1)} \\ d_{i+2,i+2}^{(-1)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{5.7}$$

We know that $d_{ii} = d_{i+1,i+1} = 1$ since $D$ has 1's along the leading diagonal. Also, $d_{i+2,i+2}^{(-1)} = 1$ for the same reason. Moreover, by 5.3(ii), $\theta_i \preceq \theta_{i+2}$, $\theta_{i+1} \preceq \theta_{i+2}$ but $\theta_i \npreceq \theta_{i+1}$. Thus, $d_{i,i+2} = 1$, $d_{i+1,i+2} = 1$ but $d_{i,i+1} = 0$, and (5.7) now reads;

$$(5.8) \qquad \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} d^{(-1)}_{i,i+2} \\ d^{(-1)}_{i+1,i+2} \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

which implies that $d^{(-1)}_{i,i+2} = d^{(-1)}_{i+1,i+2} = -1$.

Finally, from the supposition that $\pi = uR$, we have $u = \pi R^{-1}$. The $j$-th component of $u$ is given by;

$$(5.9) \qquad u_j = \sum_{i=1}^{m} \pi_i \, r^{(-1)}_{ij} = \sum_{i=1}^{m} \pi_i \, d^{(-1)}_{ij} \frac{\pi(\Phi_j)}{\pi_i} = \pi(\Phi_j) \sum_{i=1}^{m} d^{(-1)}_{ij}$$

Consider the $(i+2)$-th column of $D^{-1}$. We have shown above that there are at least two entries in this column which take the value $-1$. Now, either this column has a positive entry other than $d^{(-1)}_{i+2,i+2}$ or it does not. If it *does*, then the matrix $P$ has a *negative* entry, since $P = I - QR^{-1}$ and an entry of $D^{-1}$ has the same sign as the corresponding entry of $QR^{-1}$. If it does *not*, then the sum of entries in this column is at most $-1$, so that $u_{i+2} \leq -\pi(\Phi_{i+2}) < 0$ by (5.9). In either case, we have a contradiction with our initial supposition that $P$ is a matrix of probabilities and $u$ is a vector of probabilities.

Our conclusion that either $P$ or $u$ has a negative element is robust to the renaming of states. Recall that when we exchange two *rows* of $R$, we exchange the corresponding *columns* of $R^{-1}$. Thus, when we swap the index between $\theta_i$ and $\theta_j$, we exchange the $i$-th and $j$-th rows and $i$-th and $j$-th columns of $R$ and $R^{-1}$. Since our conclusion rests only on the sign of the elements in $D^{-1}$ and the column sums of $D^{-1}$, we conclude that our result is robust to the renaming of states.

We started with the supposition that $(\Omega, \varphi, \pi)$ has a dynamic representation but that $\varphi$ is not nested and have obtained a contradiction. We therefore conclude that nestedness is a necessary condition for the existence of a dynamic representation.

## 6. Speculation

We conclude by exploring the connection between the possibility of speculative trade and the dynamic representation of individuals' information structures. In order to keep the exposition simple, we shall assume in this section that $\Omega$ is finite, and that $\pi(\{\omega\}) > 0$ for all states $\omega$.

We define rational choice as follows. For a given individual, his payoffs are determined by his payoff function $h: A \times \Omega \rightarrow \mathbb{R}$, where $A$ is the set of *actions*. Thus, $h(a,\omega)$ is the payoff to this individual at $\omega$ when he takes action $a$. Any function $f: \Omega \rightarrow A$ such that $\varphi(\omega) = \varphi(\omega') \Rightarrow f(\omega) = f(\omega')$ is called a *strategy*. We say that the strategy $f$ is *optimal at $\omega$ given $\varphi$* if;

$$(6.1) \qquad f(\omega) \ \underset{a \in A}{\text{maximizes}} \sum_{\omega' \in \varphi(\omega)} h(a,\omega') \frac{\pi(\{\omega'\})}{\pi(\varphi(\omega))}$$

That is, $f$ is optimal at $\omega$ given $\varphi$ if the action $f(\omega)$ maximizes the expected payoff conditional on the information $\varphi(\omega)$. We say that $f$ is optimal given $\varphi$ if it is optimal at all states given $\varphi$.

