Torus Actions and Singularities in Symplectic Geometry

by

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CHAPTER I

Introduction

This thesis includes the fruits of diverse research projects in both symplectic geometry and toric geometry. We begin with algebraic toric geometry, segue through toric symplectic geometry, and end with symplectic geometry. The results from these different disciplines are mostly self-contained; the reader interested only in one field is invited to read just the corresponding section in this introduction and the relevant chapter. The only exception to this invitation concerns the reader interested only in the symplectic geometry of b^k-manifolds, who must skim Section 1.2 and Section 3.1 to understand completely Chapter IV.

1.1 Outline: geometry of T-varieties

A T-variety is a normal complex algebraic variety with an effective action of an algebraic torus. This definition matches the definition of a toric variety, except that the dimension of the torus may be less than the dimension of the variety on which it acts. In particular, any normal algebraic variety is a T-variety when endowed with the trivial action of $(\mathbb{C}^*)^0$. We therefore can't expect to prove much about general T-varieties; we usually restrict our attention to *complexity-one* T-varieties, where the dimension of the torus is exactly one less than the dimension of the variety. In Chapter II, we study the T-invariant curves of a complete complexity-one T-variety, find formulas for their intersection with T-invariant divisors (using the theory of T-invariant divisors developed by Petersen and Süß in [PS11]), and prove that the numerical equivalence classes of these curves generate the Mori cone of the T-variety.

We review the basics of T-varieties in Section 2.1. Informally speaking, a complexity-one T-variety is encoded by a family (parametrized by a projective curve Y) of polyhedral subdivisions of a vector space, all with the same tailfan. In Section 2.2, we describe two kinds of T-invariant curves in a T-variety, *vertical* curves and *horizontal* curves. The vertical curves correspond to walls (codimension-one strata) of one of these polyhedral subdivisions, while the horizontal curves correspond to certain maximal-dimensional cones of the tailfan. We give formulas that calculate the intersection of these curves with a T-invariant divisor using the language of Cartier support functions from [PS11].

In Section 2.3, we generalize the toric cone theorem, which states that the Mori cone of a toric variety is generated as a cone by the classes of T-invariant curves corresponding to the walls of its fan. In our generalization, we show that the Mori cone of a complete complexity-one T-variety is generated as a cone by the classes of a finite collection of vertical curves and horizontal curves. We end the chapter with examples in Section 2.4.

1.2 Outline: toric geometry of b-manifolds

In Chapter III we leave behind the topic of algebraic toric varieties for the corresponding objects in symplectic geometry, called symplectic toric manifolds. In the same way that we studied in Chapter II a generalization of toric varieties called T-varieties, we study in Chapter III a generalization of symplectic toric manifolds called *b-symplectic toric manifolds.*

A *b-symplectic manifold* is a manifold M^{2n} together with a symplectic form ω on M that has a particular kind of order-one singularity along a hypersurface Z. We define these objects in section 3.1, and review classic results concerning their geometry. In section 3.1.1 we review Delzant's theorem, which states that a symplectic toric manifold is classified by the image of its moment map. Through examples we demonstrate why two of the central definitions from Delzant theory, the *moment map* and *hamiltonian action*, must be modified to study torus actions on b-symplectic manifolds. In particular, we define the notion of a b-function, which is a smooth function with (at worst) logarithmic singularities along Z; these b-functions will play the role of Hamiltonian functions in the Delzant theory of b-manifolds. We also construct in section 3.2 an enlarged version of the dual of the Lie Algebra of $\mathbb{T}^n$ in order to make possible a moment map for these Hamiltonian b-functions. With these new tools, We define in section 3.4 a *b-symplectic toric manifold* as a b-symplectic manifold together with a Hamiltonian action and a choice of moment map, and show that they are classified by certain combinatorial objects, called *Delzant b-polytopes*, in the enlarged version of the Lie algebra of $\mathbb{T}^n$.

1.3 Outline: geometry of b^k-manifolds

Although we study in Chapter III the geometry of b-manifolds in the context of toric geometry, b-manifolds are interesting even outside this context. In fact, Melrose originally developed the b-calculus to study pseudodifferential operators on noncompact manifolds ([Mel93], [Gri01]). Considering the manifold in question as the interior of a manifold M with boundary, he constructed the *b-tangent bundle* ${}^bTM$ whose sections are vector fields on M tangent to ∂M, and the *b-cotangent bundle* ${}^bT^*M$, whose sections are differential forms with a specific kind of order-one singularity at ∂M. Modern treatments of the subject study these objects on a manifold M with a distinguished hypersurface Z rather than on a manifold with boundary[1], and sections of ${}^bTM$ (and ${}^bT^*M$) are vector fields (and differential forms) tangent to Z (or singular at Z). In Chapter IV, we generalize this construction so that vector fields and differential forms with higher order tangency and higher order singularity may also be realized as sections of bundles.

The construction of these bundles in Section 4.1 is subtle: although we wish to define a b^k-vector field as a vector field with an "order k tangency to Z," there is no straightforward way to rigorously define this notion. To do so, we must include in the definition of a b^k-manifold the data of a $(k-1)$-jet of Z (and insist that the morphisms in the b^k-category preserve this jet). We then define a b^k-vector field as a vector field such that $\mathcal{L}_v(f)$ vanishes to order k for functions f that represent the jet data. Then we define the b^k-tangent bundle ${}^{b^k}TM$ as the vector bundle whose sections are b^k-vector fields, and the b^k-cotangent bundle ${}^{b^k}T^*M$ as its dual. When $k=1$, these are the familiar constructions from [Mel93] and [GMP13].

In Section 4.2 we study the geometry of the fibers of ${}^{b^k}TM$ and ${}^{b^k}T^*M$. Recall from [GMP13] that the fibers of ${}^bTM$ and ${}^bT^*M$ satisfy

$$ {}^bT_pM \cong \begin{cases} T_pM & \text{for } p \notin Z \\ T_pZ + \langle y\frac{\partial}{\partial y}\rangle & \text{for } p \in Z \end{cases} \qquad {}^bT_p^*M \cong \begin{cases} T_p^*M & \text{for } p \notin Z \\ T_p^*Z + \langle \frac{dy}{y}\rangle & \text{for } p \in Z \end{cases} $$

where y is a defining function for Z. Similarly, we show that the fibers of ${}^{b^k}TM$ and ${}^{b^k}T^*M$ satisfy

$$ {}^{b^k}T_pM \cong \begin{cases} T_pM & \text{for } p \notin Z \\ T_pZ + \langle y^k\frac{\partial}{\partial y}\rangle & \text{for } p \in Z \end{cases} \qquad {}^{b^k}T_p^*M \cong \begin{cases} T_p^*M & \text{for } p \notin Z \\ T_p^*Z + \langle \frac{dy}{y^k}\rangle & \text{for } p \in Z \end{cases} $$

[1]These competing perspectives can be reconciled by viewing a manifold with boundary M as one half of its double. In doing so, the boundary of M corresponds to a hypersurface of the double. In our treatment, we follow the precedent of [GMP13] and define our bundles over manifolds with distinguished hypersurfaces.

where y is a defining function for Z that represents the jet data of the underlying b^k-manifold.

In Section 4.3 we define a differential on the complex of b^k*-forms* (sections of the exterior algebra of ${}^{b^k}T^*M$) and prove a Mazzeo-Melrose type theorem for the cohomology ${}^{b^k}H^*(M)$ of this complex.

$$ {}^{b^k}H^p(M) \cong H^p(M) \oplus \left(H^{p-1}(Z)\right)^k \tag{1.1} $$

However, this isomorphism (like that of the classic Mazzeo-Melrose theorem) is non-canonical. By defining the *Laurent Series* of a b^k-form, which expresses a b^k-form as a sum of simpler b^ℓ-forms (for $\ell \leq k$), we show that there *is* a way to construct the isomorphism in Equation 1.1 so that the $(H^{p-1}(Z))^k$ summand of a b^k-cohomology class *is* canonically defined.

In Section 4.4, we study the geometry of volume b^k-forms. In [Rad02], the author defined the *Liouville volume* of a volume b-form as a certain principal value of the form. This invariant was featured in her classification theorem of stable Poisson structures on compact surfaces. We extend this definition by defining the *volume polynomial* of a volume b^k-form. This polynomial encodes the asymptotic behavior of the integral of a volume b^k-form near Z. We define the *Liouville volume* as the constant term of this polynomial – it agrees with the classic definition when $k = 1$. Finally, we use these results to show that for volume forms, the isomorphism in Equation 1.1 *can* be defined canonically. In this context, the image of a volume form under Equation 1.1 is its *Liouville-Laurent decomposition.*

In Section 4.5, we define a symplectic b^k-form as a closed b^k 2-form having full rank, and prove the classic Moser theorems in the b^k category. We also revisit the classification theorems of stable Poisson structures on compact oriented surfaces from [Rad02] and [GMP13]. Radko classifies stable Poisson structures using geometric data, while the authors of [GMP13] use cohomological data; in Section 4.5.1, we show how the Liouville-Laurent decomposition relates the geometric data to the cohomological data.

Finally, in Section 4.6 we apply the theory of b^k-manifolds to answer questions from Poisson geometry. In particular, we define a Poisson structure to be *of b^k-type* if it is dual to a symplectic b^k-form. When M is a surface, this just means that the Poisson bivector Π is given by $f\Pi_0$ where Π_0 is dual to a symplectic form, and f is the k^{th} power of a local defining function. We give conditions for two such Poisson structures on a compact surface to be isomorphic in terms of the summands in their respective Liouville-Laurent decompositions.

CHAPTER II

T-varieties

2.1 Primer on T-varieties

In this section, we review the basic notation and construction of T-varieties. The presentation favors brevity over pedogogy; we encourage any reader unfamiliar with T-varieties to read the excellent survey article [AIP$^+$12] for a friendlier exposition to this beautiful topic.

2.1.1 Notation

Let $T \cong (\mathbb{C}^*)^k$ be an algebraic torus, and M, N be the character lattice of T and the lattice of 1-parameter subgroups of T respectively. These lattices embed in the vector spaces

$$N_{\mathbb{Q}} := \mathbb{Q} \otimes N \qquad\qquad M_{\mathbb{Q}} := \mathbb{Q} \otimes M$$

and are dual to one another[1]. In classic toric geometry, one studies the correspondence between cones (and fans) in $N_{\mathbb{Q}}$ and the toric varieties encoded by these combinatorial data. Analogously, we study T-varieties through the correspondence between combinatorial gadgets called p-divisors (and divisoral fans) and the T-varieties they encode. Informally speaking, a p-divisor is a Cartier divisor on a normal semiprojective variety Y with polyhedral coefficients; a divisorial fan is a collection of p-divisors

[1] In this chapter, when a picture of $N_{\mathbb{Q}}$ is juxtaposed with a picture of $M_{\mathbb{Q}}$, the reader may assume that the bases for these vector spaces have been chosen so that the pairing between them is the standard dot product.

whose polyhedral coefficients "fit together" in a suitable way. To make formal these definitions, we begin by discussing monoids of polyhedra.

Let σ be a pointed cone in $N_{\mathbb{Q}}$, and $\sigma^\vee \subseteq M_{\mathbb{Q}}$ its dual. The set $\mathrm{Pol}^+_{\mathbb{Q}}(N, \sigma)$ of all polyhedra in $N_{\mathbb{Q}}$ having σ as its tailcone (with the convention that $\emptyset \in \mathrm{Pol}^+_{\mathbb{Q}}(N, \sigma)$) is a monoid under Minkowski addition with identity element σ. Any nonempty $\Delta \in \mathrm{Pol}^+_{\mathbb{Q}}(N, \sigma)$ defines a map

$$h_\Delta : \sigma^\vee \to \mathbb{Q} \tag{2.1}$$

$$u \mapsto \min_{v \in \Delta} \langle v, u \rangle.$$

called the *support function* of Δ. A nonempty $\Delta \in \mathrm{Pol}^+_{\mathbb{Q}}(N, \sigma)$ also defines a *normal quasifan* $\mathcal{N}(\Delta)$ in $M_{\mathbb{Q}}$ consisting of a cone λ_F for each face F of Δ defined by

$$\lambda_F = \{u \in \sigma^\vee \mid \langle u, v \rangle = h_\Delta(u) \ \forall v \in F\}.$$

The figure below shows an example of a polyhedron and its normal quasifan.

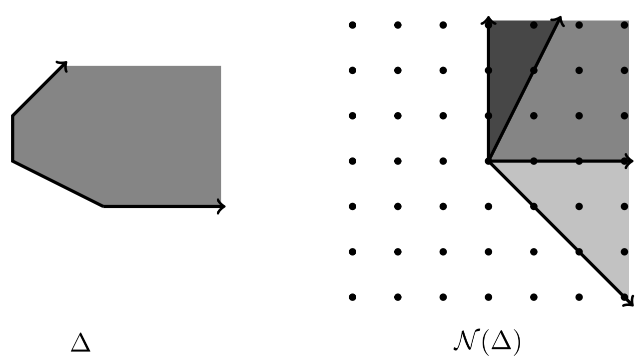

Figure 2.1: A polyhedron and its normal quasifan

Proposition II.1. ([AH06], Lemma 1.4 and Proposition 1.5) The support function h_Δ is a well-defined map whose regions of linearity are the maximal cones of $\mathcal{N}(\Delta)$. Moreover, any function in $\mathrm{Hom}(\sigma^\vee, \mathbb{Q})$ whose regions of linearity define a quasifan can be realized as h_Δ for some Δ.

Let $\mathrm{Pol}_{\mathbb{Q}}(N,\sigma)$ be the Grothendieck group of $\mathrm{Pol}^+_{\mathbb{Q}}(N,\sigma)$. Let Y be a normal semiprojective variety, with $\mathrm{CaDiv}(Y)$ its group of Cartier divisors. An element

$$\mathcal{D} \in \mathrm{Pol}_{\mathbb{Q}}(N,\sigma) \otimes_{\mathbb{Z}} \mathrm{CaDiv}(Y)$$

is a *polyhedral divisor* with tailcone σ if it has a representative of the form $\mathcal{D} = \sum \mathcal{D}_P \otimes P$ for some $\mathcal{D}_P \in \mathrm{Pol}^+_{\mathbb{Q}}(N,\sigma)$ and P prime[2]. We will describe a procedure for constructing an affine T-variety from a certain kind of polyhedral divisor (called a *p-divisor*); this construction will involve taking the spectrum of the global sections of a sheaf of rings defined over a subset of Y. This subset, called the *locus* of $\mathcal{D}$, is

$$\mathrm{Loc}(\mathcal{D}) := Y \setminus \cup_{\mathcal{D}_P = \emptyset} P.$$

The *evaluation* of $\mathcal{D}$ at $u \in M \cap \sigma^\vee$ is the $\mathbb{Q}$-Cartier divisor[3]

$$\mathcal{D}(u) := \sum_{\mathcal{D}_P \neq \emptyset} h_{\mathcal{D}_P}(u) P\Big|_{\mathrm{Loc}(\mathcal{D})}.$$

We say that $\mathcal{D}$ is a *p-divisor* if $\mathcal{D}(u)$ is semiample for all $u \in \sigma^\vee$ and big for all u in the interior of $\sigma^\vee$. The direct sum of the sheaves defined by the evaluations $\mathcal{D}(u)$ is an M-graded sheaf of rings

$$\mathcal{O}(\mathcal{D}) := \bigoplus_{u \in \sigma^\vee \cap M} \mathcal{O}_{\mathrm{Loc}(\mathcal{D})}(\mathcal{D}(u))\chi^u$$

over $\mathrm{Loc}(\mathcal{D})$. There are two different T-varieties encoded by the p-divisor $\mathcal{D}$

$$\widetilde{TV}(\mathcal{D}) := \mathrm{Spec}_{\mathrm{Loc}(\mathcal{D})}\mathcal{O}(\mathcal{D}) \qquad \text{and} \qquad TV(\mathcal{D}) := \mathrm{Spec}\,\Gamma(\mathrm{Loc}(\mathcal{D}), \mathcal{O}(\mathcal{D}))$$

where the torus action is given by the M-grading on $\mathcal{O}(\mathcal{D})$. All affine T-varieties can be constructed this way.

[2]Because σ (not $\emptyset$) is the identity element of $\mathrm{Pol}_{\mathbb{Q}}(N,\sigma)$, the summation notation in this sentence implies that only finitely of the polyhedral coefficients $\mathcal{D}_P$ differ from σ

[3]Some authors define a "$\mathbb{Q}$-Cartier" divisor to be a Weil divisor with a Cartier multiple. Our $\mathbb{Q}$-Cartier divisors are elements of $\mathbb{Q} \otimes Div(Y)$ having a Cartier multiple (so may have rational coefficients). The pedantic reader is invited to replace all instances of "$\mathbb{Q}$-Cartier divisor" in this chapter with "$\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor".

Theorem II.2. *([AH06], Corollary 8.14) Every normal affine variety with an effective torus action can be realized as $TV(\mathcal{D})$ for some p-divisor $\mathcal{D}$*

Similar to the way that a fan of a non-affine toric variety can be obtained by "gluing together" the cones constituting an affine cover, so too can a non-affine T-variety be encoded by "gluing together" the p-divisors constituting an affine cover. To make formal these concepts, we first define the *intersection* of two p-divisors $\mathcal{D}, \mathcal{D}'$ on Y as the p-divisor

$$\mathcal{D} \cap \mathcal{D}' := \sum (\mathcal{D}_P \cap \mathcal{D}'_P) \otimes P.$$

We say that $\mathcal{D}'$ is a *face* of $\mathcal{D}$ if $\mathcal{D}'_P \subseteq \mathcal{D}_P$ for each P and the induced map $TV(\mathcal{D}') \to TV(\mathcal{D})$ is an open embedding. A finite set $\mathcal{S}$ of p-divisors on Y is a *divisoral fan* if for any $\mathcal{D}, \mathcal{D}' \in \mathcal{S}$, $\mathcal{D} \cap \mathcal{D}'$ is an element of $\mathcal{S}$ and is a face of both $\mathcal{D}$ and $\mathcal{D}'$. We define $TV(\mathcal{S})$ and $\widetilde{TV}(\mathcal{S})$ to be the T-varieties obtained by gluing together the T-varieties $\{TV(\mathcal{D})\}_{\mathcal{D}\in\mathcal{S}}$ and $\{\widetilde{TV}(\mathcal{D})\}_{\mathcal{D}\in\mathcal{S}}$ according to these face relations. This process is detailed in [AHS08].

2.1.2 Geometry of $TV(\mathcal{S})$ and $\widetilde{TV}(\mathcal{S})$

Because $\widetilde{TV}(\mathcal{D})$ is defined as the relative spectrum of a sheaf of rings on $\mathrm{Loc}(\mathcal{D})$, there is a natural projection map $\pi : \widetilde{TV}(\mathcal{D}) \to \mathrm{Loc}(\mathcal{D}) \subseteq Y$. Because $TV(\mathcal{D})$ is defined as the spectrum of the global sections of the structure sheaf on $\widetilde{TV}(\mathcal{D})$, we also have a natural map $p : \widetilde{TV}(\mathcal{D}) \to \Gamma(\widetilde{TV}(\mathcal{D}), \mathcal{O}_{\widetilde{TV}(\mathcal{D})}) \cong TV(\mathcal{D})$. Given a divisoral fan $\mathcal{S}$, the maps π, p corresponding to the different $\mathcal{D} \in \mathcal{S}$ glue into maps

$$\begin{tikzcd}
\widetilde{TV}(\mathcal{S}) \arrow[r, "p"] \arrow[d, "\pi"] & TV(\mathcal{S}) \\
Y &
\end{tikzcd}$$

In this subsection, we describe the fibers of p and π. In particular, we will notice that for $y \in Y$, the reduced fiber $\pi^{-1}(y)$ is a union of irreducible toric varieties, and that the contraction map p identifies certain disjoint torus orbits of $\widetilde{TV}(\mathcal{S})$. Many of these results simplify when $TV(\mathcal{S})$ is a complexity-one T-variety; because this is the only case we will need for later sections, we will henceforth assume that Y is a smooth projective curve. The reader interested in higher-complexity T-varieties should read [AIP+12] for the more general results.

In [Pet10], the author describes the reduced fibers of π using the language of *dappled toric bouquets*. We begin by reviewing this language.

Definition II.3. The **fan ring** of a quasifan Λ in $M_{\mathbb{Q}}$ is

$$\mathbb{C}[\Lambda] := \bigoplus_{u \in |\Lambda| \cap M} \mathbb{C}\chi^u$$

with multiplication defined by

$$\chi^u \chi^v = \begin{cases} \chi^{u+v} & \text{if } u, v \in \lambda \text{ for some cone } \lambda \in \Lambda \\ 0 & \text{otherwise} \end{cases}$$

For a nonempty $\Delta \in \mathrm{Pol}^+_{\mathbb{Q}}(N, \sigma)$ and a cone λ_F of its inner normal quasifan $\mathcal{N}(\Delta)$, let

$$M_{\lambda_F} := \{u \in \lambda_F \cap M \mid h_\Delta(u) \in \mathbb{Z}\}.$$

Remark II.4. In some papers, M_{λ_F} is defined differently: when $\Delta \otimes [P]$ appears as a summand in a p-divisor, the elements $u \in M_{\lambda_F}$ are required to satisfy the condition that $h_\Delta(u)[P]$ is locally principal at P. In the complexity-one case, this condition coincides with our condition that $h_\Delta(u) \in \mathbb{Z}$.

Finally, let $S_\Delta \subseteq |\mathcal{N}(\Delta)| \cap M$ consist of those u such that $S_\Delta \cap \lambda_F = M_{\lambda_F}$ for every cone $\lambda_F \in \mathcal{N}(\Delta)$. S_Δ can be thought of as a conewise-varying sublattice of M. The figure below shows an example of S_Δ for a given Δ; the elements of $S_\Delta \subseteq M$ are in bold.

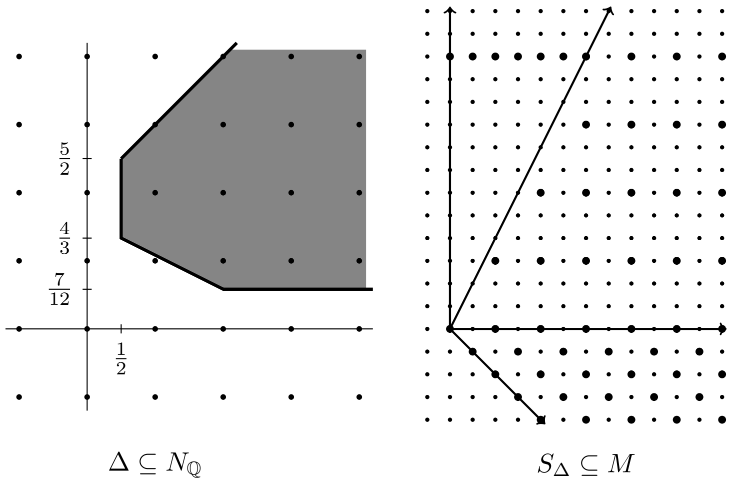

Figure 2.2: S_Δ is a conewise-varying sublattice of M

Definition II.5. The **dappled fan ring** of Δ is the following subring of $\mathbb{C}[\mathcal{N}(\Delta)]$

$$\mathbb{C}[\mathcal{N}(\Delta), S_\Delta] := \bigoplus_{u \in S_\Delta} \mathbb{C}\chi^u$$

Definition II.6. The **dappled toric bouquet** encoded by Δ is the variety $TB(\Delta) := \mathrm{Spec}(\mathbb{C}[\mathcal{N}(\Delta), S_\Delta])$. Given a polyhedral complex $\Sigma = \{\Delta\}$ in $N_{\mathbb{Q}}$, the dappled toric bouquet encoded by Σ is the variety $TB(\Sigma)$ obtained by gluing the $\{TB(\Delta)\}_{\Delta\in\Sigma}$ according to the face relations among the polyhedra.

Observe that $TB(\Delta)$ and $TB(\Sigma)$ have a natural torus action induced by the M-grading of the dappled fan rings. For a T-variety $TV(\mathcal{S})$ over Y and a point $y \in Y$, the polyhedra $\{\mathcal{D}_y\}_{\mathcal{D}\in\mathcal{S}}$ fit together into a polyhedral complex $\mathcal{S}_y$ of $N_{\mathbb{Q}}$.

Proposition II.7. [[Pet10], Prop 1.39] Let $\mathcal{S}$ be a divisorial fan on the smooth projective curve Y. The reduced fiber $\pi^{-1}(y)$ of $\pi : \widetilde{TV}(\mathcal{S}) \to Y$ is equivariantly isomorphic to $TB(\mathcal{S}_y)$.

Motivated by Proposition II.7 to study the geometry of non-affine toric bouquets, we construct a fan for each vertex of a polyhedral subdivision Σ of $N_{\mathbb{Q}}$; the toric varieties they encode will be precisely the irreducible components of $TB(\Sigma)$. For a vertex $v \in \Sigma$, define the lattice

$$M_v = \{u \in M \mid \langle u, v \rangle \in \mathbb{Z}\}$$

Because M_v is a sublattice of M, N is a sublattice of $N_v := M_v^\vee \subseteq N_\mathbb{Q}$. Let $i_v : N_\mathbb{Q} \to (N_v)_\mathbb{Q}$ be the map induced by this inclusion. As Δ ranges over all polyhedra in Σ containing v, the cones $i_v(\mathbb{Q}_{\geq 0} \cdot (\Delta - v))$ form a fan F_v in $(N_v)_\mathbb{Q}$. For any cone σ of F_v, the semigroup $\sigma^\vee \cap N_v^\vee$ is isomorphic to the semigroup $\lambda_\Delta \cap S_\Delta$. Because this isomorphism commutes with the gluing data induced by the face relations, we have the following description of the irreducible components of $TB(\Sigma)$.

Proposition II.8. The irreducible components of $TB(\Sigma)$ are equivariantly isomorphic to the toric varieties $\{TV(F_v)\}$ where the set ranges over the vertices v of Σ.

For example, the polyhedral complex in Figure 2.3 encodes a toric bouquet with three irreducible toric components. We have drawn the lattices N_v not as a square grid, but in a way that the sublattice $N \subseteq N_v$ (in bold) is a square grid so that the angles between the polyhedra are preserved. In the example, one fan encodes $\mathbb{P}^2$ and the other fans encode weighted projective spaces.

Given a divisorial fan $\mathcal{S}$, its *tailfan* $\mathrm{tail}(\mathcal{S})$ is the fan consisting of the tailcones of the p-divisors comprising $\mathcal{S}$. Because the coefficients $\mathcal{D}_y$ of a p-divisor $\mathcal{D}$ differ from its tailcone for only finitely many y, the polyhedral subdivisions $\mathcal{S}_y$ differ from $\mathrm{tail}(\mathcal{S})$ for only finitely many y. By Proposition II.8, the fiber of π over $y \in Y$ is equal to $TV(\mathrm{tail}(\mathcal{S}))$ for all but finitely many y and specializes to a (possibly non-reduced) union of toric varieties at these finitely many points.

By the discussion above, the familiar orbit-cone correspondence for toric varieties translates into a correspondence between T-orbits in $\widetilde{TV}(\mathcal{S})$ and pairs (y, F) where $y \in Y$ and $F \in \mathcal{S}_y$. To understand $TV(\mathcal{S})$, we will describe how the map p identifies

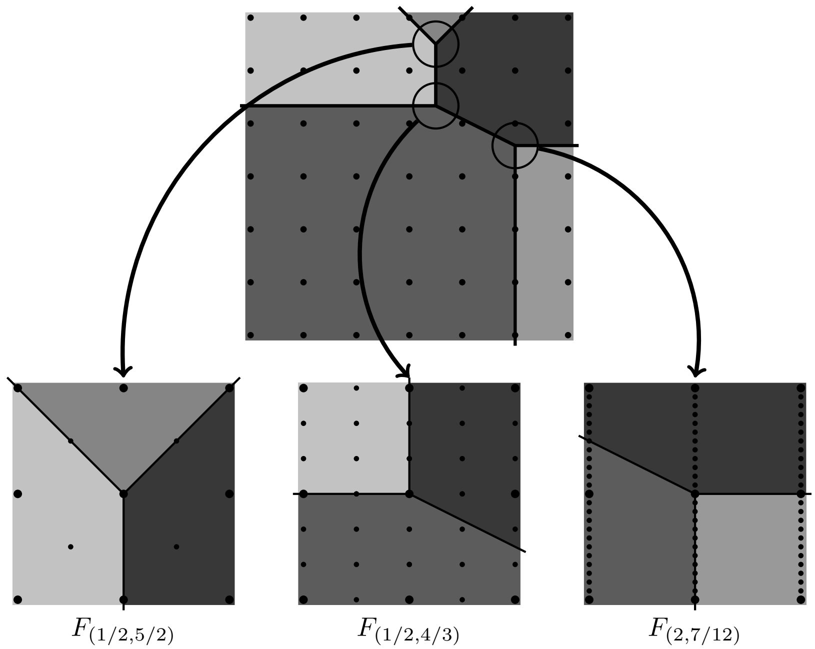

Figure 2.3: Components of a toric bouquet

certain of these orbits in different fibers. We first consider the case of an affine T-variety. For a p-divisor $\mathcal{D}$ with tailcone σ and a $u \in \sigma^\vee \cap M$, the semiample divisor $\mathcal{D}(u)$ defines a map

$$\xi_u : \mathrm{Loc}(\mathcal{D}) \to \mathrm{Proj}\left(\bigoplus_{k\geq 0} \Gamma(\mathrm{Loc}(\mathcal{D}), \mathcal{D}(ku))\right).$$

Theorem II.9. *([AH06], Theorem 10.1) The map $p : \widetilde{TV}(\mathcal{D}) \to TV(\mathcal{D})$ induces a surjection*

$$\{(y, F) : y \in Y, F \textit{ is a face of } \mathcal{D}_y\} \to \{T - \text{orbits in TV}(\mathcal{D})\}$$

that identifies the orbits corresponding to (y, F) and (y', F') iff $\lambda_F = \lambda_{F'} \subseteq M_{\mathbb{Q}}$ and $\xi_u(y) = \xi_u(y')$ for some (equivalently, for any) $u \in \mathrm{relint}(\lambda_F)$.

In the non-affine case, the gluing maps among $\{TV(\mathcal{D})\}_{\mathcal{D}\in\mathcal{S}}$ are prescribed by the face relations between the p-divisors, which identifies precisely those T-orbits in $TV(\mathcal{D})$ and $TV(\mathcal{D}')$ corresponding to the faces $\{(y, \mathcal{D}_y \cap \mathcal{D}'_y)\}_{y\in Y}$.

2.2 T-invariant curves and intersection theory

In this section, we study the intersection theory of complete complexity-one T-varieties over a projective curve Y. For the rest of Chapter II, *all* T-varieties are complete, complexity-one T-varieties over a projective curve Y. The "completeness" condition translates into the combinatorial requirement that $|\mathcal{S}_y| = N_{\mathbb{Q}}$ for all y. Motivated by the correspondence between T-invariant Cartier divisors and Cartier support functions introduced in [PS11], we define the notion of a $\mathbb{Q}$-Cartier support function to encode $\mathbb{Q}$-Cartier torus invariant divisors. We will describe two kinds of T-invariant curves – *vertical* curves and *horizontal* curves – then give formulas that compute the intersection of these curves with a T-invariant $\mathbb{Q}$-Cartier divisor.

Definition II.10. Given a nontrivial $\Delta \in \mathrm{Pol}^+_{\mathbb{Q}}(N, \sigma)$ and an affine $\varphi : \Delta \to \mathbb{Q}$, the **linear part** of φ is the function

$$\begin{aligned} \mathrm{lin}\varphi : \sigma &\to \mathbb{Q} \\ n &\mapsto \varphi(p+n) - \varphi(p) \end{aligned}$$

where p is any point in Δ. The function $\mathrm{lin}\varphi$ extends uniquely to a linear function $(\mathbb{R} \cdot \sigma) \to \mathbb{Q}$, which will also be written $\mathrm{lin}\varphi$ without risk of confusion.

Definition II.11. Let $\mathcal{S}$ be the divisorial fan of a complexity-one T-variety over Y. A $\mathbb{Q}$**-Cartier support function** is a collection of affine functions

$$\{h_{\mathcal{D},y} : |\mathcal{D}_y| \to \mathbb{Q}\}_{\substack{\mathcal{D}\in\mathcal{S} \\ y\in Y}}$$

with rational slope and rational translation such that

1. For a fixed $y \in Y$, the functions $\{h_{\mathcal{D},y}\}_{\mathcal{D}\in\mathcal{S}}$ define a continuous function $h_y : |\mathcal{S}_y| \to \mathbb{Q}$. That is, $h_{\mathcal{D},y}$ and $h_{\mathcal{D}',y}$ agree on $\mathcal{D}_y \cap \mathcal{D}'_y$ for $\mathcal{D}, \mathcal{D}' \in \mathcal{S}$.

2. For each $\mathcal{D} \in \mathcal{S}$ with complete locus, there exists $u \in M, f \in K(Y)$ and $N \in \mathbb{Z}_{>0}$ such that $Nh_{\mathcal{D},y}(v) = -\mathrm{ord}_y(f) - \langle u, v\rangle$ for all $y \in Y, v \in N_{\mathbb{Q}}$.

3. If $\mathcal{D}_y, \mathcal{D}'_{y'}$ have the same tailcone, then $\mathrm{lin}h_{\mathcal{D},y} = \mathrm{lin}h_{\mathcal{D}',y'}$.

4. For a fixed $\mathcal{D}$, $h_{\mathcal{D},y}$ differs from $\mathrm{lin}h_{\mathcal{D},y}$ for only finitely many y.

A $\mathbb{Q}$-Cartier support function is called a **Cartier support function** if each $h_{\mathcal{D},y}$ has integral slope and integral translation and $N = 1$ in condition (2). We write $CaSF(\mathcal{S})$ and $\mathbb{Q}CaSF(\mathcal{S})$ to denote the abelian group (under standard addition of functions) of Cartier support functions and $\mathbb{Q}$-Cartier support functions respectively.

For any T-invariant Cartier divisor D on $TV(\mathcal{S})$ and any p-divisor $\mathcal{D} \in \mathcal{S}$, we can always find an open cover $\{U_i\}$ of Y for which there exists Cartier data for $D\big|_{TV(\mathcal{D})}$ of the form $(TV(\mathcal{D}\big|_{U_i}), f_i\chi^{u_i})$ (see proof of [PS11], Prop 3.10 for details). These Cartier data define functions

$$\{h_{\mathcal{D},y}(v) = -\mathrm{ord}_y(f_i) - \langle u_i, v\rangle\}_{y\in U_i}$$

which agree on the overlaps of the U_i to define $h_{\mathcal{D},y}$ for all y. In this way, we can define a Cartier support function for any Cartier divisor on $TV(\mathcal{S})$.

