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# Klein 4 Group: Beth Olem Cemetery Application

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## *Introduction: A Cemetery Inside the Grounds of an Auto Plant*

A number of years ago, in the mid-1980s, General Motors Corporation built the Detroit/Hamtramck Assembly plant near the intersections of major Detroit freeways and major rail lines. Proximity to transportation links made sense from a variety of viewpoints. To acquire the land for the large new plant (eventually to cover 362 acres), a combination of deals were employed (eminent domain, purchase, and so forth); some met with more favor than did others (Wikipedia).

The Detroit/Hamtramck Assembly Plant, has extensive security surrounding it. Figure 1 shows a secured entrance gate. Figure 2 shows the general location of the plant, at the north end of Chene Street, in the contemporary context of Google Earth.

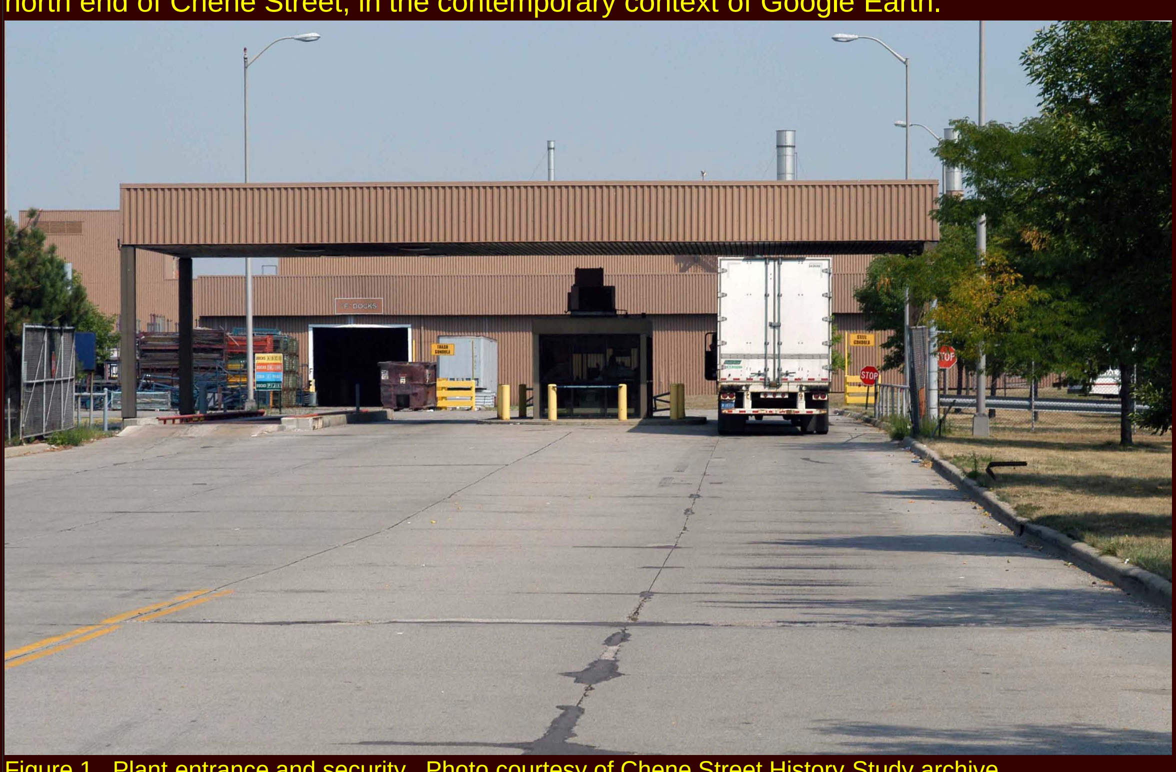

Figure 1. Plant entrance and security. Photo courtesy of Chene Street History Study archive.