Geanakoplos (1988) has shown that when $\varphi$ satisfies (C1), (C2) and (C3), an optimal strategy $f$ given $\varphi$ satisfies the following important property.

$$(6.2) \qquad \sum_{\omega} h(f(\omega),\omega)\, \pi(\{\omega\}) \geq \sum_{\omega} h(a,\omega)\, \pi(\{\omega\}), \quad \forall a \in A.$$

In other words, an optimal strategy $f$ does at least as well as any single action $a$. Alternatively, we can read (6.2) as stating that the information conveyed by $\varphi$ allows the individual to improve upon the situation where he has the trivial information $\Omega$ at all states. Geanakoplos shows that this property is instrumental in deriving no speculation results, and he generalizes a non-speculation result to a context in which individuals hold non-partitional information.

Since we have demonstrated that the existence of a dynamic representation is equivalent to (C1), (C2) and (C3), Geanakoplos's results on speculation carry over to our dynamic framework. In particular, it is instructive to see the role played by the dynamic representation in precluding speculation. It turns out that each of the clauses defining a dynamic representation has an intuitive interpretation. We shall illustrate the role of the three clauses (D1), (D2) and (D3) by showing what happens when any one of these conditions fail. It is most natural to start with (D3).

Role of (D3). Let $x$ be a column vector of $m$ numbers $x_1, x_2, \ldots, x_m$. Think of $x$ as the payoffs associated with a lottery ticket which yields the prize $x_i$ when $\theta_i$ obtains. Then (D3) implies that there is a probability distribution $u$ over $\Theta$ such that;

$$\pi x = u R x. \tag{6.3}$$

Let us denote by $\tilde{x}$ the random variable defined on $\Theta$ whose value at $\theta_i$ is $x_i$. Then, the left hand side of (6.3) is simply the expectation of $\tilde{x}$. For the right hand side, notice that the product $Rx$ is a column vector whose $i$-th component is the conditional expectation of $\tilde{x}$ given the event $\Phi_i$ (denoted by $E(\tilde{x} | \Phi_i)$). Thus, (6.3) can be expressed as;

$$E(\tilde{x}) = \sum_i u_i E(\tilde{x} | \Phi_i). \tag{6.4}$$

Thus, (D3) ensures that for any random variable $\tilde{x}$, its expectation is a convex combination of the conditional expectations. This brings to mind the rule in probability theory that the expectation of a random variable is given by the expectation of the conditional expectations of that random variable. (6.4) is a generalization of this rule in which the weight $u_i$ need not correspond to the probability of $\Phi_i$.

An important consequence of (6.4) is that it guarantees a generalized form of the dominance principle. This principle is defined in the obvious way — that is, for any random variables $\tilde{x}$ and $\tilde{y}$,

$$E(\tilde{x} \mid \Phi_i) \geq E(\tilde{y} \mid \Phi_i),\ \forall i \ \Rightarrow\ E(\tilde{x}) \geq E(\tilde{y}) \tag{6.5}$$

That is, if $\tilde{x}$ is preferred to $\tilde{y}$ given any possible information, then $\tilde{x}$ is preferred to $\tilde{y}$ ex ante. When (D3) fails, we can construct examples where dominance no longer holds, and this leads to speculative trade. Consider the following example due to Geanakoplos (1988). Rubinstein and Wolinsky (1988) has similar examples.

Let $\Omega = \{\omega_1, \omega_2, \omega_3\}$ and $\pi(\{\omega_i\}) = \frac{1}{3}$, for all $i$. Consider two lottery tickets $x$, $y$ which yield the following prizes.

| | $\omega_1$ | $\omega_2$ | $\omega_3$ |
|---|---|---|---|
| $x$ | 0 | 0 | 10 |
| $y$ | 6 | 6 | 0 |

Suppose there are two individuals, 1 and 2. 1 has the trivial information $\varphi^1(\omega_i) = \Omega$ for all $i$, while 2 has the information defined by the correspondence $\varphi^2$, where $\varphi^2(\omega_1) = \{\omega_1, \omega_3\}$, $\varphi^2(\omega_2) = \{\omega_2, \omega_3\}$, $\varphi^2(\omega_3) = \{\omega_3\}$.