Proposition II.12. ([PS11], Prop 3.10) Let $T - CaDiv(\mathcal{S})$ denote the group of T-invariant Cartier divisors on $TV(\mathcal{S})$. This association of a Cartier support function to a T-invariant Cartier divisor defines an isomorphism of groups

$$T - CaDiv(\mathcal{S}) \cong CaSF(\mathcal{S})$$

If $\{h_{\mathcal{D},y}\}$ is the Cartier support function for ND, where $N > 0$ and D is a T-invariant $\mathbb{Q}$-Cartier divisor, then $\{N^{-1}h_{\mathcal{D},y}\}$ is a $\mathbb{Q}$-Cartier support function. In this way, we can associate a $\mathbb{Q}$-Cartier support function to any T-invariant $\mathbb{Q}$-Cartier divisor on $TV(\mathcal{S})$. The following is an immediate corollary of Proposition II.12.

Corollary II.13. *Let $T - \mathbb{Q}CaDiv(\mathcal{S})$ denote the group of T-invariant $\mathbb{Q}$-Cartier divisors on $TV(\mathcal{S})$. Then the association described above is an isomorphism of groups*

$$T - \mathbb{Q}CaDiv(\mathcal{S}) \cong \mathbb{Q}CaSF(\mathcal{S})$$

2.2.1 Vertical curves

Toward the goal of describing the intersection theory of a T-variety, we study its T-invariant curves. We start with *vertical curves*, which are images (under p) of a T-invariant curve contained in a single fiber of π.

Recall from Proposition II.8 that for $y \in Y$, the reduced fiber $\pi^{-1}(y)$ has as its irreducible components a toric variety for each vertex v of $\mathcal{S}_y$. A toric variety has a T-invariant curve corresponding to each wall[4] of its fan (by taking the closure of the corresponding torus orbit). Translating this fact into the context of toric bouquets, we call a codimension-one element of a polyhedral complex a *wall* if it can be realized as the intersection of two top-dimensional polyhedra; there is a T-invariant curve in a toric bouquet for each wall of the corresponding polyhedral complex. In this section, we study the curves in $\widetilde{TV}(\mathcal{S})$ and $TV(\mathcal{S})$ corresponding to these T-invariant curves.

Fix a T-variety $TV(\mathcal{S})$ and a point $y \in Y$. Let $\tau \in \mathcal{S}_y$ be a wall of the polyhedral complex $\mathcal{S}_y$, let $\mathcal{D}, \mathcal{D}' \in \mathcal{S}$ be two p-divisors for which $\tau = \mathcal{D}_y \cap \mathcal{D}'_y$, let $\lambda_{\tau,\mathcal{D}} \subseteq M_{\mathbb{Q}}$ be the cone in $\mathcal{N}(\mathcal{D}_y)$ dual to τ, and let $u_{\tau,\mathcal{D}}$ be the semigroup generator of $M_{\lambda_{\tau,\mathcal{D}}}$. As usual, unweildy notation obfuscates a simple picture: if $\mathcal{D}$ and $\mathcal{D}'$ have polyhedral coefficients at y as shown in Figure 2.4 (τ is the horizontal plane in a single orthant at a height of 2/3), then the sublattice $\mathbb{Z} \cdot u_{\tau,\mathcal{D}} = M_{\lambda_{\tau,\mathcal{D}}} \cup M_{\lambda_{\tau,\mathcal{D}'}}$ consists of the bold elements of the vertical axis of $M \cong \mathbb{Z}^3$ shown on the right.

[4]A *wall* of a fan is a codimension-one cone that can be realized as the intersection of two top-dimensional cones.

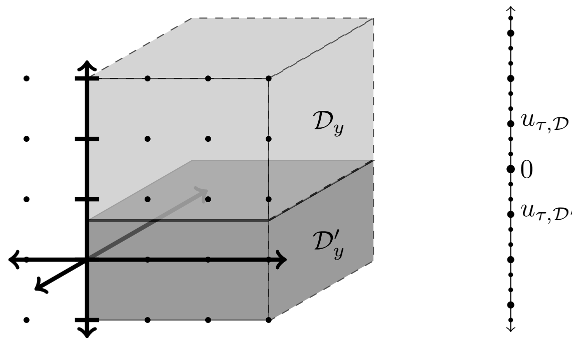

Figure 2.4: The sublattice $\mathbb{Z} \cdot u_{\tau,\mathcal{D}}$ corresponding to the wall $\tau = \mathcal{D}_y \cap \mathcal{D}'_y$

If $g \in K(Y)$ is Cartier data for $\mathcal{D}(u_{\tau,\mathcal{D}})$ in some neighborhood of y, the maps

$$\Gamma(\mathrm{Loc}(\mathcal{D}), \mathcal{O}(\mathcal{D})) \to \mathbb{C}[z]$$
$$f\chi^u \mapsto \begin{cases} 0 & u \notin M_{\lambda_\tau,\mathcal{D}} \\ (g^k f)(y) z^k & u = k u_{\tau,\mathcal{D}} \end{cases}$$

$$\Gamma(\mathrm{Loc}(\mathcal{D}'), \mathcal{O}(\mathcal{D}')) \to \mathbb{C}[z^{-1}]$$
$$f\chi^u \mapsto \begin{cases} 0 & u \notin M_{\lambda_\tau,\mathcal{D}'} \\ (g^{-k} f)(y) z^{-k} & u = -k u_{\tau,\mathcal{D}} \end{cases}$$

glue together to induce a map

$$\mathbb{P}^1 \to TV(\mathcal{S}) \tag{2.2}$$

the image of which we will call the *vertical curve* $C_{\tau,y}$.

Proposition II.14. The vertical curve $C_{\tau,y}$ is the image under p of the closure of the torus orbit in $TB(\mathcal{S}_y) \subseteq \widetilde{TV}(\mathcal{S})$ corresponding to the wall τ.

Proof. For any affine open $U \subseteq Y$ containing y, Map 2.2 factors

$$\mathbb{P}^1 \to \pi^{-1}(U) \subseteq \widetilde{TV}(\mathcal{S}) \xrightarrow{p} TV(\mathcal{S}) \tag{2.3}$$

where $\mathbb{P}^1 \to \pi^{-1}(U)$ is given by

$$\begin{aligned} \Gamma(U, \mathcal{O}(\mathcal{D})) &\to \mathbb{C}[z] \\ f\chi^u &\mapsto \begin{cases} 0 & u \notin M_{\lambda_\tau, \mathcal{D}} \\ (g^k f)(y) z^k & u = k u_{\tau, \mathcal{D}} \end{cases} \end{aligned} \tag{2.4}$$

$$\begin{aligned} \Gamma(U, \mathcal{O}(\mathcal{D}')) &\to \mathbb{C}[z^{-1}] \\ f\chi^u &\mapsto \begin{cases} 0 & u \notin M_{\lambda_\tau, \mathcal{D}'} \\ (g^{-k} f)(y) z^{-k} & u = -k u_{\tau, \mathcal{D}} \end{cases} \end{aligned}$$

Therefore, it suffices to show that the map $\mathbb{P}^1 \to \pi^{-1}(U)$ has, as its closure, the torus orbit in $TB(\mathcal{S}_y) \subseteq \widetilde{TV}(\mathcal{S})$ corresponding to the wall τ. To do so, we recall some relevant details about the isomorphism between the reduced fibers of π and a dappled toric bouquet (see [AH06], Proposition 7.10 for details). This isomorphism, for $\Delta = \mathcal{D}_y$, is constructed by first choosing a collection of functions $\{g_{\mathcal{D}(u)} \in K(Y)\}_{u \in S_\Delta}$ such that, after possibly shrinking U,

$$\operatorname{div}(g_{\mathcal{D}(u)})\big|_U = \mathcal{D}(u)\big|_U \qquad \text{and} \qquad g_{\mathcal{D}(u+u')} = g_{\mathcal{D}(u)} g_{\mathcal{D}(u')}$$

Then the isomorphism between the fiber and the dappled toric bouquet is induced by

$$\begin{aligned} \Gamma(U, \mathcal{O}(\mathcal{D})) &\to \mathbb{C}[\mathcal{N}(\Delta), S_\Delta] \\ f\chi^u &\mapsto \begin{cases} (g_{\mathcal{D}(u)} f)(y) \chi^u & \text{if } u \in S_\Delta \\ 0 & \text{otherwise} \end{cases} \end{aligned} \tag{2.5}$$

(and similarly for $\mathcal{D}'$). On the other hand, the closure of the torus orbit corresponding to τ in the toric bouquet is parametrized by gluing the maps

$$\begin{aligned}
(2.6)\qquad \mathbb{C}[\mathcal{N}(\Delta), S_\Delta] &\to \mathbb{C}[z] \\
\chi^u &\mapsto \begin{cases} z^k & \text{if } u = ku_{\tau,\mathcal{D}}, k \in \mathbb{Z} \\ 0 & \text{otherwise} \end{cases} \\
\mathbb{C}[\mathcal{N}(\Delta'), S_{\Delta'}] &\to \mathbb{C}[z^{-1}] \\
\chi^u &\mapsto \begin{cases} z^{-k} & \text{if } u = -ku_{\tau,\mathcal{D}}, k \in \mathbb{Z} \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

The composition of Equation 2.5 and Equation 2.6 yields Equation 2.4, proving the claim. □

To find a formula that calculates the intersection between $C_{\tau,y}$ and a T-invariant Cartier divisor D, we pick Cartier data for D that includes two sets of the form

$$\{(TV(\mathcal{D}\big|_V), f\chi^u), (TV(\mathcal{D}'\big|_{V'}), f'\chi^{u'})\}$$

where $V, V' \subseteq Y$ are open sets containing y. The Cartier support function for D includes

$$h_{\mathcal{D},y} = -\mathrm{ord}_y(f) - \langle u, v\rangle \qquad \text{and} \qquad h_{\mathcal{D}',y} = -\mathrm{ord}_y(f') - \langle u', v\rangle.$$

Because $h_{\mathcal{D},y}$ and $h_{\mathcal{D}',y}$ agree on τ, it must be the case that

$$\mathrm{ord}_y(f) - \mathrm{ord}_y(f') + \langle u - u', v\rangle = 0$$

for all $v \in \tau$. In particular, $u - u' \in \mathbb{Q} \cdot u_{\tau,\mathcal{D}}$. Moreover, since $\langle u - u', v\rangle = \mathrm{ord}_y(f') - \mathrm{ord}_y(f) \in \mathbb{Z}$, it must be the case that $\langle u - u', v\rangle \in \mathbb{Z}$ for $v \in \tau$. Therefore, $u - u' = ku_{\tau,\mathcal{D}}$ for some $k \in \mathbb{Z}$, and the quotient of the two Cartier data is $f\chi^u/f'\chi^{u'} = (f/f')\chi^{ku_{\tau,\mathcal{D}}}$. Under the parametrization of $C_{\tau,y}$ in Equation 2.2, this rational function pulls back to $(g^k f/f')(y)z^k$ on $C_{\tau,y} \cong \mathbb{P}^1$, where g is Cartier data for $\mathcal{D}(u_{\tau,\mathcal{D}})$. Therefore, the degree of the pullback of D onto $C_{\tau,y}$ is k. This is precisely $\mu_\tau^{-1}\langle u - u', n_{\tau,\mathcal{D}}\rangle$, where μ_τ is the index of $\mathbb{Z} \cdot u_{\tau,\mathcal{D}}$ in $\mathbb{Q} \cdot u_{\tau,\mathcal{D}} \cap M$ and $n_{\tau,\mathcal{D}}$ is any representative of the generator of $N/(u_{\tau,\mathcal{D}})^\perp$ that pairs positively with $u_{\tau,\mathcal{D}}$ (equivalently, $n_{\tau,\mathcal{D}}$ is any element of N such that $\langle n_{\tau,\mathcal{D}}, u_{\tau,\mathcal{D}}\rangle = \mu_\tau$).

$$\langle D, C_{\tau,y}\rangle = \mu_\tau^{-1}\langle u - u', n_{\tau,\mathcal{D}}\rangle$$

or, using the language of Cartier support functions,

$$\langle D, C_{\tau,y}\rangle = \mu_\tau^{-1}(\mathrm{lin} h_{\mathcal{D}',y} - \mathrm{lin} h_{\mathcal{D},y})(n_{\tau,\mathcal{D}}) \tag{2.7}$$

By linearity, the same formula applies when D is a T-invariant $\mathbb{Q}$-Cartier divisor.

Example II.15. Let $\mathcal{D}, \mathcal{D}' \in \mathcal{S}$ be p-divisors that have the slices shown in Figure 2.4. Suppose that with respect to the standard basis given by the coordinate axes in the picture, a T-invariant divisor D has the following Cartier support functions

$$h_{\mathcal{D},y}(v) = -10 + \langle (9,4,17), v\rangle \quad \text{and} \quad h_{\mathcal{D}',y}(v) = 0 + \langle (9,4,2), v\rangle$$

Then

$$\langle D, C_{\tau,y}\rangle = 3^{-1}\langle (0,0,-15), (0,0,1)\rangle = -5$$

2.2.2 Horizontal curves

Let σ be a full-dimensional cone of tail$(\mathcal{S})$. Because the T-varieties we study are complete, every $\mathcal{S}_y$ contains a polyhedron with tailcone σ. Such a polyhedron corresponds to a fixed point in the fiber $\pi^{-1}(y)$. Taking the union (as y varies) of these fixed points defines a curve $\widetilde{C}_\sigma \subseteq \widetilde{TV}(\mathcal{S})$. Theorem II.9 shows that p contracts $\widetilde{C}_\sigma$ precisely if there is some $\mathcal{D} \in \mathcal{S}$ with tailcone σ and complete locus. In this case, we say that σ is *marked*.

Definition II.16. A cone σ of $\mathrm{tail}(\mathcal{S})$ is **marked** if σ is the tailcone of a p-divisor $\mathcal{D} \in \mathcal{S}$ with complete locus.

When σ is unmarked, Theorem II.9 shows that no distinct points of $\widetilde{C}_\sigma$ are identified by p. Toward the goal of finding an intersection formula for these *horizontal curves* $C_\sigma := p(\widetilde{C}_\sigma)$, we parametrize them. Let $TV(\mathcal{S})$ be a T-variety and let σ be an unmarked full-dimensional cone of $\mathrm{tail}(\mathcal{S})$. For $\mathcal{D} \in \mathcal{S}$ with tailcone σ, we have maps of rings

$$\begin{aligned} \varphi_{\mathcal{D}} : \Gamma(TV(\mathcal{D}), \mathcal{O}(\mathcal{D})) &\to \Gamma(\mathrm{Loc}(\mathcal{D}), \mathcal{O}_Y) \\ f\chi^u &\mapsto \begin{cases} f & \text{if } u = 0 \\ 0 & \text{otherwise} \end{cases} \end{aligned} \tag{2.8}$$

Because each $\{\mathrm{Loc}(\mathcal{D}) \mid \mathrm{tail}(\mathcal{D}) = \sigma\}$ is affine, these glue into a map

$$s_\sigma : Y \hookrightarrow TV(\mathcal{S})$$

where we used the fact that $\mathcal{S}$ is complete (so $|\mathcal{S}_y| = N_{\mathbb{Q}}$ for all y) to deduce that Y is covered by $\{\mathrm{Loc}(\mathcal{D}) \mid \mathrm{tail}(\mathcal{D}) = \sigma\}$. The map s_σ factors through $\widetilde{TV}(\mathcal{S})$. By carefully following the isomorphism between the fibers of π and the corresponding toric bouquets (as in the proof of Proposition II.14), we see that the image of s_σ indeed equals the horizontal curve $p(\widetilde{C}_\sigma)$.

We can use this parametrization to find an intersection formula for T-invariant divisors and horizontal curves; fix a cone σ of $\mathrm{tail}(\mathcal{S})$ of full dimension and a T-invariant Cartier divisor D with Cartier support function $\{h_{\mathcal{D},y}\}$. Because σ has full dimension, there is a unique $u_\sigma \in M$ and collection of integers $\{a_y \in \mathbb{Z}\}_{y \in Y}$ such that for each $\mathcal{D}$ with tailcone σ,

$$h_{y,\mathcal{D}}(v) = -a_y - \langle u_\sigma, v \rangle$$

We can find Cartier data for D whose open sets and rational functions are of the form

$$(TV(\mathcal{D}\big|_U), f_{\mathcal{D},U}\chi^{u_\sigma})$$

for open sets $U \subseteq Y$. Then $\operatorname{ord}_y(f_{\mathcal{D},U}) = a_y$ for all $\mathcal{D}$ with tailcone σ and $y \in U$. When σ is unmarked, the open sets U appearing in the Carter data are affine, and the pullback of the transition function $f_{\mathcal{D},U} f_{\mathcal{D}',U'}^{-1}\chi^0$ onto the curve $C_\sigma \cong Y$ is the function $f_{\mathcal{D},U} f_{\mathcal{D}',U'}^{-1}$ on $U \cap U'$. That is, the functions $f_{\mathcal{D},U}$ appearing in the Cartier data for D are themselves the Cartier data for the pullback of D onto $C_\sigma \cong Y$. As a Weil divisor, the pullback of D onto C_σ is $\sum a_y[y]$; we call this divisor D_σ.

Definition II.17. Given a $\mathbb{Q}$-Cartier support function $\{h_{\mathcal{D},y}\}$, a cone σ of full dimension in $\operatorname{tail}(\mathcal{S})$, and a point y, there is a unique $a_y \in \mathbb{Z}$ such that for every $\mathcal{D}$ with tailcone σ and $\operatorname{Loc}(\mathcal{D}) \ni y$,

$$h_{\mathcal{D},y} = -a_y - \operatorname{lin}(h_{\mathcal{D},y}).$$

Then define

$$D_\sigma = \sum_{y \in Y} a_y[y]$$

Remark II.18. The definition of D_σ makes sense even when σ is marked. However, if $\mathcal{D}$ has complete locus, then by ([PS11], Proposition 3.1) every invariant Cartier divisor on $TV(\mathcal{D})$ is principal. It follows that $\deg(D_\sigma) = 0$ for every marked σ.

Remark II.19. Compare this definition to ([PS11], Definition 3.26). In our notation, $D_\sigma = -h\big|_\sigma(0)$.

With this new definition, we can summarize the discussion above with the following equation for the intersection theory of a T-invariant divisor with a horizontal curve.

$$\langle D, C_\sigma \rangle = \deg(D_\sigma)$$

By linearity, the same formula applies when D is a T-invariant $\mathbb{Q}$-Cartier divisor.

2.3 The T cone theorem

Given a normal variety X, let $Z_1(X)$ be the proper 1-cycles, and define

$$N^1(X) := (\mathrm{CaDiv}(X)/\sim) \otimes_{\mathbb{Z}} \mathbb{R} \qquad N_1(X) := (Z_1(X)/\sim) \otimes_{\mathbb{Z}} \mathbb{R}$$

where $\sim$ denotes numerical equivalence of divisors in the first definition, and numerical equivalence of curves in the second. The vector space $N^1(X)$ contains the cone $\mathrm{Nef}(X)$ generated by classes of nef divisors, and the vector space $N_1(X)$ contains the cone $NE(X)$ generated by classes of irreducible complete curves. The *Mori cone* $\overline{NE}(X)$ is the closure of $NE(X)$. With respect to the intersection product, $N_1(X)$ and $N^1(X)$ are dual vector spaces, and the cones $\mathrm{Nef}(X), \overline{NE}(X)$ are dual cones.

When X is the toric variety of a fan Σ, the closure of the torus orbit corresponding to a wall of Σ defines an element of $\overline{NE}(X)$. The celebrated toric cone theorem ([CLS11], Theorem 6.3.20(b)) states that $\overline{NE}(X)$ is generated as a cone by these classes. In this section, we prove the corresponding result for T-varieties. We continue to assume that all T-varieties are complete complexity-one T-varieties over a projective curve Y.

Theorem II.20. *Let $TV(\mathcal{S})$ be an n-dimensional T-variety, and let $y' \in Y$ be any point for which $\mathcal{S}_{y'} = \mathrm{tail}(\mathcal{S})$. Then*

$$\overline{NE}(TV(\mathcal{S})) = \sum_{\substack{y\in Y \\ \tau \text{ a wall of } \mathcal{S}_y \\ \dim(\mathrm{tail}(\tau))<n-1}} \mathbb{R}_{\geq 0}[C_{\tau,y}] + \sum_{\substack{\tau \text{ a wall} \\ \text{of } \mathrm{tail}(\mathcal{S})}} \mathbb{R}_{\geq 0}[C_{\tau,y'}] + \sum_{\substack{\sigma\in\mathrm{tail}(\mathcal{S}) \\ \dim(\sigma)=n-1 \\ \sigma \text{ unmarked}}} \mathbb{R}_{\geq 0}[C_\sigma] \tag{2.9}$$

For the proof, we review two important facts about divisors on T-varieties, Propositions II.21 and II.22.

Proposition II.21. Any Cartier divisor D on a T-variety $TV(\mathcal{S})$ is linearly equivalent to a T-invariant Cartier divisor.

Different authors have different definitions of concavity; to us, a function $\varphi : N_{\mathbb{Q}} \to \mathbb{Q}$ is concave if $\varphi(tv + (1-t)w) \geq t\varphi(v) + (1-t)\varphi(w)$ for all $v, w \in N_{\mathbb{Q}}$ and all $t \in [0, 1]$

Proposition II.22. ([PS11], Corollary 3.29) A T-Cartier divisor $D \in T-CaDiv(\mathcal{S})$ with Cartier support function $\{h_{\mathcal{D},y}\}$ is nef iff all h_y are concave and $\deg(D_\sigma) \geq 0$ for every maximal cone σ of the tailfan.

Toward our goal of proving Theorem II.20, we will use Proposition II.22 to show that a Cartier divisor is nef if it intersects all vertical and horizontal curves nonnegatively. The proof of this fact requires a combinatorial lemma. Given a Cartier support function $\{h_{\mathcal{D},y}\}$ and any $\mathcal{D} \in \mathcal{S}, y \in Y$ such that $\dim(\mathcal{D}_y) = n-1$, define $\widetilde{h}_{\mathcal{D},y} : N_{\mathbb{Q}} \to \mathbb{Q}$ to be the unique affine function that extends $h_{\mathcal{D},y} : |\mathcal{D}_y| \to \mathbb{Q}$.

Lemma II.23. *Let $\{h_{\mathcal{D},y}\}$ be a Cartier support function. The following are equivalent*

- $h_y : N_{\mathbb{Q}} \to \mathbb{Q}$ *is concave.*
- *For every wall $\tau = \mathcal{D}_y \cap \mathcal{D}'_y$ of $\mathcal{S}_y$, there is some $v \in \mathcal{D}'_y \backslash \mathcal{D}_y$ with $h_{\mathcal{D}',y}(v) \leq \widetilde{h}_{\mathcal{D},y}(v)$.*

Proof. This is a straightforward extension of ([CLS11], Lemma 6.1.5 $(a) \iff (d)$) (where it is proved for Cartier support functions on a fan). $\square$

Proposition II.24. A Cartier divisor $D \in \mathrm{CaDiv}(TV(\mathcal{S}))$ is nef iff $\langle D, C\rangle \geq 0$ for all vertical and horizontal curves C.

Proof. The forward direction follows from the definition of nef. To prove the reverse direction, let $D \in \mathrm{CaDiv}(\mathcal{S})$ satisfy the condition that $\langle D, C\rangle \geq 0$ for all vertical and horizontal curves. Replace D with a linearly equivalent T-invariant divisor and let $\{h_{\mathcal{D},y}\}$ be its Cartier support function. Let $\tau = \mathcal{D}_y \cap \mathcal{D}'_y$ be a wall of $\mathcal{S}_y$. Fix any $n_{\tau,\mathcal{D}} \in N$ with $\langle n_{\tau,\mathcal{D}}, u_{\tau,\mathcal{D}}\rangle = \mu_\tau$ and any $v_\tau \in \mathrm{relint}(\tau)$. Then pick $\epsilon > 0$ such that $v := v_\tau + \epsilon n_{\tau,\mathcal{D}} \in \mathcal{D}_y \backslash \mathcal{D}'_y$. Then

$$h_{\mathcal{D},y}(v) = h_{\mathcal{D},y}(v_\tau) + \mathrm{lin} h_{\mathcal{D},y}(\epsilon n_{\tau,\mathcal{D}})$$
$$\widetilde{h}_{\mathcal{D}',y}(v) = h_{\mathcal{D}',y}(v_\tau) + \mathrm{lin} h_{\mathcal{D}',y}(\epsilon n_{\tau,\mathcal{D}}).$$

Because $h_{\mathcal{D},y}$ and $h_{\mathcal{D}',y}$ agree on τ,

$$\widetilde{h}_{\mathcal{D}',y}(v) - h_{\mathcal{D},y}(v) = (\mathrm{lin} h_{\mathcal{D}',y} - \mathrm{lin} h_{\mathcal{D},y})(\epsilon n_{\tau,\mathcal{D}}) \geq 0$$

where the final inequality comes from applying Equation 2.7 to the fact that $\langle D, C_{\tau,y}\rangle \geq 0$. Because this holds for all walls in all slices $\mathcal{S}_y$, we conclude by Lemma II.23 that each h_y is concave.

To show that $\deg(D_\sigma) \geq 0$ for every maximal cone σ of the tailfan, observe that if σ is marked, then $\deg(D_\sigma) = 0$ by Remark II.18; if σ is unmarked, then $\deg D_\sigma = \langle D, C_\sigma\rangle \geq 0$. □

To put Proposition II.24 in context, remember that a T-variety has infinitely many distinct vertical curves. Indeed, if τ is a wall of $\mathrm{tail}(\mathcal{S})$, then for every $y \in Y$ there is (by completeness) a vertical curve $C_{\tau',y}$ where τ' is a wall of $\mathcal{S}_y$ with tailcone τ. The next proposition shows that the classes of all such curves lie on a single ray of $N_1(TV(\mathcal{S}))$.

Proposition II.25. Let $\tau = \sigma \cap \sigma'$ be a wall of $\mathrm{tail}(\mathcal{S})$, where σ, σ' are full dimensional cones of $\mathrm{tail}(\mathcal{S})$. The classes

$$\mathcal{C}_\tau = \left\{ [C_{\tau',y}] \;\middle|\; \begin{array}{c} \tau' = \mathcal{D}_y \cap \mathcal{D}'_y \text{ for some } \mathcal{D}, \mathcal{D}' \text{ with} \\ \mathrm{tail}(\mathcal{D}) = \sigma, \mathrm{tail}(\mathcal{D}') = \sigma' \end{array} \right\} \subseteq N_1(TV(\mathcal{S}))$$

are positive multiples of each other. Specifically,

$$[C_{\tau_1,y_1}] = \mu_{\tau_1}^{-1}\mu_{\tau_2}[C_{\tau_2,y_2}]$$

for $[C_{\tau_1,y_1}], [C_{\tau_2,y_2}] \in \mathcal{C}_\tau$.

Proof. For any $D \in \text{T-CaDiv}(TV(\mathcal{S}))$ with Cartier support function $\{h_{\mathcal{D},y}\}$, all $h_{\mathcal{D},y}$ with $\mathrm{tail}(\mathcal{D}) = \sigma$ (respectively σ') will have the same linear part, say $-u_\sigma \in M_\mathbb{Q}$ (respectively $-u_{\sigma'} \in M_\mathbb{Q}$). Then for two classes $[C_{\tau_1,y_1}], [C_{\tau_2,y_2}] \in \mathcal{C}_\tau$, Equation 2.7 calculates the intersections as

$$\langle D, C_{\tau_1,y_1}\rangle = \mu_{\tau_1}^{-1}\langle u_\sigma - u_{\sigma'}, n_{\tau_1,\mathcal{D}}\rangle \qquad \langle D, C_{\tau_2,y_2}\rangle = \mu_{\tau_2}^{-1}\langle u_\sigma - u_{\sigma'}, n_{\tau_2,\mathcal{D}}\rangle$$

Since we can choose $n_{\tau_1,\mathcal{D}} = n_{\tau_2,\mathcal{D}}$, it follows that $\langle D, C_{\tau_1,y_1}\rangle = \mu_{\tau_1}^{-1}\mu_{\tau_2}\langle D, C_{\tau_2,y_2}\rangle$ for all D. □

We are finally ready to prove Theorem II.20. Using the propositions above, the proof is nearly identical to the proof of the toric cone theorem in ([CLS11], Theorem 6.3.20(b)).

Proof. (Theorem II.20) Let Γ be the rational polyhedral cone in $NE(TV(\mathcal{S}))$ defined by the right hand side of Equation 2.9. By definition, Γ includes all horizontal curves; by Proposition II.25, it also includes all vertical curves. Therefore, Proposition II.24 implies that $\Gamma^\vee = \mathrm{Nef}(TV(\mathcal{S}))$, so $\Gamma = \Gamma^{\vee\vee} = \overline{NE}(TV(\mathcal{S}))$. □

2.4 Examples

2.4.1 Example 1

Consider the divisoral fan $\mathcal{S}$ shown in Figure 2.5. $TV(\mathcal{S})$ is the projectivized cotangent bundle of the first Hirzebruch surface. All horizontal divisors[5] in $\widetilde{TV}(\mathcal{S})$ are contracted. For each vertical divisor $D_{[y],v}$ and each maximal p-divisor $\mathcal{D}_i \in \mathcal{S}$, we write the Weil divisor $\sum a_y[y]$ and an element $u \in M$ in Table 2.1 to encode the Cartier support function $\{h_{\mathcal{D}_i,y}(w) = -a_y - \langle u, w\rangle\}$ of $D_{[y],v}$. For example, the Cartier support function for $D_{[0],(0,0)}$ includes $h_{\mathcal{D}_4,\infty}(v) = 1 - \langle(-2,-1), v\rangle$.

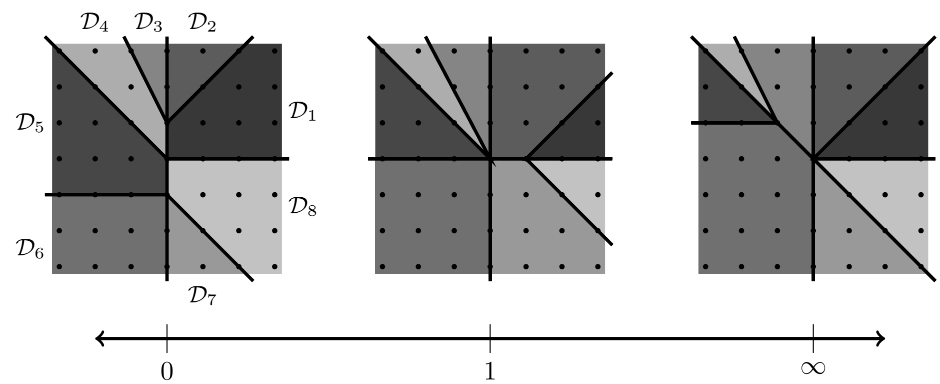

Figure 2.5: The divisorial fan $\mathcal{S}$

	$\mathcal{D}_1$	$\mathcal{D}_2$	$\mathcal{D}_3$	$\mathcal{D}_4$	$\mathcal{D}_5$	$\mathcal{D}_6$	$\mathcal{D}_7$	$\mathcal{D}_8$
$D_{[0],(0,1)}$	0 $(0,1)$	0 $(0,1)$	0 $(1,1)$	0 $(1,1)$	0 $(0,0)$	0 $(0,0)$	0 $(0,0)$	0 $(0,0)$
$D_{[0],(0,0)}$	$[0]-[1]$ $(1,-1)$	0 $(0,0)$	0 $(0,0)$	$[0]-[\infty]$ $(-2,-1)$	$[0]-[\infty]$ $(0,1)$	0 $(0,0)$	0 $(0,0)$	$[0]-[1]$ $(1,1)$
$D_{[0],(0,-1)}$	0 $(0,0)$	0 $(0,0)$	0 $(0,0)$	0 $(0,0)$	0 $(-1,-1)$	0 $(-1,-1)$	0 $(0,-1)$	0 $(0,-1)$
$D_{[1],(1,0)}$	0 $(1,0)$	0 $(1,0)$	0 $(0,0)$	0 $(0,0)$	0 $(0,0)$	0 $(0,0)$	0 $(1,0)$	0 $(1,0)$
$D_{[1],(0,0)}$	0 $(0,0)$	$[1]-[0]$ $(-1,1)$	$[1]-[0]$ $(1,1)$	$[1]-[\infty]$ $(-1,0)$	$[1]-[\infty]$ $(-1,0)$	$[1]-[0]$ $(-1,-1)$	$[1]-[0]$ $(-1,-1)$	0 $(0,0)$
$D_{[\infty],(-1,1)}$	0 $(0,0)$	0 $(0,0)$	0 $(-1,0)$	0 $(-1,0)$	0 $(-1,0)$	0 $(-1,0)$	0 $(0,0)$	0 $(0,0)$
$D_{[\infty],(0,0)}$	$[\infty]-[1]$ $(1,0)$	$[\infty]-[0]$ $(0,1)$	$[\infty]-[0]$ $(2,1)$	0 $(0,0)$	0 $(0,0)$	$[\infty]-[0]$ $(0,-1)$	$[\infty]-[0]$ $(0,-1)$	$[\infty]-[1]$ $(1,0)$

Table 2.1: Torus invariant divisors on $TV(\mathcal{S})$

[5]See [PS11] for a definition and description of horizontal and vertical divisors

Because every maximal-dimensional cone of $\mathrm{tail}(\mathcal{S})$ is marked, $TV(\mathcal{S})$ has no horizontal curves. Let $\tau_{i,j,y}$ be the wall of $\mathcal{S}_y$ realized as the intersection between $\mathcal{D}_i$ and $\mathcal{D}_j$ (if such a wall exists). Using Proposition II.25, we see that the numerical equivalence class of $C_{\tau_{i,j,y},y}$ only depends on i and j; to save space, we abbreviate $C_{\tau_{i,j,y},y}$ as $C_{i,j}$.

As an example of a calculation, consider the curve $C_{1,2}$ and the T-invariant divisor $D_{[0],(0,0)}$ with Cartier support function $\{h_{\mathcal{D},y}\}$. Using notation from Section 2.2.1, $n_{\tau,\mathcal{D}_2} = (0,1)$. The relevant linear parts of the Cartier support function are $\mathrm{lin} h_{\mathcal{D}_1,0} = -(1,-1) \in M$ and $\mathrm{lin} h_{\mathcal{D}_2,0} = (0,0) \in M$. The intersection can then be calculated using Equation 2.7

$$\langle D_{[0],(0,0)}, C_{1,2} \rangle = 1^{-1} \langle (-1,1) - (0,0), (0,1) \rangle = 1$$

The complete list of intersections is in Table 2.2. The canonical divisor is also listed; it can be expressed as a sum of the vertical divisors using the formula from ([PS11], Theorem 3.21).