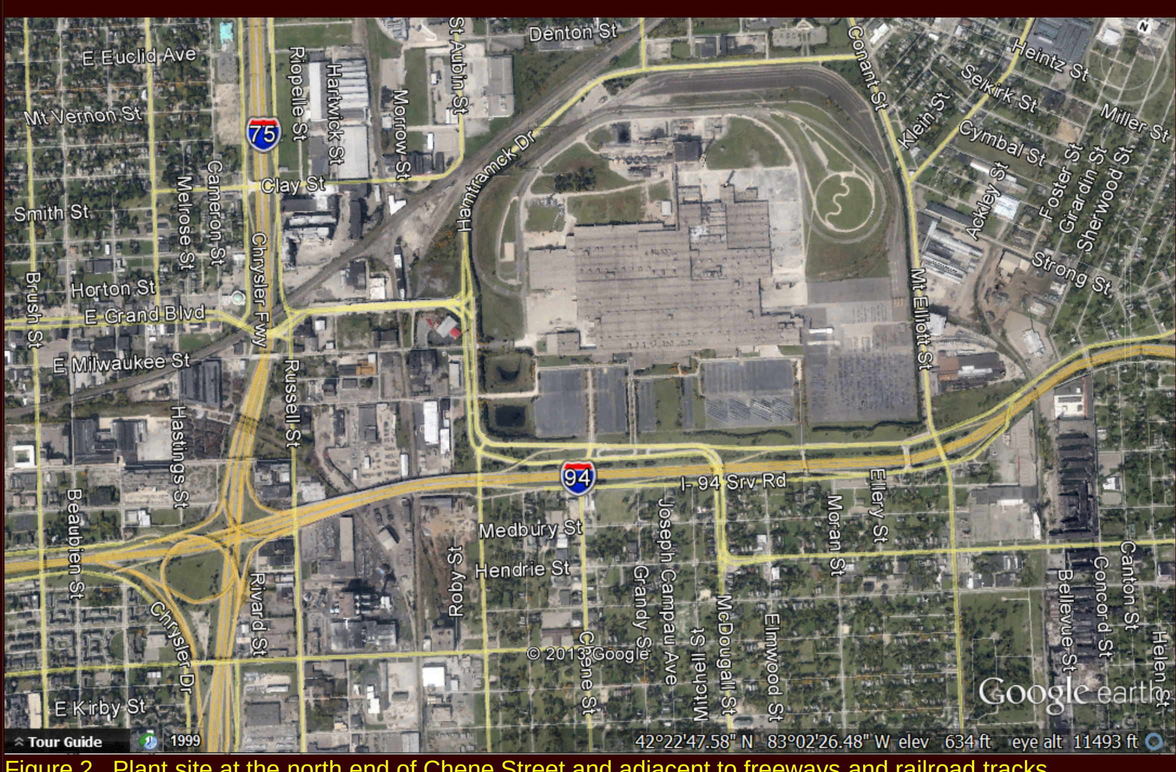


Figure 2. Plant site at the north end of Chene Street and adjacent to freeways and railroad tracks.

Take a closer look; the area to the north end of the plant contains quite a bit of grass adjacent to the giant parking lot. Figure 3 shows a small patch of trees that appear more mature than the others on the plant site. The trees appear walled into a rectangular area.


Figure 3. Rectangular patch of mature trees behind a wall.

The patch of trees is, in fact, part of a cemetery that predated, by almost a century, the Detroit/Hamtramck Assembly Plant. General Motors was not able to acquire that small patch of land because of zoning and easement restrictions already in place in association with the cemetery. Figure 4 shows a closer look at the cemetery.


Figure 4. Cemetery on the grounds of the Detroit/Hamtramck Assembly Plant. Note tombstones. Entrance gate is to the left of the white car.

Records in the Chene Street History Study (CSHS) and elsewhere show that this cemetery is named Beth Olem and that it is a Jewish cemetery that is one of the oldest in Michigan. It is open for only a few hours a year, in association with selected Jewish holidays. To visit the grave of a loved one, it is required to enter through GM security first (Figure 1) and then through cemetery security which requires the gates of the walled cemetery to be open. The walls are 8 feet tall. Naturally, this high level of security makes it difficult for visitors to gain access.