[Figure 6.1]

Consider the allocation where 1 has $x$ and 2 has $y$. This allocation is ex ante Pareto optimal since both individuals prefer $y$ to $x$. However, the conditional valuation of $x$ and $y$ by 2 is as follows.

| | $\omega_1$ | $\omega_2$ | $\omega_3$ |
|---|---|---|---|
| $E(x \mid \varphi^2)$ | 5 | 5 | 10 |
| $E(y \mid \varphi^2)$ | 3 | 3 | 0 |

Thus, at the interim stage, 2 prefers $x$ to $y$. This opens up the possibility of trade between 1 and 2 in which the lottery tickets are swapped. This trade can be seen as a bet between the two individuals as to what the true state of the world is. 2 has finer information than 1 but he does worse on average by trading his $y$ for $x$. This failure on 2's part is due to the fact that dominance is violated. In turn, we can trace this to a failure of (D3). By letting $\theta_i = \{\omega_i\}$, we see that the solution to $\pi = uR$ yields $u = (\frac{2}{3}, \frac{2}{3}, -\frac{1}{3})$. The third component is negative, violating (D3).

Role of (D2). The notion of dominance described above is a rather weak requirement on rational choice, and there are instances of the failure of rationality similar to the one examined above, which manage to "slip through". The role of (D2) is complementary to that of (D3), and consists in reinforcing the notion of dominance. It is best to illustrate this with an example. Consider the following modification of the example above. $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$ and $\pi(\{\omega_i\}) = \frac{1}{4}$, for all $i$. Consider again two lottery tickets $x$ and $y$ with the following prizes.

| | $\omega_1$ | $\omega_2$ | $\omega_3$ | $\omega_4$ |
|---|---|---|---|---|
| $x$ | 0 | 0 | 0 | 10 |
| $y$ | 0 | 6 | 6 | 0 |

There are two individuals who share the prior $\pi$. The first individual has the trivial information $\varphi^1(\omega_i) = \Omega$ for all $i$. The second individual's correspondence $\varphi^2$ is given as follows. $\varphi^2(\omega_1) = \Omega$, $\varphi^2(\omega_2) = \{\omega_2, \omega_4\}$, $\varphi^2(\omega_3) = \{\omega_3, \omega_4\}$, $\varphi^2(\omega_4) = \{\omega_4\}$.

[Figure 6.2 here]

This example is a minor modification of the previous example in which we have added the state $\omega_1$ at which both lotteries pay zero. Notice that (D3) is now satisfied for $u = (1, 0, \ldots, 0)$, so that dominance now holds. However, our intuition would suggest that the same considerations which caused problems for 2 in the previous example would also play a role here. This is indeed the case. 2's conditional evaluations of the lotteries $x$ and $y$ are as follows.

| | $\omega_1$ | $\omega_2$ | $\omega_3$ | $\omega_4$ |
|---|---|---|---|---|
| $E(x \mid \varphi^2)$ | $2\frac{1}{2}$ | 5 | 5 | 10 |
| $E(y \mid \varphi^2)$ | 3 | 3 | 3 | 0 |

The allocation where 1 has $x$ and 2 has $y$ is ex ante Pareto optimal, since both individuals prefer $y$ to $x$ ex ante. However, at the interim stage, 2 is willing to trade $y$ for $x$ at $\omega_2$, $\omega_3$ and $\omega_4$. At $\omega_1$, individual 2 prefers to keep his $y$. Individual 1 is always willing to trade. The payoffs associated with 2's strategy of trading at $\omega_2$, $\omega_3$ and $\omega_4$ and not trading at $\omega_1$ are (0, 0, 0, 10), so that the expected payoff to this strategy is $2\frac{1}{2}$. However, this falls short of the expected payoff of not trading at any state, which is 3.

Again, we have an example where finer information leads to a worse outcome. Notice that (D3) has no force in this example. What has happened is that the failure of dominance has been obscured by the introduction of the state $\omega_1$. The role of (D2) is precisely to expose this sort of masking of the failure of dominance. To see this in the example above, note that $\varphi^2$ satisfies (D1) and (D2) but is not nested. Thus, from our theorem, we know that (D2) must be violated. Let $\theta_i = \{\omega_i\}$. Then,

$$R = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad R^{-1} = \begin{bmatrix} 4 & -2 & -2 & 1 \\ 0 & 2 & 0 & -1 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

We know that $P = I - QR^{-1}$. Since Q is a diagonal matrix of positive elements, $p_{14} < 0$ since $r_{14}^{(-1)} = 1 > 0$. Thus, $P$ has a negative entry, so that (D2) fails.