	$C_{1,2}$	$C_{2,3}$	$C_{3,4}$	$C_{4,5}$	$C_{5,6}$	$C_{6,7}$	$C_{7,8}$	$C_{8,1}$	$C_{1,4}$	$C_{5,8}$	$C_{2,7}$	$C_{3,6}$
$D_{[0],(0,1)}$	0	-1	0	1	0	0	0	1	-1	0	1	1
$D_{[0],(0,0)}$	1	0	1	-2	1	0	1	-2	3	1	0	0
$D_{[0],(0,-1)}$	0	0	0	1	0	1	0	1	0	1	1	1
$D_{[1],(1,0)}$	0	1	0	0	0	1	0	0	1	1	0	0
$D_{[1],(0,0)}$	1	-2	1	0	1	0	1	0	1	1	2	2
$D_{[\infty],(-1,1)}$	0	1	0	0	0	1	0	0	1	1	0	0
$D_{[\infty],(0,0)}$	1	-2	1	0	1	0	1	0	1	1	2	2
K_X	-2	2	-2	0	-2	-2	-2	0	-4	-4	-4	-4

Table 2.2: Intersections of divisors and curves on $TV(\mathcal{S})$

2.4.2 Example 2

Let $\sigma_1, \sigma_2, \sigma_3$ be the cones

$$\sigma_1 = \mathbb{Q}_{\geq 0} \cdot (1,0) + \mathbb{Q}_{\geq 0} \cdot (0,1)$$

$$\sigma_2 = \mathbb{Q}_{\geq 0} \cdot (0,1) + \mathbb{Q}_{\geq 0} \cdot (-1,-1)$$

$$\sigma_3 = \mathbb{Q}_{\geq 0} \cdot (1,0) + \mathbb{Q}_{\geq 0} \cdot (-1,-1))$$

and let $\mathcal{S}$ be the divisorial fan on $\mathbb{P}^1$ having the following maximal p-divisors

$$\mathcal{D}_1 = ((2/3, 1/2) + \sigma_1)[0] + ((-2/3, -1/2) + \sigma_1)[1] + \emptyset[\infty]$$

$$\mathcal{D}_2 = ((2/3, 1/2) + \sigma_2)[0] + ((-2/3, -1/2) + \sigma_2)[1] + ((-1,-1) + \sigma_2)[\infty]$$

$$\mathcal{D}_3 = ((2/3, 1/2) + \sigma_3)[0] + ((-2/3, -1/2) + \sigma_3)[1] + ((-1,-1) + \sigma_3)[\infty]$$

$$\mathcal{D}_4 = \emptyset[0] + \emptyset[1] + ((-1,-1) + \sigma_1)[\infty]$$

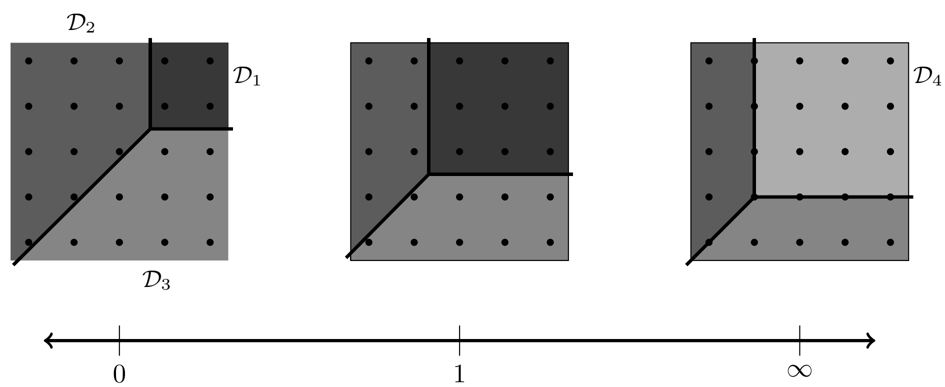

Figure 2.6: The divisorial fan $\mathcal{S}$

The T-variety corresponding to $\mathcal{S}$ is a deformation of $\mathbb{P}^3$. The T-invariant divisors and their intersections with T-invariant curves are encoded in Table 2.3 and 2.4 respectively, using the same notation as in the previous example.

	$\mathcal{D}_1$	$\mathcal{D}_2$	$\mathcal{D}_3$	$\mathcal{D}_4$
$D_{[0],(2/3,1/2)}$	$1/6[0]$ $(0,0)$	$5/18[0]-1/9[1]-1/6[\infty]$ $(-1/6,0)$	$1/4[0]-1/12[1]-1/6[\infty]$ $(0,-1/6)$	0 $(0,0)$
$D_{[1],(-2/3,-1/2)}$	$1/6[1]$ $(0,0)$	$1/9[0]+1/18[1]-1/6[\infty]$ $(-1/6,0)$	$1/12[0]+1/12[1]-1/6[\infty]$ $(0,-1/6)$	0 $(0,0)$
$D_{[\infty],(-1,-1)}$	0 $(0,0)$	$2/3[0]-2/3[1]$ $(-1,0)$	$1/2[0]-1/2[1]$ $(0,-1)$	$[\infty]$ $(0,0)$
$D_{\mathbb{Q}_{\geq 0}\cdot(1,0)}$	$-2/3[0]+2/3[1]$ $(0,0)$	0 $(-1/6,0)$	$-1/6[0]+1/6[1]$ $(0,-1/6)$	$[\infty]$ $(0,0)$
$D_{\mathbb{Q}_{\geq 0}\cdot(0,1)}$	$-1/2[0]+1/2[1]$ $(0,1)$	$1/6[0]-1/6[1]$ $(-1,1)$	0 $(0,0)$	$[\infty]$ $(0,1)$

Table 2.3: Torus invariant divisors on $TV(\mathcal{S})$

	$C_{\tau_{1,2}}$	$C_{\tau_{2,3}}$	$C_{\tau_{1,3}}$	C_{σ_1}
$D_{[0],(2/3,1/2)}$	$1/6$	$1/6$	$1/6$	$1/6$
$D_{[1],(-2/3,-1/2)}$	$1/6$	$1/6$	$1/6$	$1/6$
$D_{[\infty],(-1,-1)}$	1	1	1	1
$D_{\mathbb{Q}_{\geq 0}\cdot(1,0)}$	1	1	1	1
$D_{\mathbb{Q}_{\geq 0}\cdot(0,1)}$	1	1	1	1

Table 2.4: Intersections on $TV(\mathcal{S})$

CHAPTER III

b-symplectic Toric Manifolds

3.1 Introduction

To motivate the study of b-symplectic manifolds, the main object of study in Chapter III, we remind the reader of the definition of a Poisson manifold.

Definition III.1. A **Poisson manifold** is a smooth manifold M with a bilinear operation

$$\{\cdot,\cdot\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$$
$$f, g \mapsto \{f, g\}$$

satisfying the following conditions

- $\{f, g\} = -\{f, g\}$
- $\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$
- $\{fg, h\} = f\{g, h\} + g\{f, h\}$

The **Poisson bivector** of a Poisson manifold $(M, \{\cdot,\cdot\})$ is the unique bivector $\Pi \in \Gamma(M, \wedge^2 TM)$ such that

$$\langle \Pi, df \wedge dg \rangle = \{f, g\}$$

for all smooth functions f and g.

In the same way that a symplectic form on M defines an isomorphism $TM \to T^*M$ by contracting the symplectic form with a vector, so too does a Poisson bivector define a map $T^*M \to TM$ by contracting the Poisson bivector with a covector. Unlike the symplectic case, this map is not necessarily an isomorphism. In the Poisson case, the image of $T^*M \to TM$ defines an integrable distribution. The singular foliation it defines is called the *symplectic foliation*, and the *rank* of a Poisson bivector Π at p is the dimension of $\mathrm{im}(T_p^*M \to T_pM)$. The Poisson bivector induces a canonical symplectic form on each leaf of the symplectic foliation; one way to think of a Poisson manifold is as a collection of symplectic manifolds of different dimensions glued together.

The main difficulty in studying Poisson manifolds lies in the exotic kinds of behavior that can occur when the Poisson bivector drops rank. With this in mind, the simplest Poisson manifolds are those that have only a single leaf. This occurs precisely when M is even dimensional and Π has full rank (so Π is dual to a symplectic form), which occurs when the top exterior power of Π is a nonvanishing section of $\bigwedge^{\dim(M)}(TM)$. The next simplest case are the *stable* Poisson manifolds, where this top exterior power intersects the zero section transversely along a closed embedded hypersurface.

Definition III.2. A Poisson bivector on M^{2n} is **stable** if $\Pi^n = \Pi \wedge \cdots \wedge \Pi$ vanishes transversally. The hypersurface $\{\Pi^n = 0\}$ is called the **critical hypersurface** or **singular hypersurface**.

Example III.3. On $M = \mathbb{R}^2$, $\Pi = y\partial_y \wedge \partial_x$ is a stable Poisson structure. On $M = \mathbb{R}^4$, $\Pi = y\partial_y \wedge \partial_{x_1} + \partial_{x_2} \wedge \partial_{x_3}$ is a stable Poisson structure. Let (M, ω) be a symplectic surface, $f \in C^\infty(M)$ transverse to 0, then $\Pi = f\Pi_{\mathrm{symp}}$ is a stable Poisson structure, where Π_{symp} is the Poisson bivector dual to the symplect form ω.

Away from the critical hypersurface, a stable Poisson structure has maximal rank and is therefore dual to a symplectic form. The behavior of the bivector along Z corresponds to a particularly tame order-one singularity along Z of this dual symplectic form. In fact, given a fixed manifold M and hypersurface Z, we can construct a bundle on M whose global sections are precisely differential forms with these types of tame singularities along Z; then, a stable Poisson structure on M having singluar hypersurface Z dualizes to a *global* section of this new bundle, called a b-symplectic form. Many techniques and results from symplectic geometry, such as Moser's trick and Darboux's theorem generalize to these b-symplectic forms, allowing us to use symplectic geometric techniques to prove theorems about stable Poisson structures. Motivated in this way, we begin by reviewing the basic objects and results of b-geometry, which are introduced in detail in [GMP11].

Definition III.4. A **b-manifold** is a pair (M, Z) of a smooth oriented manifold M and a closed embedded hypersurface $Z \subseteq M$. A **b-map** from (M, Z) to (M', Z') is a map $\varphi : M \to M'$ such that $\varphi^{-1}(Z') = Z$ and φ is transverse to Z'.

Definition III.5. A **b-vector field** on (M, Z) is a vector field v on M such that $v_p \in T_pZ$ for all $p \in Z$.

Definition III.6. The **b-tangent bundle** ${}^bTM$ on (M, Z) is the vector bundle whose sections are the b-vector fields on (M, Z). The **b-cotangent bundle** ${}^bT^*M$ is the dual bundle of ${}^bTM$. The smooth sections of $\Lambda^k({}^bT^*M)$ are called **b-de Rham k-forms** or simply **b-forms**. The space of all such forms is written ${}^b\Omega^k(M)$.

The restriction of any b-form to $M\backslash Z$ is a classic differential form on $M\backslash Z$, and there is a differential $d : {}^b\Omega^k(M) \to {}^b\Omega^{k+1}(M)$ that extends the classic differential on $M\backslash Z$. With respect to this differential, we extend the standard definitions of closed and exact differential forms to *closed* b-forms and *exact* b-forms. A *b-symplectic* form is a closed b-form of degree 2 that has maximal rank (as a section of $\Lambda^2({}^bT^*M)$) at every point of M. A **b-symplectic manifold** consists of the data of a b-manifold (M, Z) together with a b-symplectic form ω. A *b-symplectomorphism* between two b-symplectic manifolds (M, ω) and (M', ω') is a b-map $\varphi : M \to M'$ such that $\varphi^*\omega' = \omega$.

Although a b-form can be thought of as a differential form with a singularity along Z, the singularity is so tame that it is even possible to define the integral of a form of top degree by taking its principal value near Z.

Definition III.7. For any b-form $\eta \in {}^b\Omega^n(M)$ on a n-dimensional b-manifold and any local defining function y of Z, the **Liouville Volume** of η is

$$ {}^b\!\!\int_M \eta := \lim_{\varepsilon \to 0} \int_{M \setminus \{-\varepsilon \le y \le \varepsilon\}} \eta $$

The fact that the limit in Definition III.7 exists and is independent of y is explained in [Rad02] (for surfaces) and [Sco13] (in the general case). Similarly, if $i_N : N \subseteq M$ is a k-dimensional submanifold transverse to Z, it inherits from M a b-manifold structure $(N, i_N^{-1}(Z))$ and for any $\eta \in {}^b\Omega^k(M)$, we define

$$ {}^b\!\!\int_N \eta := {}^b\!\!\int_N i_N^*\eta. $$

In [GMP13], the authors prove that every b-form $\eta \in {}^b\Omega^p(M)$ can be written in a neighborhood of $Z = \{y = 0\}$ as

$$ \eta = \frac{dy}{y} \wedge \alpha + \beta $$

for smooth forms $\alpha \in \Omega^{p-1}(M)$ and $\beta \in \Omega^p(M)$. Although the forms α and β in this expression are not unique, the pullback $i_Z^*(\alpha)$ is unique, where i_Z is the inclusion $Z \subseteq M$. The resulting differential form on Z admits an alternative description: if

v is a vector field on M such that $dy(v)\big|_Z = 1$, then the vector field $\mathbb{L} := yv$ is a b-vector field, $\mathbb{L}\big|_Z$ is independent of the choices of v and y, the b-form $\iota_{\mathbb{L}}\eta$ is a smooth form, and $i_Z^*(\alpha) = i_Z^*\iota_{\mathbb{L}}\eta$. For this reason, we adopt the notation $\iota_{\mathbb{L}}\eta$ for this smooth $(p-1)$-form on Z.

Every b-symplectic form dualizes to a stable Poisson bivector ([GMP13]). The symplectic foliation corresponding to this Poisson structure consists of a symplectic leaf for each component of $M \backslash Z$, and a codimension-one symplectic foliation of Z itself. One important tool in the study of the geometry of this foliated hypersurface is the *modular vector field* on M. We review its definition.

Definition III.8. Fix a volume form Ω on a b-symplectic manifold. The **modular vector field** $v_{\mathrm{mod}}^{\Omega}$ on M (or simply v_{mod} if Ω is clear from the context) is the vector field defined by the derivation

$$f \mapsto \frac{\mathcal{L}_{u_f}\Omega}{\Omega},$$

where u_f is the Hamiltonian vector field of the smooth function f on M defined by $df = \iota_{u_f}\omega$.

Although the modular vector field depends on Ω, different choices of Ω yield modular vector fields that differ by Hamiltonian vector fields. On a b-symplectic manifold, the modular vector field is tangent to the exceptional hypersurface Z and its flow preserves the symplectic foliation of Z, and Hamiltonian vector fields are tangent to the symplectic foliation.[1] In fact, in [GMP13] it is shown that corresponding to each modular vector field v_{mod} and compact leaf $\mathcal{L}$ of a component Z' of Z, there is a $k \in \mathbb{R}_{>0}$ and a symplectomorphism $f : \mathcal{L} \to \mathcal{L}$ such that Z' is the mapping torus

$$\frac{\mathcal{L} \times [0,k]}{(\ell, 0) \sim (f(\ell), k)}$$

[1] The reader should be aware that we will soon change our definition of "Hamiltonian vector fields" and this will no longer be true.

and the time-t flow of v_{mod} is translation by t in the second coordinate. The number k, which depends only on the choice of component $Z' \subseteq Z$, is called the *modular period* of Z'. This definition generalizes the one given in [Rad02] for b-symplectic surfaces. Intuitively, the modular period of Z' is the time required for the modular vector field to flow a leaf of the foliation of Z' the entire way around the $\mathbb{S}^1$ base of the mapping torus.

Let $\mathcal{F}$ be the symplectic foliation induced by ω on Z, and for each symplectic leaf $\mathcal{L}$ let $i_{\mathcal{L}} : \mathcal{L} \hookrightarrow Z$ be the inclusion. A *defining one-form* for $\mathcal{F}$ (or more simply, for Z) is an $\alpha \in \Omega^1(Z)$ such that $\ker(\alpha_z) = T_z\mathcal{L} \subseteq T_zZ$ for each $z \in Z$. The authors of [GMP13] prove that $\iota_{\mathbb{L}}\omega$ is the unique defining one-form for Z that is both closed and satisfies $\alpha(v_{\text{mod}}) = 1$ for every modular vector field.

A *defining two-form* for Z is a non-vanishing $\beta \in \Omega^1(Z)$ such that $i_{\mathcal{L}}^*\beta$ is the symplectic form induced by ω on the leaf $\mathcal{L}$. We may always choose a defining two-form that is closed and satifies $\iota_{v_{\text{mod}}}\beta = 0$.

Not all closed b-forms on a b-manifold are locally exact. For example, if y is a local defining function for Z, then $\frac{dy}{y}$ is closed, but it is not exact in any neighborhood of any point of Z. Poincaré's lemma is such a fundamental property of the (smooth) de Rham complex that we are motivated to enlarge the sheaf C^∞ on a b-manifold to include functions such as $\log|y|$ so that we have a Poincaré lemma in b-geometry.

Definition III.9. Let (M, Z) be a b-manifold. The sheaf[2] ${}^bC^\infty$ is defined by

$$
{}^bC^\infty(U) := \left\{ c\log|y| + f \;\middle|\; \begin{array}{l} c \in \mathbb{R} \\ y \text{ is any defining function for } U \cap Z \subseteq U \\ f \in C^\infty(U) \end{array} \right\}
$$

[2]Some authors require the hypersurface of a b-manifold to have a *global* defining function; other authors do not. If no global defining function for Z exists (for example, if Z is a meridian of $\mathbb{T}^2$), then this definition yields only a presheaf and ${}^bC^\infty$ is defined as its sheafification.

Global sections of ${}^bC^\infty$ are called **b-functions**.

Replacing C^∞ with ${}^bC^\infty$ also enlarges the possible Hamiltonian torus actions on b-manifolds. For example, the action of $\frac{\partial}{\partial\theta}$ on $(\mathbb{S}^2, \{h=0\}, \frac{dh}{h}\wedge d\theta)$ is generated by the Hamiltonian function $-\log|h| \in {}^bC^\infty(M)$, but is not generated by any function in $C^\infty(M)$. In fact, in Corollary III.36 we show that there are no examples of effective Hamiltonian $\mathbb{T}^n$-actions on $2n$-dimensional b-symplectic manifolds with all their Hamiltonians in $C^\infty(M)$ except those with $Z=\emptyset$. We prove a simple relationship between the modular period and b-functions which will be useful in later sections.

Proposition III.10. Let (M, Z, ω) be a b-symplectic manifold and let Z' be a connected component of Z with modular period k. Let $\pi : Z' \to \mathbb{S}^1 \cong \mathbb{R}/k$ be the projection to the base of the corresponding mapping torus. Let $\gamma : \mathbb{S}^1 = \mathbb{R}/k \to Z'$ be any loop with the property that $\pi\circ\gamma$ is the positively-oriented loop of constant velocity 1. The following numbers are equal.

- The modular period of Z'.
- $\int_\gamma \iota_{\mathbb{L}}\omega$.
- The value of $-c$ for any ${}^bC^\infty$ function $H = c\log|y| + f$ in a neighborhood of Z' such that the corresponding Hamiltonian X_H has 1-periodic orbits homotopic in Z' to some γ.

Proof. Recall from [GMP13] that $\iota_{\mathbb{L}}\omega(v_{\text{mod}})$ is the constant function 1. Let $s : [0,k] \to Z'$ be a trajectory of the modular vector field. Because the modular period is k, it follows that $s(0)$ and $s(k)$ are in the same leaf $\mathcal{L}$ of the foliation. Let $\hat{s} : [0, k+1] \to Z'$ be a smooth extension of s such that $s\big|_{[k,k+1]}$ is a path in $\mathcal{L}$ joining

$\hat{s}(k) = s(k)$ to $\hat{s}(k+1) = s(0)$, making $\hat{s}$ a loop. Then

$$k = \int_0^k 1dt = \int_s \iota_{\mathbb{L}}\omega = \int_{\hat{s}} \iota_{\mathbb{L}}\omega = \int_\gamma \iota_{\mathbb{L}}\omega.$$

This shows that the first two numbers are equal.

Next, let $r : [0,1] \mapsto Z'$ be a trajectory of X_H, and notice that X_H satisfies $\iota_{X_H}\omega = c\frac{dy}{y} + df$. Let $y\frac{\partial}{\partial y}$ be a representative of $\mathbb{L}$. Because X_H is 1-periodic and homotopic to γ, it follows from the previous computation that

$$k = \int_r \iota_{\mathbb{L}}\omega = \int_0^1 \iota_{y\frac{\partial}{\partial y}}\omega(X_H\big|_{r(t)})dt = \int_0^1 -(c\frac{dy}{y} + df)(y\frac{\partial}{\partial y})\Big|_{r(t)} dt = -c$$

completing the proof. $\square$

3.1.1 Hamiltonian actions on symplectic and b-symplectic manifolds.

Let G be a compact connected Lie group which acts on a symplectic manifold M by symplectomorphisms, and denote by $\mathfrak{g}$ and $\mathfrak{g}^*$ its Lie algebra and corresponding dual, respectively. When $G = \mathbb{T}^n$, we write $\mathfrak{t}$ and $\mathfrak{t}^*$ instead of $\mathfrak{g}$ and $\mathfrak{g}^*$. We say that the action is *Hamiltonian* if there exists a map $\mu : M \to \mathfrak{g}^*$ which is equivariant with respect to the coadjoint action on $\mathfrak{g}^*$ such that for each element $X \in \mathfrak{g}$,

$$d\mu^X = \iota_{X^\#}\omega, \tag{3.1}$$

where $\mu^X =< \mu, X >$ is the component of μ in the direction of X, and $X^\#$ is the vector field on M generated by X:

$$X^\#(p) = \frac{d}{dt}\left[\exp(tX) \cdot p\right].$$

The map μ is called the *moment map*. Delzant showed that the image of a moment map of an effective Hamiltonian $\mathbb{T}^n$-action on (M^{2n}, ω) was a certain kind of polytope in $\mathfrak{t}^* \cong \mathbb{R}^n$, and that these polytopes classified all such actions.

Definition III.11. A **symplectic toric manifold** is a compact connected symplectic manifold (M^{2n}, ω) together with an effective Hamiltonian $\mathbb{T}^n$ action and a choice of moment map $\mu : M \to \mathfrak{t}^*$.

Definition III.12. A polytope in $\mathfrak{t}^*$ is **Delzant** if for every vertex v of P, there is a lattice basis $\{u_i\}$ of $\mathfrak{t}^*$ such that the edges incident to v can be written near v in the form $v + tu_i$ for $t \geq 0$.

Theorem III.13. *([Del88]) Symplectic toric manifolds are classified by Delzant polytopes. That is, there is a bijection*

$$\left\{ \begin{array}{c} \textit{Symplectic Toric} \\ \\ \textit{Manifolds} \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \textit{Delzant} \\ \\ \textit{Polytopes} \end{array} \right\}$$

$$(M, \omega, \mu) \mapsto \mu(M)$$

where two symplectic toric manifolds are considered equivalent if there is an equivariant symplectomorphism between the two that commutes with the moment map.

An excellent treatment of Theorem III.13 is given in [dS08].

In this chapter, we study actions of tori on b-symplectic manifolds by b-symplectomorphisms. We will notice that the definition of a *Hamiltonian* action and of a *moment map* must be modified to accommodate the singularity of the symplectic form. To motivate the appropriate definitions we study two examples in detail.

Example III.14. Consider the b-symplectic manifold $(\mathbb{S}^2, Z = \{h = 0\}, \omega = \frac{dh}{h} \wedge d\theta)$, where the coordinates on the sphere are the usual ones: $h \in [-1, 1]$ and $\theta \in [0, 2\pi]$. For the usual $\mathbb{S}^1$-action given by the flow of $-\frac{\partial}{\partial \theta}$,

$$\iota_{-\frac{\partial}{\partial \theta}} \omega = \frac{dh}{h} = d(\log |h|),$$

so a moment map on $M \setminus Z$ is $\mu(h, \theta) = \log|h|$. The image of μ is drawn in Figure 3.1 as two superimposed half-lines depicted slightly apart to emphasize that each point in the image has two connected components in its preimage: one in the northern hemisphere, and one in the southern hemisphere. This phenomenon is dissimilar to classic symplectic geometry, where the level sets of Hamiltonians are connected and the moment map image of a symplectic toric manifold serves as a parameter space for the orbits of the $\mathbb{T}^n$-action. We also notice that the map μ is not defined on Z, even though the vector field from whence it came is defined on Z. In a later section, we will show that by interpreting the Hamiltonian as a section of ${}^bC^\infty$ and by enlarging the codomain of our moment map to include points "at infinity," we can define moment maps for torus actions on a b-manifold that enjoy many of the same properties as classic moment maps. In particular, they will be everywhere defined and their image will be a parameter space for the orbits of the action.

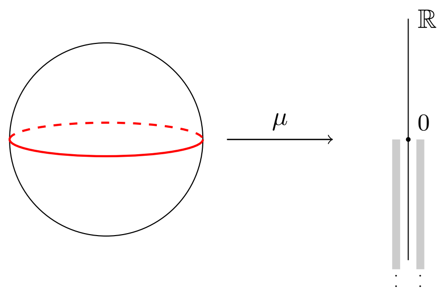

Figure 3.1: The image of μ on $\mathbb{S}^2 \setminus Z$.

Example III.15. Consider the b-symplectic manifold

$$(\mathbb{T}^2, Z = \{\theta_1 \in \{0, \pi\}\}, \omega = \frac{d\theta_1}{\sin\theta_1} \wedge d\theta_2)$$

where the coordinates on the torus are the usual ones: $\theta_1, \theta_2 \in [0, 2\pi]$. The exceptional hypersurface Z is the union of two disjoint circles. For the circle action of rotation on the θ_2 coordinate, because

$$\iota_{\frac{\partial}{\partial\theta_2}}\omega = -\frac{d\theta_1}{\sin\theta_1} = d\left(\log\left|\frac{1+\cos\theta_1}{\sin\theta_1}\right|\right),$$

the $\mathbb{S}^1$-action on $M\backslash Z$ is given by the ${}^bC^\infty$ Hamiltonian $\log\left|\frac{1+\cos\theta_1}{\sin\theta_1}\right|$.

The image of this function on $M\backslash Z$ is drawn in Figure 3.2. Each of the two connected components of $M \setminus Z$ is diffeomorphic to an open cylinder and maps to one of these lines. Again, notice that the preimage of a point in the image consists of two orbits.

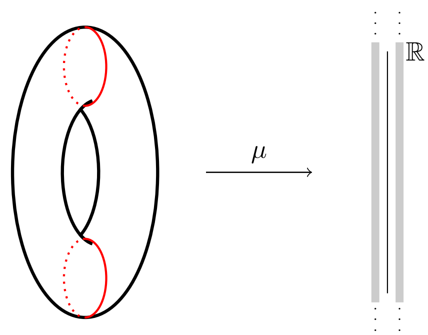

Figure 3.2: The image of μ on $\mathbb{T}^2\backslash Z$.

In both examples above, notice that although the Hamiltonian for the action on $M\backslash Z$ did not extend to a smooth function on all of M, it nevertheless extends to a ${}^bC^\infty$ function on all of M.

Definition III.16. An action of $\mathbb{T}^n$ on a b-symplectic manifold (M, ω) is **Hamiltonian** if:

- for any $X \in \mathfrak{t}$, the one-form $\iota_{X^\#}\omega$ is exact, i.e., has a primitive $H_X \in {}^bC^\infty(M)$, and
- for any $X, Y \in \mathfrak{t}$, $\omega(X^\#, Y^\#) = 0$.

A Hamiltonian action is **toric** if it is effective and the dimension of the torus is half the dimension of M.

3.2 The b-Line and b-dual of the Lie algebra

When b-functions are the Hamiltonians of a torus action, we cannot expect to be able to gather them into a moment map $\mu : M \to \mathfrak{t}^*$ the same way we do in classic symplectic geometry: it would be impossible to define μ along Z. In this section, we define a moment map for a torus action on a b-manifold. To do so, we add points "at infinity" to the codomain $\mathfrak{t}^*$ to account for the singularities of b-functions. We begin our discussion with the simplest case: when the torus is simply a circle, we enlarge the line $\mathfrak{t}^* \cong \mathbb{R}$ into "the b-line" ${}^b\mathbb{R}$.

3.2.1 The b-Line

The b-line is constructed by gluing copies of the extended real line $\overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$ together in a zig-zag pattern, then using $\mathbb{R}_{>0}$-valued labels ("weights") on the points at infinity to prescribe a smooth structure, and finally truncating the result to discard unneccessary copies of $\overline{\mathbb{R}}$. Figure 3.3 should help to put the technical details of the formal definition into a visual context.

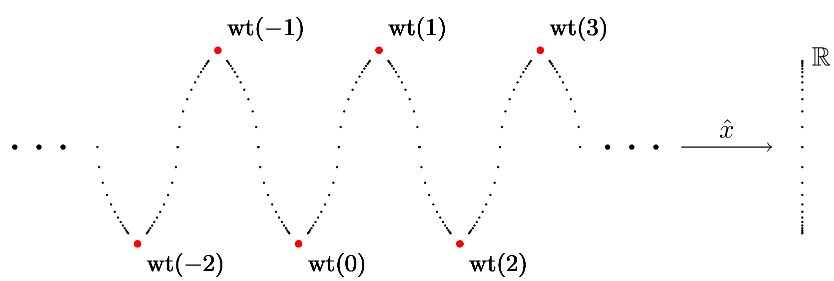

Figure 3.3: A weighted b-line with $I = \mathbb{Z}$.

Definition III.17. Let $\mathrm{wt} : I \to \mathbb{R}_{>0}$, where I can be $\mathbb{Z}$ or $[1, N] \cap \mathbb{Z}$ or $[0, N] \cap \mathbb{Z}$. When $I = \mathbb{Z}$, the **b-line with weight function wt** is described as a topological space by

$${}_{\mathrm{wt}}^{\;b}\mathbb{R} \cong (\mathbb{Z} \times \overline{\mathbb{R}})/\{(a, (-1)^a\infty) \sim (a+1, (-1)^a\infty)\}.$$

Let $Z_{{}^b\mathbb{R}} = \mathbb{Z} \times \{\pm\infty\} \subseteq {}_{\mathrm{wt}}{}^{b}\mathbb{R}$, this set will function as a exceptional hypersurface of the manifold ${}_{\mathrm{wt}}{}^{b}\mathbb{R}$. Notice that ${}_{\mathrm{wt}}{}^{b}\mathbb{R}$ is homeomorphic to $\mathbb{R}$. The weight function prescribes a smooth structure[3] on ${}_{\mathrm{wt}}{}^{b}\mathbb{R}$ in the following way. Define

$$\hat{x} : ({}_{\mathrm{wt}}{}^{b}\mathbb{R} \backslash Z_{{}^b\mathbb{R}}) = \mathbb{Z} \times \mathbb{R} \to \mathbb{R}$$

$$(a, x) \mapsto x$$

and $\hat{y}_a : ((a-1, 0), (a, 0)) \to \mathbb{R}$ as

$$\hat{y}_a = \begin{cases} -\exp\left((-1)^a \hat{x}/\mathrm{wt}(a)\right) & \text{on } ((a-1,0),(a-1,(-1)^{a-1}\infty)) \\ 0 & \text{at } (a-1,(-1)^{a-1}\infty) \\ \exp\left((-1)^a \hat{x}/\mathrm{wt}(a)\right) & \text{on } ((a,(-1)^{a-1}\infty),(a,0)) \end{cases}.$$

The coordinate maps $\{\hat{x}\big|_{\{a\}\times\mathbb{R}}, \hat{y}_a\}_{a\in\mathbb{Z}}$ define the structure of a smooth manifold on ${}_{\mathrm{wt}}{}^{b}\mathbb{R}$. When $I = [1, N] \cap \mathbb{Z}$ (respectively $[0, N] \cap \mathbb{Z}$), the weighted b-line ${}_{\mathrm{wt}}{}^{b}\mathbb{R}$ is defined as the open subset $((0, -\infty), (N, (-1)^N\infty))$ (respectively, $((-1, \infty), (N, (-1)^N\infty))$) of ${}_{\mathrm{wt}'}{}^{b}\mathbb{R}$, where $\mathrm{wt}' : \mathbb{Z} \to \mathbb{R}_{>0}$ is any function extending wt.

We will often abbreviate ${}_{\mathrm{wt}}{}^{b}\mathbb{R}$ by ${}^{b}\mathbb{R}$ when the weight function is understood from the context. To motivate the functions $\{\hat{y}_a\}$ in Definition III.17, observe that

$$\hat{x}\big|_{((a-1,0),(a,0))} = (-1)^a \mathrm{wt}(a) \log|\hat{y}_a|$$

This makes it possible to realize any $f \in {}^{b}C^{\infty}(M)$ on a b-manifold (M, Z) locally as a smooth map to a b-line.

Lemma III.18. *Let (M, Z) be a b-manifold and Z' a connected component of Z. Any $f \in {}^{b}C^{\infty}(M)$ with a singularity at Z' can be expressed in a neighborhood of Z' as a smooth function F to a b-line ${}^{b}\mathbb{R}$. That is, $F^{-1}(Z_{{}^b\mathbb{R}}) = Z'$ and $\hat{x} \circ F = f$ on the complement of Z'.*

[3]The reader may wonder why attention is being paid to define the smooth structure on ${}_{\mathrm{wt}}{}^{b}\mathbb{R}$ when a topological 1-manifold admits a unique smooth structure up to homeomorphism. The reason behind the care is because a homeomorphism intertwining two different smooth structures will not in general preserve the intrinsic affine structure present on each $\{a\} \times \mathbb{R} \subseteq {}_{\mathrm{wt}}{}^{b}\mathbb{R}$. This affine structure will be essential in the theory that follows.

Proof. Let y be a local defining function for Z', and let U be a neighborhood of Z' on which $f\big|_U = c\log|y| + g$ for some $c \in \mathbb{R}, g \in C^\infty(U)$ and for which $U \backslash Z'$ has two connected components $\{U_+, U_-\}$. Because f is singular at Z', it follows that $c \neq 0$. If c is positive, let $\mathrm{wt} : \{0\} \mapsto c$ and define $F : U \to {}^b\mathbb{R}$ by the equation

$$\hat{y}_0 \circ F := \begin{cases} \exp(f/c) & \text{on } U_+ \\ -\exp(f/c) & \text{on } U_- \\ 0 & \text{on } Z \end{cases}$$

where the function $\hat{y}_0$ is defined in Definiton III.17. If c is negative, let $\mathrm{wt} : \{1\} \mapsto -c$ and define $F : U \to {}^b\mathbb{R}$ by

$$\hat{y}_1 \circ F := \begin{cases} \exp(f/c) & \text{on } U_+ \\ -\exp(f/c) & \text{on } U_- \\ 0 & \text{on } Z \end{cases}$$

In both cases, the function F satisfies the conditions of the lemma. $\square$

Remark III.19. The function $\hat{y}_i \circ F$ constructed in the proof of Lemma III.18 is a defining function for the hypersurface Z' that depends only on the original $f \in {}^bC^\infty(M)$ and the choice of which component of $U\backslash Z'$ to label U_+ and which to label U_-. Had we chosen this labelling differently, the resulting $\hat{y}_i \circ F$ would be replaced by its negative. Therefore, given a b-function f which is singular at Z', there is a canonical choice of defining function for Z' up to sign.

Remark III.20. Not every b-function on every b-manifold can be *globally* expressed as a smooth function to a b-line. Consider when $M = \mathbb{S}^2$ and Z consists of two disjoint circles C_1 and C_2. Let y be a global defining function for Z, and pick a b-function on (M, Z) which restricts to $\log|y|$ and $2\log|y|$ in neighborhoods of C_1 and C_2 respectively. This b-function cannot be realized as a global map to any ${}^b\mathbb{R}$.

The following example illustrates Lemma III.18 in the context of Hamiltonian torus actions.