## *Comtemporary Visualization: Virtual Beth Olem Cemetery*

Google Earth or other contemporary visualization technology could make it possible, however, to overcome the frustrating security situation. Imagine a 3D model of the cemetery, complete with geo-referenced images/models of tombstones. Click on a grave marker and get taken to materials from the archive (insofar as privacy concerns permit). Link from the tombstone to a blog of associated materials. The process of building a virtual Beth Olem is underway. When complete, it will serve not only to overcome access and distance issues for loved ones to visit 24/7, but it will also serve as a basic study in the systematic use (by blog associations) of the CSHS archive, added to the present 'GEOMAT' (Geographic Events Ordering: Maps, Archives, Timelines; Arlinghaus, Haug, and Larimore) methodology.

The archives of the Chene Street History Study have many photos taken from inside Beth Olem. The image in Figure 5 is one example that shows clearly the proximity of the industrial complex to the otherwise peaceful resting ground; the juxtaposition of the different worlds is really quite startling.

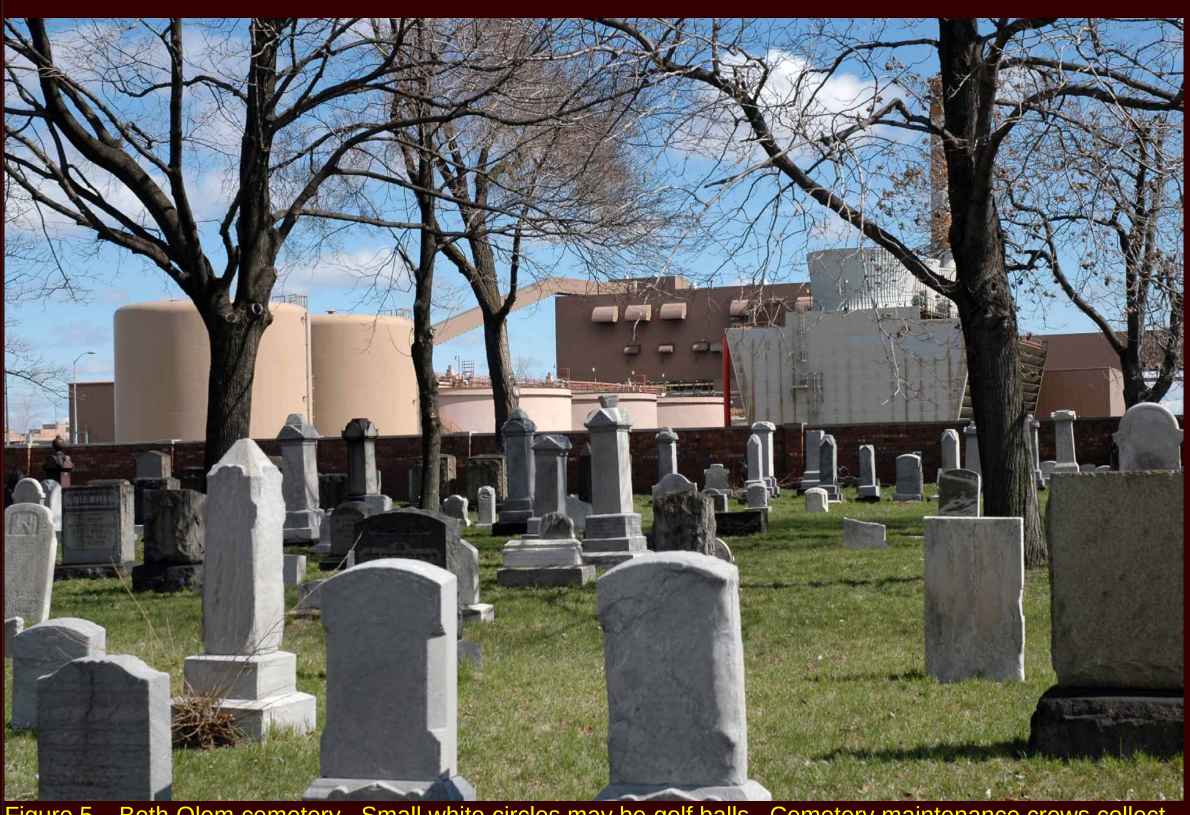

Figure 5. Beth Olem cemetery. Small white circles may be golf balls. Cemetery maintenance crews collect golf balls from the grounds that executives apparently hit at lunchtime into the cemetery from nearby parking lots. Photo courtesy of Chene Street History Study archives.