When taken together, (D2) and (D3) can be seen as working in concert to ensure dominance within each substructure of $(\Omega, \varphi)$. (D3) requires that dominance be satisfied for the structure as a whole, while (D2) complements this by applying a similar criterion to each connected substructure of $(\Omega, \varphi)$. The combination of (D2) and (D3) is closely related to Geanakoplos's (1988) notion of positive balancedness.

Role of (D1). Finally, we illustrate an instance of the failure of (D1). Let $\Omega = \{\omega_1, \omega_2, \omega_3\}$ and $\pi(\{\omega_i\}) = \frac{1}{3}$, for all $i$. Consider lotteries $x$, $y$ below.

| | $\omega_1$ | $\omega_2$ | $\omega_3$ |
|---|---|---|---|
| $x$ | 0 | 0 | 5 |
| $y$ | 2 | 2 | 2 |

There are two individuals, 1 and 2, who share the prior $\pi$. 1 has the trivial information $\varphi^1(\omega_i) = \Omega$ for all $i$. 2's correspondence $\varphi^2$ is such that $\varphi^2(\omega_1) = \varphi^2(\omega_3) = \Omega$ and $\varphi^2(\omega_2) = \{\omega_2, \omega_3\}$. Consider the allocation in which 1 has $x$ and 2 has $y$. This allocation is ex ante Pareto optimal since both individuals prefer $y$ to $x$. However, at the interim stage trade will take place. 2's conditional valuations of $x$ and $y$ are;

| | $\omega_1$ | $\omega_2$ | $\omega_3$ |
|---|---|---|---|
| $E(x\mid\varphi^2)$ | $\frac{5}{3}$ | $\frac{5}{2}$ | $\frac{5}{3}$ |
| $E(y\mid\varphi^2)$ | 2 | 2 | 2 |

At $\omega_2$, individual 2 is willing to trade, while at $\omega_1$ and $\omega_3$, he is not willing to trade. Since individual 1 is always willing to swap his $x$ for $y$, trade will take place at the interim stage at $\omega_2$. However, the payoffs associated with 2's strategy are $(2,0,2)$ and this leaves 2 worse off on average than keep his $y$.

Let us identify the reasons behind 2's failure. 2's problem is that his information at $\omega_2$ makes trade appear attractive even though his behaviour at other states betrays this appearance. The decision to trade at $\omega_2$ is essentially a gamble with individual 1 concerning the likelihood of $\omega_3$. However, this decision to trade will only be profitable for 2 when 2 is actually in possession of $x$ at $\omega_3$. However, at $\omega_3$, 2's information is no better than 1's information there, and 1 ends up by preferring $y$ instead.

Thus, there is a failure of forsight on 2's part. At $\omega_2$, 2 ought to anticipate what his information would be at $\omega_3$, and hence what his action would be there, but fails to do so. This failure results in 2's wishful thinking at $\omega_2$ that he can win the bet with his opponent.

The above example is an instance of the failure of (D1). The partition $\Theta$ is given by $\left\{\{\omega_1,\omega_3\},\ \{\omega_2\}\right\}$, while $\varphi^2(\omega_2)=\left\{\omega_2,\omega_3\right\}$, so that $\varphi^2(\omega_2)$ cannot be expressed as a union of elements of $\Theta$. (D1) stipulates that an individual can only rely upon information which will actually be received (in the form of the signal $\sigma$). When (D1) fails as above, the individual is relying on information which will not be delivered. In short, the individual is guilty of wishful thinking. This wishful thinking can lead to ill-advised trade, as we see above.

# REFERENCES


Brandenburger A., E. Dekel and J. Geanakoplos (1988) "Correlated Equilibrium with Generalized Information Structures", Harvard Business School discussion paper.

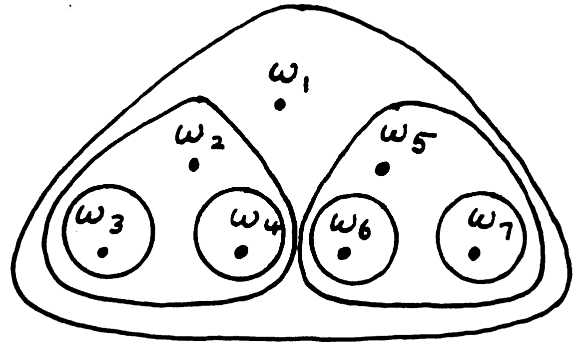
Gale, David. (1960) *Theory of Linear Ecnomic Models*, McGraw-Hill, New York.