Example III.21. Let (h, θ) be the standard coordinates on $\mathbb{S}^2$. For any $c \in \mathbb{R}_{>0}$, the form $\omega_c = c\frac{dh}{h} \wedge d\theta$ is a b-symplectic form on $(\mathbb{S}^2, Z := \{h = 0\})$. Because $\iota_{-\frac{\partial}{\partial\theta}}\omega_c = c\frac{dh}{h}$, it follows that the b-function $c\log|h|+k$ for any $k \in \mathbb{R}$ is a Hamiltonian function generating the $\mathbb{S}^1$-action given by the flow of $-\frac{\partial}{\partial\theta}$. Figure 3.4 shows the map $\mu : \mathbb{S}^2 \to {}^b\mathbb{R}$ (with weight function $\mathrm{wt} : \{0\} \mapsto c$) corresponding to the Hamiltonian $c\log|h|$, and another μ' corresponding to $c\log|h| - 2$. In both cases, we have drawn ${}^b\mathbb{R}$ twice – the first is vertically so that μ can be visualized as a projection, the second is bent so that it looks visually similar to the ${}^b\mathbb{R}$ in Figure 3.3.

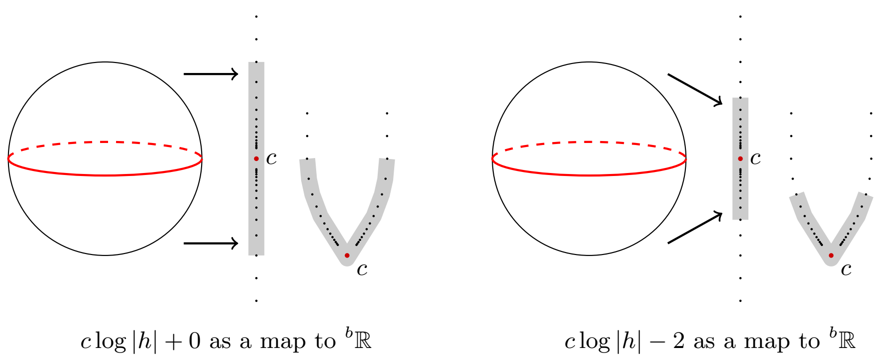

Figure 3.4: Two Hamiltonians generating the same $\mathbb{S}^1$-action.

There are two important observations to make about this example. The first is that the image of the moment maps $\mu = c\log|h|$ for different values of c have visually similar images – the only feature that distinguishes them is the numerical weight on the "point at infinity." This observation emphasizes the necessity of the weights: for different values of c, the b-manifolds (M, Z, ω_c) are *not* symplectomorphic. Were it not for the weight label, their moment map images would be indistinguishable. The second observation is that μ differs from μ' by changing the corresponding ${}^bC^\infty$ function by a constant. This shows that the picture of a "translation" of a b-line differs from the picture of a translation of $\mathbb{R}$.

Definition III.22. Let ${}^b\mathbb{R}$ be a weighted b-line. A **translation** of ${}^b\mathbb{R}$ by $c \in \mathbb{R}$ is a map ${}^b\mathbb{R} \to {}^b\mathbb{R}$ which maps (a, b) to $(a, b + c)$ for finite values of b, and $(a, \pm\infty)$ to $(a, \pm\infty)$.

Using Definition III.22, one can check that the images of μ and μ' shown in Figure 3.4 are translates of one another.

3.2.2 b-dual of the Lie algebra

Example 3.4 motivates the use of the b-line as a codomain for the moment map of a Hamiltonian $\mathbb{S}^1$-action on a b-surface. For a Hamiltonian $\mathbb{T}^n$-action on a symplectic b-manifold (M^{2n}, Z, ω) with $n > 1$, we will eventually prove that there always exists a subtorus $\mathbb{T}^{n-1}_Z \subseteq \mathbb{T}^n$ whose action is generated by vector fields tangent to the symplectic foliation of Z (even when Z is disconnected). The Lie algebra of this subtorus defines a hyperplane $\mathfrak{t}_Z$ in $\mathfrak{t}$ and dually a 1-dimensional subspace $(\mathfrak{t}_Z)^\perp$ in $\mathfrak{t}^*$. We will construct the codomain for the moment map of a toric action by replacing $(\mathfrak{t}_Z)^\perp \cong \mathbb{R}$ with a copy of ${}^b\mathbb{R}$, obtaining a space (non-canonically) isomorphic to ${}^b\mathbb{R} \times \mathbb{R}^{n-1}$.

Definition III.23. Let $\mathfrak{t}$ be the Lie algebra of $\mathbb{T}^n$ and fix a primitive lattice vector $z \in \mathfrak{t}^*$ and a weight function $\mathrm{wt} : I \to \mathbb{R}_{>0}$ (again as in Definition III.17, $I = \mathbb{Z}$ or $[0, N] \cap \mathbb{Z}$ or $[1, N] \cap \mathbb{Z}$). Write $\mathfrak{t}_Z$ for the hyperplane in $\mathfrak{t}$ perpendicular to z. When $I = \mathbb{Z}$, we define the **b-dual of the Lie algebra** ${}_{\mathrm{wt}}{}^b\mathfrak{t}^*$ (written ${}^b\mathfrak{t}^*$ when the weight function is clear from the context) to be the set

$$ {}_{\mathrm{wt}}{}^b\mathfrak{t}^* = (\mathbb{Z} \times \mathfrak{t}^*) \sqcup (\mathbb{Z} \times \mathfrak{t}^*_Z). $$

A choice of integral element $X \in \mathfrak{t}$ satisfying $\langle X, z \rangle = 1$ defines a set bijection

$$ {}_{\mathrm{wt}}{}^b\mathfrak{t}^* = (\mathbb{Z} \times \mathfrak{t}^*) \sqcup (\mathbb{Z} \times \mathfrak{t}^*_Z) \to {}_{\mathrm{wt}}{}^b\mathbb{R} \times \mathfrak{t}^*_Z \tag{3.2} $$

$$\begin{aligned} (a,\xi) &\longmapsto ((a,\langle \xi, X\rangle),[\xi]) \\ (a,[\xi]) &\longmapsto ((a,(-1)^{a+1}\infty),[\xi]) \end{aligned}$$

where the square brackets denote the image of an element of $\mathfrak{t}^*$ in $\frac{\mathfrak{t}^*}{\langle z\rangle} \cong \mathfrak{t}_Z^*$. The target space of the map (3.2) has a smooth b-manifold structure from Definition III.17. This induces a smooth b-manifold structure on ${}_{\mathrm{wt}}{}^{b}\mathfrak{t}^*$. We will show in Proposition III.24 that this structure is independent of the choice of X. When the domain of wt is a subset of $\mathbb{Z}$, we choose any $\mathrm{wt}' : \mathbb{Z} \to \mathbb{R}_{>0}$ that extends wt and define ${}_{\mathrm{wt}}{}^{b}\mathfrak{t}^*$ as the preimage (under the map (3.2)) of ${}_{\mathrm{wt}}{}^{b}\mathbb{R} \times \mathfrak{t}^* \subseteq_{\mathrm{wt}'}^{b} \mathbb{R} \times \mathfrak{t}^*$.

Proposition III.24. The smooth structure on ${}^b\mathfrak{t}^*$ is independent of the choice of X in its definition.

Proof. Let X_1 and X_2 be integral elements of $\mathfrak{t}$ satisfying $\langle X_1, z\rangle = \langle X_2, z\rangle = 1$. This gives the following isomorphisms, where $\xi \in \mathfrak{t}^*$.

$$\begin{array}{ccccc} {}^b\mathbb{R}\times\mathfrak{t}_Z^* & \xleftarrow{\varphi_1} & (\mathbb{Z}\times\mathfrak{t}^*)\sqcup(\mathbb{Z}\times\mathfrak{t}_Z^*) & \xrightarrow{\varphi_2} & {}^b\mathbb{R}\times\mathfrak{t}_Z^* \\ ((a,\langle\xi,X_1\rangle),[\xi]) & \longleftarrow\!\shortmid & (a,\xi) & \longmapsto & ((a,\langle\xi,X_2\rangle),[\xi]) \\ ((a,(-1)^{a+1}\infty),[\xi]) & \longleftarrow\!\shortmid & (a,[\xi]) & \longmapsto & ((a,(-1)^{a+1}\infty),[\xi]) \end{array}$$

Because $X_2 - X_1 \in \mathfrak{t}_Z$, the map $\varphi_2 \circ \varphi_1^{-1}$ is given on $\mathbb{R} \times \mathfrak{t}_Z^*$ by

$$((a,x),[\xi]) \mapsto ((a, x + \langle [\xi], X_2 - X_1\rangle),[\xi])$$

which is linear in the coordinate charts $(\{a\}\times\mathbb{R})\times\mathfrak{t}_Z^*$ of ${}^b\mathfrak{t}$. In the $\hat{y}_a$ coordinates $\varphi_2 \circ \varphi_1^{-1}$ is given by

$$(\hat{y}_a,[\xi]) \mapsto (\hat{y}_a \exp((-1)^a\langle[\xi], X_2 - X_1\rangle/\mathrm{wt}(a)),[\xi])$$

which shows that the entire map $\varphi_2 \circ \varphi_1^{-1}$ is a diffeomorphism, proving that the smooth structures on ${}^b\mathfrak{t}^*$ induced by φ_1 and φ_2 are the same. $\square$

In practice, a Hamiltonian torus action on a b-manifold will not determine a natural choice of $z \in \mathfrak{t}^*$, but only the hypersurface $\mathfrak{t}_Z = z^\perp$. The reader may therefore find inelegant that the definition of ${}^b\mathfrak{t}^*$ depends on an arbitrary choice of z in Definition III.23. However, a similar issue arises in classic symplectic geometry. Namely, given a Hamiltonian $\mathbb{S}^1$-action, the moment map $M \to \mathfrak{t}^*$ cannot be realized as a Hamiltonian function $M \to \mathbb{R}$ until an arbitrary choice has been made of which of the (two) lattice generators of $\mathfrak{t}^*$ to send to 1 in the identification $\mathfrak{t}^* \cong \mathbb{R}$. Choosing the opposite generator amounts to replacing the Hamiltonian function by its negative – in other words, postcomposing the Hamiltonian function with $\mathbb{R} \to \mathbb{R}$, $a \mapsto -a$. This situation is complicated in b-geometry by the sad fact that there is no automorphism φ of ${}^b\mathbb{R}$ that satisfies $\hat{x} \circ \varphi = -\hat{x}$. This can be seen from the fact that the b-line in Figure 3.3 does not have horizontal symmetry – you must follow your flip by a "horizontal shift" in order to realize an automorphism satisfying $\hat{x} \circ \varphi = -\hat{x}$. In other words, there is an automorphism of ${}_{\mathrm{wt}}{}^b\mathfrak{t}^*$ (using $z \in \mathfrak{t}^*$ as the distinguished lattice vector) and ${}_{\widetilde{\mathrm{wt}}}{}^b\mathfrak{t}^*$ (using $-z$ as the distinguished lattice vector), where $\widetilde{\mathrm{wt}}$ is defined by $\widetilde{\mathrm{wt}}(a) = \mathrm{wt}(a+1)$ (or $\widetilde{\mathrm{wt}}(a) = \mathrm{wt}(a-1)$, if the domain of wt is $[0, N]$). This is illustrated for the case when $\mathfrak{t}$ is 1-dimensional and wt has domain $[1, 3]$ in Figure 3.5. The reader who continues to find inelegant the choice of z in Definition

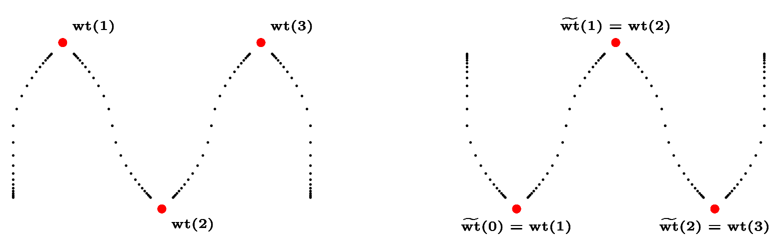

Figure 3.5: The effect of choosing a different distinguished direction.

III.23 may prefer to write more general definitions of weight functions and of ${}^b\mathfrak{t}^*$ so that the two pictures in Figure 3.5 correspond to the same object.

Remark III.25. Notice that for any $X \in \mathfrak{t}$ the map

$$^{b}\mathfrak{t}^* \supseteq (\mathbb{Z} \times \mathfrak{t}^*) \to \mathbb{R}, \quad (a, \xi) \mapsto \langle \xi, X \rangle$$

extends to a b-function on $^{b}\mathfrak{t}^*$. This observation motivates the definition of a moment map.

Definition III.26. Consider a Hamiltonian $\mathbb{T}^n$-action on a b-symplectic manifold (M, Z, ω), and let $\mu : M \to {}^{b}\mathfrak{t}^*$ be a smooth $\mathbb{T}^n$-invariant b-map. We say that μ is a **moment map** for the action if $X \mapsto \mu^X$ is linear and

$$\iota_{X^\#}\omega = d\mu^X$$

where μ^X is the b-function $\mu^X(p) = \langle \mu(p), X \rangle$ described in Remark III.25.

Example III.27. Consider the b-symplectic manifold

$$(M = \mathbb{S}^2 \times \mathbb{S}^2, Z = \{h_1 = 0\}, \omega = 3\frac{dh_1}{h_1} \wedge d\theta_1 + dh_2 \wedge d\theta_2)$$

where $(h_1, \theta_1, h_2, \theta_2)$ are the standard coordinates on $\mathbb{S}^2 \times \mathbb{S}^2$. The $\mathbb{T}^2$-action

$$(t_1, t_2) \cdot (h_1, \theta_1, h_2, \theta_2) = (h_1, \theta_1 - t_1, h_2, \theta_2 - t_2)$$

is Hamiltonian. Let X_1 and X_2 be the elements of $\mathfrak{t}$ satisfying $X_1^\# = -\frac{\partial}{\partial \theta_1}$ and $X_2^\# = -\frac{\partial}{\partial \theta_2}$ respectively. Then $\mathfrak{t}_Z = \langle X_2 \rangle$. Letting $\mathrm{wt} : \{0\} \mapsto 3$ be the weight function, and $v = (X_1)^*$ be the distinguished direction in $\mathfrak{t}^*$, then we have a moment map $\mu : M \to {}^{b}\mathfrak{t}^*$ which can be described (using the basis $\{X_1, X_2\}$) as

$$M \to {}^{b}\mathbb{R} \times \mathbb{R}, \quad (h_1, \theta_1, h_2, \theta_2) \mapsto (\log |h_1|, h_2),$$

the image of which is illustrated in Figure 3.6.

The image on the left of Figure 3.6 shows the similarity between the moment map image and that of the standard action of $\mathbb{T}^2$ on $\mathbb{S}^2 \times \mathbb{S}^2$ from classic symplectic

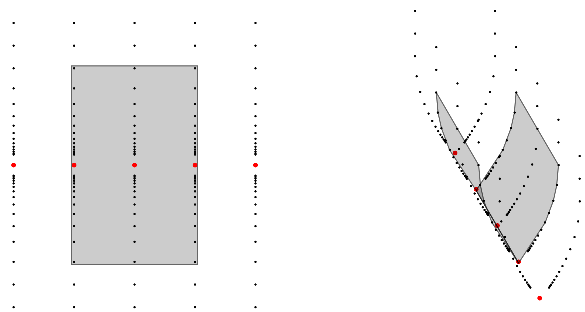

Figure 3.6: The moment map image $\mu(\mathbb{S}^2 \times \mathbb{S}^2)$, drawn two different ways.

geometry. In the right image, the ${}^b\mathbb{R}$ factor of ${}^b\mathfrak{t}^*$ is bent to be visually similar to Figure 3.3.

In some cases, we must first quotient the codomain ${}^b\mathfrak{t}^*$ by a discrete group action in order to have a well-defined moment map.

Definition III.28. Let $N \in \mathbb{Z}_{>0}$ be even and $\mathrm{wt} : [1, N] \to \mathbb{R}_{>0}$ be a weight function. Let $\mathrm{wt}' : \mathbb{Z} \to \mathbb{R}_{>0}$ be the N-periodic weight function that extends wt. Then ${}_{\mathrm{wt}}^{b}\mathbb{R}/\langle N\rangle$ (or just ${}^b\mathbb{R}/\langle N\rangle$) is defined as the quotient of ${}^b\mathbb{R}$ by the $\mathbb{Z}$ action $k \cdot (a, x) = (kN + a, x)$. Similarly, ${}_{\mathrm{wt}}^{b}\mathfrak{t}^*/\langle N\rangle$ is defined as the quotient of ${}_{\mathrm{wt}'}^{b}\mathfrak{t}^*$ by the smooth extension of the $\mathbb{Z}$ action $k \cdot (a, \xi) = (kN + a, \xi)$ on $\mathbb{Z} \times \mathfrak{t}^*$ to ${}_{\mathrm{wt}'}^{b}\mathfrak{t}^*$.

Topologically, the spaces ${}^b\mathbb{R}/\langle N\rangle$ and ${}^b\mathfrak{t}^*/\langle N\rangle$ are homeomorphic to a circle. The subset $Z_{{}^b\mathbb{R}}$ is preserved by the action described in Definition III.28; its image in ${}^b\mathbb{R}/\langle N\rangle$ will be called $Z_{{}^b\mathbb{R}/\langle N\rangle}$. Similarly, the function $\hat{x}$ is well-defined on the complement of $Z_{{}^b\mathbb{R}/\langle N\rangle}$, and it still is the case that for any smooth b-map $\mu : M \to {}^b\mathfrak{t}^*$ and any $X \in \mathfrak{t}$, the function $p \mapsto \langle \mu(p), X\rangle$ on $M\backslash Z$ extends to a b-function on all of M. We define a moment map to the quotient spaces ${}^b\mathfrak{t}^*/\langle N\rangle$ in the same way as in Definition III.26.

Definition III.29. Consider a Hamiltonian $\mathbb{T}^n$-action on a b-symplectic manifold (M, Z, ω), and let $\mu : M \to {}^b\mathfrak{t}^*/\langle N\rangle$ be a smooth $\mathbb{T}^n$-invariant b-map. We say that μ

is a **moment map** for the action if $X \mapsto \mu^X$ is linear and

$$\iota_{X^{\#}}\omega = d\mu^X$$

where μ^X is the b-function $\mu^X(p) = \langle \mu(p), X\rangle$.

Example III.30. Consider the b-symplectic manifold

$$(\mathbb{T}^2 = \{(\theta_1,\theta_2) \in (\mathbb{R}/2\pi)^2\}, Z = \{\theta_1 \in \{0,\pi\}\}, \omega = \frac{d\theta_1}{\sin\theta_1} \wedge d\theta_2)$$

with $\mathbb{S}^1$-action given by the flow of $\frac{\partial}{\partial\theta_2}$. Let $X \in \mathfrak{t}$ be the element satisfying $X^{\#} = \frac{\partial}{\partial\theta_2}$. The weight function $\{0,1\} \mapsto 1$ and distinguished vector X^* define ${}^b\mathfrak{t}^*/\langle 2\rangle$, which we identify with ${}^b\mathbb{R}/\langle 2\rangle$ using the isomorphism induced by $X \in \mathfrak{t}$. A moment map for the $\mathbb{S}^1$-action is

$$\mu : \mathbb{T}^2 \to {}^b\mathbb{R}/\langle 2\rangle, \qquad (\theta_1,\theta_2) \mapsto \begin{cases} (0,\infty) & \text{if } \theta_1 = 0 \\ \left(1, \log\left|\frac{1+\cos\theta_1}{\sin\theta_1}\right|\right) & \text{if } 0 < \theta_1 < \pi \\ (1,-\infty) & \text{if } \theta_1 = \pi \\ \left(0, \log\left|\frac{1+\cos\theta_1}{\sin\theta_1}\right|\right) & \text{if } \pi < \theta_1 < 2\pi \end{cases}$$

The reader is invited to check that μ is smooth. The image is shown in Figure 3.7.

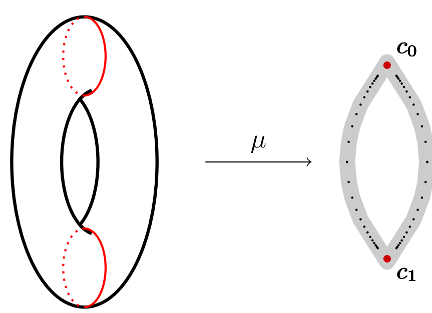

Figure 3.7: The moment map μ surjects onto ${}^b\mathfrak{t}^*/\langle 2\rangle$.

3.3 The moment map of a toric b-symplectic manifold

3.3.1 Local picture: in a neighborhood of Z

Our first goal towards understanding toric actions on b-symplectic manifolds is to study their behavior near each connected component of Z. To simplify our exposition, we will assume throughout Section 3.3.1 that Z is connected; in general, the results from this section will hold in a neighborhood of each connected component of Z.

Proposition III.40 is the main result of this section, which states that a toric action near Z is locally a product of a codimension-1 torus action on a symplectic leaf of Z with an circle action whose flow is transverse to the leaves. This $\mathbb{T}^{n-1} \times \mathbb{S}^1$-action has a moment map whose image is the product of a Delzant polytope (corresponding to the action on the symplectic leaf) with an interval of ${}^b\mathbb{R}$.

The codimension-1 subtorus $\mathbb{T}^{n-1}$ will consist of those elements of $\mathbb{T}^n$ that preserve the symplectic foliation of Z. Toward the goal of showing that this subtorus is well-defined, we remind the reader of the following standard fact from Poisson geometry.

Remark III.31. Let (M, Z, ω) be a b-symplectic manifold. Since Z is a Poisson submanifold of M, a Hamiltonian vector field X_f is tangent to the symplectic leaves of Z if and only if $f\big|_U \in C^\infty(U)$ for some neighborhood U of Z. In this case, if $i_{\mathcal{L}} : (\mathcal{L}, \omega_{\mathcal{L}}) \to Z$ is the inclusion of a symplectic leaf into Z, then $X_f\big|_{\mathcal{L}} = X_{f \circ i_{\mathcal{L}}}$.

Given a Hamiltonian $\mathbb{T}^k$-action on (M^{2n}, Z, ω) and any $X \in \mathfrak{t}$, the b-form $\iota_{X^\#}\omega$ has a ${}^bC^\infty$ primitive that can be written in a neighborhood of Z as $c \log|y| + g$, where y is a local defining function for Z, the function g is smooth, and $c \in \mathbb{R}$ depends on X.

Definition III.32. The map $X \mapsto c$ is an element v_Z of $\mathfrak{t}^* = \mathrm{Hom}(\mathfrak{t}, \mathbb{R})$ (we invite the reader to verify that c does not depend on the choices involved and that $X \mapsto c$ is a homomorphism). We will denote by $\mathfrak{t}_Z$ the kernel of v_Z.

By Proposition III.10, the values of $\langle v_Z, X \rangle$ are integer multiples of the modular period of Z when X is a lattice vector. Therefore, we may conclude that v_Z is rational. First, we prove an equivariant Darboux theorem for compact group actions in the neighborhood of a fixed point; this will prepare us to prove that v_Z is nonzero. Given a fixed point p of an action $\rho : G \times M \to M$, we denote by $d\rho$ the linear action defined via the exponential map in a neighborhood of the origin in T_pM. That is, $d\rho(g, v) = d_p(\rho(g))(v)$.

Theorem III.33. *Let ρ be a b-symplectic action of a compact Lie group G on the b-symplectic manifold (M, Z, ω), and let $p \in Z$ be a fixed point of the action, i.e., $\rho(g, p) = p, \quad \forall g \in G$. Then there exist local coordinates $(x_1, y_1, \ldots, x_{n-1}, y_{n-1}, z, t)$ centered at p such that the action is linear in these coordinates and*

$$\omega = \sum_{i=1}^{n-1} dx_i \wedge dy_i + \frac{1}{z}\, dz \wedge dt.$$

Proof. After choosing a metric near p, the exponential map gives a diffeomorphism ϕ from a neighborhood U of $0 \in T_pM$ to a neighborhood of $p \in M$. By choosing the metric wisely we can guarantee that $\phi(U \cap T_pZ) \subseteq Z$. Pulling back under ϕ the group action and symplectic form on M to a group action and symplectic form on T_pM, it suffices to prove the theorem for the b-manifold (T_pM, T_pZ). Therefore, assume that ω and ρ live on (T_pM, T_pZ).

By Bochner's theorem [Boc45], the action of ρ is locally equivalent to the action of $d\rho$. That is, there is a system of coordinates $(x_1, y_1, \ldots, x_{n-1}, y_{n-1}, z, t)$ centered at $\mathbf{0} = (0, 0, \ldots, 0)$ on which the action is linear. By studying the construction of φ in [Boc45], we see that the coordinates can be chosen so that T_pZ is the coordinate hyperplane $\{z = 0\}$. Also, after a linear change of these coordinates, we may assume that

$$\omega\big|_{\mathbf{0}} = \sum_{i=1}^{n-1} dx_i \wedge dy_i + \frac{1}{z}\, dz \wedge dt.$$

Next, we will perform an equivariant Moser's trick. Let $\omega_0 = \omega$,

$$\omega_1 = \sum_{i=1}^{n-1} dx_i \wedge dy_i + \frac{1}{z}\, dz \wedge dt, \quad \text{and} \quad \omega_s = s\omega_1 + (1-s)\omega_0, \text{ for } s \in [0,1].$$

Because ω_s has full rank at $\mathbf{0}$ for all s, we may assume (after shrinking the neighborhood) that ω_s has full rank for all s. Let α be a primitive for $\omega_1 - \omega_0$ that vanishes at $\mathbf{0}$ (α is a b-form), and let X_s be the b-vector field defined by the equation

$$\iota_{X_s}\omega_s = -\alpha.$$

Since X_s is a b-vector field that vanishes at $\mathbf{0}$, its flow preserves Z and fixes $\mathbf{0}$. The time one flow of X_s is a symplectomorphism $(T_pM, \omega_0) \to (T_pM, \omega_1)$, but this symplectomorphism will not in general be equivariant, and so there is no guarantee that the action is still linear. We therefore pick a Haar measure μ on G and consider the vector field

$$X_s^G = \int_G \rho(g)_*(X_s) d\mu.$$

The vector field X_s^G commutes with the group action. Since $\rho(g)$ preserves ω_0 and ω_1, it also preserves ω_s for all s. Therefore, the averaged vector field satisfies the equation

$$\iota_{X_s^G}\omega_s = -\int_G \rho(g)^*(\alpha) d\mu.$$

Observe also that the new invariant b-form $\alpha_G = \int_G \rho(g)^*(\alpha) d\mu$ is also a primitive for $\omega_1 - \omega_0$ due to $d\rho$-invariance of the family of b-forms ω_s. Thus, the flow of X_s^G commutes with the linear action and satisfies the equation

$$\iota_{X_s^G}\omega_s = -\alpha_G.$$

Therefore the time one flow of X_s^G takes ω_0 to ω_1 in an equivariant way. $\square$

In the particular case where the group considered is a torus we obtain the following:

Corollary III.34. *Consider a fixed point $z \in Z$ of a symplectic $\mathbb{T}^k$-action on (M, Z, ω). If the isotropy representation on T_zM is trivial, then the action is trivial in a neighborhood of z.*

Claim III.35. For a toric action on (M^{2n}, Z, ω), v_Z is nonzero. As a consequence, $\mathfrak{t}_Z$ is a hyperplane.

Proof. Consider a toric action on (M^{2n}, Z, ω) with the property that $\iota_{X^\#}\omega \in \Omega^1(M)$ for every $X \in \mathfrak{t}$. It suffices to prove that such an action is not effective. Let $(\mathcal{L}, \omega_{\mathcal{L}})$ be a leaf of the symplectic foliation of Z. By Remark III.31 the action on M induces a toric action on the symplectic manifold $(\mathcal{L}, \omega_{\mathcal{L}})$. Because $\dim(\mathcal{L}) = 2n - 2$, there must be a subgroup $\mathbb{S}^1 \subseteq \mathbb{T}^n$ that acts trivially on $\mathcal{L}$.

For any $z \in \mathcal{L}$, the isotropy representation of this $\mathbb{S}^1$-action on T_zM restricts to the identity on $T_z\mathcal{L} \subseteq T_zM$ and preserves the subspace T_zZ. It therefore induces a linear $\mathbb{S}^1$-action on the 1-dimensional vector space $T_zZ/T_z\mathcal{L}$. Any such action is trivial, so it follows that the isotropy representation restricts to the identity on T_zZ. Following the same argument, the isotropy representation on all of T_zM is the identity. By Corollary III.34, this shows that the $\mathbb{S}^1$-action is the identity on a neighborhood of z, so the action is not effective. $\square$

In the general case (when Z is not connected), there will be different elements $v_{Z'}$ for different connected components $Z' \subseteq Z$. However, we will see later in Claim III.44 that these $v_{Z'}$ will be nonzero scalar multiples of each other, and so the hyperplane $\mathfrak{t}_{Z'}$ is independent of Z' (in other words, the elements $v_{Z'}$ are nonzero scalar multiples of one another).

Corollary III.36. *If the b-manifold (M, Z, ω) admits a toric action with the property that each $\iota_{X^\#}\omega \in \Omega^1(M)$ for every $X \in \mathfrak{t}$, then $Z = \emptyset$.*

Proposition III.37. Let (M^{2n}, Z, ω) be a b-symplectic manifold with a toric action. Let X be a representative of a primitive lattice vector of $\mathfrak{t}/\mathfrak{t}_Z$ that pairs positively with v_Z. Then $\langle X, v_Z \rangle$ equals the modular period of Z.

Proof. By Proposition III.10, it suffices to prove that a time-1 trajectory of $X^\#$ that starts on Z, when projected to the $\mathbb{S}^1$ base of the mapping torus of Z, travels around the loop just once. Let $p \in \mathbb{R}_{>0}$ be the smallest number such that $\Phi_p^X(\mathcal{L}) = \mathcal{L}$, where $\Phi_p^{X^\#}$ is the time-p flow of $X^\#$. The condition that $\omega(X^\#, Y^\#) = 0$ for all $Y \in \mathfrak{t}_Z$ implies that the symplectomorphism $\Phi_p^{X^\#}\big|_{\mathcal{L}}$ preserves the $\mathbb{T}_Z$-orbits of $\mathcal{L}$. We can realize any such symplectomorphism as the time-1 flow of a Hamiltonian vector field v on the symplectic leaf $(\mathcal{L}, \omega_{\mathcal{L}})$ (see, for example, the proof of Proposition 6.4 in [LT97]). The product of the $\mathbb{T}_Z$ action with the flow of $p^{-1}v$ defines a Hamiltonian $\mathbb{T}_Z \times \mathbb{S}^1 \cong \mathbb{T}^n$ action on the $(\mathcal{L}, \omega_{\mathcal{L}})$, so there exists $\mathbb{S}^1 \subseteq \mathbb{T}_Z \times \mathbb{S}^1$ that acts trivially on $\mathcal{L}$. Since the $\mathbb{T}_Z$ action is effective, this $\mathbb{S}^1$ is not a subset of $\mathbb{T}_Z$. Therefore we may assume, after replacing X with $X + Y$ for some $Y \in \mathfrak{t}_Z$, that the time-p flow of $X^\#$ is the identity on $\mathcal{L}$. Then, for any $z \in \mathcal{L}$, the isotropy representation of the time-p flow of $X^\#$ would be the identity on T_zM, proving (by Corollary III.34) that the time-p flow of $X^\#$ is the identity in a neighborhood of z. By effectiveness, $p = 1$. □

In particular, Proposition III.37 proves that the trajectories of $X^\#$ inside Z, when projected to the $\mathbb{S}^1$ base of the mapping torus, travel around the loop just once. Because $X^\#$ is periodic and preserves the symplectic foliation, the flow of $X^\#$ defines a product structure on Z

Corollary III.38. *Let (M, Z, ω) be a b-symplectic manifold with a toric action. Let $\mathcal{L}$ be a symplectic leaf of the foliation of Z. Then*

$$Z \cong \mathcal{L} \times \mathbb{S}^1.$$

In the general case (where Z may have more than one component), this result implies that each connected component Z' of Z is of the form $\mathcal{L}' \times \mathbb{S}^1$ for possibly distinct $\mathcal{L}'$. We will see however that the existence of a global toric action forces all $\mathcal{L}'$ to be identical.

We are nearly ready to prove Proposition III.40, which states that a toric action near Z splits as the product of a $\mathbb{T}^{n-1}_Z$-action and an $\mathbb{S}^1$-action. We preface its proof by studying a related example in classic symplectic geometry – the intuition gained from this informal discussion will prepare the reader for the proofs of Lemma III.39 and Proposition III.40.

Consider the symplectic manifold $M = \mathbb{S}^2 \times \mathbb{S}^2, \omega = dh_1 \wedge d\theta_1 + dh_2 \wedge d\theta_2$ with a Hamiltonian $\mathbb{T}^2$-action defined by

$$(t_1, t_2) \cdot (h_1, \theta_1, h_2, \theta_2) = (h_1, \theta_1 + t_1, h_2, \theta_2 + t_2).$$

Let $\{X_1, X_2\}$ be the basis of $\mathfrak{t}$ such that $X_1^\# = \frac{\partial}{\partial \theta_1}$ and $X_2^\# = \frac{\partial}{\partial \theta_2}$. After identifying $\mathfrak{t}^*$ with $\mathbb{R}^2$ using the dual basis, the $\mathbb{T}^2$-action is given by the moment map (h_1, h_2) with image $\Delta = [-1, 1]^2$.

Consider the two hypersurfaces $Z_1 = \{h_2 = 0\}, Z_2 = \{h_1 + h_2 = -1\}$ in M as shown in Figure 3.8. Near L_1, Δ is locally the product $L_1 \times (-\varepsilon, \varepsilon)$; near L_2, Δ is not locally a product. The vector field $u = \frac{\partial}{\partial h_1}$ in a neighborhood of Z_1 has the property that $dh_1(u) = 1$, and $\omega(Y^\#, u) = 0$ for all Y in the hypersurface of $\mathfrak{t}$ spanned by X_2. If we flow Z_1 along the vector field u, the image under μ would consist of the line segment L_1 moving with constant velocity in the direction perpendicular to $\langle X_2 \rangle$,

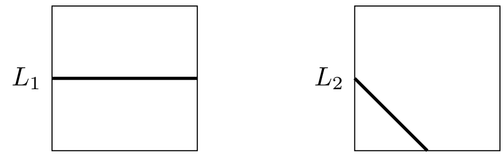

Figure 3.8: Hypersurfaces in $\mathbb{S}^2 \times \mathbb{S}^2$: $Z_1 = \mu^{-1}(L_1)$ and $Z_2 = \mu^{-1}(L_2)$.

showing once again that Δ is locally the product $L_1 \times (-\varepsilon, \varepsilon)$ near L_1. In contrast, there is no vector field u' in a neighborhood of Z_2 such that $d(h_1 + h_2)(u') = 1$ and $\omega(Y^{\#}, u') = 0$ for all Y in a hypersurface of $\mathfrak{t}$, reflecting the fact that Δ is not locally a product near L_2. The reason that no such u' exists is because every hypersurface of $\mathfrak{t}$ contains some Y such that $\iota_{Y^{\#}}\omega$ is a multiple of $d(h_1 + h_2)$ somewhere along Z_2 (making the condition that $d(h_1 + h_2)(u') = 1$ incompatible with $\omega(Y^{\#}, u') = 0$). In other words, the fact that Δ is locally a product near L_1 is reflected in the fact that $\ker(\iota_{Y^{\#}}\omega_z) \neq T_z Z_1$ for all $z \in Z_1$ and all Y in some hyperplane of $\mathfrak{t}_{Z_1}$.