The cemetery is no longer taking new 'residents.' In that regard, it offers to researchers an advantage similar to the opportunity offered to foreign language students who begin by studying Latin (or another 'dead' language). There is no (or little) change--the 'syntax' and 'grammar' of the situation are frozen. These are true anchors for process and a fine place to begin study, prior to moving out, in this case to the more dynamic setting of the changing urban Chene Street scene.

## *A First Step in Creating the Virtual Beth Olem: The Walls*

The cemetery is a compact entity that is easy to deal with geometrically: it is a rectangle. The walls around it delineate it clearly and make it quickly recognizable. In terms of creating a virtual cemetery, the walls serve as a good starting point. Once the walled boundary is created, then infill can proceed with the walls as guides to reduce placement error. Accuracy in placement of the walls is straightforward: it is easy to read off the latitude and longitude from a smartphone camera used to take a photo next to the wall. General placement is straightforward from tracing the footprint in Google Earth. What is a challenge with modeling the walls is getting the surface to look correct so that the created visualization is realistic.

## *Surface Pattern*

It is a simple matter to capture a swatch of the pattern on the walls from a photograph. However, it is not possible to use that swatch, only, to create the full wall--at least not in a realistic manner. In Figure 6a, a single swatch of an arbitrary pattern is used to tile a broad area; the visual effect is not satisfactory. One has a sense that the single tile might be employed to greater advantage; blue and orange pattern do not align as the walls are tiled (Figure 6b). The tiling of a plane using geometric shapes is called a tessellation (see Wikipedia reference).

Figure 6a: single pattern tile, based on a background from

MicroSoft PowerPoint.

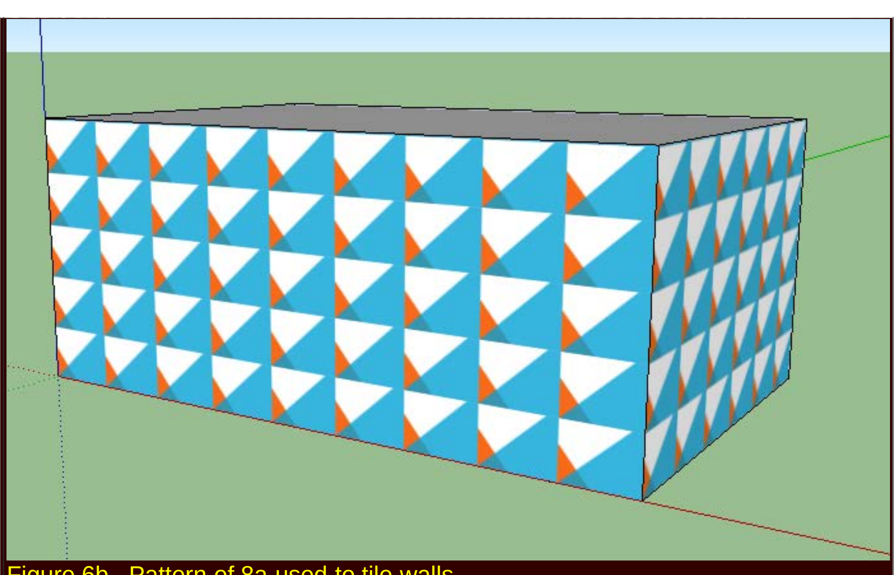

Figure 6b. Pattern of 8a used to tile walls.

To improve alignment and consequent appearance and visual impression, one might flip the single tile or rotate it to create different patterns and then align it with the base tile of Figure 6a to create a larger single tile to tile the walls with. Figure 7 shows a flip about a vertical axis in animated fashion. Figure 8a shows the flipped tile appended to the base tile; Figures 8b, and 8c illustrate the resulting pattern when 8a is applied to the walls of a box.

Figure 7. Flip of tile about a vertical axis.

In Figure 8b, the applied pattern has alignment issues resolved in horizontal strips but not from top to bottom. Figure 8c suggests that this single flip is sufficient to optimize visual alignment if the tile covers the wall from top to bottom.