Geanakoplos J. (1988) "Game Theory without Partitions, and an Application to Speculation", mimeo, Cowles Foundation, Yale University.

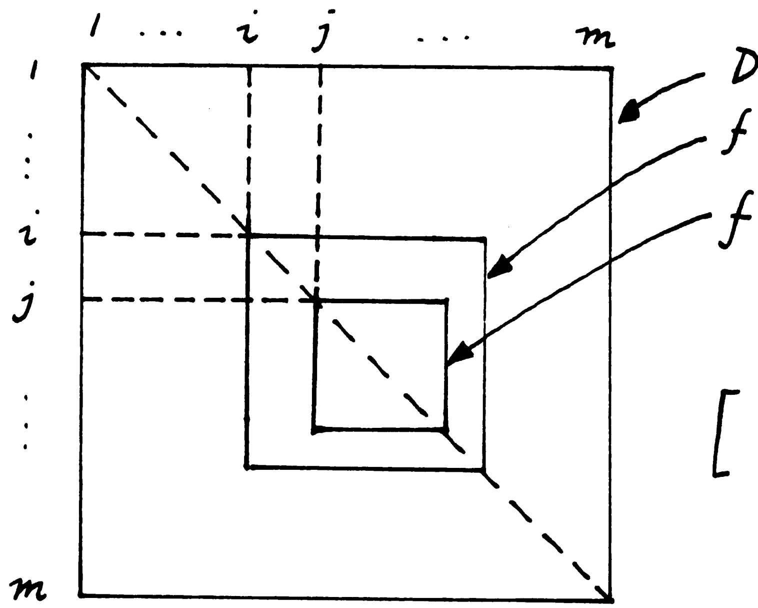
Rubinstein A. and A. Wolinsky (1988) "The Logic of 'Agreeing to Disagree' Results", ICERD discussion paper, London School of Economics.

Samet D. (1987) "Ignoring Ignorance and Agreeing to Disagree", MEDS discussion paper, Northwestern University.

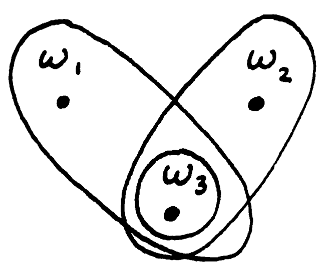
Shin H. (1987) "Logical Structure of Common Knowledge", mimeo, Nuffield College, Oxford University.



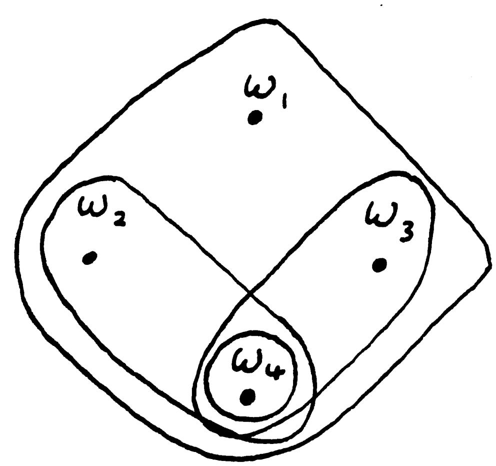

[Figure 3.1]




[Figure 4.1]




[Figure 6.1]




[Figure 6.2]

**Recent CREST Working Papers**

89–01: Mark Bagnoli, Severin Borenstein, "Carrot and Yardstick Regulation: Enhancing Market Performance with Output Prizes," October, 1988.

89–02: Ted Bergstrom, Jeffrey K. MacKie–Mason, "Some Simple Analytics of Peak-Load Pricing," October, 1988.

89–03: Ken Binmore, "Social Contract I: Harsanyi and Rawls," June, 1988.

89–04: Ken Binmore, "Social Contract II: Gauthier and Nash," June, 1988.

89–05: Ken Binmore, "Social Contract III: Evolution and Utilitarianism," June, 1988.

89–06: Ken Binmore, Adam Brandenburger, "Common Knowledge and Game Theory," July, 1989.