In a neighborhood of the exceptional hypersurface Z of a b-manifold, a toric action will always behave similarly to Z_1, in the sense that the hyperplane $\mathfrak{t}_Z \subseteq \mathfrak{t}$ satisfies the property $\ker(\iota_{Y^{\#}}\omega_z) \neq T_z(Z)$ for all $z \in Z$ and all $Y \in \mathfrak{t}_Z$. This fact is the content of Lemma III.39 and will play an important role in the proof of Proposition III.40.

Lemma III.39. *Let $k < n$ and consider a Hamiltonian $\mathbb{T}^k$-action on (M^{2n}, Z, ω) for which $\iota_{X^{\#}}\omega \in \Omega^1(M)$ for each $X \in \mathfrak{t}$. Then for any $z \in Z$ and $X \in \mathfrak{t}$, $\ker(\iota_{X^{\#}}\omega_z) \neq T_z Z$.*

Proof. Let u be a vector field in a neighborhood of Z with the property that u is transverse to Z and is $\mathbb{T}^k$-invariant (for example, by picking any transverse vector field and averaging). Let Φ_t^u be the time-t flow along u. For sufficiently small ε

$$\phi : Z \times (-\varepsilon, \varepsilon) \to U, \quad (z, t) \mapsto \Phi_t^u(z)$$

is a diffeomorphism onto a neighborhood U of Z. Let p and y be the projections of $Z \times (-\varepsilon, \varepsilon)$ onto Z and $(-\varepsilon, \varepsilon)$ respectively. Then

$$\phi^*(\omega) = \frac{dt}{t} \wedge p^*(\alpha) + \beta$$

where $\alpha \in \Omega^1(Z)$ is given by $\iota_{\mathbb{L}}(\omega)$ and β is a smooth 2-form on $Z \times (-\varepsilon, \varepsilon)$.

Let $V \subseteq Z$ be a neighborhood of $z \in Z$ for which $\alpha\big|_V$ has a primitive $\theta' \in C^\infty(V)$, and define $\theta := p^*(\theta')$. Pick functions $\{x_i\}$ such that $\{y, \theta, x_1, \ldots, x_{2n-2}\}$ are coordinates in a neighborhood of $(z, 0) \in Z \times (-\varepsilon, \varepsilon)$. Then we can write $X^\#$ and $\phi^*\omega$ in these coordinates

$$X^\# = v_\theta \frac{\partial}{\partial \theta} + v_t \frac{\partial}{\partial t} + \sum_i v_i \frac{\partial}{\partial x_i}$$
$$\phi^*\omega = \frac{dt}{t} \wedge d\theta + w_{t\theta} dt \wedge d\theta + \sum_i (w_{ti} dt \wedge dx_i + w_{\theta i} d\theta \wedge dx_i) + \sum_{ij} w_{ij} dx_i \wedge dx_j$$

where the subscripted v's and w's are smooth functions. Because the kernel of the covector $\iota_{X^\#}\omega_z$ has dimension either $2n$ or $2n-1$, it is enough to show that if $z \in Z$ and $X \in \mathfrak{t}$ are such that $\ker(\iota_{X^\#}\omega_z) \supseteq T_z Z$, then actually $\iota_{X^\#}\omega_z = 0$, which happens exactly if the dt term of $\iota_{X^\#}\omega$ vanishes at z. The coefficient of the dt term of $\iota_{X^\#}\omega$ is

$$-\left(\frac{v_\theta}{t} + v_\theta w_{t\theta} + \sum_i v_i w_{ti}\right) \tag{3.3}$$

Because u was chosen to be $\mathbb{T}^k$-invariant, $\frac{\partial}{\partial t}$ is also $\mathbb{T}^k$-invariant, so

$$0 = \left[\frac{\partial}{\partial t}, X^\#\right](\theta) = \frac{\partial}{\partial t}\left(X^\#(\theta)\right) - X^\#\left(\frac{\partial}{\partial t}(\theta)\right) = \frac{\partial}{\partial t}\left(d\theta(X^\#)\right) = \frac{\partial}{\partial t}(v_\theta)$$

This shows that $\frac{\partial v_\theta}{\partial t}$ also vanishes at z. Because $X^\#$ is the Hamiltonian vector field of a smooth function H_X, it is tangent to the symplectic leaf at z and can be calculated (by Remark III.31) as the Hamiltonian vector field (using the symplectic form on the leaf) of the pullback of H_X to the leaf. Since we are assuming that $\ker(\iota_{X^\#}\omega_z)$

contains T_zZ (and in particular $T_z\mathcal{L}$), it follows that the pullback of H_X has a critical point at z. Therefore, $(X^{\#})_z = 0$, which proves that v_θ and all v_i vanish at z. This shows that the term (3.3) also vanishes at z, proving the claim. □

Proposition III.40. Let (M^{2n}, Z, ω) be a b-symplectic manifold with a toric action. Let c be the modular period of Z and $\mathcal{L}$ a leaf of its symplectic foliation. Pick a lattice element $X_b \in \mathfrak{t}$ that represents a generator of $\mathfrak{t}/\mathfrak{t}_Z$ and pairs positively with v_Z.

Then there is a neighborhood $\mathcal{L} \times \mathbb{S}^1 \times (-\varepsilon, \varepsilon) \cong U \subseteq M$ of Z such that the $\mathbb{T}^n$-action has moment map

$$\mu : \mathcal{L} \times \mathbb{S}^1 \times (-\varepsilon, \varepsilon) \to {}^b\mathfrak{t}^* \cong {}^b\mathbb{R} \times \mathfrak{t}_Z^*, \qquad (\ell, \rho, t) \mapsto (y_0 = t, \mu_{\mathcal{L}}(\ell)) \tag{3.4}$$

where the weight function on ${}^b\mathbb{R}$ is given by $\{0\} \mapsto c$, the map $\mu_{\mathcal{L}} : \mathcal{L} \to \mathfrak{t}_Z^*$ is a moment map for the $\mathbb{T}_Z^{n-1}$-action on $\mathcal{L}$, and the isomorphism ${}^b\mathfrak{t}^* \cong {}^b\mathbb{R} \times \mathfrak{t}_Z^*$ is the one described in Definition III.23 using X_b as the primitive lattice element.

Proof. Observe that the splitting $\mathfrak{t} \cong \langle X_b \rangle \oplus \mathfrak{t}_Z$ induces a splitting $\mathbb{T}^n \cong \mathbb{S}^1 \times \mathbb{T}^{n-1}$. Pick a primitive f_b of $\iota_{X_b}\omega$. Let $y_Z : U \to M$ be a defining function for Z corresponding to f_b (as defined in Remark III.19) in some neighborhood U of Z. Because f_b is $\mathbb{T}^n$-invariant, so too is y_Z, since the level sets of y_Z coincide with those of f_b. Our first goal is to pick a vector field u in a neighborhood of Z with the following three properties.

1. $dy_Z(u) = 1$
2. $\iota_{Y^{\#}}\omega(u) = 0$ for all $Y \in \mathfrak{t}_Z$
3. u is $\mathbb{T}^n$-invariant

To show that a vector field exists that satisfies conditions (1) and (2) simultaneously, it suffices to observe that for each $z \in Z$ and $Y \in \mathfrak{t}_Z$, $\ker(\iota_{Y^\#}\omega_z) \neq T_zZ$ by Lemma III.39. Let u be a vector field satisfying (1) and (2). Because dy_Z and each $\iota_{Y^\#}\omega$ are $\mathbb{T}^n$-invariant, we can average u by the $\mathbb{T}^n$-action without disturbing properties (1) and (2). Therefore, by replacing u with its $\mathbb{T}^n$-average we may assume that u is $\mathbb{T}^n$-invariant. Let Φ_t^u and $\Phi_t^{X_b^\#}$ be the time-t flows of u and $X_b^\#$ respectively. Then, using Corollary III.38, the map

$$\phi : \mathcal{L} \times \mathbb{S}^1 \times (-\varepsilon, \varepsilon) \to U, \quad (\ell, \rho, t) \mapsto \Phi_t^u \circ \Phi_\rho^{X_b^\#}(\ell)$$

is a diffeomorphism for sufficiently small ε. Let p and t be the projections of $\mathcal{L} \times \mathbb{S}^1 \times (-\varepsilon, \varepsilon)$ onto $Z \cong \mathcal{L} \times \mathbb{S}^1$ and $(-\varepsilon, \varepsilon)$ respectively. To study the induced $\mathbb{T}^n$-action on the domain of ϕ, fix some $(s, g) = (\exp(kX_b), \exp(Y)) \in \mathbb{S}^1 \times \mathbb{T}^{n-1}$ and recall that since u is $\mathbb{T}^n$-invariant, its flows commute with the flows of all $\{X^\# \mid X \in \mathfrak{t}\}$. If we denote the $\mathbb{T}^{n-1}$-action on $\mathcal{L}$ by $g \cdot_{\mathcal{L}} \ell$, then

$$\begin{aligned} \phi(g \cdot_{\mathcal{L}} \ell, \rho + s, t) &= \Phi_t^u \circ \Phi_{\rho+s}^{X_b^\#}(g \cdot_{\mathcal{L}} \ell) = \Phi_t^u \circ \Phi_\rho^{X_b^\#} \circ \Phi_s^{X_b^\#} \circ \Phi_1^{Y^\#}(\ell) \\ &= \Phi_s^{X_b^\#} \circ \Phi_1^{Y^\#} \phi(\ell, \rho, t) = (s, g) \cdot \phi(\ell, \rho, t) \end{aligned}$$

which shows that the induced $\mathbb{T}^n$-action on the domain is given by

$$(s, g) \cdot (\ell, \rho, t) = (g \cdot_{\mathcal{L}} \ell, \rho + s, t).$$

We will show that the moment map for this is given by (3.4). Notice that $\mu^{X_b} \in {}^bC^\infty(\mathcal{L} \times \mathbb{S}^1 \times (-\varepsilon, \varepsilon))$ is given by $c \log |t|$, and $X_b^\#$ is $\frac{\partial}{\partial \rho}$. Then

$$\iota_{X_b^\#}\phi^*\omega = \phi^*(\iota_{X_b^\#}\omega) = \phi^*(df_b) = d\phi^*(c \log |y_Z|) = (-1)^a c \frac{dt}{t} = d\mu^{X_b}$$

as required. To prove that $\iota_{Y^\#}(\phi^*\omega) = d\mu^Y$ for $Y \in \mathfrak{t}_Z$, first we define the map

$$p_{\mathcal{L}} : U \to \mathcal{L}, \quad \phi(\ell, \rho, t) \mapsto \ell$$

and observe that $p_{\mathcal{L}} \circ \phi(\ell, \rho, t) = \ell$. Also, since the map $p_{\mathcal{L}}$ can be realized at $\phi(\ell, \rho, t)$ as the time-$(-t)$ flow of u followed by the time-$(-\rho)$ flow of $X_b^{\#}$, both of which preserve $\iota_{Y^{\#}}\omega$, it follows that $p_{\mathcal{L}}^*(\iota_{Y^{\#}}\omega) = \iota_{Y^{\#}}\omega$. Then

$$\iota_{Y^{\#}}(\phi^*\omega) = \phi^*(\iota_{Y^{\#}}\omega) = \phi^* p_{\mathcal{L}}^*(\iota_{Y^{\#}}\omega) = (p_{\mathcal{L}} \circ \phi)^*(d\mu_{\mathcal{L}}^Y) = d\mu^Y$$

where the final equality follows from the fact that $p_{\mathcal{L}} \circ \phi(\ell, \rho, t) = \ell$. $\square$

Notice that if we had chosen X_b to be a generator of $\mathfrak{t}/\mathfrak{t}_Z$ that pairs negatively with v_Z, then by the discussion following Proposition III.24, the moment map for the action would be exactly the same, with the exception with y_1 appearing in the place of y_0, and the weight function given by $\{1\} \mapsto c$ instead of $\{0\} \mapsto c$. Also, notice that given a moment map $\mu : M \to {}^b\mathfrak{t}^*$, the restriction of μ to a single connected component W of $M \backslash Z$ gives a moment map (in a classic sense) by identifying each $\{a\} \times \mathfrak{t}^* \subseteq {}^b\mathfrak{t}^*$ with $\mathfrak{t}^*$.

$$\mu_W : W \to \mathfrak{t}^*$$

Restricting the moment map described in Proposition III.40 in this way gives the following result.

Corollary III.41. *Let (M^{2n}, Z, ω) be a b-symplectic manifold with a toric action and assume that Z is connected. Let W be a connected component of $M \backslash Z$. Then there is a neighborhood $U \subseteq M$ of Z such that the $\mathbb{T}^n$-action on $U \cap W$ has moment map with image equal to the Minkowski sum.*

$$\Delta + \{kv_Z \mid k \in \mathbb{R}^-\}$$

where $\Delta \subseteq \mathfrak{t}^*$ is an affinely embedded copy of the image of $\mu_{\mathcal{L}} : \mathcal{L} \to \mathfrak{t}_Z^*$ into $\mathfrak{t}^*$.

The next proposition describes a local model for the b-symplectic manifold in a neighborhood of Z. We will use it in the upcoming generalization of Delzant's theorem to show that the moment map is unique.

Proposition III.42. (Local Model) Let (M^{2n}, Z, ω) be a b-symplectic manifold with a toric action and assume Z is connected. Fix ${}^b\mathfrak{t}^*$ with $wt(1) = c$ and some $X \in \mathfrak{t}$ representing a lattice generator of $\mathfrak{t}/\mathfrak{t}_Z$ that pairs positively with the distinguished vector v, inducing an isomorphism ${}^b\mathfrak{t}^* \cong {}^b\mathbb{R} \times \mathfrak{t}_Z^*$. For any Delzant polytope $\Delta \subseteq \mathfrak{t}_Z^*$ with corresponding symplectic toric manifold $(X_\Delta, \omega_\Delta, \mu_\Delta)$, define the *local model* b-symplectic manifold as

$$M_{\mathrm{lm}} = X_\Delta \times \mathbb{S}^1 \times \mathbb{R} \qquad \omega_{\mathrm{lm}} = \omega_\Delta \times c\frac{dt}{t} \wedge d\theta$$

where θ and t are the coordinates on $\mathbb{S}^1$ and $\mathbb{R}$ respectively. The $\mathbb{S}^1 \times \mathbb{T}_Z$ action on M_{lm} given by $(\rho, g)\cdot(x, \theta, t) = (g\cdot x, \theta+\rho, t)$ has moment map $\mu_{\mathrm{lm}}(x, \theta, t) = (y_0 = t, \mu_\Delta(x))$.

For any toric action on a b-manifold (M, Z, ω) with moment map μ such that $\mu(U) = (-\epsilon \le y_0 \le \epsilon) \times \Delta$ in a neighborhood U of Z, there is an equivariant b-symplectomorphism $\varphi : M_{\mathrm{lm}} \to M$ in a neighborhood of $X_\Delta \times \mathbb{S}^1 \times \{0\}$ satisfying $\mu \circ \varphi = \mu_{\mathrm{lm}}$.

Proof. Fix a symplectic leaf $\mathcal{L} \subseteq Z$. Because μ maps Z surjectively to $\{y_0 = 0\} \times \Delta$ and μ is $\mathbb{T}^n$-invariant, it must be the case that $\mathrm{im}(\mu\big|_{\mathcal{L}}) = \{y_0 = 0\} \times \Delta$. Define $\mu_{\mathcal{L}} : \mathcal{L} \to \mathfrak{t}_Z^*$ to be the projection of $\mu\big|_{\mathcal{L}}$ onto its second coordinate. By the classic Delzant theorem there is an equivariant symplectomorphism $\varphi_\Delta : (X_\Delta, \omega_\Delta) \to (\mathcal{L}, \omega_{\mathcal{L}})$ such that $\mu_\Delta = \mu_{\mathcal{L}} \circ \varphi_\Delta$. As in the proof of Proposition III.40, let y_Z be a local defining function for Z corresponding to a primitive of $\iota_X\omega$ and let u be a $\mathbb{T}^n$-equivariant vector field in a neighborhood of Z, such that $dy_Z(u) = 1$ and $\iota_{Y^\#}\omega(u) = 0$ for all $Y \in \mathfrak{t}_Z$. Then the map

$$\varphi : M_{\mathrm{lm}} = X_\Delta \times \mathbb{S}^1 \times \mathbb{R} \to M, \quad (x, \theta, t) \mapsto \Phi_t^u \circ \Phi_\theta^{X^\#} \circ \varphi_\Delta(x)$$

is defined in a neighborhood of $X_\Delta \times \mathbb{S}^1 \times \{0\}$.

It follows by the equivariance of $u, X^\#$, and φ_Δ that φ itself is equivariant. Next, observe that

$$\mu \circ \varphi(x, \theta, t) = \mu \circ \Phi_\theta^{X^\#} \circ \Phi_t^u \circ \varphi_\Delta(x) = \mu \circ \Phi_t^u \circ \varphi_\Delta(x)$$

since μ is $\mathbb{T}^n$-invariant. Observe that the $\mathfrak{t}_Z^*$-component of $\mu \circ \Phi_t^u \circ \varphi_\Delta(x)$ will equal $\varphi_\Delta(x)$, since $\iota_{Y^\#}\omega(u) = 0$ for all $Y \in \mathfrak{t}_Z$. The ${}^b\mathbb{R}$-component of $\mu \circ \Phi_t^u \circ \varphi_\Delta(x)$ will equal t, since the $X^\#$ action is generated by the b-function $(y_0 = y_Z)$ and the vector field u satisfies $dy_Z(u) = 1$. Therefore, $\mu \circ \varphi = \mu_{\text{lm}}$.

This shows that on the b-symplectic manifolds $(M_{\text{lm}}, \omega_{\text{lm}})$ and $(M_{\text{lm}}, \varphi^*(\omega))$, the same moment map μ_{lm} corresponds to the same action. Our next goal is to show that $\varphi^*\omega\big|_Z = \omega_{\text{lm}}\big|_Z$. For $z \in Z$, let $A \subseteq {}^bT_zM$ be the symplectic orthogonal to $(X^\#)_z$. Restriction of the canonical map ${}^bT_zM \to T_zM$ to A leaves its image unchanged (since the kernel of the canonical map, $\mathbb{L}$, is not in A). By picking a basis for $T_zL \subseteq T_zZ$ and pulling it back to A, and then adding $(X^\#)_z$ and $(t\frac{\partial}{\partial t})_z$, we obtain a basis of ${}^bT_zZ$. By calculating the value of ω_z with respect to this basis, and using the facts that φ_Δ is a symplectomorphism and that

$$\varphi^*\omega(t\frac{\partial}{\partial t}, \frac{\partial}{\partial \theta}) = \omega(y_Z u, X^\#) = d(c \log |y_Z|)(y_Z u) = c,$$

we conclude that $\varphi^*\omega\big|_Z = \omega_{\text{lm}}\big|_Z$. To complete the proof, we will carefully apply Moser's path method to construct a symplectomorphism between $\varphi^*\omega$ and ω_{lm}.

Note that $\varphi^*\omega - \omega_{\text{lm}}$ is $\mathbb{T}^n$-invariant and has the property that the tangent space to each orbit is in the kernel of $\varphi^*\omega - \omega_{\text{lm}}$. Therefore, we can write $\varphi^*\omega - \omega_{\text{lm}}$ as the pullback under μ_{lm} of a smooth form ν on ${}^b\mathbb{R} \times \mathfrak{t}_Z^*$. Let α be the pullback (under μ_{lm}) of a primitive of ν. Then α is a primitive of $\varphi^*\omega - \omega_{\text{lm}}$ with the property that the vector fields defined using Moser's path method will be tangent to the orbits of the torus action, and also with the property that α is torus invariant. Therefore, the equivariant symplectomorphism it defines leaves the moment map unchanged, completing the proof.

□

3.3.2 Global picture

Let $(M^{2n}, Z, \omega, \mathbb{T}^n)$ be a b-symplectic manifold with a toric action. As before, for a connected component W of $M \backslash Z$, we write $\mu_W : W \to \mathfrak{t}^*$ for the moment map on W induced by μ.

Claim III.43. The image $\mu_W(W)$ is convex.

Proof. Let $Z_1, \dots, Z_r$ be the connected components of Z which are in the closure of W. By Proposition III.40, we can find a function t_i in a neighborhood of Z_i for which an $\mathbb{S}^1$ factor of the $\mathbb{T}^n$-action is generated by the Hamiltonian $c \log |t_i|$ for some $c \neq 0$. Define $W_{\geq \varepsilon} \subseteq W$ to be $W \backslash \{|t_i| < \varepsilon\}$, let $W_{=\varepsilon}$ be its boundary, and let $W_{>\varepsilon} = W_{\geq\varepsilon} \backslash W_{=\varepsilon}$. Figure 3.9 shows W with $W_{>\varepsilon}$ shaded.[4]

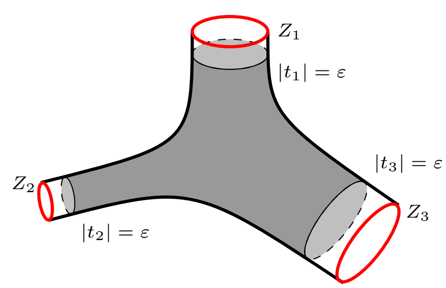

Figure 3.9: A connected component W of $M \backslash Z$ and the open subset $W_{>\varepsilon}$.

Performing a symplectic cut at $W_{=\varepsilon}$ gives a compact symplectic toric manifold $\overline{W_{\geq\varepsilon}}$ which has an open subset canonically identified with $W_{>\varepsilon}$. Let $\mu_{W,\varepsilon} : \overline{W_{\geq\varepsilon}} \to \mathfrak{t}^*$ be the moment map for the toric action on $\overline{W_{\geq\varepsilon}}$ that agrees with μ_W on $W_{>\varepsilon}$.

[4] With Claims III.44 and III.45 we will see that the number of connected components of Z adjacent to W can be at most two. We have drawn three connected components of Z adjacent to W in Figure 3.9 so that the figure is more pedagogically effective at the expense of accuracy.

To show that $\mu_W(W)$ is convex, pick points $\mu_W(p), \mu_W(q)$ in $\mu_W(W)$ and fix some $\varepsilon > 0$ small enough that that $p, q \in W_{>\varepsilon}$. Because $\overline{W_{\geq\varepsilon}}$ is compact, $\mu_{W,\varepsilon}(\overline{W_{\geq\varepsilon}})$ contains the straight line joining $\mu_W(p) = \mu_{W,\varepsilon}(p)$ and $\mu_W(q) = \mu_{W,\varepsilon}(q)$. Since $\mu_{W,\varepsilon}(\overline{W_{\geq\varepsilon}}) \subseteq \mu_W(W)$, the image $\mu_W(W)$ also contains the straight line joining $\mu_W(p)$ and $\mu_W(q)$. $\square$

By Corollary III.41, we know that for each connected component $Z' \subseteq Z$ adjacent to W, there is a neighborhood U of Z' such that $\mu_W(U_i \cap W)$ is the product of a Delzant polytope with the ray generated by $-v_{Z'}$. By performing symplectic cuts near the hypersurfaces adjacent to W (as in the proof of Claim III.43) to partition the image of μ_W into a convex set and these infinite prisms, we see that the convex set $\mu_W(W)$ extends indefinitely in precisely the directions

$$\{-v_{Z'}\}_{Z' \text{ is adjacent to } W}. \tag{3.5}$$

Claim III.44. Each of these directions occupy the same one-dimensional subspace of $\mathfrak{t}^*$. That is, $\mathfrak{t}_{Z'}$ is independent of the choice of component $Z' \subseteq Z$.

Proof. Pick $x_1, x_2 \in \mu_W(W)$ such that the rays

$$\{x_1 + tv_1 \mid t \in \mathbb{R}_{>0}\} \qquad \text{and} \qquad \{x_2 + tv_2 \mid t \in \mathbb{R}_{>0}\}$$

are both in $\mu_W(W)$ (for example, by taking x_1 and x_2 to be images of points in the neighborhoods of Z_1 and Z_2 described in in Corollary III.41). By convexity of $\mu_W(W)$ the point y_t below is in $\mu_W(W)$ for all $t \geq 0$ and any $\lambda \in [0, 1]$.

$$y_t = \lambda(x_1 + tv_1) + (1-\lambda)(x_2 + tv_2) = \lambda x_1 + (1-\lambda)x_2 + t(\lambda v_1 + (1-\lambda)v_2)$$

which proves that there is a ray in $\mu_W(W)$ that extends infinitely far in the $(\lambda v_1 + (1-\lambda)v_2)$ direction. Because there are only finitely many directions in which $\mu_W(W)$ extends indefinitely far, x_1 must be a scalar multiple of x_2. $\square$

As a consequence of Claim III.44 and of convexity, we must have that $\mu_W(W)$ extends indefinitely in one direction or in two opposite directions. The next claim shows that each of these “infinite directions” corresponds to only one connected component of Z.

Claim III.45. Suppose that Z_1 and Z_2 are two different connected components of Z both adjacent to the same connected component W of $M \backslash Z$. Then $v_{Z_1} = kv_{Z_2}$ for some $k < 0$.

Proof. By Claim III.44, $v_{Z_1} = kv_{Z_2}$ for some $k \in \mathbb{R}$, and by Claim III.35, $k \neq 0$. It suffices, therefore, to prove that k cannot be positive. Assume towards a contradiction that k is positive, and pick a lattice element $X \in \mathfrak{t}$ such that $\langle X, v_{Z_1} \rangle = 1$, and let $H : W \to \mathbb{R}$ be a Hamiltonian for the $\mathbb{S}^1$-action generated by X. By performing symplectic cuts sufficiently close to the components of Z adjacent to W (as in the proof of Claim III.43) and using the fact that the level sets of moment maps on compact connected symplectic manifolds are connected, it follows that the level set $H^{-1}(\lambda)$ is connected for any $\lambda \in \mathbb{R}$. In a neighborhood of Z_1 and of Z_2, the function H approaches negative infinity. Therefore, for sufficiently large values of N, the level set $H^{-1}(-N)$ has a connected component completely contained in a neighborhood of Z_1 and another connected component completely contained in a neighborhood of Z_2. Because $H^{-1}(-N)$ has just one connected component, $Z_1 = Z_2$. $\square$

In particular, this means that in M, each component of $M \backslash Z$ is adjacent to at most two connected components of Z.

Definition III.46. The **adjacency graph** G_M of a b-manifold (M, Z) is a graph with a vertex for each component of $M \backslash Z$ and an edge for each connected component of Z that connects the vertices corresponding to the components of $M \backslash Z$ that it separates.

When (M^{2n}, Z, ω) has an effective toric action, this graph is either a loop or a line, as illustrated in Figure 3.10. If it is a loop, Claim III.45 implies that it must have an even number of vertices.

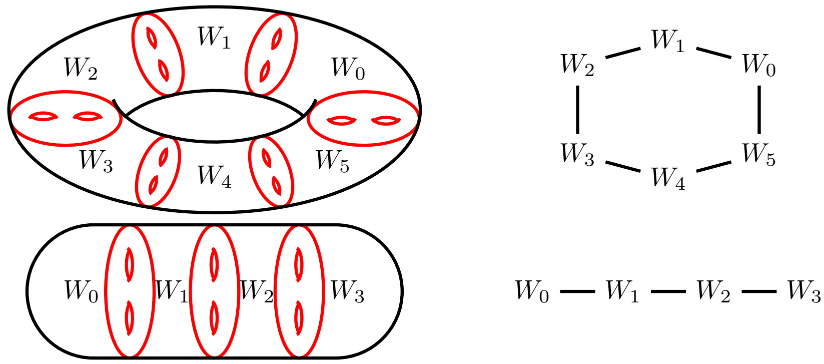

Figure 3.10: The adjacency graph is either a cycle of even length or a line.

We are finally ready to prove the main theorem of of this section: that every b-symplectic manifold with a toric action has a moment map.

Theorem III.47. *Let (M, Z, ω) be a b-symplectic manifold with a toric action. For an appropriately-chosen ${}^b\mathfrak{t}^*$ or ${}^b\mathfrak{t}^*/\langle N\rangle$, there is a moment map $\mu : M \to {}^b\mathfrak{t}^*$ or $\mu : M \to {}^b\mathfrak{t}^*/\langle N\rangle$.*

Proof. Consider the adjacency graph of the connected components of $M\backslash Z$ as described in Figure 3.10. We first consider the case when the graph is a line. Number the components $W_0, \ldots, W_{N-1}$ as described, and let Z_i be the connected hypersurface between W_{i-1} and W_i. Let c_i be the modular period of Z_i. Then consider the ${}^b\mathfrak{t}^*$ defined using weight function $\mathrm{wt}(i) = c_i$ and the primitive lattice vector in the direction of $-v_{Z_1}$. Fix any moment map $\mu_{W_0} : W_0 \to \mathfrak{t}^*$ for the action on W_0. By identifying the codomain $\mathfrak{t}^*$ of this moment map with $\{0\} \times \mathfrak{t}^* \subseteq {}^b\mathfrak{t}^*$, we get a moment map $\mu_{W_0} : W_0 \to {}^b\mathfrak{t}^*$. By Proposition III.40, there is a moment map μ_{U_1} for the $\mathbb{T}^n$-action in a neighborhood U_1 of Z_1. The two moment maps

$$\mu_{W_0}\big|_{W_0 \cap U_1} \quad \text{and} \quad \mu_{U_1}\big|_{W_0 \cap U_1}$$

correspond to the same $\mathbb{T}^n$-action on $W_0 \cap U_1$, so by postcomposing μ_{U_1} with a translation we may glue μ_{W_0} and μ_{U_1} into a moment map defined on all of $W_0 \cup U_1$. We continue extending the moment map in this manner until it is a moment map μ defined on all of M. As a consequence of this construction, notice that μ maps the component W_i into $\{i\} \times \mathfrak{t}^* \subseteq {}^b\mathfrak{t}^*$; this motivates the decision to label the components of $M \backslash Z$ starting with 0 instead of 1.

When the adjacency graph is a cycle, consider performing the above construction using the weight function defined on $\mathbb{Z}$ which is N-periodic with $\mathrm{wt}(i) = c_i$ for $0 \leq i \leq N-1$. The construction breaks down in the final stage; after choosing the correct translation of the moment map μ_{U_N} so that it agrees with $\mu_{W_{N-1}}$ on the overlap of their domains, it will not be the case that μ_{U_N} agrees with μ_{W_0} on the overlap of their domains. Pick some $p \in U_N \cap W_0$, and define

$$x_{\text{start}} = \mu_{W_0}(p) \quad \text{and} \quad x_{\text{end}} = \mu_{U_N}(p)$$

and assume without loss of generality that $x_{\text{start}} = (0,0) \in \mathbb{Z} \times \mathfrak{t}^* \subseteq {}^b\mathfrak{t}^*$. Let $\gamma : \mathbb{S}^1 = \mathbb{R}/\mathbb{Z} \to M$ be a loop with $\gamma(0) = \gamma(1) = p$ that visits the sets $W_0, U_1, W_1, \ldots, W_{N-1}, U_N$ in that order. Then, for any $X \in \mathfrak{t}$, we have

$$x_{\text{end}} = (N, x) \quad \text{where } \mu^X(x) = \begin{cases} \int_\gamma \iota_{X^\#}\omega & X \in \mathfrak{t}_Z \\ {}^b\!\!\int_\gamma \iota_{X^\#}\omega & X \notin \mathfrak{t}_Z \end{cases}$$

When $X \in \mathfrak{t}_Z$, the 1-form $\iota_{X^\#}\omega$ has a smooth primitive, so this integral equals zero. When $X \notin \mathfrak{t}_Z$, the 1-form $\iota_{X^\#}\omega$ does not have a smooth primitive, but still has a ${}^bC^\infty$ primitive, and the Liouville volume of the pullback is still zero. And therefore the integral equals zero. Therefore, $x_{\text{end}} = (N, 0)$ and the moment maps for each the sets W_i and U_i glue into a moment map $\mu : M \to {}^b\mathfrak{t}^*/\langle N \rangle$. $\square$

Theorem III.47 proves that every toric action on a b-symplectic manifold has a moment map. However, as in the classic case (where different translations of the moment map correspond to the same action), the moment map is not uniquely determined by the action. To better understand the diversity of moment maps that can correspond to a torus action on a b-manifold, we review the arbitrary choices made during the construction of the moment map. Clearly, the adjacency graph as well as the modular periods are determined uniquely by the b-symplectic manifold, but the labelling of the vertices is not. When the graph is a line, we chose which leaf of the vertex to label W_0 and which to label W_{N-1}; when the graph is a cycle, we chose which vertex to label W_0 and in which direction around the cycle the graph should increase (or when $N = 2$, which lattice generator of $\mathfrak{t}^*_Z$ to distinguish in the construction of ${}^b\mathfrak{t}^*$). As such, the moment map is unique not only up to translation, but also up to certain permutations of the domain of the weight function and possibly a different choice of distinguished element of $\mathfrak{t}^*_Z$. The effect of changing the distinguished lattice vector necessitates a notational juggling which is described in Proposition III.24 and illustrated in Figure 3.5. For the upcoming Delzant theorem, we will incorporate into our definition of a *b-symplectic toric manifold* the data of a moment map. Not only does this follow the precedent of the classic definition of a symplectic toric manifold, but it also relieves from us the burden of following and notating these choices made in constructing a moment map.

3.4 Delzant theorem

In this section, we prove a Delzant theorem in b-geometry. We first define the notion of a b-symplectic toric manifold, and a Delzant b-polytope.

Definition III.48. A **b-symplectic toric manifold** is

$$(M^{2n}, Z, \omega, \mu : M \to {}^b\mathfrak{t}^*) \text{ or } (M^{2n}, Z, \omega, \mu : M \to {}^b\mathfrak{t}^*/\langle a\rangle)$$

where (M, Z, ω) is a b-symplectic manifold and μ is a moment map for a toric action on (M, Z, ω).

Notice that the definition of a b-symplectic toric manifold also implicitly includes the information of a weight function and a distinguished lattice vector used to construct ${}^b\mathfrak{t}^*$. As in the classic case, the definition of a polytope in ${}^b\mathfrak{t}^*$ will use the definition of a half-space in ${}^b\mathfrak{t}^*$. The definition of a half-space is an intuitive concept obfuscated by notation; we encourage the reader to look at the examples in Figure 3.11 before reading the formal definition. Although the boundaries of the half-spaces in Figure 3.11 appear curved, they are actually straight lines when restricted to each $\{k\} \times \mathfrak{t}^* \cong \mathfrak{t}^*$; they appear curved only because of the way ${}^b\mathfrak{t}^*$ is drawn. Notice that the boundary of a half-space will not intersect $Z_{{}^b\mathfrak{t}^*}$ unless it is perpendicular to it.