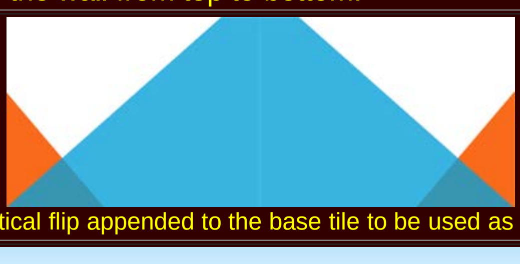

Figure 8a. Tile with a vertical flip appended to the base tile to be used as a new tile for tiling a wall.

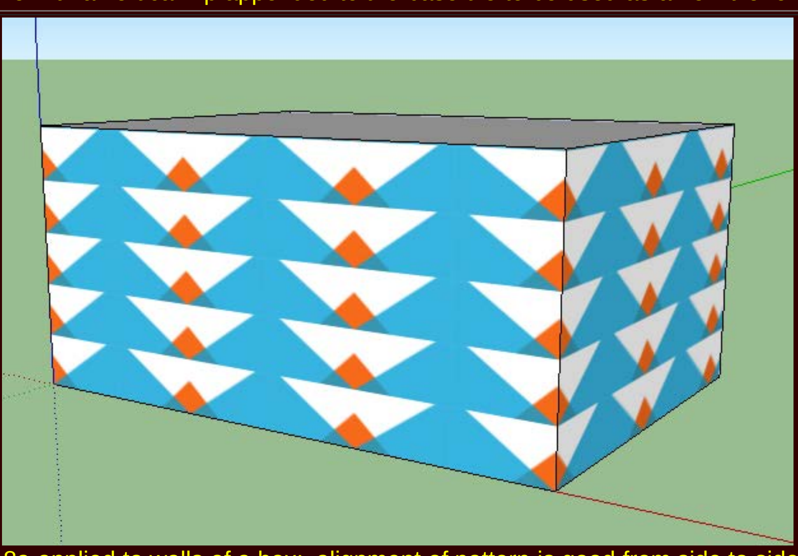

Figure 8b. Tile of 8a applied to walls of a box: alignment of pattern is good from side to side but not from top to bottom.

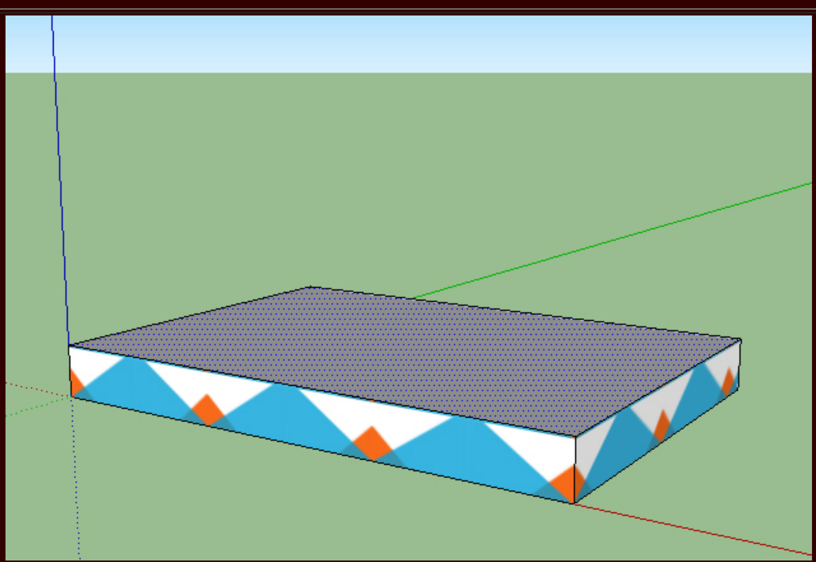

Figure 8c. Tile of 8a applied to walls of a box: height of tile is same as height of wall. Alignment is good both from side to side and from top to bottom.