89–07: Jeffrey A. Miron, "A Cross Country Comparison of Seasonal Cycles and Business Cycles," November, 1988.

89–08: Jeffrey A. Miron, "The Founding of the Fed and the Destabilization of the Post-1914 Economy," August, 1988.

89–09: Gérard Gaudet, Stephen W. Salant, "The Profitability of Exogenous Output Contractions: A Comparative-Static Analysis with Application to Strikes, Mergers and Export Subsidies," July, 1988.

89–10: Gérard Gaudet, Stephen W. Salant, "Uniqueness of Cournot Equilibrium: New Results from Old Methods," August, 1988.

89–11: Hal R. Varian, "Goodness-of-fit in Demand Analysis," September, 1988.

89–12: Michelle J. White, "Legal Complexity," October, 1988.

89–13: Michelle J. White, "An Empirical Test of the Efficiency of Liability Rules in Accident Law," November, 1988.

89–14: Carl P. Simon, "Some Fine-Tuning for Dominant Diagonal Matrices," July, 1988.

89–15: Ken Binmore, Peter Morgan, "Do People Exploit Their Bargaining Power? An Experimental Study," January, 1989.

89–16: James A. Levinsohn, Jeffrey K. MacKie–Mason, "A Simple, Consistent Estimator for Disturbance Components in Financial Models," April 25, 1989.

89–17: Hal R. Varian, "Sequential Provision of Public Goods," July, 1989.

89–18: Hal R. Varian, "Monitoring Agents with Other Agents," June, 1989.

89–19: Robert C. Feenstra, James A. Levinsohn, "Distance, Demand, and Oligopoly Pricing," July 17, 1989.

89–20: Mark Bagnoli, Shaul Ben-David, Michael McKee, "Voluntary Provision of Public Goods," August, 1989.

89–21: N. Gregory Mankiw, David Romer, Matthew D. Shapiro, "Stock Market Forecastability and Volatility: A Statistical Appraisal," August, 1989.

89–22: Arthur J. Robson, "Efficiency in Evolutionary Games: Darwin, Nash and the Secret Handshake," 1989.

89–23: Mark Bagnoli, Ted Bergstrom, "Log-Concave Probability and Its Applications," September 7, 1989.

89–24: Gérard Gaudet, Stephen W. Salant, "Towards a Theory of Horizontal Mergers," July, 1989.

89–25 (evolved from 87-35): Stephen W. Salant, Eban Goodstein, "Predicting Committee Behavior in Majority-Rule Voting Experiments," July, 1989.

89–26: Ken Binmore, Martin J. Osborne, Ariel Rubinstein, "Noncooperative Models of Bargaining," 1989.

89–27: Avery Katz, "Your Terms or Mine? The Duty to Read the Fine Print in Contracts," February 19, 1989.

**Recent CREST Working Papers**

90–01: Hal R. Varian, "A Solution to the Problem of Externalities and Public Goods when Agents are Well–Informed," January 17, 1990.

90–02: Hal R. Varian, "Sequential Provision of Public Goods," January 17, 1990.

90–03: Hal R. Varian, "Goodness-of-Fit Optimizing Models," December 11, 1989.

90–04: Paul N. Courant, Alan Deardorff, "International Trade with Lumpy Countries," August 22, 1989.

90–05: Dilip K. Bhattacharyya, Jan Kmenta, "Testing Hypotheses about Regression Coefficients in Misspecified Models," December 15, 1989.

90–06: Greg Shaffer, "Slotting Allowances and Resale Price Maintenance: A Comparison of Facilitating Practices," November 1989.

90–07: Michelle J. White, "Why are Taxes So Complex and Who Benefits?" January 19, 1990.

90–08: Daniel P. O'Brien, "Endogenous Price Leadership: A Bargaining Model of International Telecommunications Settlements," May 1989.

90–09: Mark Bagnoli, Barton L. Lipman, "Stock Price Manipulation Through Takeover Bids," November 1989.

90–10: Ken Binmore, "Bargaining Theory Without Tears," December 1, 1989.

90–11: Hyun Song Shin, "Non-Partitional Information On Dynamic State Spaces and the Possibility of Speculation," December 1989.

Shin, Hyun Song
"Non-partitonal information on dynamic state spaces and the possibility of speculation."