Definition III.49. For a fixed ${}^b\mathfrak{t}^*$ with weight function with domain $[a, N]$ for $a \in \{0, 1\}$ and distinguished vector $v \in \mathfrak{t}^*$, consider the two following kinds of hypersurfaces in ${}^b\mathfrak{t}^*$, where $X \in \mathfrak{t}$, $Y \in v^\perp$, $k \in \mathbb{R}$ and $c \in [a-1, N]$:

$$A_{X,k,c} = \{(c, \xi) \mid \langle \xi, X\rangle = k\} \subseteq \{c\} \times \mathfrak{t}^* \subseteq {}^b\mathfrak{t}^*,$$

$$B_{Y,k} = \overline{\{(c, \xi) \mid \langle \xi, Y\rangle = k, c \in [a-1, N]\}} \subseteq \overline{[a-1, N] \times \mathfrak{t}^*} = {}^b\mathfrak{t}^*.$$

The complement of any such hypersurface is two connected components in ${}^b\mathfrak{t}^*$. The closure of any such connected component is a **half-space** in ${}^b\mathfrak{t}^*$. The same definitions

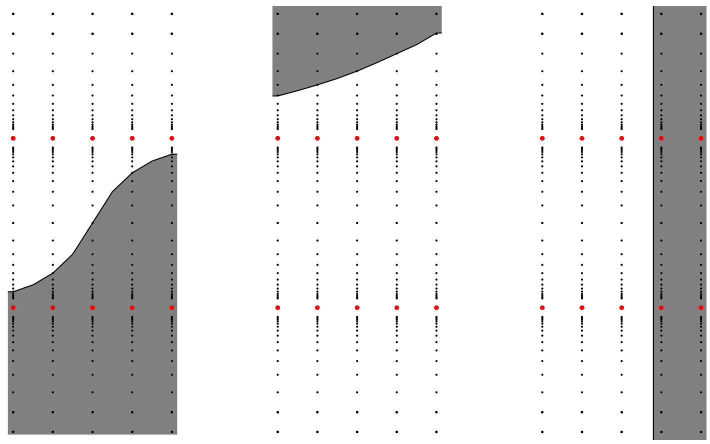

Figure 3.11: Examples of half-spaces in ${}^{b}\mathfrak{t}^*$.

of $A_{X,k,c}$ and $B_{X,k}$ also define hypersurfaces in ${}^{b}\mathfrak{t}^*/\langle N\rangle$ when N is even. But in this case, the hypersurfaces of type $A_{X,k,c}$ do not separate the space. Therefore, only the closure of a connected component of the complement of some $B_{X,k} \subseteq {}^{b}\mathfrak{t}^*/\langle N\rangle$ is called a **half-space** in ${}^{b}\mathfrak{t}^*/\langle N\rangle$.

In Figure 3.11, the first two images are examples of a half-space corresponding to some $A_{X,k,c}$, while the rightmost image is an example of a half-space corresponding to $B_{X,k}$.

Definition III.50. A ***b*-polytope** in ${}^{b}\mathfrak{t}^*$ (or ${}^{b}\mathfrak{t}^*/\langle N\rangle$) is a bounded subset P that intersects each component of $Z_{{}^{b}\mathfrak{t}^*}$ (or $Z_{{}^{b}\mathfrak{t}^*/\langle N\rangle}$) and can be expressed as a finite intersection of half-spaces.

If the condition that P must intersect each component of $Z_{{}^{b}\mathfrak{t}^*}$ were removed from the definition of a polytope, then for any pair of weight functions $\mathrm{wt}' : [a, N'] \to \mathbb{R}_{>0}, \mathrm{wt} : [a, N] \to \mathbb{R}_{>0}$ such that wt' extends wt, any polytope in ${}_{\mathrm{wt}}^{b}\mathfrak{t}^*$ would also be a polytope in ${}_{\mathrm{wt}'}^{b}\mathfrak{t}^*$ under the inclusion ${}_{\mathrm{wt}}^{b}\mathfrak{t}^* \subseteq {}_{\mathrm{wt}'}^{b}\mathfrak{t}^*$. The upcoming statement of Theorem III.54, which generalizes the Delzant theorem, is easier to state when we disallow this non-uniqueness of the weight function.

Example III.51. Figure 3.12 shows two examples of b-polytopes. In both cases, the torus has dimension two. The polytope on the left is a subset of ${}^b\mathfrak{t}^* \cong {}^b\mathbb{R} \times \mathbb{R}$, and the polytope on the right is a subset of ${}^b\mathfrak{t}^*/\langle 2 \rangle$ (the top of the picture on the right is identified with the bottom of the picture).

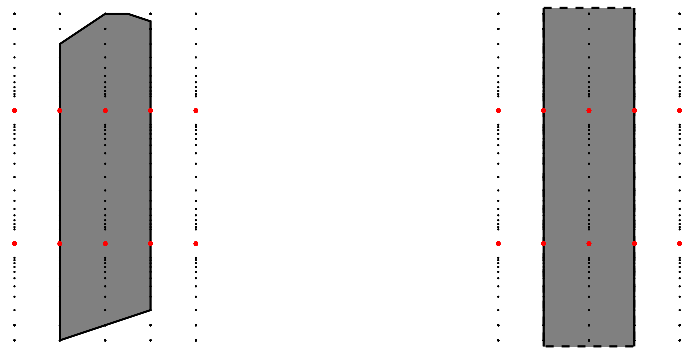

Figure 3.12: Examples of a polytope in ${}^b\mathfrak{t}^*$ and one in ${}^b\mathfrak{t}^*/\langle N \rangle$.

The definitions of many features of classic polytopes, such as *facets*, *edges*, and *vertices*, generalize in a natural way to b-polytopes, as does the notion of a *rational* polytope (one in which the X's and k's in Definition III.49 are rational). We state some properties of b-polytopes, all of which are straightforward consequences of the definition.

- The hypersurface $A_{X,k,c}$ separates $Z_{{}^b\mathfrak{t}^*} = [a, N] \times \mathfrak{t}_Z^*$ into $[a, c-1] \times \mathfrak{t}_Z^*$ and $[c, N] \times \mathfrak{t}_Z^*$. Because of the condition that P must intersect each component of $Z_{{}^b\mathfrak{t}^*}$, the only hypersurfaces of type $A_{X,k,c}$ that will appear as boundaries of half-spaces constituting P will have $c = a - 1$ or $c = N$.

- No vertex of P lies on $Z_{\mathfrak{t}^*}$.

- Given a polytope $P \subseteq {}^b\mathfrak{t}^*$, there is a (classic) polytope $\Delta_Z \subseteq \mathfrak{t}_Z^*$ having the property that the intersection of P with each component of $Z_{{}^b\mathfrak{t}^*}$ is Δ_Z.

- P is locally isomorphic to $\{-\varepsilon \leq y_i \leq \varepsilon\} \times \Delta_Z \subseteq {}^b\mathbb{R} \times \mathfrak{t}_Z^*$ near each component of $Z_{^b\mathfrak{t}^*}$, and is isomorphic to $\Delta_Z \times \mathbb{R}$ in any component $\{i\} \times \mathfrak{t}^* \cong \mathfrak{t}^*$ except $i \in \{a-1, N\}$.

- Any polytope in ${}^b\mathfrak{t}^*$ is isomorphic to $\Delta_Z \times {}^b\mathbb{R}$.
- For $i \in \{a-1, N\}$, the restriction of P to $\{i\} \times \mathfrak{t}^*$ is a polyhedron with recession cone equal to $\mathbb{R}_{\geq 0} \cdot v$ (if i is even) or $\mathbb{R}_{\geq 0} \cdot (-v)$ (if i is odd), where v is the distinguished direction in $\mathfrak{t}^*$ used to define ${}^b\mathfrak{t}^*$.

Because no vertex of P lies on $Z_{\mathfrak{t}^*}$, the definition of a Delzant polytope generalizes easily to the context of b-polytopes.

Definition III.52. A b-polytope $P \subseteq {}^b\mathfrak{t}^*$ is **Delzant** if for every vertex v of P, there is a lattice basis $\{u_i\}$ of $\mathfrak{t}^*$ such that the edges incident to v can be written near v in the form $v + tu_i$ for $t \geq 0$. A b-polytope $P \subseteq {}^b\mathfrak{t}^*/\langle N \rangle$ (which has no vertices) is **Delzant** if the polytope $\Delta_Z \subseteq \mathfrak{t}_Z^*$ is Delzant.

The left polytope in Figure 3.12 is not Delzant – the Delzant condition is not satisfied at the vertex at the top of the picture in the center column of lattice points. However, the Delzant condition is satisfied at all other vertices. The right polytope in Figure 3.12 is Delzant. Given a Delzant b-polytope P, the intersection of P with a component of $Z_{^b\mathfrak{t}^*}$ (or $Z_{^b\mathfrak{t}^*/\langle N \rangle}$) is a Delzant polytope in $\mathfrak{t}_Z^*$. By the properties of b-polytopes, it follows that this Delzant polytope does not depend on which component of $Z_{^b\mathfrak{t}^*}$ (or $Z_{^b\mathfrak{t}^*/\langle N \rangle}$) is chosen.

Definition III.53. Given a b-polytope P, the **extremal polytope** Δ_P is the Delzant polytope in $\mathfrak{t}_Z^*$ given by $P \cap Z'$, where Z' is any connected component of $Z_{^b\mathfrak{t}^*}$ (or $Z_{^b\mathfrak{t}^*/\langle N \rangle}$).

Theorem III.54. *For a fixed primitive lattice vector $v \in \mathfrak{t}^*$ and weight function $wt : [1, N] \to \mathbb{R}_{>0}$, the maps*

$$\left\{ \begin{array}{c} b\text{−}symplectic\ toric\ manifolds \\ (M, Z, \omega, \mu : M \to {}^b\mathfrak{t}^*) \end{array} \right\} \to \left\{ \begin{array}{c} \textit{Delzant b-polytopes} \\ \textit{in } {}^b\mathfrak{t}^* \end{array} \right\} \tag{3.6}$$

and

$$\left\{ \begin{array}{c} b\text{−}symplectic\ toric\ manifolds \\ (M, Z, \omega, \mu : M \to {}^b\mathfrak{t}^*/\langle N \rangle) \end{array} \right\} \to \left\{ \begin{array}{c} \textit{Delzant b-polytopes} \\ \textit{in } {}^b\mathfrak{t}^*/\langle N \rangle \end{array} \right\} \tag{3.7}$$

that send a b-symplectic toric manifold to the image of its moment map are bijections. Here, the sets on the left should be considered as equivalent up to equivariant b-symplectomorphisms that preerve the moment map.

Proof. To prove surjectivity in the ${}^b\mathfrak{t}^*$ case, let P be a Delzant b-polytope, and first construct the (classic) symplectic toric manifold $(X_Z, \omega_Z, \mu_Z : X_Z \to \mathfrak{t}_Z^*)$ associated with the extremal polytope Δ_P. Pick some $X \in \mathfrak{t}$ that pairs positively with the distinguished vector in the definition of ${}^b\mathfrak{t}^*$, which induces an identification ${}^b\mathfrak{t}^* \cong {}^b\mathbb{R} \times \mathfrak{t}_Z^*$. Let I be a closed interval in ${}^b\mathbb{R}$ large enough that $I \times \Delta_{Z_{{}^b\mathfrak{t}^*}} \supseteq P$. Let $(\mathbb{S}^2, Z_S, \omega_S, \mu_S : \mathbb{S}^2 \to {}^b\mathbb{R})$ be a symplectic toric manifold having $I \subseteq {}^b\mathbb{R}$ as its moment map image. Then

$$(\mathbb{S}^2 \times X_Z, \omega_S \times \omega_Z, (\mu_S, \mu_Z))$$

is a symplectic toric b-manifold having $I \times \Delta_{Z_{{}^b\mathfrak{t}^*}}$ as the image of its moment map. By performing a sequence of symplectic cuts inside $\{0\} \times \mathfrak{t}^*$ and $\{N-1\} \times \mathfrak{t}^*$, we arrive at a symplectic toric manifold having P as its moment map image.

To prove surjectivity in the ${}^b\mathfrak{t}^*/\langle N \rangle$ case, we again begin by constructing the (classic) symplectic toric manifold $(X_Z, \omega_Z, \mu_Z : X_Z \to \mathfrak{t}_Z^*)$ associated with the polytope $\Delta_{Z_{{}^b\mathfrak{t}^*}}$. Pick some $X \in \mathfrak{t}$ that pairs positively with the distinguished vector in the definition of ${}^b\mathfrak{t}^*$, which induces an identification ${}^b\mathfrak{t}^*/\langle N\rangle \cong {}^b\mathbb{R}/\langle N\rangle \times \mathfrak{t}_Z^*$. Let $(\mathbb{T}^2, Z_T, \omega_T, \mu_T : T^2 \to {}^b\mathbb{R}/\langle N\rangle)$ be a symplectic toric manifold having all of ${}^b\mathbb{R}/\langle N\rangle$ as its moment map image. Then

$$(\mathbb{T}^2 \times X_Z, \omega_T \times \omega_Z, (\mu_T, \mu_Z))$$

is a symplectic toric b-manifold having P as the image of its moment map.

The proof of injectivity is inspired by the proof of Proposition 6.4 in [LT97]. We prove the statement when the adjacency graph is a line; the proof is the same (with different notation) in the case when the adjacency graph is a cycle. Let (M, Z, ω, μ) and (M', Z', ω', μ') be two different symplectic toric manifolds having the same moment map image. Pick $X \in \mathfrak{t}$ that pairs positively with the distinguished vector $v \in \mathfrak{t}^*$. For each component $\{i\} \times \mathfrak{t}_Z^*$ of $Z_{{}^b\mathfrak{t}^*}$, by Proposition III.42 there is an $\varepsilon_i > 0$ such that there is an equivariant isomorphism $\varphi_{Z_i} : \mu^{-1}(P_{Z_i}) \to \mu'^{-1}(P_{Z_i})$, where

$$P_{Z_i} = \{-\varepsilon \le y_a \le \varepsilon\} \times \Delta_Z \subseteq P \subseteq {}^b\mathbb{R} \times \mathfrak{t}_Z^*$$

Similarly, for any $N > 0$, there is an equivariant isomorphism $\varphi_{W_i} : \mu^{-1}(P_{W_i}) \to \mu'^{-1}(P_{W_i})$, where $P_{W_i} = \{i\} \times (-N, N) \times \mathfrak{t}_Z^* \subseteq {}^b\mathfrak{t}^*$. Pick N sufficiently large that the open sets $\{P_{W_i}\} \cup \{P_{Z_i}\}$ cover P: see Figure 3.13.

If the equivariant isomorphisms φ_{Z_i} and φ_{W_j} agreed on $U_{ij} := \mu^{-1}(P_{W_i} \cap P_{Z_j})$ for all i, j, then we could glue these isomorphisms together and the proof of injectivity would be complete. Therefore, it suffices to show for every U_{ij} that there is an equivariant automorphism ψ_{W_i} of $\mu^{-1}(P_{W_i})$ such that

$$\varphi_{W_i} \circ \psi_{W_i}\big|_{U_{ij}} = \varphi_{Z_j}\big|_{U_{ij}} \quad \text{and} \quad \varphi_{W_i} \circ \psi_{W_i}\big|_{U_{ik}} = \varphi_{W_j}\big|_{U_{ik}}$$

for $k \neq j$. Then by replacing φ_{W_i} with $\varphi_{W_i} \circ \psi_{W_i}$, the isomorphisms φ_{Z_i} and φ_{W_j} can be glued. Repeating this process for each U_{ij} gives the desired global equivariant isomorphism.

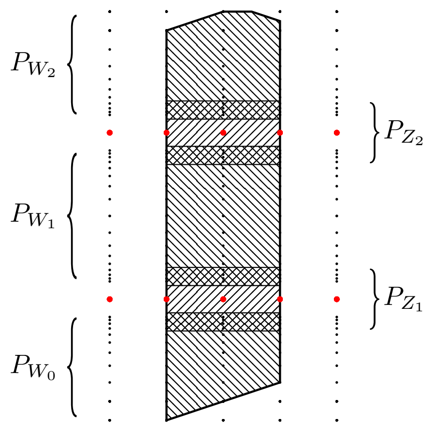

Figure 3.13: The subsets P_{Z_i} and P_{W_i} of a Delzant b-polytope.

Let ϕ be the automorphism of U_{ij} given by $\varphi_{W_i}^{-1} \circ \varphi_{Z_j}$. We must extend this automorphism to an automorphism of $\mu^{-1}(P_{W_i})$ which is the identity outside an arbitrarily small neighborhood of U_{ij}. Notice that ϕ is a $\mathbb{T}$-equivariant symplectic diffeomorphism that preserves orbits. Therefore, by Theorem 3.1 in [HS91], there exists a smooth $\mathbb{T}$-invariant map $h : U_{ij} \to \mathbb{T}^n$ such that $\phi(x) = h(x) \cdot x$. By the $\mathbb{T}$-invariance of h and the contractibility of $\mu(U_{ij}) = P_{W_i} \cap P_{Z_j}$, there is a map $\theta : U_{ij} \to \mathfrak{t}$ such that $\exp \circ \theta = h$. Define the vector field X_θ to be $X_\theta(x) = \frac{d}{ds}\big|_{s=0} \exp(s\theta(x)) \cdot x$. Observe that X_θ is a symplectic vector field whose time one flow is the symplectomorphism ϕ. By Proposition III.55, the vector field is Hamiltonian. Pick a $\hat{f}$ such that $d\hat{f} = \iota_{X_\theta}\omega$. Extend $\hat{f}$ to be a function f on all of $\mu^{-1}(P_{W_i})$ that is locally constant outside a small neighborhood of U_{ij}. Then the time-1 flow of the Hamiltonian vector field corresponding to f will be the desired symplectic automorphism of $\mu^{-1}(P_{W_i})$. $\square$

Proposition III.55. Let $(X_\Delta, \omega_\Delta, \mathbb{T}^{n-1}, \mu_\Delta : X_\Delta \to \Delta)$ be a (classic) compact connected symplectic toric manifold, and $a < b \in \mathbb{R}$. Consider the non-compact symplectic toric manifold

$$(M = (a,b) \times \mathbb{S}^1 \times X_\Delta, \omega_M = dy \wedge d\theta + \omega_\Delta, \mathbb{S}^1 \times \mathbb{T}^{n-1}, (y, \mu_\Delta) : (a,b) \times \Delta)$$

where y and θ are the standard coordinates on (a,b) and $\mathbb{S}^1$ respectively. This symplectic toric manifold has the property that any vector field which is both symplectic and tangent to the fibers of the moment map is a Hamiltonian vector field.

Proof. Choose any $y_0 \in (a,b), x_0 \in X_\Delta$, and consider the loop

$$\gamma : \mathbb{S}^1 \to (a,b) \times \mathbb{S}^1 \times X_\Delta, \quad t \mapsto (y_0, t, x_0).$$

Integration of a 1-form on γ represents an element of $H^1(M)^*$ which pairs nontrivially with $[d\theta]$ and hence is itself nontrivial. By the Künneth formula,

$$H^1(M) \cong (H^0((a,b) \times \mathbb{S}^1) \otimes H^1(X_\Delta)) \oplus (H^1((a,b) \times \mathbb{S}^1) \otimes H^0(X_\Delta))$$

which is one-dimensional due to the fact that the cohomology of a compact symplectic toric manifold is supported in even degrees. Therefore, a closed 1-form on M is exact precisely if its integral along γ is zero.

Let v be a symplectic vector field on M tangent to the fibers of the moment map. Because the fibers of the moment map are isotropic and because the image of γ is contained in a single such fiber, it follows that $\omega_M(v, \gamma_*(\partial/\partial t)) = 0$ at all points in the image of γ. Therefore, the integral of $\iota_v\omega$ along γ vanishes, so $\iota_v\omega$ is exact and therefore v is Hamiltonian. $\square$

Notice that the proof of the surjectivity of the bijections in Theorem III.54 is unlike the proof of surjectivity in the classical Delzant theorem, since we do not construct the b-symplectic manifold globally through a symplectic cut in some large $\mathbb{C}^N$. However, we suspect that such a construction is possible by replacing an appropriate direction in $\mathbb{C}^n$ with a suitable b-object, similar to how we replaced a copy of $\mathbb{R} \subseteq \mathfrak{t}^*$ with a copy of ${}^b\mathbb{R}$ in our construction of ${}^b\mathfrak{t}^*$. We invite the interested reader to write down the details.

3.5 Further directions

3.5.1 Cylindrical moment map

In classic symplectic geometry, several generalizations of the standard moment map have been studied (Chapter 5 of [OR04]). We suspect that many of these generalizations extend to the b-geometry setting as well. One such generalization of the standard moment map, called the *cylinder valued moment map* (introduced in [CDM88], an English reference is Section 5.2 of [OR04]), is defined for any symplectic Lie group action, even when the action is not Hamiltonian. When the Lie group is a torus and the action is especially well-behaved (specifically, when the holonomy group of a certain connection related to the action is a closed subgroup of $\mathfrak{t}^*$), the cylinder valued moment map enjoys many of the same properties as the standard moment map ([OR04], Prop. 5.4.4). In Example III.56, we give an example without details of what a cylinder-valued b-moment map might look like.

Example III.56. Let $f : \mathbb{S}^1 = \mathbb{R}/2\pi \to \mathbb{R}$ be a smooth nonnegative bump function, supported on $(\pi/4, 3\pi/4)$, with $\int_{\mathbb{S}^1} f(\theta)d\theta = 3$. Consider the b-symplectic manifold

$$(\mathbb{T}^2 = \{(\theta_1, \theta_2) \in (\mathbb{R}/2\pi)^2\}, Z = \{\theta_1 \in \{0, \pi\}\}, \omega = (\csc\theta_1 + f(\theta_1))\, d\theta_1 \wedge d\theta_2)$$

with $\mathbb{S}^1$-action given by the flow of $v = \frac{\partial}{\partial\theta_2}$. The graph of $(\csc\theta_1 + f(\theta_1))$ is shown in Figure 3.14; observe that ω differs from the b-symplectic form from Example III.30 by the presence of f in the formula of ω, which appears as the "bump" in the graph in Figure 3.14 near $\theta_1 = \pi/2$.

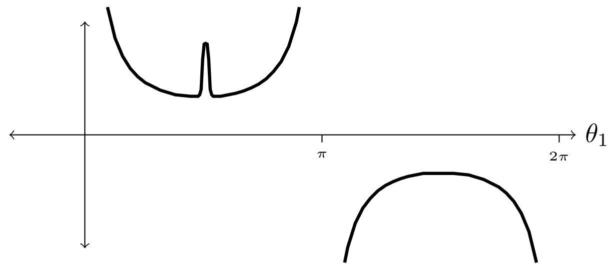

Figure 3.14: The graph of $\csc\theta_1 + f(\theta_1)$.

Because of this bump, the b-form

$$\iota_v\omega = -\left(\csc\theta_1 + f(\theta_1)\right)d\theta_1$$

has no ${}^bC^\infty$ primitive and there is no globally-defined function moment map for the action to any ${}^b\mathbb{R}$ or ${}^b\mathbb{R}/\langle N\rangle$. However, if ${}^b\mathbb{R}/(2,-3)$ denotes the quotient of ${}^b\mathbb{R}$ (with weight function $\mathbb{Z}\to\{1\}$) by the $\mathbb{Z}$-action $(a,x)\mapsto(a+2,x-3)$, there *is* a well-defined moment map $\mu:\mathbb{T}\to{}^b\mathbb{R}/(2,-3)$ as shown in Figure 3.15.

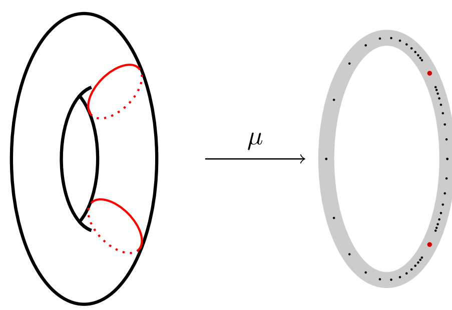

Figure 3.15: A cylindrical moment map.

3.5.2 Case of Z self-intersecting transversally

In most definitions of a b-manifold, Z is required to be an embedded submanifold. However, many of the constructions and results from b-geometry apply even when Z is a union of embedded submanifolds $\{Z_i\}$ which pairwise intersect transversally. To begin with, we can define a bundle ${}^bTM$ over M whose sections are vector fields that are tangent to each Z_i. If $\{x_1,\ldots,x_n\}$ are coordinates for an open $U\subseteq M$ with the property that $Z\cap U=\{x_1=0\}\cup\cdots\cup\{x_r=0\}$, then a trivialization of ${}^bTM$

is given by the sections

$$\left\{ x_1 \frac{\partial}{\partial x_1}, \ldots, x_r \frac{\partial}{\partial x_r}, \frac{\partial}{\partial x_{r+1}}, \ldots, \frac{\partial}{\partial x_n} \right\}.$$

We can generalize the notions of the b-deRham complex, a b-symplectic form, and a b-function in a straightforward manner. Example III.57 shows what the moment map might look like for a toric action on one of these objects.

Example III.57. If we allow the components of Z to intersect transversally, the following is a b-symplectic manifold

$$(M = \mathbb{S}^2 \times \mathbb{S}^2, Z = \{h_1 = 0\} \cup \{h_2 = 0\}, \omega = \frac{dh_1}{h_1} \wedge d\theta_1 + \frac{dh_2}{h_2} \wedge d\theta_2)$$

where $(h_1, \theta_1, h_2, \theta_2)$ are the standard coordinates on $\mathbb{S}^2 \times \mathbb{S}^2$. The $\mathbb{T}^2$-action

$$(t_1, t_2) \cdot (h_1, \theta_1, h_2, \theta_2) = (h_1, \theta_1 - t_1, h_2, \theta_2 - t_2)$$

is Hamiltonian. Let X_1 and X_2 be the elements of $\mathfrak{t}$ satisfying $X_1^{\#} = -\frac{\partial}{\partial \theta_1}$ and $X_2^{\#} = -\frac{\partial}{\partial \theta_2}$ respectively. With weight function $\{0\} \mapsto 1$, there is a smooth moment map

$$M \to {}^b\mathbb{R} \times {}^b\mathbb{R}, \quad (h_1, \theta_1, h_2, \theta_2) \mapsto (\log |h_1|, \log |h_2|),$$

the image of which is illustrated below.

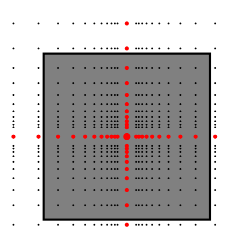

Figure 3.16: A moment map image in $({}^b\mathbb{R})^2$.

CHAPTER IV

b^k-manifolds

4.1 Preliminaries

In this section, we establish notation pertaining to jet bundles, review definitions from the theory of b-manifolds, and generalize these definitions. All manifolds and maps and vector fields are assumed to be smooth.

4.1.1 Notation

Let $i : Z \to M$ be the inclusion of a hypersurface into a manifold, let C^∞ be the sheaf of smooth functions on M, and let $\mathcal{I}_Z \subseteq C^\infty$ be the ideal sheaf of Z.

Definition IV.1. The **sheaf of germs** at Z is $i^{-1}(C^\infty)$; a **germ** at Z is a global section of this sheaf. The **sheaf of k-jets** at Z is $\mathcal{J}_Z^k := i^{-1}(C^\infty/\mathcal{I}_Z^{k+1})$; a **$k$-jet** at Z is a global section of this sheaf.

We will write J_Z^k (or simply J^k) to denote the k-jets at Z, and I_Z (or simply I) to denote the global sections of $i^{-1}(\mathcal{I}_Z)$. We write $[f]_Z^k$ (or simply $[f]^k$) to denote the k-jet represented by a smooth function f defined in a neighborhood of Z. Also, if j is a k-jet, we write $f \in j$ if f represents j and $f \in I^k$ if f represents an element of I^k (equivalently, if $[f]^{k-1} = 0$).

4.1.2 Definitions

In Chapter III, we used b-manifolds to study symplectic forms having order-one singularities along a hypersurface. In general, we defined the b-cotangent bundle ${}^bT^*M$, and saw that sections of its exterior power were differential forms on M with a certain kind of order-one singularity at Z. Towards the goal of constructing similar bundles to study differential forms with higher-order singularities, we wish to define a b^k-vector field as a vector field "tangent to order k on Z." However, the next example shows that the naïve definition of being "tangent to order k on Z" (as a vector field v such that $\mathcal{L}_v(f) \in I^k$ for a defining function f of Z) is ill-defined.

Example IV.2. On the b-manifold $(M, Z) = (\{(x, y) \in \mathbb{R}^2\}, \{y = 0\})$, two different defining functions for Z are given by y and $e^x y$. The vector field $v = \frac{\partial}{\partial x}$ satisfies

$$\mathcal{L}_v(y) = 0 \in I^2 \qquad \text{and} \qquad \mathcal{L}_v(e^x y) = e^x y \notin I^2$$

so the order of vanishing of the Lie derivative of a defining function depends on the choice of defining function.

This phenomenon prevents us from reproducing the definitions and results of [GMP13] mutatis mutandis; we must endow our b-manifolds with additional data to make possible the definition of a b^k-vector field.

Definition IV.3. For $k \geq 1$, a b^k**-manifold** is a triple (M, Z, j_Z) where

- M is an oriented manifold.
- $Z \subseteq M$ is a closed embedded oriented hypersurface.
- j_Z is an element of J_Z^{k-1} that can be represented by a positively oriented local defining function y for Z (that is, if Ω_Z is a positively oriented volume form of Z, then $dy \wedge \Omega_Z$ is positively oriented for M)

If $k > 1$ and a function shares the same $(k-1)$-jet as a positively oriented local defining function for Z, then it itself is a positively oriented local defining function for Z. In this case, any $f \in j_Z$ is a positively oriented local defining function for Z. When $k = 1$, the jet data $\{j_Z\}$ is vacuous (because any local defining function for Z represents the trivial 0-jet), so the definition of a b^1-manifold nearly[1] agrees with that of a b-manifold.

Definition IV.4. A b^k**-map** from (M, Z, j_Z) to $(M', Z', j_{Z'})$ is a map $\varphi : M \to M'$ such that $\varphi^{-1}(Z') = Z$, φ is transverse to Z', and $\varphi^*(j_{Z'}) = j_Z$.

The interested reader is invited to check that b^k-manifolds and b^k-maps form a category.

Remark IV.5. Given an embedded hypersurface $Z \subseteq M$, a function $f \in C^\infty(M)$, and a vector field v on M satisfying $v_p \in T_pZ$ for all $p \in Z$, the jet $[\mathcal{L}_v(f)]^{k-1}$ depends only on $[f]^{k-1}$.

Proof. If $[f_2]^{k-1} = [f_1]^{k-1}$, then $f_2 - f_1 = y^k g$ for a local defining function y and some smooth g. For a vector field v satisfying $v_p \in T_pZ$, $\mathcal{L}_v(y) \in I$, so

$$[\mathcal{L}_v(f_2)]^{k-1} = [\mathcal{L}_v(f_1) + y^k\mathcal{L}_v(g) + kgy^{k-1}\mathcal{L}_v(y)]^{k-1} = [\mathcal{L}_v(f_1)]^{k-1}.$$

$\square$

Remark IV.5 shows that the following definition makes sense.

Definition IV.6. A b^k**-vector field** on (M, Z, j_Z) is a vector field v with $v_p \in T_p(Z)$ for $p \in Z$ such that for any $f \in j_Z$, $\mathcal{L}_v(f) \in I^k$.

[1] We do not demand that $Z = \{f = 0\}$ for some *globally*-defined f, so the definition of a b^1-manifold is slightly more general than the definition of a b-manifold given in [GMP13]. However, any symplectic b^1-manifold *will* have the property that Z is defined by a global function, so the symplectic geometry of b^1-manifolds coincides with the symplectic geometry of b-manifolds.

To check whether a vector field v is a b^k-vector field, it suffices (by Remark IV.5) to check that $\mathcal{L}_v(f) \in I^k$ for just one local defining function $f \in j_Z$. The following example shows that Definition IV.6 formalizes the notion of a vector field being "tangent to order k" along a hypersurface.

Example IV.7. On the b^k manifold $(\mathbb{R}^n, Z = \{x_n = 0\}, [x_n]^{k-1})$, a vector field $v = \sum_{i=1}^n v_i \frac{\partial}{\partial x_i}$ is a b^k-vector field iff

$$\mathcal{L}_v(x_n) \in I^k$$

which occurs iff $v_n \in I^k$. That is, the b^k-vector fields are precisely those of the form

$$\phi_n x_n^k \frac{\partial}{\partial x_n} + \sum_{i=1}^{n-1} \phi_i \frac{\partial}{\partial x_i}$$

for smooth functions ϕ_i.

On a b^k-manifold (M, Z, j_Z), each $p \notin Z$ is contained in a coordinate neighborhood $(U, \{x_1, \ldots, x_n\})$ on which $\{\frac{\partial}{\partial x_i}\}$ generate the space of b^k-vector fields over U as a free $C^\infty(U)$-module. For points $p \in Z$, Example IV.7 shows that on a coordinate neighborhood $(U, \{x_1, \ldots, x_n\})$ of p with $x_n \in j_Z$, the vector fields

$$\left\{ \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_{n-1}}, x_n^k \frac{\partial}{\partial x_n} \right\}$$

generate the space of b^k-vector fields over U as a $C^\infty(U)$-module. Consequently, b^k-vector fields form a projective C^∞ module over M, as well as a Lie subalgebra of the algebra of vector fields on M, so we can realize b^k-vector fields as the sections of a bundle on M.

We call this bundle ${}^{b^k}TM$ the b^k-*tangent bundle*. The dual of this bundle ${}^{b^k}T^*M$ is the b^k-*cotangent bundle*. When $k = 1$ we recover the classical definitions of a b-vector field and the b-(co)tangent bundle. We write ${}^{b^k}\Omega^p(M)$ for sections of $\wedge^p({}^{b^k}T^*M)$. Elements of ${}^{b^k}\Omega^p(M)$ are b^k-*forms*.

4.2 Geometry of the b^k-(co)tangent bundle

In this section, we describe the fibers of the b^k-(co)tangent bundles and study maps between b^k-(co)tangent bundles as k varies. These results will prepare us to study the de Rham theory and the symplectic geometry of b^k-manifolds.

Let ${}^{b^k}\mathrm{Vect}(M)$ be the space of b^k-vector fields and $C_p^\infty(M)$ be the ideal of functions vanishing at $p \in M$. We can define ${}^{b^k}T_pM$ intrinsically as

$$ {}^{b^k}T_pM \cong {}^{b^k}\mathrm{Vect}(M)/(C_p^\infty(M) \cdot {}^{b^k}\mathrm{Vect}(M)). $$

There is a canonical map that relates the fibers of ${}^{b^k}TM$ to those of TM.

$$ {}^{b^k}T_pM \cong \frac{{}^{b^k}\mathrm{Vect}(M)}{C_p^\infty(M) \cdot {}^{b^k}\mathrm{Vect}(M)} \to \frac{\mathrm{Vect}(M)}{C_p^\infty(M) \cdot \mathrm{Vect}(M)} \cong T_pM \tag{4.1} $$

The results of this section will show that for $p \in Z$, there is a canonical element in the kernel of Map 4.1 (and dually a canonical element in the quotient ${}^{b^k}T_p^*M/T_p^*M$). Instead of proving these results using this intrinsic description of individual fibers, we will take a more global perspective in order to follow more closely the exposition and results of [GMP13].