In the case of the walls at Beth Olem, the situation of Figure 8c prevails; it is possible to find a swatch from top to bottom. Figure 9 shows the results of a model created in Trimble SketchUp. The edges along the tops of the walls, as well as the dots in the walls, align across the entire wall. Look at the grass stains on the bottom to see where the vertical flip was made. Figure 10 confronts the model with the reality of a photograph. There are no grass stains on the outsides of the walls. The reason there are grass stains on the modeled walls is that images from the interior side of the walls were used as textures on the outsides; use of actual images of the exterior required excessive removal of tree limbs not present when the interior images were used. Evidently, there are varying degrees of wetness at different times of the year. A similar strategy of using a view from the inside, and then flipping it, was employed with the sign for the cemetery, again so that it too might be disentangled from the tree limbs. One might further refine the detail of images; that action, however, has nothing to do with establishing process.

Figure 9. Beth Olem walls, model.

Figure 10. Beth Olem Cemetery entrance. Photo courtesy of Chene Street History Study archives.

In the situation above, a flipped tile was appended to one side of a base tile. Naturally, the flipped tile might also be applied to each of the other three sides to create other tiling patterns. The one selected to be exhibited is one that works well for modeling the Beth Olem walls. Thus, the first step in wall completion is solved. But, to learn more from this 'anchor' case, consider other possibilities.

***The Klein 4 Group: Pattern Alignment Issues***

The case above employed a vertical flip of a rectangular (non-square) base tile to create a new larger tile by appending the flipped tile to one side of the base tile; it was an exercise in 'spatial mathematics' (Arlinghaus and Kerski). What other transformations of the base tile might be employed to create other larger tiles that improve tiling alignment issues on a wall? Clearly, one might flip the base tile about a horizontal axis. Further, one might rotate the base tile through 180 degrees and still maintain tile orientation. Figure 11a-d illustrates these possibilities.

Figure 11a. Base tile

Figure 11b. Vertical flip

Figure 11c. Horizontal flip

Figure 11d. 180 degree flip

Because the base tile is non-square, rotation through 90 degrees will not maintain tile orientation; the 'landscape' tile will rotate to a 'portrait' tile under such a transformation. Are there, however, other rigid motions (see Wikipedia reference) of a non-square rectangle that will yield new pattern? Intuitively, the answer appears to be 'no'. It is possible to prove that answer using a structure from a branch of mathematics called group theory.

To introduce appropriate notation, replace the visual pattern in the non-square rectangles with numerical pattern, labelling the vertices of the rectangles as 1, 2, 3, 4. Thus the sequence in Figure 13a-d is replaced by the sequence in Figure 12a-d.


Figure 12a. Base tile

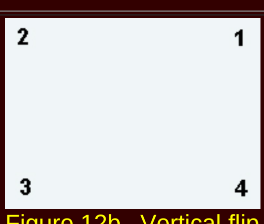


Figure 12b. Vertical flip

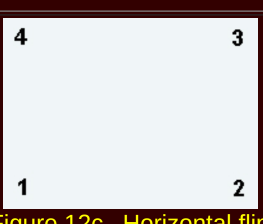


Figure 12c. Horizontal flip

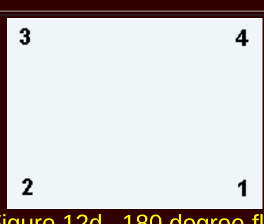


Figure 12d. 180 degree flip

To illustrate how to use the numbers, represent the base tile as the identity permutation on these four numbers: (1)(2)(3)(4). Represent the vertical flip as: (12)(34), read perhaps as '1 goes to 2' then once the end of a parenthetical notational phrase is reached, the last element 'goes' to the first one, so here '2 goes to 1'. Similarly, represent the horizontal flip as: (14)(23). Finally, represent the 180 rotational flip as (13)(24). Does this set of permutations form a closed system? If so, then there are no other possible rigid motions to use to generate larger tiles from the base tile.

A 'group' is a mathematical structure that may exist on a set of elements with one operation. When elements are combined using that operation, the system is said to be a group if it is closed (no new elements are generated), if it is associative (grouping using parentheses is clear: a(bc)=(ab)c), if there is an identity property: a*I = a, and if there is a unique inverse for each element: $a*a^{(-1)}$ = I. In the case of the permutations representing rigid motions of a non-square rectangle I = (1)(2)(3)(4). The operation, *, involves combining permutations as below.