4.2.1 Fibers of the b^k-(co)tangent bundle

Similar to the b-manifold case, there are maps between the (co)tangent bundles of Z and the b^k-(co)tangent bundles of M restricted to Z.

$$ {}^{b^k}TM\big|_Z \twoheadrightarrow TZ \tag{4.2} $$

$$ {}^{b^k}T^*M\big|_Z \hookleftarrow T^*Z \tag{4.3} $$

Map 4.2 is induced by the map of sections $\Gamma(M, {}^{b^k}TM) \to \Gamma(Z, TZ)$ given by restricting a b^k-vector field to Z. Map 4.3 is dual to Map 4.2. We study the (co)kernel of these maps, starting with a technical remark.

Remark IV.8. Let v be a b^k-vector field that vanishes on Z when viewed as a section of TM, and let $x_n \in j_Z$ be a local defining function for Z. Then v also vanishes on Z as a section of ${}^{b^k}TM$ at precisely at those points where the k-jet $[\mathcal{L}_v(x_n)]^k$ vanishes.

Proof. In local coordinates $\{x_1, \ldots, x_n\}$

$$v = \phi_n x_n^k \frac{\partial}{\partial x_n} + \sum_{i<n} \phi_i \frac{\partial}{\partial x_i}$$

where $\{\phi_i\}_{i \leq n}$ are smooth functions and $\{\phi_i\}_{i<n}$ vanish on Z. Because the functions $\{\phi_i\}_{i \leq n}$ constitute the trivialization of ${}^{b^k}TM$ induced by the local coordinates, v vanishes on Z as a section of ${}^{b^k}TM$ at those points of Z where ϕ_n vanishes, which are precisely the points where $[\mathcal{L}_v(x_n)]^k = [\phi_n x_n^k]^k$ vanishes.

□

Proposition IV.9. The kernel of Map 4.2 has a canonical nowhere vanishing section.

Proof. Pick a local defining function $y \in j_Z$ and a vector field v satisfying $dy(v)\big|_Z = 1$. Then $[\mathcal{L}_{y^k v}(y)]^k = [y^k \mathcal{L}_v(y)]^k$ is nonvanishing. By Remark IV.8, $y^k v$ is a b^k-vector that vanishes on Z as a section of TM but is nowhere vanishing as a section of ${}^{b^k}TM$.

To prove that $y^k v$ is canonical, suppose $y_2 \in j_Z$ and v_2 are different choices of defining function and vector field. Then $y_2 = y(1 + gy^{k-1})$ for some smooth g and

$$\begin{aligned} \mathcal{L}_{v_2}(y_2) &= \mathcal{L}_{v_2}(y) + gy^{k-1}\mathcal{L}_{v_2}(y) + y\mathcal{L}_{v_2}(gy^{k-1}) \\ &= (1 + kgy^{k-1})\mathcal{L}_{v_2}(y) + y^k \mathcal{L}_{v_2}(g) \end{aligned}$$

and

$$\begin{aligned} [\mathcal{L}_{y^k v - y_2^k v_2}(y)]^k &= [y^k \mathcal{L}_v(y) - y^k(1 + gy^{k-1})^k \mathcal{L}_{v_2}(y)]^k \\ &= \left[y^k \mathcal{L}_v(y) - y^k(1 + gy^{k-1})^k \frac{\mathcal{L}_{v_2}(y_2) - y^k \mathcal{L}_{v_2}(g)}{1 + kgy^{k-1}} \right]^k \\ &= 0 \end{aligned}$$

By Remark IV.8, $y^k v - y_2^k v_2$ vanishes on Z as a section of ${}^{b^k}TM$, so $y^k v$ and $y_2^k v_2$ represent the same section of ${}^{b^k}TM\big|_Z$. $\square$

Turning our attention to the cotangent bundle, observe that although $y^{-k}dy$ is not defined on Z as a section of T^*M, its pairing with any b^k-vector field extends smoothly over Z. Therefore, $y^{-k}dy$ extends smoothly over Z as a section of ${}^{b^k}T^*M$. By pairing $y^{-k}dy$ with a representative of a nowhere vanishing section of $\ker({}^{b^k}TM\big|_Z \to TZ)$, we see that $y^{-k}dy$ is nowhere vanishing. This proves the following claim.

Claim IV.10. The cokernel of map (4.3) has a nowhere vanishing section.

The preceding discussion describes of the fibers of the b^k-(co)tangent bundle of a b^k-manifold $(M, Z, [y]^{k-1})$ as follows.

$$
{}^{b^k}T_pM \cong \begin{cases} T_pM & \text{for } p \notin Z \\ T_pZ + \langle y^k \frac{\partial}{\partial y} \rangle & \text{for } p \in Z \end{cases}
$$
$$
{}^{b^k}T_p^*M \cong \begin{cases} T_p^*M & \text{for } p \notin Z \\ T_p^*Z + \langle \frac{dy}{y^k} \rangle & \text{for } p \in Z \end{cases}
$$

4.2.2 Properties of b^k-forms

From the above description of the fibers of ${}^{b^k}T_pM$, we see that $\Omega^p(M\backslash Z) \cong {}^{b^k}\Omega^p(M\backslash Z)$. That is, every b^k-form restricts to an ordinary differential form on $M\backslash Z$. We can therefore interpret a b^k-form as a differential form on $M\backslash Z$ that satisfies certain asymptotic properties (prescribed by the jet data) around Z. We also see that for any defining function $y \in j_Z$, every b^k-form can be written in a neighborhood U of Z in the form

$$
\omega = \frac{dy}{y^k} \wedge \alpha + \beta \tag{4.4}
$$

for differential forms $\alpha \in \Omega^{p-1}(U), \beta \in \Omega^p(U)$. Although the forms α and β appearing in Equation 4.4 are not uniquely defined by ω, we will show that $i^*(\alpha)$ is independent of the choice of y, α and β, where $i : Z \to M$ is the inclusion.

Proposition IV.11. On a b^k-manifold, if $f_1, f_2 \in j_Z$ are local defining functions for Z, then in a neighborhood U of Z

$$\frac{df_1}{f_1^k} = \frac{df_2}{f_2^k} + \beta$$

where $\beta \in \Omega^1(U)$.

Proof. The proof is technical. See Section 4.7 for the details. □

Corollary IV.12. *Given a decomposition of $\omega \in {}^{b^k}\Omega(M)$ as in Equation 4.4, $i^*(\alpha)$ is independent of the decomposition.*

Proof. Let α_1 and α_2 be the α terms of two such decompositions. Setting the decompositions equal and applying the preceding proposition shows that

$$\frac{dy}{y^k} \wedge (\alpha_2 - \alpha_1)$$

is a smooth form for some local defining function $y \in j_Z$, so $i^*(\alpha_2 - \alpha_1) = 0$. □

This proves the well-definedness of the map

$$\begin{aligned} \iota_{\mathbb{L}} : {}^{b^k}\Omega^p(M) &\to \Omega^{p-1}(Z) \\ \frac{dy}{y^k} \wedge \alpha + \beta &\mapsto i^*(\alpha) \end{aligned} \tag{4.5}$$

Alternatively, this map can be defined by restricting a form to Z, then contracting with the canonical section $\mathbb{L}$ described in Proposition IV.9. This motivates the notation $\iota_{\mathbb{L}}$ for the map.

Equation 4.4 might give us hope that we can define a b^k-form without reference to any jet data as "a form ω on $M \backslash Z$ which admits a decomposition $\omega = y^{-k} dy \wedge \alpha + \beta$

in a neighborhood of Z for *some* local defining function y". However, for a fixed ω the existence of a decomposition $\omega = y^{-k}dy \wedge \alpha + \beta$ depends strongly on $[y]^{k-1}$. It turns out that the set of $\omega \in \Omega(M \backslash Z)$ which extends over Z with respect to *some* $[y]^{k-1}$ is not even closed under addition. This hopefully motivates (for a second time) the necessity of the jet data in the definition of a b^k-manifold.

4.2.3 Viewing a b^ℓ-form as a b^k-form

To prepare for the next section, we consider a new family of maps between the b^k-(co)tangent bundles. These maps generalize the fact that any smooth differential form is naturally a b^k-form.

For any $0 < \ell \leq k$, the natural map $J_Z^{k-1} \to J_Z^{\ell-1}$ allows us to canonically endow a b^k-manifold (M, Z, j_Z) with a b^ℓ-manifold structure. Defining ${}^{b^0}TM := TM$ and ${}^{b^0}T^*M := T^*M$ for notational convenience, a b^k-manifold structure on M defines $2k+2$ different bundles ${}^{b^\ell}TM$, ${}^{b^\ell}T^*M$ over M for $0 \leq \ell \leq k$. A b^k-vector field will also be a b^ℓ-vector field for the induced b^ℓ-manifold structure. This induces a map

$$ {}^{b^k}TM \to {}^{b^\ell} TM \tag{4.6} $$

and its dual

$$ {}^{b^\ell}T^*M \to {}^{b^k} T^*M, \tag{4.7} $$

the latter of which can be described explicitly in terms of the decompositions from Equation 4.4 as

$$ \frac{dy}{y^\ell} \wedge \alpha + \beta \mapsto \frac{dy}{y^k} \wedge (y^{k-\ell}\alpha) + \beta. $$

4.3 De Rham Theory and Laurent Series of b^k-forms

We define a differential $d : {}^{b^k}\Omega^p(M) \to {}^{b^k}\Omega^{p+1}(M)$ by

$$d\left(\frac{dy}{y^k} \wedge \alpha + \beta\right) = \frac{dy}{y^k} \wedge d\alpha + d\beta.$$

This definition does not depend on the decomposition. Indeed, $d(\omega)$ is the unique extension of the image of the classic de Rham differential $d(\omega\big|_{M\setminus Z}) \in \Omega^p(M\setminus Z) \cong {}^{b^k}\Omega^p(M\setminus Z)$ over Z.

Definition IV.13. The **b^k-de Rham complex** is $({}^{b^k}\Omega^p(M), d)$, with ${}^{b^k}\Omega^0(M) := C^\infty(M)$. The **$b^k$-cohomology** ${}^{b^k}H^*(M)$ is the cohomology of this complex.

Proposition IV.14. The sequence below, with g given by Map (4.7), is exact

$$0 \to {}^{b^{k-1}}\Omega^p(M) \xrightarrow{g} {}^{b^k}\Omega^p(M) \xrightarrow{\iota_{\mathbb{L}}} \Omega^{p-1}(Z) \to 0. \tag{4.8}$$

Moreover, for any closed $\alpha \in \Omega^{p-1}(Z)$ and collar neighborhood $(y, \pi) : U \to \mathbb{R} \times Z$ of Z with $y \in j_Z$, there is a closed form $\omega \in \iota_{\mathbb{L}}^{-1}(\alpha)$ such that

$$\omega = \frac{dy}{y^k} \wedge \pi^*(\alpha)$$

in a neighborhood of Z.

Proof. The only nontrivial part of the exactness claim is that $\ker(\iota_{\mathbb{L}}) \subseteq \mathrm{im}(g)$. The kernel of $\iota_{\mathbb{L}}$ consists precisely of those ω that admit some decomposition

$$\omega = \frac{dy}{y^k} \wedge \alpha + \beta$$

in a neighborhood of Z for which $i^*(\alpha) = 0$. Locally around Z, T^*M splits as $T^*Z + \langle dy \rangle$, so we may replace α by a form that vanishes on Z without changing ω. Then $y^{-1}\alpha$ is a smooth form, and

$$\frac{dy}{y^{k-1}} \wedge \frac{\alpha}{y} + \beta$$

extends over M to a b^{k-1} form in $g^{-1}(\omega)$. Therefore, Sequence 4.8 is exact.

Given a closed $\alpha \in \Omega^{p-1}(Z)$ and a collar neighborhood $(y, \pi) : U \to (-R, R) \times Z$ of Z with $y \in j_Z$, let $\widetilde{y} \in C^\infty(M)$ be a function that agrees with y on $(-R/2, R/2) \times Z$ and is locally constant outside U. Then the b^k-form $\omega = d\widetilde{y}/\widetilde{y}^k \wedge \pi^*(\alpha)$ extends to a closed b^k-form on M that vanishes outside U and satisfies $\iota_{\mathbb{L}}(\omega) = \alpha$. In $(-R/2, R/2) \times Z$,

$$\omega = \frac{dy}{y^k} \wedge \pi^*(\alpha).$$

□

One can check that the short exact sequence from Proposition IV.14 is a chain map of complexes, hence induces a long exact sequence

$$\cdots \to {}^{b^{k-1}}H^*(M) \to {}^{b^k}H^*(M) \to H^{*-1}(Z) \to {}^{b^{k-1}}H^{*+1}(M) \to \dots$$

By Proposition IV.14, the maps ${}^{b^k}H^*(M) \to H^{*-1}(Z)$ are surjective, so the long exact sequence is a collection of short exact sequences

$$0 \to {}^{b^{k-1}}H^p(M) \to {}^{b^k}H^p(M) \to H^{p-1}(Z) \to 0 \tag{4.9}$$

Using induction on k, this proves the following proposition.

Proposition IV.15.

$${}^{b^k}H^p(M) \cong H^p(M) \oplus \left(H^{p-1}(Z)\right)^k$$

Proof. From the remarks above. □

So far, this isomorphism is non-canonical: although we can lift every $[\alpha] \in H^{p-1}(Z)$ in Equation 4.9 to an element of ${}^{b^k}H^p(M)$, we do not yet have a preferred choice of lifting, and different choices yield genuinely different isomorphisms.

Results in Subsection 4.3.1, where we show that the $(H^{p-1}(Z))^k$ summand of the image of any $[\omega] \in {}^{b^k}H^p(M)$ *can* be canonically defined, will give us partial relief from this uncomfortable state of affairs. Finally, in Section 4.4 we will give an explicit canonical map for the isomorphism in Proposition IV.15, and in doing so we will see a geometric interpretation for the terms on the right side of the isomorphism.

4.3.1 The Laurent series of a closed b^k-form

Definition IV.16. A **Laurent Series** of a closed b^k-form ω is an expression for ω in a neighborhood of Z of the form

$$\omega = \sum_{i=1}^{k} \frac{dy}{y^i} \wedge \alpha_{-i} + \beta$$

where $y \in j_Z$ is a positively oriented local defining function and each α_{-i} is closed.

Remark IV.17. Every closed b^k-form has a Laurent series. In fact, Proposition IV.14 shows that given a collar neighborhood $(y, \pi) : U \to (-R, R) \times Z$ of Z with $y \in j_Z$, every closed b^k-form ω can be written (in a neighborhood of Z) as the sum of a closed b^{k-1} form and

$$\frac{dy}{y^k} \wedge \pi^*(\iota_{\mathbb{L}}\omega).$$

By applying induction on the b^{k-1} form, we arrive at a Laurent series of the form

$$\omega = \sum_{i=1}^{k} \frac{dy}{y^i} \wedge \pi^*(\gamma_{-i}) + \beta$$

for closed forms γ_{-i} on Z.

Example IV.18. Consider the b^k-manifold $(S^1 \times S^1, Z_1 \cup Z_2, [y]^{k-1})$ pictured in Figure 4.1.

where a collar neighborhood $U = U_1 \cup U_2$ of Z is shaded. Let $\{(\theta_i, y)\}$ be coordinates on U_i. Then $d\theta_1$ (respectively $d\theta_2$) extends trivially over U_2 (respectively U_1) to a

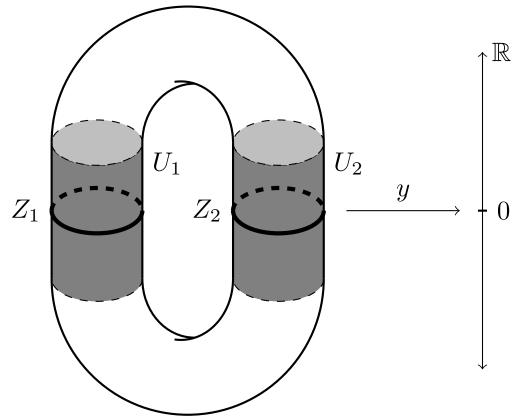

Figure 4.1: A b^k-manifold with disconnected Z

smooth form on all of U. Let ω be a b^k 2-form on M. On U, it admits a decomposition

$$\omega = \frac{dy}{y^k} \wedge (f d\theta_1 + g d\theta_2) + \beta$$

for smooth functions f, g and a smooth form β. Let $\pi : U \to Z$ be the vertical projection, and for $-k \leq i \leq -1$, let

$$f_i := \frac{1}{(k+i)!} \frac{\partial^{k+i} f}{\partial y^{k+i}}\bigg|_Z \qquad g_i := \frac{1}{(k+i)!} \frac{\partial^{k+i} g}{\partial y^{k+i}}\bigg|_Z .$$

Then

$$f = \pi^*(f_{-k}) + \pi^*(f_{-k+1})y + \cdots + \pi^*(f_{-1})y^{k-1} + \widetilde{f}$$
$$g = \pi^*(g_{-k}) + \pi^*(g_{-k+1})y + \cdots + \pi^*(g_{-1})y^{k-1} + \widetilde{g}$$

for $\widetilde{f}, \widetilde{g} \in I^k$. Then ω has a Laurent series

$$\omega = \sum_{i=1}^{k} \frac{dy}{y^i} \wedge (\pi^*(f_i) d\theta_1 + \pi^*(g_i) d\theta_2) + \beta'$$

where β' is smooth form.

Proposition IV.19. The cohomology classes $[i^*(\alpha_{-i})] \in H^{p-1}(Z)$ appearing in a Laurent series of $\omega \in {}^{b^k}\Omega^p(M)$ depend only on $[\omega]$.

Proof. By Proposition IV.11, we may assume that all our Laurent series are written with respect to the same local defining function $y \in j_Z$. When $k = 1$, then for $\omega \in {}^{b^1}\Omega^p(M)$, the class $[i^*(\alpha_{-1})]$ is the image of $[\omega]$ in the map appearing in Equation 4.9, and therefore depends only on $[\omega]$.

For $k > 1$, assume the proposition is true for $k-1$, and let $\omega \in {}^{b^k}\Omega^p(M)$. Consider Laurent series of two representatives of $[\omega]$,

$$\omega_0 = \sum_{i=1}^{k} \frac{dy}{y^i} \wedge \alpha_{-i} + \beta \quad \text{ and } \quad \omega_1 = \sum_{i=1}^{k} \frac{dy}{y^i} \wedge \alpha'_{-i} + \beta'$$

Both $[i^*(\alpha_{-k})]$ and $[i^*(\alpha'_{-k})]$ are the image of $[\omega]$ in Equation 4.9, so are equal. If we can show that

$$\sum_{i=1}^{k-1} \frac{dy}{y^i} \wedge \alpha_{-i} + \beta \quad \text{ and } \quad \sum_{i=1}^{k-1} \frac{dy}{y^i} \wedge \alpha'_{-i} + \beta'$$

are cohomologous b^{k-1}-forms, then we will be done by induction. That is, we must show that

$$\omega_1 - \frac{dy}{y^k} \wedge \alpha'_{-k} - \left(\omega_0 - \frac{dy}{y^k} \wedge \alpha_{-k}\right) \tag{4.10}$$

is an exact b^{k-1}-form. Because $[\omega_0] = [\omega_1]$, there is a b^k-form η with $d\eta = \omega_1 - \omega_0$. Moreover, because $\alpha_{-k} - \alpha'_{-k}$ is a closed form with $i^*(\alpha_{-k} - \alpha'_{-k})$ exact, the relative Poincaré lemma implies that it has a primitive μ. Then

$$\eta + \frac{dy}{y^k} \wedge \mu$$

is a primitive for the form (4.10). However, this primitive is a b^k-form; to prove that (4.10) is exact as a b^{k-1}-form (and in doing so complete the induction), simply observe that the map

$${}^{b^{k-1}}H^p(M) \to {}^{b^k}H^p(M)$$

from Sequence (4.9) is injective, so any b^{k-1}-form exact as a b^k-form is also exact as a b^{k-1}-form. $\square$

Corollary IV.20. *Let*

$$\omega = \sum_{i=1}^{k} \frac{dy}{y^i} \wedge \alpha_{-i} + \beta$$

be a Laurent series of the closed b^k-form ω. The map

$$\begin{aligned} {}^{b^k}H^p(M) &\to (H^{p-1}(Z))^k \\ [\omega] &\mapsto ([i^*(\alpha_{-1})], [i^*(\alpha_{-2})], \dots, [i^*(\alpha_{-k})]) \end{aligned} \tag{4.11}$$

is independent of the choice of Laurent series.

Definition IV.21. Given a b^k-form ω, the image of $[\omega]$ under Map (4.11) is the **Laurent Decomposition** of $[\omega]$.

The result below strengthens Theorem IV.15.

Theorem IV.22. *The sequence below, with g, f given by Map 4.7 and Map 4.11 respectively, is exact.*

$$0 \to H^p(M) \xrightarrow{g} {}^{b^k}H^p(M) \xrightarrow{f} (H^{p-1}(Z))^k \to 0 \tag{4.12}$$

Proof. The map g is the composition of the inclusions

$${}^{b^{\ell-1}}H^n(M) \to {}^{b^\ell}H^n(M)$$

appearing in the short exact sequence (4.9) for $\ell \leq k$. Therefore, it itself is an inclusion. The proof that f is surjective follows from the same trick used to create a preimage of a closed $\alpha \in \Omega^{p-1}(Z)$ in the proof of Proposition IV.14. Exactness at the middle is straightforward. $\square$

4.4 Volume Forms on a b^k-manifold

Let (M, Z, j_Z) be a compact b^k-manifold, and let $\omega \in {}^{b^k}\Omega^{\dim(M)}(M)$. Because ω "blows up" along Z, we cannot expect its integral to be finite. If we remove from

M a neighborhood of Z, then the integral of ω over the remainder is finite, but obviously depends on the choice of neighborhood. In this section, we extract a useful invariant of ω by studying the behavior of this integral as the size of the removed neighborhood shrinks. We will use this invariant to split the short exact sequence (4.12), and in doing so make the isomorphism (IV.15) canonical.

The results from this section apply even to non-compact manifolds; so that we may state these results in full generality, we begin by introducing notation for compactly supported de Rham theory.

Definition IV.23. The subset ${}^{b^k}\Omega_c^p(M) \subseteq {}^{b^k}\Omega^p(M)$ consists of b^k-forms with compact support. They form a subcomplex of the b^k-de Rham complex, the homology of which is called the **compact b^k-cohomology** ${}^{b^k}H_c^*(M)$

4.4.1 Liouville volume of a b^k-form

Definition IV.24. Let (M, Z, j_Z) be an n-dimensional b^k-manifold. Given $\omega \in {}^{b^k}\Omega_c^n(M)$, $\epsilon > 0$ small, and a local defining function $y \in j_Z$, define $U_{y,\epsilon} = y^{-1}((-\epsilon, \epsilon))$ and

$$\mathrm{vol}_{y,\epsilon}(\omega) = \int_{M \setminus U_{y,\epsilon}} \omega$$

In [Rad02], Radko proved that when M is a surface[2] and $k = 1$, $\lim_{\epsilon\to 0}\mathrm{vol}_{y,\epsilon}(\omega)$ converges and is independent of y. This limit, the *Liouville volume* of ω, was a key ingredient in her classification of stable Poisson structures on compact surfaces. When $k > 1$, this limit will not necessarily converge to a number, but rather to a polynomial in ϵ^{-1}. After proving the existence and well-definedness of this polynomial, we will define the *Liouville volume* of a b^k-cohomology class of top degree as the constant term of this polynomial.

[2]Although Radko studied b-forms only on surfaces, her proof of the fact stated here works for all n.

Theorem IV.25. *For a fixed $[\omega] \in {}^{b^k}H^n_c(M)$ on a b^k-manifold (M, Z, j_Z) with Z compact, there is a polynomial $P_{[\omega]}$ for which*

$$\lim_{\epsilon \to 0} \left(P_{[\omega]} \left(\epsilon^{-1} \right) - \mathrm{vol}_{y,\epsilon}(\omega) \right) = 0 \tag{4.13}$$

for any $y \in j_Z$ and any ω representing $[\omega]$.

Proof. We first prove that there is a polynomial $P_{[\omega]}$ that satisfies Equation 4.13 for a specific y and ω, then we prove that the polynomial is independent of y, then that the polynomial vanishes for exact ω (so depends only on the b^k-cohomology class of ω).

Fix a local defining function $y \in j_Z$ and a closed collar neighborhood $(y, \pi) : U \to [-R, R] \times Z$ of Z. Because ω is compactly supported, $\int_{M \setminus U} \omega < \infty$, so to prove the existence of $P_{[\omega]}$ it suffices to construct a polynomial for the case $M = U$. By Remark IV.17, there exists a Laurent series of ω of the form

$$\omega = \sum_{i=1}^{k} \frac{dy}{y^i} \wedge \pi^*(\alpha_{-i}) + \beta$$

where $\alpha_{-i} \in \Omega^{n-1}(Z)$. Then

$$\mathrm{vol}_{y,\epsilon}(\omega) = \int_{U \setminus U_{y,\epsilon}} \sum_{i=1}^{k} \frac{dy}{y^i} \wedge \pi^*(\alpha_{-i}) + \int_{U \setminus U_{y,\epsilon}} \beta$$

Applying Fubini's theorem (and cancelling log terms), the first term simplifies to

$$\begin{aligned}
&\sum_{i=2}^{k} \frac{-1}{i-1} \left(\left(\frac{1}{R}\right)^{i-1} + \left(\frac{-1}{\epsilon}\right)^{i-1} - \left(\frac{-1}{R}\right)^{i-1} - \left(\frac{1}{\epsilon}\right)^{i-1} \right) \int_Z \alpha_{-i} \\
&\qquad = \sum_{\substack{i=2 \\ i \text{ even}}}^{k} \left(\frac{-2R^{1-i}}{i-1} \right) \int_Z \alpha_{-i} + \sum_{\substack{i=2 \\ i \text{ even}}}^{k} \left(\frac{2}{i-1} \int_Z \alpha_{-i} \right) (\epsilon^{-1})^{i-1}
\end{aligned}$$

and the last term simplifies to

$$\int_U \beta - \int_{[-\epsilon,\epsilon] \times Z} \beta$$

so the polynomial

$$P(t) = \left(\int_U \beta + \sum_{\substack{i=2 \\ i \text{ even}}}^{k} \left(\frac{-2R^{1-i}}{i-1} \right) \int_Z \alpha_{-i} \right) + \sum_{\substack{i=2 \\ i \text{ even}}}^{k} \left(\frac{2}{i-1} \int_Z \alpha_{-i} \right) t^{i-1}$$

satisfies the conditions of a volume polynomial for this specific choice of y and ω.

The proof that this polynomial does not depend on y is techincal; the details can be found in Section 4.7. To show that the polynomial associated to any exact form is trivial, suppose ω is exact and let

$$\eta = \sum_{i=1}^{k} \frac{dy}{y^i} \wedge \pi^* \eta_{-i} + \beta_\eta$$

be a Laurent series of a primitive of ω. Then

$$\int_{M \setminus U_{y,\epsilon}} \omega = \int_{\partial(M \setminus U_{y,\epsilon})} \eta = \int_{\partial(M \setminus U_{y,\epsilon})} \beta_\eta$$

which approaches 0 as $\epsilon \to 0$. $\square$

Definition IV.26. The polynomial $P_{[\omega]}$ described in Theorem IV.25 is the **Volume Polynomial** of $[\omega]$. The constant term of $P_{[\omega]}$ is the **Liouville Volume** of $[\omega]$.

The Liouville volume of $[\omega]$ can be thought of as the volume that remains of $[\omega]$ after its singular parts have been carefully discarded. For arbitrary (non-b^k) singularities of a form of top degree, no similar concept exists. In the b^k case, the definition is made possible by how well-behaved b^k singularities are, as well as by how we use the jet data (when $k > 1$) to prescribe the asymptotic manner in which $U_{y,\epsilon}$ approaches Z as $\epsilon \to 0$.

We may also define the *Liouville volume* of a $p < \dim(M)$ dimensional b^k-form ω along a compact p-dimensional submanifold $Y \subseteq M$ transverse to Z: the pullback of ω will be a b^k-form of top degree for the induced b^k-structure on Y and therefore has a Louville volume. By Poincaré duality, this remark inspires the definition of the *smooth part* of a b^k-form.

Definition IV.27. Let $[\omega] \in {}^{b^k}H^p(M)$. The image of $[\omega]$ under the map

$$
\begin{aligned}
(4.14) \qquad {}^{b^k}H^p(M) &\to (H_c^{n-p}(M))^* \cong H^p(M) \\
[\omega] &\mapsto \big([\eta] \mapsto P_{[\omega\wedge\eta]}(0)\big)
\end{aligned}
$$

is its **smooth part** $[\omega_{\mathrm{sm}}] \in H^p(M)$.

If $[\omega]$ is smooth (that is, $[\omega] \in H^n(M) \subseteq {}^{b^k}H^n(M)$), then so too is $[\omega\wedge\eta]$ smooth for all $[\eta] \in (H_c^{n-p}(M))^*$. In this case, it follows that $P_{[\omega\wedge\eta]}(0)$ equals $\int_M \omega \wedge \eta$ and that $[\omega] = [\omega_{\mathrm{sm}}]$. This remark shows that Equation 4.14 splits the short exact sequence from Equation 4.12, which yields a canonical isomorphism, the **Liouville-Laurent** isomorphism, that realizes the (abstract) isomorphism from Proposition IV.15.

$$
\begin{aligned}
(4.15) \qquad \varphi : {}^{b^k}H^n(M) &\cong H^n(M) \oplus \big(H^{n-1}(Z)\big)^k \\
[\omega] &\mapsto ([\omega_{\mathrm{sm}}], [\alpha_{-1}], \dots, [\alpha_{-k}])
\end{aligned}
$$

Definition IV.28. Let ω be a b^k-form of top degree. The **Liouville-Laurent** decomposition of $[\omega]$ is its image under Equation 4.15.

The following proposition shows that taking the Liouville-Laurent decomposition of a b^k-form commutes with taking its pullback under a b^k-map.

Proposition IV.29. Let $\varphi : (M, Z, j_Z) \to (M', Z', j_{Z'})$ be a b^k-map. If $[\omega'] \in {}^{b^k}H^p(M')$ has Liouville-Laurent decomposition $([\omega'_{\mathrm{sm}}], [\alpha'_{-1}], \dots, [\alpha'_{-k}])$, then $[\varphi^*(\omega')]$ has Liouville-Laurent decomposition

$$([\varphi^*(\omega'_{\mathrm{sm}})], [\varphi\big|_Z^{\,*}(\alpha'_{-1})], \dots, [\varphi\big|_Z^{\,*}(\alpha'_{-k})]).$$

Proof. Let $y' \in j_{Z'}$, and $i_Z : Z \to M$, $i_{Z'} : Z' \to M'$ be the inclusions. By the definition of a b^k-map, $y := \varphi^*(y')$ represents j_Z. Then for a Laurent series of ω',

$$\omega' = \sum_{i=1}^{k} \frac{dy'}{y'^i} \wedge \pi^* \alpha'_{-i} + \beta',$$

the pullback of ω' has Laurent series

$$\varphi^*(\omega') = \sum_{i=1}^{k} \frac{dy}{y^i} \wedge \varphi^*(\pi^*\alpha'_{-i}) + \varphi^*(\beta').$$

and we see that $[\varphi^*(\omega')]$ has Laurent decomposition

$$\begin{aligned}([i_Z^*(\varphi^*(\pi^*\alpha'_{-1}))], \ldots, &[i_Z^*(\varphi^*(\pi^*\alpha'_{-k}))]) \\ &= ([\varphi\big|_Z{}^*(i_{Z'}^*(\pi^*\alpha'_{-1}))], \ldots, [\varphi\big|_Z{}^*(i_{Z'}^*(\pi^*\alpha'_{-k}))])\end{aligned}$$

which proves that the Laurent decomposition commutes with pullback.

Let $[\eta] \in H_c^{n-p}(M)$. To prove that $[\varphi^*(\omega')_{\mathrm{sm}}] = [\varphi^*(\omega'_{\mathrm{sm}})]$, it suffices to show that

$$P_{[\varphi^*(\omega')\wedge\eta]}(0) = \int_M \varphi^*(\omega'_{\mathrm{sm}}) \wedge \eta. \tag{4.16}$$

Our strategy for proving Equation 4.16 will be to introduce an auxiliary family of smooth closed differential forms $\omega'_\epsilon \in \Omega^p(M')$ with the property that the Liouville volume of $\varphi^*(\omega') \wedge \eta$ can be calculated in terms of the asymptotic behavior of $\int_M \varphi^*(\omega'_\epsilon) \wedge \eta$ instead of $\int_{M \setminus U_{y,\epsilon}} \varphi^*(\omega') \wedge \eta$.

For $\epsilon > 0$ small, let $f_\epsilon : \mathbb{R} \to [0,1]$ be a smooth function such that

$$f_\epsilon\big|_{\mathbb{R}\setminus(-\epsilon,\epsilon)} = 1 \qquad \text{and} \qquad f_\epsilon\big|_{(-\epsilon+\exp(-\epsilon^{-1}),\epsilon-\exp(-\epsilon^{-1}))} = 0$$

and assume that f_ϵ varies smoothly with ϵ. Define

$$\omega'_\epsilon = \sum_{i=1}^{k} f_\epsilon(y') \frac{dy'}{y'} \wedge \pi^*\alpha'_{-i} + \beta'$$

and observe that ω'_ϵ is closed and that $\int_M \varphi^*(\omega'_\epsilon) \wedge \eta$ approaches

$$\mathrm{vol}_{y,\epsilon}(\varphi^*(\omega') \wedge \eta)$$

as $\epsilon \to 0$.

Next, recall that the pullback map in de Rham cohomology induces (by Poincaré duality) a pushforward map in compactly supported cohomology; we will use the notation $\varphi_*\eta$ for a representative of the pushforward of $[\eta] \in H_c^{n-p}(M)$. Using this notation,

$$\begin{aligned}
0 &= \lim_{\epsilon\to 0}\left(P_{[\varphi^*(\omega')\wedge\eta]}(\epsilon^{-1}) - \mathrm{vol}_{y,\epsilon}(\varphi^*(\omega')\wedge\eta)\right)\\
&= \lim_{\epsilon\to 0}\left(P_{[\varphi^*(\omega')\wedge\eta]}(\epsilon^{-1}) - \int_M \varphi^*(\omega'_\epsilon)\wedge\eta\right)\\
&= \lim_{\epsilon\to 0}\left(P_{[\varphi^*(\omega')\wedge\eta]}(\epsilon^{-1}) - \int_{M'} \omega'_\epsilon\wedge\varphi_*\eta\right)\\
&= \lim_{\epsilon\to 0}\left(P_{[\varphi^*(\omega')\wedge\eta]}(\epsilon^{-1}) - P_{[\omega'\wedge\varphi_*\eta]}(\epsilon^{-1})\right)
\end{aligned}$$

so

$$P_{[\varphi^*(\omega')\wedge\eta]}(0) = \int_{M'} \omega'_{\mathrm{sm}}\wedge\varphi_*\eta = \int_M \varphi^*(\omega'_{\mathrm{sm}})\wedge\eta$$

which proves Equation 4.16. □

4.4.2 b^k-orientation

The notion of orientability of a smooth manifold generalizes in an obvious way to the b^k-world.

Definition IV.30. A **volume b^k-form** on a b^k manifold is a nowhere vanishing b^k-form of top degree. A b^k-manifold is **b^k-orientable** if it admits a volume b^k-form. A **b^k-orientation** on a connected orientable b^k-manifold is a choice of one of the two connected components of the space of volume b^k-forms.

Although the underlying smooth manifold of every b^k-manifold is orientable (an orientation for M is included in the data of a b^k-manifold), not all b^k-manifolds are b^k-orientable. For example, if $Z \subseteq M$ is a meridian of the torus $S^1 \times S^1$ (so $M \backslash Z$ is connected), the corresponding b^1-manifold admits no volume b^1-form even though M is orientable. The opposite is also true: if you remove from the definition of a b^k-manifold the condition that M is oriented, then it remains possible to define the b^k-(co)tangent bundles, and according to these new definitions there would exist b^k-manifolds that admit a b^k-orientation even though the underlying manifold is unorientable. For example, if $Z \subseteq M$ is a meridian of the Klein bottle, there exists a volume b^1-form on the corresponding b^1-manifold even though M is not orientable. Although it is possible to study the b^k-geometry of non-orientable manifolds by modifying the definition of a b^k-manifold in this way, omitting the data of an orientation makes it impossible to define the Liouville volume of a b^k-form of top degree. It is for this reason that we have restricted our attention to b^k-structures on oriented manifolds in this paper.

Notice that the image under $\iota_{\mathbb{L}}$ of a volume b^k-form ω will be a smooth volume form on Z. In this way, a b^k-orientation on (M, Z, j_Z) induces an orientation on Z which may or may not agree with the orientation of Z given in the data of a b^k-manifold.

Definition IV.31. Let ω be a volume b^k-form on (M, Z, j_Z). If the smooth form $\iota_{\mathbb{L}}(\omega)$ is positively oriented, we say that ω is a **positively oriented** volume b^k-form.

Notice that if ω is a volume b^k-form which is *not* positively oriented, one can replace the b^k structure on (M, Z, j_Z) with a different b^k structure for which ω is a positively oriented volume b^k-form. To do so, reverse the orientations of those components Z' of Z for which $\iota_{\mathbb{L}}(\omega)\big|_{Z'}$ is negatively oriented, and replace the jet data for those Z' with their negatives.

4.5 Symplectic and Poisson Geometry of b^k-Forms

We begin this section by introducing the notion of a symplectic b^k-form and proving Moser's theorems in the b^k-category. We then classify symplectic b^k-surfaces, and show how the Liouville-Laurent decomposition of a b-symplectic form on a surface reconciles a classification theorem from [GMP13] with one from [Rad02].

Definition IV.32. A **symplectic b^k-form** on a b^k-manifold is a closed b^k 2-form having maximal rank at every $p \in M$.

Definition IV.33. A **symplectic b^k-manifold** (M, Z, j_Z, ω) is a b^k-manifold (M, Z, j_Z) with a symplectic b^k-form ω.

Definition IV.34. A **b^k-symplectomorphism** is a b^k-map

$$\varphi : (M, Z, j_Z, \omega) \to (M', Z', j_{Z'}, \omega')$$

satisfying $\varphi^*(\omega') = \omega$.

Theorem IV.35. *(relative Moser's theorem) If ω_0, ω_1 are symplectic b^k-forms on (M, Z, j_Z) with Z compact, $\omega_0\big|_Z = \omega_1\big|_Z$, and $[\omega_0] = [\omega_1]$, then there are neighborhoods U_0, U_1 of Z and a b^k-symplectomorphism $\varphi : (U_0, Z, j_Z, \omega_0) \to (U_1, Z, j_Z, \omega_1)$ such that $\varphi\big|_Z = \mathrm{id}$.*

Proof. Pick a local defining function $y \in j_Z$ and Laurent series of ω_0, ω_1

$$\omega_0 = \sum_{i=1}^{k} \frac{dy}{y^i} \wedge \alpha_{-i} + \beta \qquad\qquad \omega_1 = \sum_{i=1}^{k} \frac{dy}{y^i} \wedge \alpha'_{-i} + \beta'.$$

Then $i^*(\alpha'_{-i} - \alpha_{-i}) \in \Omega^1(Z)$ is exact for all i, and $i^*(\alpha'_{-k} - \alpha_{-k}) = i^*(\beta' - \beta) = 0$. By the relative Poincaré lemma there are primitives μ_i of $(\alpha'_{-i} - \alpha_{-i})$ and μ_β of $(\beta' - \beta)$ with $\mu_{-k}\big|_Z = \mu_\beta\big|_Z = 0$. Then $\omega_1 - \omega_0 = d\mu$, where

$$\mu = \sum_{i=1}^{k} \frac{dy}{y^i} \wedge \mu_{-i} + \mu_\beta.$$

Let $\omega_t = t\omega_1 + (1-t)\omega_0$, and observe that $d\omega_t/dt = d\mu$. By shrinking our neighborhood around Z, we can assume that ω_t has full rank for all t, giving a pairing between b^k-vector fields and b^k 1-forms. Because μ is a b^k 1-form vanishing on Z (since $\mu_{-k}\big|_Z = 0$ and $\mu_\beta\big|_Z = 0$), the vector field v_t defined by Moser's equation

$$\iota_{v_t}\omega_t = -\mu$$

is a b^k-vector field that vanishes on Z, the time-one flow of which is the desired b^k-symplectomorphism. □

Theorem IV.36. *(global Moser's theorem) Let (M, Z, j_Z) be a compact b^k-manifold, and $\omega_t := t\omega_1 + (1-t)\omega_0$ a symplectic b^k-form for $t \in [0,1]$, with $[\omega_0] = [\omega_1]$. Then there is an isotopy ρ_t of b^k-maps with $\rho_t^*(\omega_t) = \omega_0$ for $t \in [0,1]$.*

Proof. Because $\frac{d\omega_t}{dt} = \omega_1 - \omega_0$ is exact, there is a smooth b^k-form μ such that $d\mu = \omega_1 - \omega_0$. Because ω_t is a b^k-form, it defines an pairing between b^k 1-forms and b^k-vector fields. Therefore, the vector field v_t defined by Moser's equation

$$\iota_{v_t}\omega_t = -\mu$$

is a b^k-vector field, so its flow defines an isotopy ρ_t of b^k-maps with $\rho_t^*(\omega_t) = \omega_0$. □

4.5.1 Classification of symplectic b^k-surfaces

In [Rad02], the author classifies the space of stable Poisson structures on a connected, compact surface in terms of geometric data. In [GMP13], the authors demonstrate a correspondence between stable Poisson structures and b-symplectic forms on a manifold, and classify b-symplectic forms on a connected, compact surface in terms of their b-cohomology class. Pictorially, we have two sides of the triangle

$$
\begin{tikzcd}
\left\{\begin{matrix} \text{L. Vol} \in \mathbb{R} \\ \{\text{pd}(\gamma_i)\}_{i=1}^{r} \in \mathbb{R}^{r}_{>0} \end{matrix}\right\} & & {}^{b}H^2(M) \\
& \left\{\begin{matrix} \text{Symplectic} \\ b\text{-forms on } (M,Z) \end{matrix}\right\} \Big/ b\text{-symp.} \arrow[ul, leftrightarrow, "\text{[Rad02]}"] \arrow[ur, leftrightarrow, "\text{[GMP13]}"'] &
\end{tikzcd}
$$

where M is a connected, compact surface, $\{\gamma_i\}$ are the r oriented circles that constitute Z, L. Vol $\in \mathbb{R}$ is the Liouville volume of (M, Z, ω), and $\text{pd}(\gamma_i)$ is the period of the modular vector field on γ_i.

Theorem IV.37 completes the triangle. That is, it exhibits a direct connection between the cohomological classification data in [GMP13] and the geometric classification data in [Rad02].

Theorem IV.37. *Let $[\omega] = ([\omega_{sm}], [\alpha_{-1}])$ be the Liouville-Laurent decomposition of a positively oriented b-symplectic form on a connected compact surface. Let $\{\gamma_r\}$ be the oriented circles that constitute Z. Then the Liouville volume of ω is $\int_M \omega_{sm}$, and the period of the modular vector field on γ_r is*

$$\int_{\gamma_r} \alpha_{-1}$$

Proof. The fact that the Liouville volume of ω equals $\int_M \omega_{\text{sm}}$ follows from the definition of the smooth part of a b^k-form. Let γ_i be a connected component of Z. We can find a collar neighborhood

$$U = \{(y, \theta), |y| < R, \theta \in [0,1]/\sim\} \qquad R > 0$$

such that on U

$$\omega = c\frac{dy}{y} \wedge d\theta \qquad c > 0$$

where $d\theta$ is a positively-oriented volume form on Z. From [Rad02], we know that the period of the modular vector field is c, and we calculate that

$$\int_{\gamma_i} \alpha_{-1} = \int_{\gamma_i} c d\theta = c.$$

□

Theorem IV.38. *Let ω_0, ω_1 be symplectic b^k-forms on a compact connected b^k-surface (M, Z, j_Z). The following are equivalent*

1. *There is a b^k-symplectomorphism $\varphi : (M, Z, j_Z, \omega_0) \to (M, Z, j_Z, \omega_1)$.*
2. $[\omega_0] = [\omega_1]$
3. *The Liouville volumes of ω_0 and ω_1 agree, as do the numbers*

$$\int_{\gamma_r} \alpha_{-i}$$

for all connected components $\gamma_r \subseteq Z$ and all $1 \leq i \leq k$, where α_{-i} are the terms appearing in the Laurent decomposition of the two forms.

Proof.

(1) $\iff$ (2) This follows from the global Moser's Theorem (Theorem IV.36) in dimension 2.

(2) $\iff$ (3) The isomorphism (4.15) shows that the cohomology class of a volume b^k form is determined by its Liouville-Laurent decomposition, which in turn is determined by its Liouville volume and the integrals $\int_{\gamma_r} \alpha_{-i}$.

□

4.6 Symplectic and Poisson structures of b^k-type

When the authors of [GMP13] studied the Poisson structures dual to symplectic b-forms, they found that b-symplectomorphisms are precisely Poisson isomorphisms of the dual Poisson manifolds. Unfortunately, this observation does not generalize to the b^k case: although every symplectic b^k-form is dual to a Poisson bivector, not every Poisson isomorphism (with respect to this bivector) is realized by a b^k-map. Similarly, if (M, Z, j_Z, ω) and (M, Z, j'_Z, ω') are two symplectic b^k-manifolds, there may be a diffeomorphism of (M, Z) that restricts to a symplectomorphism $(M\backslash Z, \omega) \to (M\backslash Z, \omega')$ even if there is no b^k-symplectomorphism $(M, Z, j_Z, \omega) \to (M, Z, j'_Z, \omega')$. In this section, we show how to use b^k-manifolds to prove statements about objects outside of the b^k-category. We begin by defining the notion of a Poisson (and symplectic) structure of b^k-type – these are the Poisson (and symplectic) structures that are dual to (or equal to) a symplectic b^k-form for *some* choice of jet data. Then we apply the theory of symplectic b^k-forms to classify these structures on compact connected surfaces.

Definition IV.39. Let Z be an oriented hypersurface of an oriented manifold M. Let Π be a Poisson structure on M having full rank on $M\backslash Z$, and let $\omega \in \Omega^2(M\backslash Z)$ be the symplectic form dual to $\Pi\big|_{M\backslash Z}$. We say that Π and ω are **of b^k type** if there is some $j_Z \in J^{k-1}$ for which (M, Z, j_Z) is a b^k-manifold on which ω extends to a symplectic b^k-form.

Remark IV.40. Notice that if Π is a Poisson structure of b^k-type on (M^{2n}, Z) with dual form ω, then there will be several distinct jets with respect to which ω is a symplectic b^k-form. For example, if $j_Z = [y]$ is one such jet and $f : \mathbb{R} \to \mathbb{R}$ satisfies $f(0) = 0$ and $f'(0) > 0$, then the jet $j'_Z := [f \circ y]$ defines exactly the same b^k-

(co)tangent bundles as j_Z. As such, ω is a symplectic form with respect to both j'_Z and j_Z. However, one can check that the condition of ω^n being positively oriented (as a volume b^k-form in the sense of Definition IV.31) does *not* depend upon the chosen jet. Therefore, we say that Π (or ω) is a **positively oriented** Poisson structure (or symplectic form) of b^k type if ω extends to a positively oriented volume b^k-form for *any* choice of jet j_Z for which ω extends to a b^k form.

To study Poisson and symplectic structures of b^k-type using the tools of b^k-geometry, we must understand how a b^k-form behaves under diffeomorphisms of (M, Z) that are not necessarily b^k-maps. Of particular interest to us will be diffeomorphisms of M that restrict to $(z, y) \mapsto (z, P(y))$ in a collar neighborhood $Z \times \mathbb{R}$ of Z, where P is a polynomial. The following proposition describes how the Liouville-Laurent decomposition behaves under pullback of such a map (compare this proposition to Proposition IV.29, where we showed that the Liouville-Laurent decomposition commutes with the pullback of a b^k-map).

Proposition IV.41. Let P be a polynomial with $P(0) = 0$ and $P'(0) > 0$. Let (M, Z, j_Z) be a b^k-manifold with positively oriented local defining function $y \in j_Z$, and let $\varphi : M \to M$ be a diffeomorphism given by $\mathrm{id} \times P(y)$ in a collar neighborhood $(\pi, y) : U \to Z \times \mathbb{R}$ of Z. Then

- If ω is a b^k-form, then $\varphi^*(\omega)$ is also a b^k-form on (M, Z, j_Z).
- If $[\omega]$ has Liouville-Laurent decomposition $([\omega_{\mathrm{sm}}], [\alpha_{-1}], \ldots, [\alpha_{-k}])$ and $[\varphi^*(\omega)]$ has Laurent decomposition $([\omega'_{\mathrm{sm}}], [\alpha'_{-1}], \ldots, [\alpha'_{-k}])$, then $[\varphi^*(\omega_{\mathrm{sm}})] = [\omega'_{\mathrm{sm}}]$ and $[\alpha_{-1}] = [\alpha'_{-1}]$.

Proof. In a collar neighborhood, let

$$\omega = \sum_{i=1}^{k} \frac{dy}{y^i} \wedge \pi^*(\alpha_{-i}) + \beta$$

be a Laurent decomposition of ω. Then

$$\varphi^*(\omega) = \sum_{i=1}^{k} \frac{P'(y)dy}{P(y)^i} \wedge \pi^*(\alpha_{-i}) + \varphi^*\beta. \tag{4.17}$$

Notice that each term $\frac{P'(y)}{P(y)^i}$ must have a Laurent series with no exponents less than $-i$: indeed,

$$y^i \frac{P'(y)}{P(y)^i} = \left(\frac{y}{P(y)}\right)^i P'(y)$$

is smooth. By replacing each $\frac{P'(y)}{P(y)^i}$ in equation 4.17 with its Laurent series, this proves the first claim. To prove the second claim, first observe that for $i \neq 1$,

$$\frac{P'(y)dy}{P(y)^i} = d\left(\frac{1}{-i+1} P(y)^{-i+1}\right)$$

so the meromorphic function $P'(y)P(y)^{-i}$ has no residue. For $i = 1$ the function $P'(y)P(y)^{-1}$ has a Laurent series with principal part $1/y$. Therefore, by replacing the $P'(y)P(y)^{-i}$ terms in Equation 4.17 with their Laurent series in the variable y, we arrive at a Laurent series of $\varphi^*(\omega)$ that has $y^{-1}dy \wedge \pi^*(\alpha_{-1})$ as its residue term, proving that $[\alpha_{-1}] = [\alpha'_{-1}]$. To prove that $[\varphi^*(\omega_{\mathrm{sm}})] = [\omega'_{\mathrm{sm}}]$, let $[\eta] \in {}^{b^k}H_c^{n-p}(M)$, where p is the degree of ω and $n = \dim(M)$. It suffices to show that

$$P_{[\varphi^*(\omega) \wedge \eta]}(0) = \int_M \varphi^*(\omega_{\mathrm{sm}}) \wedge \eta. \tag{4.18}$$

Towards this goal, observe that for $\epsilon > 0$ small, $\varphi(U_{y,\epsilon}) = U_{y,P(\epsilon)}$, so $\mathrm{vol}_{y,\epsilon}(\varphi^*(\omega \wedge (\varphi^{-1})^*\eta)) = \mathrm{vol}_{y,P(\epsilon)}(\omega \wedge (\varphi^{-1})^*\eta))$. Then letting

$$\omega \wedge (\varphi^{-1})^*\eta = \sum_{i=1}^{k} \frac{dy}{y} \wedge \pi^*(\widetilde{\alpha}_{-i}) + \widetilde{\beta}$$

be a Laurent series of $\omega \wedge (\varphi^{-1})^* \eta$,

$$\begin{aligned}\mathrm{vol}_{y,\epsilon}(\varphi^*(\omega) \wedge \eta) - \mathrm{vol}_{y,\epsilon}(\omega \wedge (\varphi^{-1})^*\eta) \\ &= \left(\int_{M \setminus U_{y,P(\epsilon)}} - \int_{M \setminus U_{y,\epsilon}} \right) \omega \wedge (\varphi^{-1})^* \eta \\ &= \int_Z \left(\int_{P(\epsilon)}^{\epsilon} - \int_{P(-\epsilon)}^{-\epsilon} \right) \sum_{i=1}^{k} \frac{dy}{y^i} \pi^*(\widetilde{\alpha}_{-i}) \\ &\quad + \left(\int_{M \setminus U_{y,P(\epsilon)}} - \int_{M \setminus U_{y,\epsilon}} \right) \widetilde{\beta}\end{aligned}$$

As $\epsilon \to 0$, this limit approaches an odd function of ϵ, proving that $P_{[\varphi^*(\omega)\wedge\eta]}(0) = P_{[\omega\wedge(\varphi^{-1})^*\eta]}(0)$, from which Equation 4.18 follows. $\square$

Lemma IV.42. *Let $(a_{-1}, \ldots, a_{-k}) \in \mathbb{R}^k$ with $a_{-k} > 0$. There is a polynomial $P = \sum p_i y^i$ with $p_0 = 0$ and $p_1 > 0$ satisfying*

$$\sum_{i=1}^{k} a_{-i} \frac{P'(y)}{P(y)^i} = \frac{1}{y^k} + \frac{a_{-1}}{y} + Q(y)$$

where $Q(y)$ is a polynomial.

Proof. The proof is technical. See Section 4.7 for the details. $\square$

The two results above are the ingredients we need to prove the main theorem of this section.

Theorem IV.43. *Let Z be an oriented hypersurface of a compact oriented surface M. Let Π, Π' be two positively oriented Poisson structures of b^k-type on (M, Z), and ω, ω' be the dual b^k-symplectic forms (with respect to possibly different b^k-structures) with Liouville-Laurent decompositions*

$$[\omega] = ([\omega_{sm}], [\alpha_{-1}], \ldots, [\alpha_{-k}])$$

$$[\omega'] = ([\omega'_{sm}], [\alpha'_{-1}], \ldots, [\alpha'_{-k}]).$$

If $[\omega'_{sm}] = [\omega_{sm}] \in H^2(M)$ *and* $[\alpha'_{-1}] = [\alpha_{-1}] \in H^1(Z)$*, then there is a Poisson isomorphism* $\varphi : (M, \Pi) \to (M, \Pi')$*.*

Proof. Let j_Z and j'_Z be the jets of Z with respect to which ω and ω' respectively are b^k-forms with the described Liouville-Laurent decompositions, and let $y \in j_Z, y' \in j'_Z$ be positively oriented local defining functions for Z. Let $\{\gamma_\ell\}$ be the oriented circles that constitute the connected components of Z. If

$$\varphi : U_\ell \to \mathbb{R} \times \mathbb{S}^1 = \{(y, \theta)\}$$

$$\varphi' : U_\ell \to \mathbb{R} \times \mathbb{S}^1 = \{(y', \theta)\}$$

are local coordinate charts for a collar neighborhood U_ℓ of γ_ℓ, then the map $(\varphi')^{-1} \circ \varphi$ is an orientation-preserving map in a neighborhood of γ_i, restricts to the identity on γ_i, and pulls j'_Z back to j_Z. As such, the collection of these maps (one for each $\gamma_\ell \subseteq Z$) defines a smooth map in a neighborhood of Z that extends to a b^k-diffeomorphism $(M, Z, j_Z) \to (M, Z, j'_Z)$. By replacing ω' with its pullback under this b^k-diffeomorphism and citing Proposition IV.29, we may assume that ω, ω' are b^k-symplectic forms on the *same* b^k-manifold (M, Z, j_Z), and that the Liouville-Laurent decompositions of ω, ω' with respect to this b^k structure are as described in the theorem statement.

Let $\pi : U_\ell = \{(y, \theta_\ell)\} \to S^1$ be projetion onto the second coordinate. We may assume (by the global Moser's theorem) that

$$\omega\big|_{U_\ell} = \sum_{i=1}^{k} \frac{dy}{y^i} \wedge a_i \pi^*(d\theta_\ell) + \beta_0$$

where $a_i \in \mathbb{R}$ and $a_{-k} > 0$ (because Π, Π' are positively oriented). Then we may apply Lemma IV.42 to choose a polynomial $P_\ell = \sum p_i y^i$ with $p_0 = 0, p_1 > 0$ satisfying

$$\sum_{i=1}^{k} a_{-i} \frac{P'(y)}{P(y)^i} = \frac{1}{y^k} + \frac{a_{-1}}{y} + Q_\ell(y)$$

for some polynomial $Q_\ell(y)$. By replacing ω with its pullback under a diffeomorphism of (M, Z) that is of the form $(y, \theta_\ell) \mapsto (P_\ell(y), \theta_\ell)$ in each U_ℓ, we may assume $[\omega]$ has Liouville-Laurent decomposition

$$([\omega_{\mathrm{sm}}], [\alpha_{-1}], 0, \ldots, 0, [d\theta])$$

where $d\theta$ is the form on Z that restricts to $d\theta_i$ on each γ_i. Similarly, we may replace ω' with a form also having this Liouville-Laurent decomposition. Finally, we apply the global Moser's theorem (Theorem IV.36) and the fact that M is a surface to complete the proof. $\square$

4.7 Proof of Technical Results

4.7.1 Proof of Proposition IV.11

Proof. The case $k = 1$ was covered in [GMP13], so we may assume $k \geq 2$. Because $[f_1]^{k-1} = [f_2]^{k-1}$, we have $f_1 = f_2(1 + g f_2^{k-1})$ for a smooth function g. Note that $(1 + g f_2^{k-1})^{-1} = (1 + g' f_2^{k-1})$ for $g' = -g(1 + g f_2^{k-1})^{-1}$. Then

$$\begin{aligned}
\frac{df_1}{f_1^k} &= \frac{df_2}{f_2^k (1 + g f_2^{k-1})^{k-1}} + \frac{d(1 + g f_2^{k-1})}{f_2^{k-1}(1 + g f_2^{k-1})^k} \\
&= (1 + g' f_2^{k-1})^{k-1} \frac{df_2}{f_2^k} + \frac{(k-1) g df_2}{(1 + g f_2^{k-1})^k f_2} + \beta' \\
&= (1 + (k-1) g' f_2^{k-1}) \frac{df_2}{f_2^k} + \frac{(k-1) g df_2}{(1 + g f_2^{k-1})^k f_2} + \beta'' \\
&= \frac{df_2}{f_2^k} + (k-1) g (-(1 + g f_2^{k-1})^{-1} + (1 + g' f_2^{k-1})^k) \frac{df_2}{f_2} + \beta'' \\
&= \frac{df_2}{f_2^k} + (k-1) g (-(1 + g' f_2^{k-1}) + (1 + g' f_2^{k-1})^k) \frac{df_2}{f_2} + \beta'' \\
&= \frac{df_2}{f_2^k} + \beta
\end{aligned}$$

where

$$\beta' = (1 + g' f_2^{k-1})^k dg$$
$$\beta'' = \beta' + \sum_{i=2}^{k-1} \binom{k-1}{i} (g' f_2^{k-1})^i \frac{df_2}{f_2^k}$$
$$\beta = \beta'' + (k-1)g(-g' f_2^{k-1} + \sum_{i=1}^{k} \binom{k}{i} (g' f_2^{k-1})^i) \frac{df_2}{f_2}$$

□

4.7.2 Proof of subclaim of Theorem IV.25

We begin the proof with two technical lemmas.

Lemma IV.44. *If $f(x) : \mathbb{R} \to \mathbb{R}$ satisfies $[f]_0^{k-1} = [x]_0^{k-1}$, then its inverse $h : (-\epsilon, \epsilon) \to \mathbb{R}$ satisfies $[h]_0^{k-1} = [x]_0^{k-1}$.*

Proof. Because $[f]^{k-1} = [x]^{k-1}$, $f = (x + g(x)x^k)$ for some smooth g. Then

$$x = f(h(x)) = h(x) + g(h(x))h(x)^k.$$

Because $h(x)$ vanishes at 0 (since f does), $x - h(x)$ vanishes to order at least k, so $[h]^{k-1} = [x]^{k-1}$. □

Lemma IV.45. *Let $f : \mathbb{R} \to \mathbb{R}$ satisfy $[f]_0^{k-1} = [x]_0^{k-1}$. Then for all $i \leq k-1$*

$$\frac{1}{x^i} - \frac{1}{(-x)^i} + \frac{1}{f(-x)^i} - \frac{1}{f(x)^i} \tag{4.19}$$

is a smooth function that vanishes at 0.

Proof. Because $[f]^{k-1} = [x]^{k-1}$, $f(x) = x(1 + gx^{k-1})$ for some smooth g. Then

$$h(x) := \frac{1}{x^i} - \frac{1}{f(x)^i} = \frac{(1 + gx^{k-1})^i - 1}{x^i(1 + gx^{k-1})^i} = \frac{(\sum_{j=1}^{i} \binom{i}{j} g^j x^{j(k-1)-i})}{(1 + gx^{k-1})^i}$$

is a smooth function. Equation 4.19 equals $h(x) - h(-x)$, so it is a smooth odd function, hence vanishes at zero. □

Proof. (of subclaim of Theorem IV.25) Let U be a tubular neighborhood (y, π) : $U \to [-R, R] \times Z$, with $y \in j_Z$. Let h be another element of j_Z. It suffices to show that

$$\lim_{\epsilon \to 0} (\mathrm{vol}_{h,\epsilon}(\omega) - \mathrm{vol}_{y,\epsilon}(\omega)) = 0$$

for the case $M = U$. To do so, let $y_{h,z} : \mathbb{R} \to \mathbb{R}$ be the function, defined near zero, inverse to $h\big|_{[-R,R]\times\{z\}}$. That is, for sufficiently small ϵ, $h(y_{h,z}(\epsilon), z) = \epsilon$ and

$$U_{h,\epsilon} = \{(y, z) \in [-R, R] \times Z \mid y_{h,z}(-\epsilon) \leq y \leq y_{h,z}(\epsilon)\}$$

Then

$$\begin{aligned}
\mathrm{vol}_{h,\epsilon}(\omega) - \mathrm{vol}_{y,\epsilon}(\omega) &= \left(\int_{U\setminus U_{h,\epsilon}} - \int_{U\setminus U_{y,\epsilon}}\right)\omega \\
&= \int_Z \left(\int_{y_{h,z}(\epsilon)}^{R} + \int_{-R}^{y_{h,z}(-\epsilon)} - \int_{\epsilon}^{R} - \int_{-R}^{-\epsilon}\right) \sum_{i=1}^{k} \frac{dy}{y^i}\pi^*(\alpha_{-i}) \\
&\quad + \left(\int_{U\setminus U_{h,\epsilon}} - \int_{U\setminus U_{y,\epsilon}}\right)\beta \\
&= \int_Z \left(\log\left|\frac{y_{h,z}(-\epsilon)}{y_{h,z}(\epsilon)}\right| + \log\left|\frac{\epsilon}{-\epsilon}\right|\right)\pi^*(\alpha_{-1}) \\
&\quad + \sum_{i=2}^{k} \frac{1}{1-i}\int_Z \left(-(y_{h,z}(\epsilon))^{1-i} + y_{h,z}(-\epsilon)^{1-i} + \epsilon^{1-i} - (-\epsilon)^{1-i}\right)\pi^*(\alpha_{-i}) \\
&\quad + \int_{U_{y,\epsilon}} \beta - \int_{U_{u,\epsilon}} \beta
\end{aligned}$$

by the previous lemmas, the limit as $\epsilon \to 0$ of the above expression is zero, proving the claim. □

4.7.3 Proof of Lemma IV.42

Proof. Recall from the proof of Proposition IV.41 that for any polynomial P and $i \neq 1$, the expression

$$\frac{P'(y)}{P(y)^i}$$

has a Laurent series in y with trivial residue term and no exponents less than $-i$. When $i = 1$, the same expression has principal part y^{-1}. Therefore, for any polynomial P,

$$\sum_{i=1}^{k} a_{-i} \frac{P'(y)}{P(y)^i} = \sum_{i=2}^{k} \frac{b_{-i}}{y^i} + \frac{a_{-1}}{y} + Q(y) \tag{4.20}$$

for some $b_{-i} \in \mathbb{R}$ and some polynomial $Q(y)$. In particular, if $P(y) = (a_{-k})^{1/(1-k)}y$, then a straightforward calculation shows that $b_{-k} = 1$ in the expression above. However, we wish to find a polynomial P such that not only does $b_{-k} = 1$ in the expression above, but $(b_{-k}, b_{-k+1}, \ldots, b_2) = (1, 0, \ldots, 0)$. The remainder of the proof will be inductive: assume that we can pick $P = \sum p_i y^i$ so that $P(0) = 0$, $P'(0) > 0$, and $(b_{-k}, b_{-k+1}, \ldots, b_{-k+j-1}) = (1, 0, \ldots, 0)$ in Equation 4.20 – we aim to find a new P so that $P(0) = 0$, $P'(0) > 0$, $(b_{-k}, b_{-k+1}, \ldots, b_{-k+j}) = (1, 0, \ldots, 0)$. For $t \in \mathbb{R}$ let $\widetilde{P} = P + tP^{j+1}$, we have for some smooth function g,

$$\begin{aligned}
\sum_{i=1}^{k} a_{-i} \frac{\widetilde{P}'(y)}{\widetilde{P}^i} &= \sum_{i=1}^{k} a_{-i} \frac{P'}{P^i} \frac{(1 + (j+1)tP^j)}{(1 + tP^j)^i} \\
&= \sum_{i=1}^{k} a_{-i} \frac{P'}{P^i} \left(1 + (j + 1 - i)tp_1^j y^j + gy^{j+1}\right) \\
&= \frac{1}{y^k} + \sum_{i=2}^{k-j} \frac{b_{-i}}{y^i} + \frac{a_{-1}}{y} + Q(y) \\
&\qquad + \sum_{i=1}^{k} a_{-i} \frac{P'}{P^i} \left((j + 1 - i)tp_1^j y^j + gy^{j+1}\right)
\end{aligned}$$

Notice that the y^{-k+j} term of the above expression has coefficient

$$b_{-k+j} + a_{-k}p_1^{1-k}(j + 1 - k)tp_1^j = 0$$

if we set $t = -b_{-k+j}p_1^{k-j-1}/(a_{-k}(j+1-k))$, the y^{-k+j} term vanishes, completing the induction. $\square$

4.8 Further directions

4.8.1 b^k-symplectic toric manifolds

In Chapter III, we defined the notion of a b-symplectic toric manifold and a Delzant b-polytope and proved a generalization of Delzant's theorem in the b-context. It would be interesting to study whether similar techniques apply to study effective torus actions on b^k-manifolds. Here, we give a simple example of a moment map on a b^2 manifold.

Example IV.46. Consider the differential form $\omega = h^{-2} dh \wedge d\theta$ on $\mathbb{S}^2$, where h, θ are the standard coordinates on $\mathbb{S}^2$. Let $Z = \{h = 0\}$. The $\mathbb{S}^1$-action given by the flow of $-\frac{\partial}{\partial \theta}$ is generated on $\mathbb{S}^2 \backslash Z$ by the Hamiltonian function $-h^{-1}$.

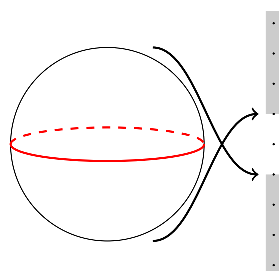

Figure 4.2: The map $-h^{-1}$ on $\mathbb{S}^2 \backslash Z$.

Like in the b-case, there are two main obstacles to the construction of a global moment map: we must enlarge the sheaf $C^\infty(\mathbb{S}^2)$ to include objects such as $-h^{-1}$, and we must enlarge the codomain so that the map is defined on Z. For this particular example,[3] we define the sheaf of b^2 **functions**

$$ {}^{b^2}C^\infty(\mathbb{S}^2) := \left\{ c_{-1}h^{-1} + c_0 \log|h| + f \;\middle|\; \begin{array}{c} c_{-1}, c_0 \in \mathbb{R} \\ f \in C^\infty(\mathbb{S}^2) \end{array} \right\} $$

and call the $\mathbb{S}^1$-action in our example **Hamiltonian** because it is generated by a b^2

[3]See [Sco13] for the general definition of this sheaf on a general b^k-manifold.

function. Next, we construct an appropriate codomain for our action by identifying[4] the points $(0, \infty)$ and $(1, -\infty)$ in $\{0, 1\} \times \overline{\mathbb{R}}$, and then discarding the points $(0, -\infty)$ and $(1, \infty)$. We endow this b^2**-line** with a smooth structure by declaring that the function

$$y_1 : ((0,0),(1,0)) \to \mathbb{R}, \quad y_1 = \begin{cases} -1/x & \text{for points } (0,x) \\ 0 & \text{at } (0,\infty) \\ -1/x & \text{for points } (1,x) \end{cases}$$

is a coordinate function. Then, we can represent the map $-h^{-1}$ as a smooth globally defined moment map $\mu : \mathbb{S}^2 \to {}^{b^2}\mathbb{R}$, which is drawn in Figure 4.3.

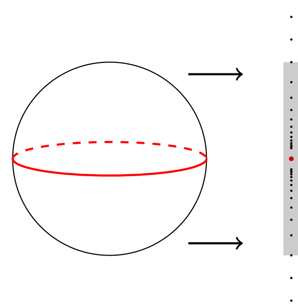

Figure 4.3: A moment map for an effective toric action on a b^2 manifold.

Although the moment map in Figure 4.3 is visually very similar to Figure 3.4, we remind the reader that the codomains of these two maps are very different, despite both being homeomorphic to $\mathbb{R}$. Also, in order to develop the theory of b^k symplectic toric manifolds in its full generality, one would need to assign weights (perhaps even $\mathbb{R}^k$-valued weights) to the components of the codomain "at infinity."

[4]Unlike the construction of the b-line, we glue $+\infty$ to $-\infty$. This reflects the fact that when k is odd (such as when $k = 1$) the singularities of b^k functions approach ∞ (or $-\infty$) on *both* sides of the singularity, but when k is even (such as this example) one side approaches ∞ and one side approaches $-\infty$.

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