> (12)(34)*(13)(24) is executed as starting on the left (for example) with 1--in the first permutation, 1 goes to 2; in the second permutation, 2 goes to 4. Thus, in the resulting product, 1 goes to 4 or (14. Now, where does 4 go? Start in the first permutation--4 goes to 3 and in the second permutation, 3 goes to 1. Thus, in the resulting product, 4 goes to 1 so it is now correct to close the parentheses (14). Now go back to the first permutation to see where 2 goes. In the first permutation, 2 goes to 1; in the second permutation, 1 goes to 3. So, in the result, 2 goes to 3: (23 . Then go back to the first permutation where 3 goes to 4 and then in the second permutation 4 goes to 2. Thus, 3 goes to 2 and it is correct to close the parentheses: (23).

Thus, (12)(34)*(13)(24) = (14)(23). With a bit of practice, one can perform this operation quickly. Look at a table composed of all possible permutation 'multiplications' (Figure 13). The column on the left is the set of 'first' permutations; the row across the top is the set of 'second' permutations

| * | (1)(2)(3)(4) | (12)(34) | (14)(23) | (13)(24) |
|---|---|---|---|---|
| (1)(2)(3)(4) | (1)(2)(3)(4) | (12)(34) | (14)(23) | (13)(24) |
| (12)(34) | (12)(34) | (1)(2)(3)(4) | (13)(24) | (14)(23) |
| (14)(23) | (14)(23) | (13)(24) | (1)(2)(3)(4) | (12)(34) |
| (13)(24) | (13)(24) | (14)(23) | (12)(34) | (1)(2)(3)(4) |

Figure 13. Group table, Klein 4 Group.

Verify that no new elements were created: all are displayed in the table. Verify that the associative law holds: for example, (12)(34)*[(13)(24)*(14)(23)] is the same as [(12)(34)*(13)(24)]*(14)(23). Show for each grouping; begin by working from within sets of parenthetically enclosed permutations. It is straightforward from the table that (1)(2)(3)(4) is an identity element; it is also straightforward from the table that there is no other identity element. Finally, read the table to see that each element is its own inverse: (12)(34)*(12)(34) = (1)(2)(3)(4), for example. Thus, this set of four permutations, representing rigid motions of a non-square rectangle, forms a group. It was discovered by Felix Klein and is referred to as the Klein 4-Group (Vierergruppe) and is often denoted V.

Thus, because the group structure is verified, there are no other patterns of the sort above, based on a non-square rectangle, that can be used to generate wall tiling. Of course, one can improve pattern alignment by using a larger tile, such as the one in Figure 8a, and flipping that to create an even larger base tile. Figures 14a-c suggest one such strategy: apply a vertical flip to the base tile, append that to the base tile (Figure 14a) then apply a horizontal flip to the tile in 14a to create a larger tile in Figure 14b. This new tile, as shown in Figure 14c, will combine both the good side-to-side alignment and top-to-bottom alignment of the vertical and horizontal flips. The tile is new, the alignment pattern of wall tiling is new and improved; however, there is no new motion involved, as the Klein 4 Group shows.

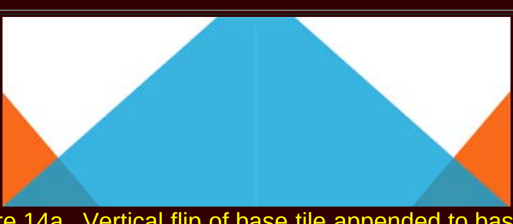

Figure 14a. Vertical flip of base tile appended to base tile

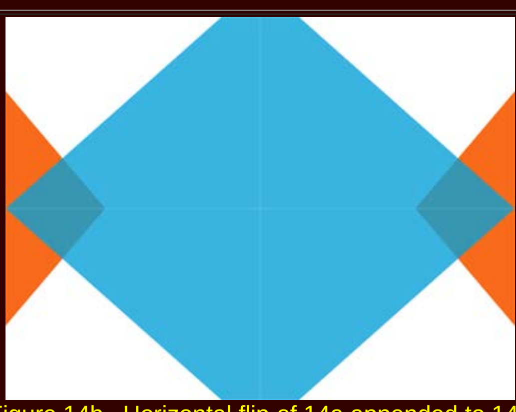

Figure 14b. Horizontal flip of 14a appended to 14a

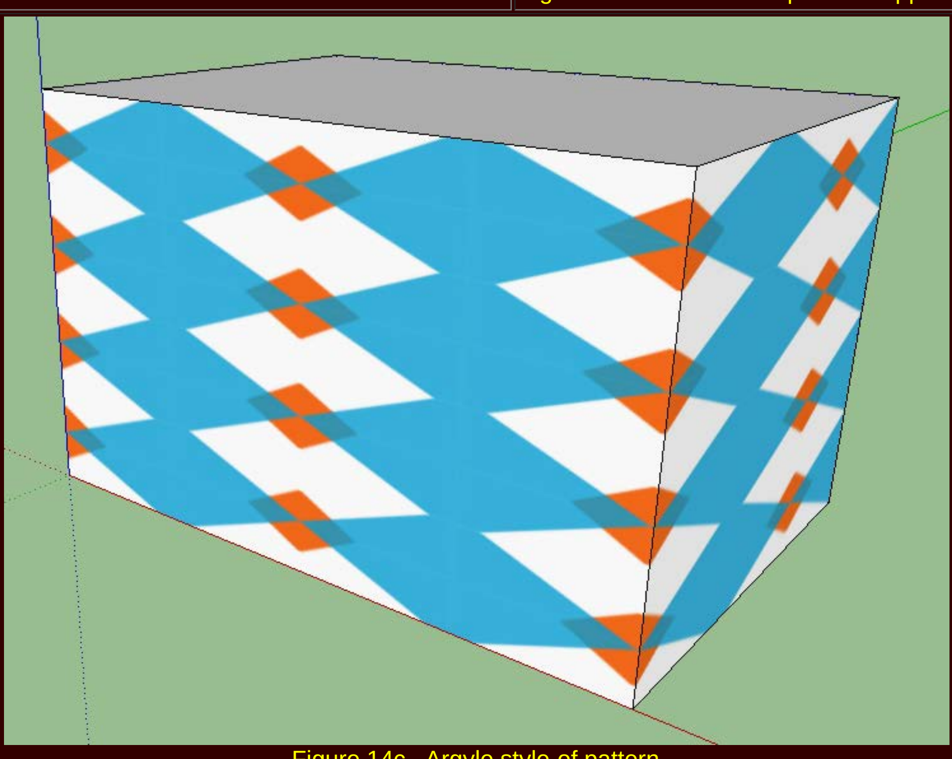

Figure 14c. Argyle style of pattern

Here, only a simple non-square rectangular tile was considered. One might carve out pieces of a rectangular tile and glue them on top or bottom or sides and create oddly-shaped Escher-like fish that fit together perfectly in two different directions and at different scales. The process is similar and employs the same general style of reasoning. One may use the chain of reasoning for non-Euclidean as well as Euclidean objects (see comments in the Escher Wikipedia reference involving the Escher 'Circle Limit series). The subjects of group theory and of tiling are deep ones--group theory lies at the theoretical root not only of simple tiling such as that shown here but also at that root of any tiling (see for example, Wikipedia, 'Wallpaper group'). References for further reading are suggested at the end of this document.

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  M. C. Escher: http://en.wikipedia.org/wiki/M._C._Escher
  Rigid Motions, Euclidean Group: http://en.wikipedia.org/wiki/Euclidean_group


### *Software Used*

Adobe PhotoShop
Google Earth
MicroSoft Office, PowerPoint, Word.
Trimble SketchUp

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Wikipedia:
Group: http://en.wikipedia.org/wiki/Group_%28mathematics%29
Group Theory, http://en.wikipedia.org/wiki/Group_theory
Klein Four-group, http://en.wikipedia.org/wiki/Klein_four-group




**Remembering those who are gone now but who contributed in various ways to *Solstice* or to IMaGe projects, directly or indirectly, during the first 28 years of IMaGe:**

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